February 4, 1980.

I want to test conjecture 3 of yesterday in the case $p = 2$. It says that to give a multiplicative operation

\[(\star) \quad \psi : H^*(X) \longrightarrow R \otimes H^*(X)\]

is the same as giving the elements $r_\nu \in R_{2^{\nu} - 1}^{\nu \geq 0}$ and that this sequence is determined from $\psi$ by

$$\psi x = \sum_{\nu \geq 0} r_\nu x^{2^\nu} \quad x \in H^1(X).$$

Using Kunneth we can prove the existence of a universal such operation as follows. Suppose $\psi$ is given as above. Then

$$\psi^0 : H^6(X) \longrightarrow [R \otimes H^*(X)]^6$$

is an additive operation. As $H^6(X) = [X, K_3]$, where $K_3 = K(\mathbb{Z}_2, 3)$, $\psi^0$ is the same as an element of degree $q$ of

$$P(R \otimes H^*(K_3)) = \text{Hom}_{\mathbb{Z}_2[\pi_0]}(\mathbb{Z}^6, R)$$

where $\psi^0$ is a place-keeping element of homological degree $-q$. The fact that the $\{\psi^0\}$ form a multiplicative operation means that

\[(\star \star) \quad \bigoplus_{q > 0} 2 H^*_c(K_3) \otimes R \longrightarrow R\]

is a ring homomorphism. So the conjecture is the following.
Hence, everything in this argument is defined in terms of the universal multiplicative $G$.

Put in different terms, this means that multiplication by $G$ on the universal ring $Z[\mathbb{Z}]$ is injective. This ring is a free $Z[\mathbb{Z}]$-module.

By virtue of the above, we will next consider $Z[\mathbb{Z}]$-modules $\mathbb{Z}$-algebras.

$\text{Proof:}$ First note that from the cohomology of EM spaces one knows that the suspension axiom induces a surjection

$$PH^*(K_0) \rightarrow PH^*(K_{n-1})$$

Let $\mathfrak{g} \in H_2(\mathbb{R}P^n)$ be a generator. Then

$$\mathfrak{g}^* : Z_2(\mathbb{R}P^n) \rightarrow H^2(\mathbb{R}P^n)$$

is an isomorphism.
\[ y(\beta x) = \sum_{\nu \geq 0} \xi_\nu (\beta x)^\nu \]
\[ y(\rho x) = \rho \left[ \sum_{\nu \geq 0} \xi_\nu (\beta x)^\nu + \xi_0 x \right]. \]

This needs some checking especially to understand what happens to the signs.

So we start with a multiplicative operation

\[ \varphi: H^*(X) \longrightarrow R \otimes H^*(X) \]

Again we get elements

\[ y_0 \in P \left[ R \otimes H^*(X) \right] \cong \text{Hom}_F \left( 2H_*(K_0) \otimes_\mathbb{Z} R \right) \]

to describe the additive operation in each degree. These maps work as follows: Given \( x \in \text{Hom}_F \left( H^0(X) \otimes_\mathbb{Z} R \right) \) and \( x \in H_0(X) = [x_0, 0] \), we consider the composition

\[ x \circ x \]

Then \( y(x) \in \left[ R \otimes H^*(X) \right] \) is the unique element such that

\[ (x \circ x)(y) = \varphi(y) \]

for all \( y \in H^*(X) \).

Again the \( y_0 \) should fit together to form a ring homomorphism. We can obviously construct the universal multiplicative operation

\[ H^*(X) \longrightarrow R_{\text{univ}} \otimes H^*(X) \]
For the prime 2 claim there is an operation

\[ H(X) \xrightarrow{\gamma} \mathbb{Z}_2[\xi_0, \ldots, \xi_{n-1}] \otimes H(X) \]

\[ \gamma x = \sum \xi_\nu x^{2^\nu} \quad \xi \in H' \]

Moreover, \( \gamma \) is a universal multiplicative operation.

**Proof of existence:** Start with

\[ p^{(n)} : H(X) \rightarrow \mathbb{Z}_2[\omega_{2^{r-2} \nu}, \ldots, \omega_{2^2 - 2}] \otimes H(X) \]

such that

\[ p^{(n)} x = \sum_{i=0}^{r-1} \omega_{2^{r-2} i} x^{2^i} \]

Setting \( \xi_\nu = \omega_{2^{r-2} \nu} \) for \( \nu = 0, \ldots, r-1 \) we get an operation

\[ p^{(n)} : H(X) \rightarrow \mathbb{Z}_2[\xi_0, \ldots, \xi_{r-1}] \otimes H(X) \]

\[ \gamma x = \sum_{\nu=0}^{r-1} \xi_\nu x^{2^\nu} + x^{2^r} \]

**Question:** Does

\[ H(X) \rightarrow \mathbb{Z}_2[\xi_0, \ldots, \xi_{n-1}] \otimes H(X) \]

commute for \( n >> \text{dim } X \)? If so then we get the operation we want. To prove commutes we can invert \( \xi_0 \) and
What is $Y$ in terms of the usual Steenrod operations?

$$H(X) \xrightarrow{\gamma} \mathbb{Z}_2[\gamma_i]_{i \geq 0} \otimes H(X)$$

Then

$$R_u = \sum_{\alpha} (\frac{\gamma}{\gamma_0})^\alpha S_{\alpha} u$$

$$R(x) = \sum_{\gamma \geq 0} \frac{\gamma}{\gamma_0} x^\gamma$$

$$F_u = \sum_{\alpha} \deg u \sum' (\frac{\gamma}{\gamma_0})^\alpha S_{\alpha} u$$

$$F_u = \sum_{\alpha} \deg u - l(\alpha) \frac{\gamma}{\gamma_0} S_{\alpha} u$$

Can you see why $\sum S_{\alpha} u = 0$ if $l(\alpha) > \deg u$?

By cobordism $u = f_\ast 1$ $f: \mathbb{Z} \to X$ deg $u$ codim $f = (\deg u)$

Theorem: $S_{\alpha} u = f_\ast c_\alpha(V_f)$ not much help.

Example: $\alpha = (i,0,0,\ldots)$ $S_{\alpha} = Sq^i$ and one knows that $Sq^i u = 0$ if $i > \deg u$!

Proof that $S_{\alpha} u = 0$ if $l(\alpha) > \deg u$. 
Take \( \alpha = (\alpha_1, \ldots, \alpha_n, 0, \ldots) \)

Then look at \( p^{(n)} u \)

\[ F u \mapsto \sum_{\alpha_1, \ldots, \alpha_n} \omega_{2^{n-1}}^{-\sum_{i=0}^{n-1} \alpha_i} \left( \omega_{n-2} \right)^{\alpha_1} \cdots \left( \omega_{n-2}^{-2n-1} \right)^{\alpha_n} S_{\beta} u . \]

\[ \deg S_{\beta} u = \deg u + \sum_{i=1}^{n} \alpha_i (2^{i-1}) \]

Take \( n \) so large that if \( \alpha_n > 0 \) then \( S_{\beta} u = 0 \) for dual. Reason \( 2^{n-1} > \dim X \). Then

\[ F u \mapsto \sum_{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}} \omega_{2^{n-1}}^{-\sum_{i=0}^{n-1} \alpha_i} \left( \omega_{n-2} \right)^{\alpha_1} \cdots \left( \omega_{n-2}^{-2n-1} \right)^{\alpha_n} S_{\beta} u \]

\[ \sum \alpha_i = \deg p^{(n)} u \]

all of these are distinct from which we conclude that that since \( p^{(n)} u \) has no denominators

\[ S_{\beta} u \neq 0 \implies \sum_{i=1}^{\infty} \alpha_i \leq \deg u \]

restrict from \( \mathbb{Z}_2^n \) to \( \mathbb{Z}_2^{n-1} \) and then \( \omega_{2^{n-1}} u \mapsto \omega_{2^{n-2} i-1} u \)

\[ \left( p^{(n-1)} u \right)^2 = \sum_{\alpha + \alpha_{n-1} = 0} \left( \omega_{2^{n-1}} \omega_{2^{n-2}} \right)^{\alpha_1} \cdots \left( \omega_{2^{n-2}^{-2n-1}} \right)^{\alpha_n} S_{\beta} u \]

\[ = \sum \omega_{2^{n-1}}^{-\sum_{i=1}^{n-1} \beta_i} \left( \omega_{2^{n-1}}^{-1} \right)^{2 \beta_1} \cdots \left( \omega_{2^{n-2}^{-2n-1}} \right)^{2 \beta_{n-1}} \left( S_{\beta} u \right)^2 \]

\[ S_{\beta} u \cdot \cdots \cdot S_{\beta} u \]

\[ \sum \beta_i = 0 \]
\[ \partial_{(\partial_0, \partial_1)} u = (S^2_0 u)^2 = u^2. \]

Check: \( n = 1 \)
\[
\partial u = \sum_{\alpha_0 + \alpha_1 = 8} S^\alpha_0 S^\alpha_1 u_{\alpha_0 \alpha_1}.
\]

The assertion is that no negative powers of \( x_0 \) occur because
\[
S^g_{u_i} u = 0 \quad \text{if} \quad i > g.
\]

And \( \partial u = S_0 u + \partial_1 u^2 \).

This tests my conjecture for \( p = 2 \) that \( \exists \) an operation
\[
\gamma: H(X) \rightarrow \mathbb{Z}_2 \left[ \mathbb{F}_v \right] \_\gamma \otimes H(X)
\]
and shows that
\[
\partial u = \sum_{\sum d_i = \deg u} S^\alpha_{\alpha_0} u = \sum \xi_{\alpha_0} \xi_{\alpha_1} S^\alpha_{\alpha_1} u.
\]

Next, I want to know that \( \gamma \) is a universal multiplicative operation. Thus I suppose given
\[
\gamma: H(X) \rightarrow R \otimes H(X)
\]
and let \( \gamma x = \sum \xi_{\alpha} x^2 \). So I want to show that
\[
H(X) \xrightarrow{\gamma} \mathbb{Z}_2 \left[ \mathbb{F}_v \right] \otimes H(X) \xrightarrow{\gamma} R \otimes H(X)
\]
commutes, or equivalently that
February 6, 1970.

Let $l$ be a prime number such that the Chow ring of $G/B$ is generated by its elements of degree 1 mod $l$. Then according to Borel it should be possible by spectral sequence arguments to show that $H^*(BG)$ is a polynomial ring and $H^*(G)$ is an exterior algebra, and the suspension homomorphism induces an isomorphism

$$q^* H(BG) \cong p^{q-1}H(G).$$

Assume all of this. Suppose $l \mid q - 1$ and that $G$ is defined over $F_q$ such that the torus $T$ of $B$ is split, i.e. isomorphic to $\mathbb{G}_m^k$ over $F_q$. Then $\mathbb{T} \cong \mu_{r_1}^* \oplus \cdots \oplus \mu_{r_m}^*$ is a finite group without Galois twisting, i.e. $F = \text{identity}$ on $\mathbb{T}$. I claim that this implies that the spectral sequence

\begin{equation}
E_{p^q}^2 = H^p(BG) \otimes H^q(T) \Rightarrow H^{p+q}(BG(k))
\end{equation}

is degenerate. Indeed this is the spectral sequence for $H^*_G(G/G(k))$. Now since $H^*(BT) \to H^*(G/T) = H^*(G/B)$ is surjective by assumption and $H^*(B_T) \to H^*(T/T)$ is surjective by direct calculation, we have descent for the map $B_T \to BG$. Thus pulling back the filtration. Thus the spectral sequence

\begin{equation}
E_{p^q}^2 = H^p(B_T) \otimes H^q(G) \Rightarrow H^{p+q}(G/G(k))
\end{equation}

is a direct sum of copies of $1)$ and so it suffices to prove that $2)$ degenerates. More precisely we have the following triangle already encountered in Spin.
\[ \mathbb{H}^*(G / G(k)) \leftarrow \mathbb{H}^*_{e^T}(G / G(k)) \leftarrow \mathbb{H}^*_{e^T}(pt) \]

Since \( \mathbb{H}^*_{e^T}(pt) \) goes to zero, one sees that one arrow is surj. off the other arrow is.

But now if we identify \( G / G(k) \) with \( G \) via the map \( gG(k) \mapsto g(Fg)^{-1} \), one sees that the left multiplication by \( e^T \) corresponds to \( (tg \mapsto tgF_gF^{-1} = t(gF_g)t^{-1}) \), the conjugation action of \( e^T \) on \( G \). Thus it suffices to show that

\[ \mathbb{H}^*_{e^T}(G) \twoheadrightarrow \mathbb{H}^*(G) \]

is surjective where \( e^T \) acts by conjugation. I claim that in fact

3) \[ \mathbb{H}^*_{e^T}(G) \twoheadrightarrow \mathbb{H}^*(G) \]

is surjective. To see this I use that \( \mathbb{H}^*(G) \) is generated as an algebra by the image of the suspension homomorphism. Geometrically, the suspension is defined by the principal bundle \( G \times G \) over \( G \Sigma \).

\[ \begin{array}{cccc}
    t g + (1-t) g_1 & G \times G & \longrightarrow & EG \\
    \downarrow & \downarrow & \downarrow \\
    t g g_1^{-1} & G \Sigma & \longrightarrow & BG \\
\end{array} \]

We note that \( G \) acts on the left of \( G \times G \) and this corresponds to the conjugation action on \( G \Sigma \). Thus the suspension
homomorphism factors

\[ H^*(BG) \longrightarrow H^*_G(G\wedge \Sigma) \longrightarrow H^*(G\wedge \Sigma) \]

proving that the image of 3) contains the image of the suspension and hence is surjective.

Conclusion: Let \( G \) be a reductive algebraic group defined over \( \mathbb{F}_q \). Let \( l \) be a prime number dividing \( q-1 \) which is good for \( G \), i.e., \( \text{Chow}(G/B) \) generated by its elements of dimension 1. Then the spectral sequence

\[ H^*(BG) \otimes H^*(G/G(F_q)) \Rightarrow H^*(B G(F_q)) \]

degenerates. \( \text{If } l \neq 2, \quad H^*(BG(F_q)) \) is isomorphic to the tensor product of \( H^*(BG) \) and \( H^*(G) \).

(\* and suppose \( G \) has a split torus defined over \( \mathbb{F}_q \))

The philosophy behind the above argument is that we can descend to a subgroup of \( G(k) \) and that then the twisted conjugation is equivalent to the conjugation action which one knows has a totally non-homologous zero fibres. When \( l \mid q-1 \) and there is a split torus \( T \) one can descend to \( T \subset G(k) \) (Chevalley groups have split tori!)
February 8, 1970:

Let $G$ be a connected algebraic group defined over $F_\ell = k$. Let $F$ be the Frobenius endomorphism of $G$ and let $G_F$ be $G$ as a variety regarded as a $G$-variety with action map:

$$G \times G_F \rightarrow G_F$$

$$g, x \mapsto g \cdot F(x)$$

Then there is a map of $G$-varieties:

$$G / G(k) \rightarrow G_F$$

$$g \cdot G(k) \mapsto g \cdot F(g)^{-1}$$

and this map is an isomorphism since $G$ is connected (thm. of Lang). Thus $G_F$ is a homogeneous space of $G$, and consequently:

$$H^*_G(G_F) \cong H^*(BG(k))$$

Now, let us compute $H^*(BG(k))$ by using the spectral sequence:

$$(*) \quad E_2^{p,q} = H^p(BG) \otimes H_q(G_F) \Rightarrow H^{p+q}_G(G_F)$$

The key result is the following:

Theorem: suppose that $H^*_G(G_F)$ is a $\ell$-adic object. $\ell$ is a good prime for $G$ in the sense of Borel. This means that
The spectral sequence for \((G, EG, BG)\) has the good form: \(H^*(G)\) has a simple system of transgressive generators \(e_i\), whence if \(c_i \in H^*(BG)\) represents \(t e_i\), then \(H^*(BG)\) is a polynomial ring with generators \(c_i\).

Then in \((*)\) the elements \(e_i\) are transgressive and \(t e_i\) is represented by \(c_i - F^*c_i\).

Comments:

1) As \(H^*(G)\) is a Hopf algebra, if \(k\) is odd, then all the \(e_i\) are of odd dimension and \(H^*(G)\) is an exterior algebra.

2) From Borel's theory one knows that \(PH(G) \xrightarrow{2} H^*(BG)\) has the transgression for its inverse. It follows that \(H^*(G)\) is generated by its primitives hence \(H^*(G)\) is commutative and killed by Frobenius.

3) If \(H^*(G)\) is an exterior algebra with odd degree generators then Eilenberg-Moore \(\xrightarrow{\text{Eilenberg-Moore}} H^*(BG)\) is a poly. ring.

Corollary: Under the conditions of the theorem suppose \(p\) is odd. Let \(P_0\) be the kernel of \(1 - F^*\) on \(PH^*(G)\) and let \(2_0\) be the kernel of \(1 - F^*\) on \(2 H^*(BG) = PH^*(G)\). Then

\[
H^*(BG(k)) \cong \bigoplus (P_0) \otimes S(2_0)
\]

When \(p = 2\), then \(gr H^*(BG(k)) \cong (P_0) \otimes S(2)\).

Proof: Consider the spectral sequence \((\ast)\). The cohomology
of the fibre $H^\ast(G^\ast) \cong H^\ast(G)$ is transgressively generated by $PH^\ast(G)$. Choose a splitting $P^\ast = P_0^\ast + P_1^\ast$ where $P_0 = \ker(d - F^\ast)$ and let $e_{i0}, e_{i1}$ be homogeneous bases for $P_0^\ast$ and $P_1^\ast$. Then we can choose representatives for the transgression $\tau$

$$\tau(e_{i0}) = 0$$

$$\tau(e_{i1})$$
forms part of a generating system for $H^\ast(BG)$.

Then I can map a perfect spectral sequence into $(\ast)$

$$\mathbb{Z}[e_{i0}] \otimes \mathbb{Z}[e_{i1}], \tau(e_{i1})] \otimes \mathbb{Z}[e_{j1}] \longrightarrow H^\ast(BG) \otimes H^\ast(G^\ast)$$

and I find that

$$E_\infty = \mathbb{Z}[e_{i0}] \otimes \mathbb{Z}[e_{j1}] \cong \Lambda P_0 \otimes S(2).$$

Thus if $l+2$, $E_\infty$ is a free graded anti-commutative algebra, hence $H^\ast(BG, \mathbb{R})$ must be isomorphic to $E_\infty$.

The theorem will follow from a more general assertion.

Theorem: Assume $H^\ast(G)$ has a simple system of transgressive generators for the spectral sequence of $(G, EG, BG)$. Consider $G$ as a homogeneous space for $G \times G$ by $(x, y) \cdot z = xyz^{-1}$. Thus $G$ is the homogeneous space for the diagonal subgroup $\Delta: G \rightarrow G \times G$. Hence the simple system of generators is also transgressive in the fibration $(G \times G/\Delta, BG, BG \times G)$ and the transgression $\tau_1$ in this last spectral sequence is represented by $\tau_1 = \tau_1 \otimes 1 + 1 \otimes \tau_1$. 


To prove the theorem on page 1, we note that $G$ acting on itself via the twisted conjugation is the pull-back of $G \times G$ acting on $G$ by left + right multiplication by means of the map $(id, F): G \rightarrow G \times G$. Thus there is a map of "fibrations" $(G, BG \times G, BG) \rightarrow (G, BG, B(G \times G))$ and as the $e_i$ transgress in the latter to $te_i \otimes 1 - 1 \otimes te_i$, they must transgress to $(te_i) \cdot F(\xi_i)$ in the former spectral sequence. q.e.d.

Proof of theorem on page 3: Consider the diagram

$$
\begin{array}{c}
\xymatrix{
H^0(G \times G) \ar[r]^-{\delta} & H^0(E(G \times G), G \times G) \\
H^0(G) \ar[u]^{\phi^*} \ar[r]^-{\delta} & H^0(BG, G) \ar[u]^{\phi^*} \ar[r]^-{\delta} & H^0(BG) \\
H^0(BG \times BG) \ar[u]^{=} \ar@{-->}[ru]_{\Delta^*}
}
\end{array}
$$

where $\phi: G \times G \rightarrow G$, $\phi(xy) = xy^{-1}$, and $\phi$ comes from the homotopy equivalence $E(G \times G) \times \mathbb{G} \sim BG$. (Note that the map $G \rightarrow BG$, coming from the inclusion of a fibre is homotopic to zero since $\phi: G \times G \rightarrow G$ admits a section.) The middle row is exact. Now given $e \in PH(G)$ we know that

$$
\phi^*(e) = e \otimes 1 - 1 \otimes e
$$

and that

$$
\delta \phi^*(e) = \phi^*(\delta e) \in \text{Im} \ \phi^* \mathbb{F}^*
$$

since $\phi^*(e)$ is transgressive in $(G \times G, E(G \times G), BG \times BG)$. 

Consider the element $e \otimes 1 - 1 \otimes e \in H^0(BG \times BG)$. Then
\[ \pi^*(e \otimes 1 - 1 \otimes e) = \delta u \]
for a unique $u$ in $H^{n-1}(G)$. Then as $\delta^{-1} \psi^* \pi^*$ is the suspension we have
\[ \delta \psi^*(u) = \psi^* \pi^*(e \otimes 1 - 1 \otimes e) \]
\[ = \delta (e \otimes 1 - 1 \otimes e) \]
and so $\psi^*(u) = e \otimes 1 - 1 \otimes e$. But $\psi^*$ is injective so $u = e$. Thus
\[ \delta e = \pi^*(e \otimes 1 - 1 \otimes e) \]
completing the proof of the theorem.
February 9, 1970

Question: Is $H^*_c(BG(k)) \to H^*_c(BT(k))$ injective?

A related question is whether

1) $H^*_c(G^c) \to H^*_c(T)$ \hspace{1cm} ($G^c = G$ with conj. action)

is injective. I conjecture that this latter is the map

2) $H^*(BG) \otimes H^*(G) \to H^*(BT) \otimes H^*(T)$

$c_i, c_j \mapsto c_i, d c_j$

which in many cases is an isomorphism with the $W$-invariants.

Note that 2) is not free, since on killing $H(BG)$ it is the map

$H^*(G) \to H^*(G/T) \otimes H^*(T)$

$G^c \leftarrow G/T \times T$

$g_t g^{-1} \leftarrow (gT, t)$

which isn't free for $G = SU(2)$. Thus injectivity of 2) is due to other reasons.

Start with the map

$f: G \times T \to G^c$

which can be rewritten $G/T \times W T \to G^c$. The map $f$ is an isomorphism on $G_{reg}$, hence $f_1 \cdot 1 = 1$ in cohomology.

I claim that in fact $f_1 \cdot 1 = 1$ in equivariant cohomology since $1 \in H^*_G(G \times T)$ has nowhere to go but into 1. Thus quite generally

3) $f^*: H^*_G(G^c) \to H^*_N(T)$
is injective. By Hochschild–Serre there is a spectral sequence

\[ H^*(N/T, H^*(T)) \Rightarrow H^*(N/T) \]

which degenerates \(\theta\) at primes not dividing the order of \(W\) giving an isomorphism

\[ H^*_N(T) = \{H^*(BT) \otimes H^*(T)\}^W \]

**Question:** Any relation between \(\varprojlim_k \text{GL}_n(F_{p^k})\) and \(\text{GL}_n(F_p)\). The conjecture is that

\[ \varprojlim_k B \text{GL}_n(F_{p^k}) \rightarrow (B \text{GL}_n(F_p))_{et} \]

is the "off-p" completion of the former. (Recall that if \(X\) is a space, it has a completion with respect to any "class" of groups. In this case we take all finite groups of order prime to \(p\). On the right we take the étale homotopy type of the classifying topos of \(\text{GL}_n(F_p)\).)

Note that the homotopy groups of the simplicial set on the left are finite. Here is an example to verify our guess:

Consider the completion à la Artin-Mazur of the group \(\hat{\mathbb{Z}}/2\mathbb{Z}\). What this means is that we must represent their functor \(X \rightarrow [K(\hat{\mathbb{Z}}/2\mathbb{Z}), X]\) as \(\varprojlim S^1 \rightarrow S^1\).
Recall that it is easy to compute the cohomology groups of a completion

\[ H^0(\hat{X}, \mathbb{Z}_{\ell^v}) = \text{Hom}[\hat{X}, K(\mathbb{Z}_{\ell^v}, 0)] = [X, K(\mathbb{Z}_{\ell^v}, 0)] = H^0(X, \mathbb{Z}_{\ell^v}) \]

Let's use the cohomology criterion! Then

\[ H^*(B \text{Gl}_n(\overline{\mathbb{F}}_p), \mathbb{Z}_{\ell^v}) = \hat{\mathbb{Z}}_{\ell^v} [c_1, \ldots, c_n] \quad \text{(take inverse of open seq over } \mathbb{Z}_{\ell^v}) \]

\[ H^*(B \text{Gl}_n(\overline{\mathbb{F}}_p), \mathbb{Z}_{\ell^v}) = \ldots \]

which shows that the conjecture is correct. (Note that \( B \text{Gl}_n(\overline{\mathbb{F}}_p) \) is simply-connected since \( \text{Gl}_n(\overline{\mathbb{F}}_p) \) has no finite quotient groups, since \( \text{SL}_n(\overline{\mathbb{F}}_p) \) would have to go to zero in such a thing by simplicity and since \( \overline{\mathbb{F}}_p^* \) has no finite quotients, being divisible.)
February 10, 1970

Proposal for a definition of algebraic K-theory:

Recall that if $G$ is a group, it has a "classifying space" simplicial set $\underline{W}(G)$. The functor $\underline{W}$ commutes with direct products.

Now given a ring $R$, let

$$X(R) = \bigoplus_{n > 0} \underline{W}(\text{Gl}_n(R)) \quad \text{Gl}_0(R) = \{1\}$$

Then $X(R)$ is a simplicial set which depends functorially on $R$ and commutes with direct products (in fact arbitrary inverse limits). Define a monoid structure on $X(R)$ by defining

$$X_p(R) \times X_q(R) \longrightarrow X_{p+q}(R)$$

to be the map induced by the natural inclusion

$$\text{Gl}_p \times \text{Gl}_q \longrightarrow \text{Gl}_{p+q}.$$ 

Note that this is associative since the diagram of groups

$$\begin{array}{ccc}
\text{Gl}_p \times \text{Gl}_q \times \text{Gl}_r & \longrightarrow & \text{Gl}_p \times \text{Gl}_{q+r} \\
\downarrow & & \downarrow \\
\text{Gl}_{p+q} \times \text{Gl}_r & \longrightarrow & \text{Gl}_{p+q+r}
\end{array}$$

is commutative.

Since $X(R)$ is a simplicial monoid, it has a "classifying
space $\overline{W}(X(R))$. We propose the following definition

$$K_i(R) = \prod_{i+1} \overline{W}(X(R)) \quad i \geq 0.$$  

(i=0 gets $\mathbb{Z}$)

This is reasonable because $\Omega \overline{W}(X(R))$ is the invertible H-space generated by the simplicial monoid $X(R)$.

The problem with this definition is that $K_0(R)$ seems to be $\mathbb{Z}$. Thus $\overline{W}(X(R)) = X(R) = \bigcup_{n \geq 0} e$. It seems that

$$[\overline{W}(X(R)), \overline{W}(\pi)] = [X(R), \pi] = \pi$$

for any group $\pi$. Thus as $\overline{W}(\pi)$ is a Kan complex one sees that

$$\pi_1 \overline{W}(X(R)) = \mathbb{Z}.$$  

This shows that we have failed to get the correct $K_0$ somehow we have failed to understand putting in direct summands of free modules. Might there be any way of obtaining the category of projective modules from the category of free modules and automorphisms? Unfortunately idempotents are not automorphisms.
Assume your calculations are correct so that

\[ M \text{ monoid} \]

\[ E_M \]

\[ B_M \]

semi-simp.

\[ \tilde{W}(M) \times M \]

\[ \tilde{W}(M) \]

\[ \tilde{W}(M) \times \pi \rightarrow G(\tilde{W}(M)) \]

\[ \tilde{W}(M) \]

\[ \tilde{W}(M) \rightarrow \tilde{W}(M) \]

if \( M \) gp \( \Rightarrow G(\tilde{W}(M)) \rightarrow M \).

\[ \tilde{w}(C) \]

\[ \tilde{W}(M) \times \pi \rightarrow \tilde{W}(M) \]

contractible

so is not clean

question: \( M \)

\[ M \rightarrow G(\tilde{W}(M)) \]

\[ \tilde{W}(M) \rightarrow E\tilde{W}(M) \]

fibration

\[ \tilde{W}(M) \rightarrow \tilde{W}(M) \]

and the question is whether this map from \( M \) to \( G(\tilde{W}(M)) \) is a homology isomorphism or if \( \cdots \)
February 12, 1970.

Conjecture: Let $R$ be a henselian local ring with finite residue field $k$ and let $l$ be a prime number different from $\text{char}(k)$. Then the map

$$\text{GL}_n(R) \rightarrow \text{GL}_n(k)$$

induces an isomorphism on group cohomology modulo $l$.

More generally, this should hold for any algebraic group defined over $R$.

Examples: 1) $n=1$. Then we have an exact sequence

$$0 \rightarrow 1 + m \rightarrow R^* \rightarrow k^* \rightarrow 0.$$

The equation $ax = 1$ is solvable because the equation $ax = 1$ has a root by Hensel's lemma, since $1+m$ is uniquely $l$-divisible it has no cohomology modulo $l$.

2) Suppose $G$ is a nilpotent algebraic group over $R$. Then $G(R)$ is a nilpotent group which is uniquely $l$-divisible. Here one sees $G(R)$ has no cohomology mod $l$ by means of a composition series for $G$ which reduces one to the abelian case.

The situation probably should be generalized: Let $G$ be a group scheme over $S$ and consider the simplicial object

$$\mathcal{W}(G): G \times G \rightarrow G \rightarrow S.$$
as a simplicial object in the category of sheaves for the etale topology on $\text{Sch}/S$. According to Deligne(?), there is a spectral sequence

$$E_2^{p0} = H^p(\nu \mapsto H^q(\mathcal{U}, G^{\geq 1})) \Rightarrow \text{something}$$

where $U$ is an object over $S$. I think the "something" is

$$H^*(BG_u)$$

which is the cohomology of the classifying topos of the group $G_u = U \times_S G$. If this is not correct, then suppose that the example suppose $U$ is an algebraically closed field. Then

$$H^*(U, G^{\geq 1}) = H^*(U \mathcal{O}^{\geq 1})$$

and $H^*(U, G)$ is an exterior algebra in good cases. This unfortunately shows that you are not getting what you want.

**Problem:** Is there any way to get a spectral sequence involving the cohomology of the group $G(S)$?

The corresponding topological problem in this: suppose $S$ is a fixed space and $G$ is a sheaf of groups over $S$. Then can one relate the cohomology of $G$ and the cohomology of $G(S)$?
February 16, 1970:

Take the spectrum of the family of symmetric groups:

\[ X = \prod_{n \geq 0} \text{Spec } H^*(B\Sigma_n) \text{ cohomology mod } p. \]

Since \( G \to \text{Spec } H(BG) \) is covariant and commutes with products, \( X \) is a monoid scheme over \( \text{Spec } \mathbb{Z}_p \) (provided we work with perfect schemes). For each integer \( a \geq 0 \) there is a map

\[ \text{Spec } \left\{ H^*(B\mathbb{Z}_p^a \text{Gl}_2(\mathbb{Z}_p)) \right\} \to X_p^a \]

SV

\[ A^a = \text{Spec } \mathbb{Z}_p[\mathbb{Z}_p, \mathbb{Z}_p, \ldots, \mathbb{Z}_p] \]

and the conjecture is that \( X \) is a free commutative monoid with these generators possibly with some relations. Example of a relation: The diagram of groups

\[
\begin{array}{c}
\mathbb{Z}_p^a \\
\downarrow \quad \downarrow \quad \downarrow \\
\Sigma_p^a \\
\Delta \quad \Delta \quad \Delta \\
\Sigma_p^a \\
\downarrow \quad \downarrow \quad \downarrow \\
\Sigma_p^{a+1} \\
\end{array}
\]

via wreath product

\[
\begin{array}{c}
\mathbb{Z}_p \\
\downarrow \quad \downarrow \quad \downarrow \\
\Sigma_p^a \times \Sigma_p^a \\
\end{array}
\]

\[ \text{gives rise to a diagram of rings} \]

\[ \mathbb{Z}_p[\mathbb{Z}_p, \mathbb{Z}_p, \ldots, \mathbb{Z}_p] \leftrightarrow H^*(B\Sigma_p^a) \leftrightarrow H(B\Sigma_p^a)^{\otimes p} \]
which shows that
\[
\begin{array}{ccc}
(x_{a-1}, \cdots, x_0) & \mathbb{A}^a & X_{p^a} \\
\downarrow & \downarrow & \downarrow \text{multiplication by } p \\
(x_{a+1}, \cdots, x_0) & \mathbb{A}^{a+1} & X_{p^{a+1}}
\end{array}
\]
commutes.

Perhaps it is better to use that there is a map
\[
\text{Spec } \left\{ H^* \mathbb{B}(\mathbb{Z}_p \Sigma_n) \right\} \longrightarrow \text{Spec } \left\{ H^* \mathbb{B}(\Sigma_n) \right\}
\]
\[
\mathbb{A}^1 \times \text{Spec } H^* \mathbb{B}(\Sigma_n) \rightarrow (\text{Spec } H^* \mathbb{B}(\Sigma_n))^p.
\]

Thus there is a basic map
\[
\mathbb{A}^1 \times X_n \longrightarrow X_{p^n}
\]
such that
\[
\begin{array}{ccc}
0 \times X_n & \longrightarrow & X_{p^n} \\
\uparrow \text{mull. by } p \text{ in } X
\end{array}
\]
commutes.

Moreover
\[
(\mathbb{A}^1)^{\otimes} \times X_1 \longrightarrow X_{p^a}
\]
divide by
\[
\text{Galois}(\mathbb{Z}_p)
\]
the map considered before.
Suppose you have numbers $\lambda_1, \ldots, \lambda_a$. Do you have any filling for Dickson's invariants

$$
\sum_{i=0}^{a} t^{P_i} c_{p^i-P_i} (\lambda_1, \ldots, \lambda_a) = \prod_{i \leq i < p} (t + \sum_{j \geq i} \lambda_j)
$$

which generate the functions $\Phi(\lambda_1, \ldots, \lambda_a)$ invariant under $GL_a(\mathbb{Z}_p)$. Thus if the operation

$$
\mathbb{A}^1 \times X \rightarrow X
$$

is denoted $\Gamma$ we know that

$$
\Gamma(\lambda_1, \ldots, \lambda_a) \cdot X
$$

is a $GL_a(\mathbb{Z}_p)$-invariant of $\lambda_1, \ldots, \lambda_a$.

Simpler example:

$$
\varprojlim_{n \rightarrow 0} \text{Spec } H^i(BGL_n) \cong \varprojlim_{n \rightarrow 0} \text{Sp}_n(\mathbb{G}_a)
$$

which is the free commutative monoid generated by $\mathbb{G}_a = \mathbb{A}^1$. 
February 17, 1970.

Question: Can you compute the cohomology or homotopy groups of \( \Omega B \{ \bigcup_n \text{BGL}_n(F_p) \} \)?

The idea suggested by conversation with Sullivan is to regard \( \text{BGL}_n(F_p) \) as the homotopy kernel of the pairs \( \text{BGL}_n \rightarrow \text{BGL}_n \) where \( \text{BGL}_n \) denotes the classifying topos of the group \( \text{GL}_n \) over \( \overline{F}_p \), and then form the "exact sequence" of monoids

\[
\bigcup_n \text{BGL}_n(F_p) \rightarrow \bigcup_n \text{BGL}_n \rightarrow \bigcup_n \text{BGL}_n
\]

which hopefully should be preserved by passage to classifying spaces. If everything works then we expect a filtration

\[
\Omega B \left( \bigcup_n \text{BGL}_n(F_p) \right)^{\mathbb{F}_p(1)} \rightarrow (\mathbb{Z} \times B \mathbb{U})^{\mathbb{F}_p(1)} \rightarrow (\mathbb{Z} \times B \mathbb{U})^{\mathbb{F}_p(1)}
\]

which on taking the long exact sequence of homotopy groups should yield the exact sequence

\[
K_1(F_p)^{\mathbb{F}_p(1)} \rightarrow 0 \rightarrow O
\]

\[
K_2(F_p)^{\mathbb{F}_p(1)} \rightarrow \mathbb{Z}^{\mathbb{F}_p(1)} \rightarrow \mathbb{Z}^{\mathbb{F}_p(1)}
\]

\[
K_3(F_p)^{\mathbb{F}_p(1)} \rightarrow 0 \rightarrow O
\]

from which we obtain the formulas.
\[ K_{2j} \left( \mathbb{F}_p \right)^j = 0 \quad j > 0 \]

\[ K_{2j-1} \left( \mathbb{F}_p \right)^j = \mathbb{Z}/q^j-1, \quad j > 0 \]

Given a space \( X \), I form
\[
\coprod_n E \Sigma_n x_{\Sigma_n} X_n
\]
which is a monoid. I conjecture that
\[
\Omega B \coprod_n (E \Sigma_n x_{\Sigma_n} X_n)
\]
is the free homotopy symmetric \( H \)-space generated by \( X \) and is naturally isomorphic to
\[
\lim_n \Omega^n \Sigma^n X_n = Q(X)
\]

One method of testing this conjecture would be to produce a map and then check that it induces an isomorphism in cohomology. But a much better method would be to produce the cohomology theory involved.

The idea might be to produce the functor represented by
\[
B \coprod_n (E \Sigma_n x_{\Sigma_n} X_n)
\]

First the functorial aspect. Given a space \( X \), I can form the monoid
\[
M(X) = \coprod_n E \Sigma_n x_{\Sigma_n} X_n
\]
and its classifying space $BM(X)$. Moreover $M$ is a triple (co?)

$$M(X) \xrightarrow{\cong} M(M(X))$$

$$X \rightarrow M(X)$$

where the first map comes from the wreath product maps

$$E \Sigma^i_j \times (E \Sigma^i_j \times \Sigma^k \leftarrow X^k)^j = E (\Sigma^i_j \times \Sigma^k) \times \Sigma^j \times \Sigma^k$$

For this it is necessary to check that

$$W(\Sigma^i_j S G) = W \Sigma^i_j \times (WG)^k$$

but in dimension $q$ we have

$$W^0_q (\Sigma^i_j S G) = (\Sigma^i_j \times G^k)^0 = \Sigma^i \times (G^k)^k = W^0_q (\Sigma^i_j) \times W^0_q (G^k)^k$$

so it's okay.

Now if $X$ is a homotopy symmetric $H$-space, we are given maps

$$M(X) \rightarrow X$$

which make $X$ is into any algebra over the triple i.e.

$$X \rightarrow M(X) \xrightarrow{\mu} X$$

$$M(M(X)) \xrightarrow{\cong} M(X)$$

Now the next step is to understand
inverting.

Suppose we admit Boardman's theorem that an invertible homotopy symmetric H-space extends uniquely to a connected generalized cohomology theory. 

Apply this to \( \Omega B M(X) \rightarrow J(X) \). I ought to be able to show that \( J(X) \rightarrow Q(X) \).

First of all \( Q(X) \) is a homotopy symmetric H-space so there is a map

\[
M(X) \rightarrow Q(X)
\]

and since \( Q(X) \) is invertible this extends to a map

\[
J(X) \rightarrow Q(X)
\]

On the other hand by Boardman one gets a connected gpc coh. theory with spectrum

\[
E_i = \Omega^{i+n} B^n M(X) \quad n \geq 1
\]

and there is a canonical map \( X \rightarrow E_0 \) whence a morphism of spectra

\[
\lim \Omega^{i+n} S^n X \rightarrow E_i
\]

defined by

\[
X \rightarrow E_0 = \Omega^n E_n \rightarrow S^n X \rightarrow E_n \rightarrow \Omega^{i+n} S^n X \rightarrow \Omega^{n+i} E_n = E_i
\]
which yields \( \mathcal{X} \) (take \( i = 0 \)) a map

\[ Q(X) \rightarrow J(X) \]

So there are maps in both directions, which is a good indication. To put this all on a good footing, all that is needed is an argument which will allow you to extend a functor with traces for coverings to a generalized theory.
February 17, 1970.

Some comments on your conjecture that $BG(R) \to BG(k)$ is a cohomology isomorphism mod $l$, where $R$ is a henselian d.v.r. with residue field $k$ of char $p \neq l$.

One technique would be to deduce it somehow from the specialization theorems for étale cohomology. Thus take $k = F_\ell$, whence we can realize $BG(k)$ as a fibre “space” over $BG_k$, or better we get our hands on $G(k)$ through its action on the scheme $G_\ell$.

The covering

$$1 \to G(k) \to G_k \to G_\ell \to 1$$

is a principal $G(k)$ bundle hence gives rise to a map

$$\pi_1(G_k) \to \pi_1(G(k))$$

which is surjective since $G$ is connected. Now if $G$ were proper over $\text{Spec } R$, then one would have an isomorphism

$$\pi_1(G) \cong \pi_1(G_k)$$

by the theorem on specialization of the fundamental groups. More precisely the categories of étale coverings of $G$ and $G_k$ are equivalent. Hence to the covering of $G_k$ by itself would correspond a covering of $G$.

$$G(k) \to \mathbb{Z} \to G$$

with group $G(k)$. Now what?
General problem. Let $G$ be a group in a topos $T$ and let $f : T_0 \to T$ be the classifying topos of $G$. Is there anything that one can say about the cohomology of the group $G(S)$ where $S$ is an object of $T$?

One has a natural map of groups in $T/S$

$$
\begin{array}{ccc}
G(S)_S & \longrightarrow & G_S \\
\downarrow & & \downarrow \\
S & \rightarrow & S
\end{array}
$$

where $G(S)_S$ is the group $S \times G(S)$ over $S$. Consequently one has a homomorphism of abelian sheaves in $T/S$

$$(\ast)
\xrightarrow[\text{just take discrete group cohomology of } M \text{ fibre over } \ast]{} R^0_f(M) \longrightarrow H^0_{G(S)}(M)
$$

for any abelian sheaf $M$ of $T_0$. If you take a point $\ast \in S$ then the stalk of the latter is just the cohomology of $G(S)$ acting on the abelian group $M_\ast$

$$
H^0_{G(S)}(M)_\ast = H^0_G(M_\ast).
$$

Example. Take $T$ to be the category of $\pi$-sets so that $G$ is a group on which $\pi$ acts and $T_0 = \text{sets}_{\pi \times G}$. Then if $M$ is a $\pi \times G$-module

$$
R^0_{\pi \times G}(M) = H^0_G(M)
$$

with its natural $\pi$-action. If $S = \pi / \pi$, then the map $\ast$ above is the restriction homomorphism.
Let $G$ be a connected algebraic group defined over $\mathbb{F}_\ell$. Then the covering $G/G(\mathbb{F}_\ell) \rightarrow G$ defines a surjection

$$\pi_1(G) \rightarrow G(\mathbb{F}_\ell)$$

If $G$ is abelian we have the following diagram

\[
\begin{array}{ccc}
G(\mathbb{F}_\ell^\infty) & \longrightarrow & G \\
\downarrow \text{norm} & & \downarrow \text{id} \\
G(\mathbb{F}_\ell) & \longrightarrow & G
\end{array}
\]

\[
\begin{array}{ccc}
\chi & \mapsto & \chi \cdot F^x \\
\sum_{i=0}^n & & \\
\chi & \mapsto & \chi \cdot F^x
\end{array}
\]

so that we end up with a homomorphism

$$\pi_1(G) \rightarrow \varprojlim_n G(\mathbb{F}_\ell^\infty)$$

(for the norm maps)

For example

$$\pi_1(G_m) \rightarrow \hat{T}_l \otimes \hat{T}_l(G_m)$$

$$\pi_1(G_a) \rightarrow \prod_{\infty} \mathbb{Z}/p\mathbb{Z}$$
In fact
\[ \text{Hom}(\pi_i \mathbb{G}_m, \mathbb{Z}/p\mathbb{Z}) \cong H^i(\mathbb{G}_m, \mathbb{Z}/p\mathbb{Z}) \cong \frac{k[T, T^{-1}]^*}{(k[T, T^{-1}])^*} \cong \mathbb{Z}/p\mathbb{Z} \]
because the units in \( k[T, T^{-1}] \) are \( \langle k^*, T^m \rangle \). Also by Artin-Schreier,
\[ \text{Hom}(\pi_i \mathbb{G}_a, \mathbb{Z}/p\mathbb{Z}) \cong \text{Coker} \left\{ k[T] \rightarrow k[T] \right\} \]

If \( G \) is non-abelian, there seems to be no good way to connect the various homomorphisms
\[ \pi_1(G) \rightarrow G(\mathbb{F}_q) \]
together. Thus for \( G = \text{PSL}_2 \) the groups \( G(\mathbb{F}_q) \) are simple and can't be mapped to each other.

New idea for non-abelian class field theory. According to the general yoga in Serre's book one obtains the abelian covering of a variety \( V \) by mapping \( f: V \rightarrow A \) where \( A \) is an abelian algebraic group (Albanese variety?) and pulling back an isogeny \( B \rightarrow A \). The next thing to do is to replace \( A \) by an homotopy symmetric animal. Serre's idea: Suppose \( K \) is a local field and \( K \rightarrow L \) is a finite extension. Then one gets an isogeny
\[ \text{Norm} : U_L \rightarrow U_K \]
of pro-algebraic groups whose kernel is canonically isomorphic to \( \text{Gal}(K/L) \) (needs checking).
From course notes on Iwasawa:

\[(\mathbb{Q}, \mathbb{Q}) \leq \infty\].

\[\mathcal{O} = \text{set of integers of } k.\]

Some \(S\) is a collection of non-archimedean prime divisors of \(k\).

For \(v \in S\), \(\mathcal{O}_v\) is the corresponding prime ideal of \(\mathcal{O}\).

\[\mathcal{O}' = \mathcal{O} \times \mathcal{O}_v, \quad \text{denominators of } (x) \text{ is a product of prime ideals of the form } \mathcal{O}_v, \quad \text{for } x \neq 0.\]

\[I = \text{ideal group of } \mathcal{O}.\]

\[I' = \text{ideal group of } \mathcal{O}' = \text{group of finitely generated non-zero } \mathcal{O}'-\text{submodules of } k.\]

**FACTS:**

1. \(\phi: I \rightarrow I'\) given by \(\phi(o) = o \mathcal{O}'\), \(o \in I\), is a surjective group homomorphism.

2. \(H = \ker(\phi) = \text{subgroup of } I\) generated by all \(\mathcal{O}_v, \forall v \in S\).

3. Let \(P\) be the subgroup of principal ideals of \(I\).

\[P' = \text{subgroup of } I'\) generated by the "principal \(S\)-ideals": \(a \mathcal{O}', \quad a \in k^\times.\]

\[C = I/P.\]

If maps \(P\) onto \(P'\) and induced surjection

\[C \rightarrow C' = I'/P'\]

with kernel \(HP/P\). So we have the exact sequence

\[
\begin{array}{c}
\rightarrow \quad HP/P \rightarrow C \rightarrow C' \rightarrow 0.
\end{array}
\]

In particular if \(\mathcal{O}_v, \forall v \in S\), is a principal ideal of \(\mathcal{O}\),

\[C \cong C'\]

which gives the answer to your query.
February 18, 1970:

Attempt to understand Boardman's theorem on the construction of \( \text{gen coh. theories for homotopy symmetric } H\text{-spaces} \).

I want to start with a functor on the category of \( C^\infty \)-manifolds to \( \text{Ab} \) which is endowed with \( \text{Lyndon homomorphisms for finite covering maps with the following properties} \):

(i) base change
(ii) composition
(iii) disjoint union

This to me is how a homotopy symmetric \( H\)-space should be defined.

Example of how you get such a \( k \): Let \( h^* \) be a gen. coh. theory, that is, a contravariant functor from the category of manifolds to the category of graded abelian groups which is endowed with \( \text{Lyndon homomorphisms for proper framed maps} \). Let \( k \) be \( h^0 \).

Boardman's theorem should say that the forgetful functor

\[
\begin{array}{ccc}
h^* & \longrightarrow & h^0 \\
(\text{gen coh. th}) & \longrightarrow & (\text{trace theories})
\end{array}
\]

is faithful has a left adjoint which allows one to identify trace
theories with the full subcategory of connected \((h^+(pt) = 0)\) gen. coh. theories.

(Or all the above add the word representable!)

Starting with the category of compact \(C^\infty\)-manifolds we form the category of motives; this should be the suspension category and its key property is that a functor to Ab. with \(Sp\)-homomorphisms for proper framed degree 0 maps on manifolds extends to an additive functor from the suspension category to Ab.
we are given a function \( h(x) = \left[ x, \otimes^g \right] \) on the category of finite complexes endowed with

(i) abelian group structure

(ii) traces for finite coverings. Axioms for coverings can be composed, unioned, and base changed. I want to show how to extend \( h \) to a generalized coh. theory.

First part of this notice being

If \( h = h^0 \) of a gen. coh. theory, then I can define \( f_* \) for any framed map of degree 0. \( f: \mathbb{Z} \to X \)

Conversely, if you have succeeded in doing this much then maybe you have a way of extending to a gen. coh. theory!

Methods

\[ h^+(x) = ?, \quad g > 0 \]

If I know \( h^{-1} \), can I extend? So I take \( X \). If \( X = SY \)

I know what to do so I look at

\[ [X, B(K)] = [\Omega X, K] \]

So therefore \( h^+(x) \) is something like \( \Phi h^0(\Omega X) \).

Similarly

\[ h^g(x) = [X, B^g(K)] = [\Omega^g X, K] \]

\[ = \Phi h^0(\Omega^g X) \]

\[ \text{hyper-primitive functor order } g. \]
now the feeling I have is that the commutative structure of $\Omega^\infty X$ is difficult to understand because of instability. Thus the thing to do is first show

\[ \pi^0(\Omega^\infty X) = \left[ \Omega(\infty), K \right] \text{ hom. sym. H-spaces} \]

\[ \pi^0(\Omega^\infty X) = \pi^0(\Omega^\infty X) \text{ to } h^0(?) \]

h^0(X) = \text{natural transf. from } \pi^0(\Omega^\infty X) \text{ to } h^0(?)

compatible with traces

[?, \Omega^\infty X] = \pi^0(\Omega^\infty X), \text{ do suppose you can prove that}

\[ h^0(X) = \text{natural transf. from } \pi^0(\Omega^\infty X) \text{ to } h^0(?) \]

compatible with traces

(The arrow ← is clear. For the arrow → one needs only to construct Gysin hom. for framed maps!)

Then I can define

\[ h^0(X) = \text{natural transf. from } \pi^0(\Omega^\infty X) \text{ to } h^0(?) \]

(\text{get signs correct})

\[ h^0(X) = h^0(S^{-\infty}X) = \text{Hom}_S(\pi^0(\Omega^\infty X), h^0(?) \text{ in } [?, S^{-\infty}X]) \]

\[ \pi^0(S^{-\infty}X) = \pi^0(S^{-\infty}X) = [S^{-\infty}, X] \]

Note that

\[ h^0(pt) = \text{natural transf. from } \pi^0(\Omega^\infty pt) \text{ to } h^0(?) \]

\[ \text{trace compatible} \]

is intuitively $h^0(S^{-\infty}) = h^0(pt) = \pi^0(E_8) = 0^\infty \geq 1$ since hopefully $E_8$ is a connected spectrum.
This aspect of the theory should be formal. Why a functor $X \mapsto h^0(X)$? Gysin for degree 0 extends.

**Lemma:** $h^0(X) = \text{Hom}_Y(\pi^0(\mathcal{G};X), h^0(\mathcal{G}))$ in functors on manifolds with Gysin for proper degree 0 framed maps.

Suppose you have an element $g \in h^0(X)$ and an $f$ element $x$ of $\pi^0(Y;X)$ i.e. a diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & X \\
\downarrow{f} & & \\
Y & & \\
\end{array}
\]

where $f$ is proper framed of degree 0. Then can define $x^*(g) = f_*g*x \in h^0(Y)$. By the axioms this should be natural.

We can suppose that $X$ is finite.

Point is that a functor with Gysin of degree is a functor on the suspension cat.

\[
\begin{cases}
\text{Obj} &= \text{f.c. } X \text{ with basepoint} \\
\text{Hom}_{\text{sup}}(X, Y) &= \{X, Y\} = \pi^0(X; Y)
\end{cases}
\]

so I have $\underline{\text{h}}^0 : \text{sup} \longrightarrow \text{Ab}$ $h^0(X) = \text{Hom}(h_X, h^0)$. 
February 18, 1970:

Sullivan claims that $BU^* \not\cong \Omega X$ isomorphic as an $h$-space to $\Omega X$, where $X$ is a space with finite $k$-invariants, and in fact

$$BU^* \not\cong \Omega X \quad \text{X has finite $k$-invariants}$$

$$\Downarrow$$

$$\exists \varphi : X \cong SU \quad \text{at each prime } p$$

$$\Downarrow$$

$$BU^* \cong \Omega X \cong \Omega SU \cong BU^* \quad \text{at each } p$$

but this is false, since $[X, BU^*] = (1 + R(X))^X$ is not isomorphic to $[X, BU^*] = R(X)$; take $X = B\mathbb{Z}_p$, former has an element of order $p$ and the latter is torsion-free, in fact it is free of rank $p-1$ over $\mathbb{Z}_p$.

Work with the prime 2 take $X = RP^4$ where

$$R(RP^4) = \mathbb{Z}_2 \oplus \mathbb{Z}_4(H-1)$$

where $H =$ Hopf bundle. Note that

$$(H-1)^2 = 1 - 2H + 1 = 2(1-H) \neq 0$$

since

$$c_2(2(1-H)) = (c_1(1-H))^2 = (-\omega)^2 = \omega^2 \neq 0$$

Now so we start with $\Omega X \cong BU^*$ and try to deduce a contradiction. Now the homotopy $h$-groups of $X$ must be
\[ \pi_8(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \]

Let \( X(0,5) \) be the part of the Postnikov system of \( X \) with the same homotopy groups \( 0 \leq g \leq 6 \), as \( X \).

\[
\begin{align*}
K(\mathbb{Z},5) & \\
\downarrow & \\
X(0,5) & \\
\downarrow & \\
K(\mathbb{Z},3) & \longrightarrow K(\mathbb{Z},6)
\end{align*}
\]

Now one knows that \( H^6(K(\mathbb{Z},3),\mathbb{Z}) = \mathbb{Z}/(\beta \mathbb{Z}) \) and hence the \( k \)-invariant of \( X \) must be different from zero, otherwise we have

\[ X(7,\infty) \longrightarrow X \cong K(\mathbb{Z},3) \times K(\mathbb{Z},5) \]

\[ \begin{align*}
H^8(K(\mathbb{Z},3) \times K(\mathbb{Z},5)) & \longrightarrow H^8(X) & \quad 8 \leq 6 \\
0 & \longrightarrow H^7(K(\mathbb{Z},3) \times K(\mathbb{Z},5)) & \longrightarrow H^7(X) & \longrightarrow H^7(X(7,\infty))
\end{align*} \]

and therefore we would have \( H^6(X) = \mathbb{Z}_2 \). Still no contradiction.
Sullivan claims that if \( BU^o \cong DX \) as \( h \)-spaces, then

\[
\begin{align*}
\pi_{2i+1}(X) &= \mathbb{Z} & i &\geq 1 \\
\pi_{2i}(X) &= 0 & i &\geq 1
\end{align*}
\]

and that

\[
H^{2i+2}(X(0, \ldots, 2i-1), \mathbb{Z})
\]

is cyclic \( i \)-generated by \( k_{2i+1} \)

\[
\begin{array}{ccc}
K(\mathbb{Z}, 2i+1) & \downarrow \\
X(0, \ldots, 2i+1) & \downarrow \\
K(\mathbb{Z}, 2i-1) & \rightarrow & X(0, \ldots, 2i-1) & \rightarrow & K(\mathbb{Z}, 2i+2) \\
& & \xrightarrow{k_{2i+1}}
\end{array}
\]

and that

\[
H^{2i+2}(K(\mathbb{Z}, 2i-1)) = \mathbb{Z} \cdot \beta S^2 \cdot \eta_{2i-1}
\]

is generated by the restriction of \( k_{2i+1} \)

---

He might get in trouble right at the bottom. Sullivan's argument consists in the following inductive set. He assumes constructed an isomorphism of \( X(0,2i-1) \) and \( SU(0,2i-1) \) at the prime 2. Then because \( SU \) has torsion-free homology one knows that the \( k \) invariant in

\[
H^{2i+2}(SU(0,2i-1), \mathbb{Z})
\]
The cyclic group and restricts to generate $H^{2i+2}(K(\mathbb{Z}, 2i-1), \mathbb{Z}) = \mathbb{Z}(3^i)$ (here must have $i \geq 2$).

The $k$ invariant for $X(0, 2i+1)$ also generates the fibre because of the isomorphism $X \simeq BU$, so by Nakayama this $k$ invariant must generate $H^{2i+2}(K(\mathbb{Z}, 2i-1), \mathbb{Z})$ (working at 2) (here must have $i \geq 3$ since the restriction of the $k$-invariant in $\Omega^2$ is in $H^{2i+4}(K(\mathbb{Z}, 2i-2), \mathbb{Z})$ which is $\mathbb{Z}_2$ only if $i \geq 3$.)

By means of a suitable 2-unit $\mathbb{Z} \rightarrow \mathbb{Z}$ we can carry one generator to another and so extend the isomorphism $SU(0, 2i+1) \simeq X(0, 2i+1)$.

Sullivan's argument would seem to imply that if $BU \simeq \Omega^2 X$ and $X$ has finite $k$-invariants, then

$$X(5, \infty) \simeq SU(5, \infty) \quad \text{(at 2)}$$

so that $BU \simeq BU(4) \times BSU^\circ$

as $H$-spaces, then perhaps

$$(BSU)^\circ \simeq (BSU)^+ \quad \text{(at 2)}.$$

Maybe this can be proved by Segal's method. This isn't correct because $SU(5, \infty)$ doesn't necessarily have torsion-free homology at 2.
February 19, 1970

To understand Brauer theory from the point of view of representation rings.

Start with a finite group $G$ of order $p^m$ where $(m, p) = 1$, and let $A$ be a d.v.r. with quotient field $K$ of char 0 and residue field $k$ of char. $p$. There is a basic homomorphism (called decomposition) or specialization

$$d: \mathbb{R}_k(G) \longrightarrow \mathbb{R}_k(G)$$

which is a $1$-ring homomorphism defined as follows. The maps

$$\text{Spec } k \xrightarrow{i} \text{Spec } A \xleftarrow{i^*} \text{Spec } K$$

should give rise to an exact sequence

$$\mathbb{R}_k(G) \xrightarrow{i_*} \mathbb{R}_A(G) \xrightarrow{i^*} \mathbb{R}_k(G) \longrightarrow 0$$

And $i_*1$ is the $A[G]$-module $k$ with trivial action and admits the resolution

$$0 \longrightarrow A \xrightarrow{\pi} A \longrightarrow k \longrightarrow 0$$

$\pi = \text{uniformizing parameter, hence}$

$$i^*i_1 = [A] - [A] = 0.$$
Thus the homomorphism \( d = i^* (j^*)^{-1} : R_k(G) \to R_k(G) \) is well-defined and is obviously a \( \lambda \)-ring homomorphism.

In more concrete terms, we take a \( K[G] \)-module \( V \) choose a lattice \( M \) inside \( V \) which is invariant and then take \( V \otimes_k M \). This is obviously \( [V] = i^* [M] \) and \( d [V] = i^* [M] \).

The next point is to show that \( \psi^g = \text{id on } R_k(G) \) for some power of \( p \).

So for simplicity I assume that \( k \) is finite and sufficiently large so that \( R_k(G) \to R_k(F) \). One knows that \( \psi^F = \text{action of Frobenius} \). But \( \psi^g = \text{id on } R_k(G) \) if \( k \), \( \text{for } g \). On the other hand one knows that if \( g = p^r \) and

\[
\begin{align*}
g &\equiv 1 \pmod{m} \\
g &\equiv 0 \pmod{p^r}
\end{align*}
\]

then \( \psi^g \) is an idempotent operation on \( R_k(G) \). Here you use the description of \( R_k(G) \) as functions on the group and the formula \( \psi^g(x)(g) = X(g^g) \). The image of \( \psi^g \) consists of those characters \( X \) such that

\[
X(g) = \prod \left( 1 + \sum_{d \mid g} \frac{X(g)}{X(g)} \right)^{d(g)} X(g)
\]

(since \( \psi^g(x)(g) = X(g^g) = X(g^{g_0}) = X(g) \), where \( g_0, g_0 \) are the regular and singular components of \( g \).)
Theorem: If $K$ has enough roots of unity, then

\[ \text{Im} \left\{ \psi_K : R_k(G) \to R_k(G) \right\} \cong R_k(G) \]

(*)

in other words a modular representation lifts to a unique virtual representation with character satisfying

\[ \chi(g) = \chi(g_0). \]

Proof: According to Serre's book p. 111-113, $d$ is surjective so we have

\[
\begin{array}{c}
R_k(G) \\
\downarrow \psi^g \\
R_k(G)
\end{array} \quad \overset{d}{\longrightarrow} \quad \begin{array}{c}
R_k(G) \\
\downarrow \psi^g = \text{id} \\
R_k(G)
\end{array}
\]

which proves that the map (*) is surjective. But now count the ranks of these free abelian groups. One knows that $R_k(G)$ has rank equal to the number of $p$-reg conjugacy classes and on the other hand $(\text{Im} \, \psi^g) \otimes K \to R(G) \otimes K \to \text{Class functions on } G$, with $(\text{Im} \, \psi^g) \otimes K$ corresponding to those class functions $\varphi \to \varphi(g) = \varphi(g_0^g)$. Thus rank $(\text{Im} \, \psi^g) \leq \text{no. of } p\text{-reg conjugacy classes}$. With surjectivity this means the map is an isomorphism.

In this way we construct a ring homomorphism

\[ s : R_k(G) \longrightarrow R_k(G). \]

which is a section of $d$. 
In fact this is a \(\mathbb{L}\)-ring homomorphism. To see this note that

\[
\psi^k \otimes (s \, dx) = \psi^k \psi^b x = \psi^b \psi^k x = s \, dx \psi^k x = s (\psi^k dx)
\]

so \(s\) commutes with \(\psi^k\). To check that \(s \, \lambda_k(x) = \lambda_k(sx)\) it's enough to embed \(R_k(G)\) into \(R_k(G) \otimes \mathbb{Q}\) whence we have the Newton formula:

\[
\lambda_k(x)^{-1} = \exp \sum_{m=1}^{\infty} \frac{1}{m} \lambda^m(x)
\]

expressing \(\lambda_k(x)^{-1}\) in terms of the \(\psi_k\).

Next I would like to understand the Brauer lifting of the standard representation of \(\text{GL}_n(\mathbb{F}_q)\). This perhaps should be entirely algebraic and contained in Serre's paper. Thus if \(A\) is a d.v.r. and if \(G_A\) is a group scheme over \(\text{Spec} \, A\) we should have a decomposition homomorphism

\[
R_A(G) \rightarrow R_k(G)
\]

defined as above. More precisely denote by \(C\) the affine coordinate ring of \(G\) so that \(C\) is a \(k\)-algebra

\[
C \rightarrow C \otimes C
\]

\[
C \rightarrow A
\]

over \(A\). I assume that \(C\) is flat (possibly free as an \(A\)-module if necessary.) Then \(G_k, \tilde{G}_k\) have coordinate rings \(C_k, \tilde{C}_k\), resp.
$R_k(G)$ and $R_k(G)$ are formed from the categories of $C_k, C_k$ comodules which are finite dimensional over $K, k$. For $R_A(G)$ we can use $C$-comodules which are either free f.g. over $A$ or just f.g. over $A$, getting two Grothendieck groups.

$$R_A(G)^* \longrightarrow R_A(G).$$

I work with the former. It is necessary to check the existence of a torus, so suppose $V$ is a $K$-module and a $C$-comodule.

$$V \xrightarrow{\Delta} C \otimes A$$

By Serre–Hébrat page 12, $V$ is a union of sub-comodules of finite type over $A$, i.e. invariant lattices. (Details: given $M$ an $A$-submodule of $V$, consider $M^0 = \{ v \mid \Delta v \in C \otimes M \} \subseteq M$). To show that $M^0$ is a sub-comodule, i.e. $\Delta(M^0) \subseteq C \otimes M^0$. But $M^0 = \text{ker} \{ V \longrightarrow C \otimes V \longrightarrow C \otimes (V/M) \}$ so

$$C \otimes M^0 \xrightarrow{\text{id} \otimes \text{inc.}} C \otimes V \xrightarrow{\text{id} \otimes (\text{id} \otimes \Delta)} C \otimes C \otimes V/M$$

so it's clear. If $N \subseteq V$ f.t. over $A$, then $\Delta(N) \subseteq C \otimes A$ for some f.t. $A$-module $M$, whence $M^0$ is a f.t. subcomodule of $V$ containing $N$.)
Thus lattices exist and so as in Serre's book one can show that the decomposition map

\[ d : R_k(G_K) \longrightarrow R_k(G_k) \]

is defined. Serre probably proves that \( d \) is surjective, which for \( G = \text{Gl}_n \) is no surprise although perhaps non-trivial.

Now I have the following situation:

\[ R_k(G(F_0)) \longrightarrow R_k(G(F)) \]

which doesn't seem to yield anything. **Conjecture:**

\[ R_k(G_K) \longrightarrow R_k(G_k) \]

is exact in the category of \( \Lambda \)-rings.

Let's go back to the case of \( \text{Gl}_n(F_3) \). The problem is to construct the virtual representation which lifts the standard representation. Its character \( \chi \) is characterized by the fact that \( \chi(g^2) = \chi(g) \) and \( \chi(g_0) = B_0 \) the Brauer character of the standard representation. Things should be easier stably. **Idea:** Start with standard representation of \( \text{Gl}_n(F_3) \), lift
it in the obvious way to $\text{Gl}_n(\mathbb{Z}[\sqrt{-1}])$ getting a representation $\psi^g$, and the hope is that this comes from $\text{Gl}_n(\mathbb{F}_p)$. The hope is therefore that if $E$ is the standard representation of $\text{Gl}_n(\mathbb{Z}[\sqrt{-1}])$, then

$$\psi^g[E] \in \text{Im} \{ R(\text{Gl}_n(\mathbb{F}_p)) \rightarrow R(\text{Gl}_n(\mathbb{Z}[\sqrt{-1}]))) \}.$$

This is a pretty concrete assertion which should be easy to settle. For $n=1$ we have $(\mathbb{Z}[\sqrt{-1}])^* \subseteq \mathbb{Q}^*$. We are asking that the $g$th power of this character dies on the units $\equiv 1 \text{ mod } p$. This is absurd when the group of units is infinite. NO good.
February 19, 1970

Brauer theory for modular "real" and "symplectic"
representations:

Let $G$ be a finite group. Denote by $R(G)$, $RO(G)$, and $RSp(G)$ the Grothendieck
groups of complex, real, and quaternionic representations of $G$, respectively.

Then there is an exact sequence

$$
\begin{align*}
\psi^{-1} - \text{id} & : R(G) \rightarrow RO(G) \oplus RSp(G) \rightarrow R(G) \\
& \rightarrow R(G)
\end{align*}
$$

$$
E \rightarrow (E, -H \otimes E) \\
E \oplus F \rightarrow (O_E)^{1+F}
$$

$$
\chi \mapsto \psi^{-1} \chi - \chi
$$

which I shall want to check is compatible with $\psi^3$ at least for $q$ odd.

The good way to think of $RO(G)$ (resp. $RSp(G)$) is as the Grothendieck group of orthogonal (resp. symplectic) representations of $G$. (An orthogonal representation over a field $k$ is a representation on a $k$-vector space $V$ endowed with a non-degenerate symmetric (resp. skew-symmetric) quadratic form). The reason is that if $E$ is a complex representation with a symmetric symmetric form $B$ and if $(,)$ is an invariant hermitian form, then we get an operator $A : E \rightarrow E$ which is conjugate linear defined by

$$
B(x,y) = (x, Ay).
$$
Moreover, $B$ symmetric $\Rightarrow (x, Ay) = (y, Ax) \Rightarrow (x, A^2x) = \|Ax\|^2 > 0$. Thus $A^2 = c \cdot \text{id}$ with $c > 0$, and so changing $\langle , \rangle$ by $e^{-\frac{1}{2}}$ we get that $B(x, y) = (x, Ay)$ and $A^2 = \text{id}$ so $A$ is a conjugation and $E' = \text{coker}(E^A)$. Similarly, if $B$ is anti-symmetric, then

$$B(x, y) = (x, Jy) \quad \text{J cong. linear}$$

implies

$$\langle x, Jy \rangle = - \langle y, Jx \rangle$$

$$\Rightarrow \langle x, J^2x \rangle = - \|Jx\|^2$$

on each of the eigenspaces of $J^2$. So by changing $\langle , \rangle$ by a scalar we can assume that $J^2 = -1$, whence $J$ defines a quaternionic vector space structure on $E$.

In these terms, the map

$$R(G) \longrightarrow RO(G)$$

is

$$E \longrightarrow E \oplus E^* \quad \text{with} \quad Q(x+y) = \langle x, y \rangle.$$ 

So if

$$A(x+y^*) = y + x^*$$

where $x^*$ is the linear functional given in terms of a hermitian product $\langle , \rangle$ on $E$ by

$$\langle z, x \rangle = \langle z, x^* \rangle$$

this means we have put

$$(x+y^*, x'+y'^*) = B(x+y^*, y'+x'^*).$$
\[ = Q(x+y^*+y^*+x^*) - Q(x+y^*) - Q(y^*+x^*) \]
\[ = \langle x+y^*, y^*+x^* \rangle - \langle x, y^* \rangle - \langle y^*, x^* \rangle \]
\[ = (x+y^*, y^*+x^*) - (x, y^*) - (y^*, x^*) \]
\[ = (x, x^*) + (y^*, y^*) \]

hence \[ ||x+y^*||^2 = ||x||^2 + ||y||^2 \] which is what we expect.

The map

\[ \mathbb{R}(G) \rightarrow \mathbb{R} \mathbb{S}(G) \]

is given by

\[ E \rightarrow E \oplus E^* \quad \text{with} \]
\[ B(x+x^*, x^*+\lambda) = \langle x, \lambda^* \rangle - \langle x^*, \lambda \rangle \]

Thus if

\[ J(x+y^*) = \overset{\text{?}}{y - x^*} \]

we have

\[ (x+y^*, x+y^*) = B(x+y^*, y^*+x^*) \]
\[ = +\langle x, x^* \rangle + \langle y^*, y^* \rangle \]
\[ = (x, x^*) + (y^*, y^*) \]

which agrees with the expected hermitian product.

The two maps
\[ \mathbb{R}(G) \rightarrow \mathbb{R}(G) \quad \text{and} \quad \mathbb{R}Sp(G) \rightarrow \mathbb{R}(G) \]

in these terms are the maps which forget the bilinear forms.

The exactness of the sequence can now be checked using the basis for \( \mathbb{R}(G) \) consisting of the irreducible representations.

Note that \( \mathbb{R}O(G) \rightarrow \mathbb{R}(G) \) and \( \mathbb{R}Sp(G) \rightarrow \mathbb{R}(G) \) are injective since on composing them with \( \varphi^! \), the natural maps the other way multiplies by 2. Consequently we have inclusions

\[
\text{Norm} \{ \mathbb{R}(G) \} \subset \mathbb{R}(G) \quad \subset \quad \mathbb{R}(G) \mathbb{Z}_2
\]

where \( \text{Norm} \) denotes the image of \( \varphi^{-1} + \text{id} \) from \( \mathbb{R}(G) \) to \( \mathbb{R}(G) \).

Now I claim that

\[
\begin{align*}
\mathbb{R}(G) \xrightarrow{\varphi^{-1} - \text{id}} &\quad \mathbb{R}(G) \xrightarrow{\varphi^{-1} + \text{id}} \quad \mathbb{R}(G)
\end{align*}
\]

is exact. This is because \( \mathbb{R}(G) \) is the free abelian group generated by the irreducible complex representations which are permuted by \( \varphi^{-1} \). Thus the exactness follows from

\[
H^4(\mathbb{Z}_2, \mathbb{Z}) = 0 \quad H^4(\mathbb{Z}_2, \text{incl} \mathbb{Z}_2 \rightarrow \mathbb{Z}_2) = 0
\]
Therefore the exactness of (\star) is equivalent to (\star\star) being bicartesian. But now we can compute. Suppose \( x \) is an invariant element of \( R(G) \); to show it's the sum of something in \( RO \) and \( RSp \). We have that \( x \) is a integral linear combination of terms of the form \([E]\), where \( E \) is irreducible and isomorphic to \( E^* \), and \([E] + [F^*] \) where \( F \) is irreducible and \( F \neq F^* \). The latter is already a norm so can be forgotten, so we can assume \( x = [E] \). Then \( (E \otimes E^*)^G = 0 \) by Schur's lemma so either \( \Lambda^2 E^* \) or \( S^2 E^* \) has no invariant; the resulting form is necessarily non-degenerate by irreducibility of \( E \). Thus \( x \) comes from either \( RO \) or \( RSp \). Note that no such \([E]\) comes from both \( RO \) and \( RSp \) so

\[
RO(G) \cap RSp(G) = \{ (g^{-1} + id) R(G) \}
\]

finishing the proof that (\star) is exact.

Remark: This argument generalizes to representations over sufficiently large fields of odd characteristic. To make things clearer you might put

\[
\hat{H}^0(Z_2, R(G)) = \mathbb{Z}_2 \text{-vector space with basis } [E] \oplus E \otimes E^*.
\]

which is the sum of the images of \( RO \) and \( RSp \). In order to see these are disjoint we need an argument allowing us to write elements of \( RO \) and \( RSp \) as sums of irreducibles whose behavior under \( RO \to R \), \( RSp \to R \) is clear. Thus I want to know that an irreducible \( \otimes \) \( RO(G) \) or \( RSp(G) \) element becomes in \( R(G) \) either a norm or an irreducible \([E] \).
February 20, 1970:

Let $k$ be a field of odd characteristic (eventually an arbitrary commutative ring). If $G$ is a group, then I can form the Grothendieck groups $RO_k(G)$ and $\text{RSp}_k(G)$ of orthogonal and symplectic representations of $G$ over $k$. (It should ultimately be possible to make this construction in an arbitrary ringed topos.) For the moment we think of $RO_k(G)$ as the free abelian group generated by the free monoid of positive elements, and a positive element is represented by an orthogonal representation $E$ under an equivalence relation to be made precise later. I want now to lay the framework for putting $\lambda$-ring structure on $RO_k(G) \oplus \text{RSp}_k(G)$.

If $E$ and $E'$ are refining with bilinear forms so is $E \otimes E'$ with the form

$$(B \otimes B)(x \otimes x', y \otimes y') = B(x, y)B(x', y').$$

If $B$ and $B'$ are skew-symmetric then $B \otimes B'$ is skew-symmetric etc. This should put a ring structure on $RO_k(G) \oplus \text{RSp}_k(G)$.

If $E$ is endowed with a form $B$, then $A B E$ is inherits the form

$$(A \otimes B)(x_1 \cdots x_q, y_1 \cdots y_r) = \det B(x_i, y_j).$$

If $B$ is skew-symmetric then changing $x_i$'s and $y_i$'s the matrix $B(x_i, y_j)$ changes to its negative transpose so the determinant changes by $(-1)^q$. Thus $A E B$ is symmetric for $q$ even and skew-symmetric for $q$ odd. This should put a $\lambda$-ring structure on
$RO_k(G) \oplus RSp_k(G)$ in such a way that the map into $R_k(G)$ is a $\Lambda$-ring homomorphism.

How to form $RO_k(G)$: Think of an orthogonal representation as a map $BG \to BO_k$. The analogue of an exact sequence should be a lifting to a parabolic subgroup of $O_k$.

Not quite. There are two basic ways of decomposing a quadratic space:

(i) If $E' \subset E$ is a subspace and $B|E'$ is non-degenerate, then $E \cong E' \oplus (E')^\perp$. (ii) If $E' \subset E$ is an isotropic subspace, then we want to have an equivalence

$$[E] \sim [(E' \oplus E/E')] + [(E')^\perp/E'].$$

Only case (ii) fits the parabolic subgroup description.

Definition: $E \mapsto [E] \in RO_k(G)$ is the universal function such that

(i) $E = E_1 \oplus E_2$ o.d.s. $\Rightarrow [E] = [E_1] \oplus [E_2]$  
(ii) $0 \to E' \to E \to E'' \to 0$ exact and $B(E',E') = 0$ $\Rightarrow$

$$[E] = [E \oplus E/(E')^\perp] + [(E')^\perp/E].$$

The above definition should work for a commutative ringed topos. It's necessary to check that we get a $\Lambda$-ring structure on $RO_k(G) \oplus RSp_k(G)$ and that over a field of odd characteristic one gets a free abelian group with irreducibles for basis.
The last point may be proved as follows. Divide up the simple \( k[G] \)-modules into three groups \( \bar{\mathbb{I}}, \mathbb{I}, \mathbb{I} \) according to whether they support a non-degenerate symmetric form, no form, or an anti-symmetric form, and then split up the \( \mathbb{O} \) group into a system of representatives. Let \( E \) be an orthogonal representation. Then by induction on the length of \( E \) as a \( k[G] \)-module one sees that

\[
[E] = \sum_{i \in I_0} n_i [E_i + E_i^*] + \sum_{i \in I_1} m_i [E_i] \quad \text{in} \quad RO_k(G)
\]

if the same formula holds in \( R_k(G) \). The inductive step consists of choosing a minimal submodule \( M \) of \( E \) whence either \( M \) is non-isotropic and \( [E] = \bigoplus [M] + [M^*] \) or \( M \) is isotropic and \( [E] = [M \oplus M^*] + [M^*/M] \). This shows that \( RO_k(G) \rightarrow R_k(G) \) is injective so \( R_k(G) \) is a free group with the desired basis.

**Conclusion:** If field \( k \) is not of characteristic 2 such that all \( k[G] \)-simple modules are absolutely irreducible, then

\[
R_k(G) \xrightarrow{\psi-\text{id}} R_k(G) \quad \rightarrow \quad RO_k(G) \oplus RSp_k(G) \quad \rightarrow \quad R_k(G) \xrightarrow{\psi-\text{id}} R_k(G)
\]

is exact.

Moreover \( RO_k(G) \rightarrow R_k(G), RSp_k(G) \rightarrow R_k(G) \) are injective with given basis.
Sullivan suggests constructing the basic representation of $\text{GL}_n(\mathbb{F}_q)$ furnished by the Brauer theory by inducing up the standard lifting of the modular representation of the normalizer $\Sigma_n \text{S}_n \mathbb{F}_q^\times$ on $\mathbb{C}^n$ furnished by the basic identification $\mathbb{F}_q^\times \cong \mu^\times_f \subset \mathbb{C}^\times$ used in the definition of the Brauer character. By Brauer theory the character $\lambda$ we are after is characterized by the conditions

$$\psi^\circ_\lambda(\chi) = \chi$$

$$\chi \left[ \begin{array}{c}
\lambda_1 \\
\vdots \\
\lambda_n
\end{array} \right] = \sum_{i=1}^n \lambda_i \chi_{i}, \quad \lambda_i \in \mu^\times_f \subset \mathbb{C}^\times$$

$\chi$ is the Brauer character of the modular repn. of $\text{GL}_n(\mathbb{F}_q)$. As the standard lifting of $\Sigma_n \text{S}_n \mathbb{F}_q^\times = N$ is fixed by $\psi^\circ_\lambda$, it follows that $\text{ind}_{N \to G}(S)$ will be invariant.

Let's compute the character $\text{ind}_{N \to G}(S)$. By the double coset formula, this will be a sum over the orbits of $T$ on $G/N$. A point of $G/N$ is the same as a family of $n$-independent lines $\{L_i\}$ in $\mathbb{F}_q^n$ and the stabilizer of this is of special form. Indeed if $(a_1, \ldots, a_n)$ is a generator of a line $L$, then the stabilizer is the subgroup of diagonal matrices $(d_1, \ldots, d_n)$ such that $d_i = d_j$ if $i \neq j$. Thus the stabilizers are all of the form $T_i$ where $\{1, \ldots, n\} = \bigcup I_j$ and $T_i = \{(d_i)\} \text{ if } i \in I_j$. The families of $n$-independent lines stabilized by the subgroup $T_i$ break up into families contained in the eigenspaces. Therefore we have a terrible sum.
\[
\text{res}^G_I \text{ ind}_N^G \sigma(S) = \sum_{I \text{ partition}} \sum_{T \in \{1, \cdots, n\}} \text{ ind}_{T \rightarrow T} f_x(S)
\]

where \( f_x(S) \) is defined as follows. \( x \cdot N \) is the same as a set of axes stabilized by \( T_I \), hence using \( x \) we get a map of these axes with the standard set and so can transform the representation to the new axes via \( x \). Where we get a new representation of \( T_I \). The important thing at the moment is to note that unless \( I \) is the finest partition then \( T_I \oplus A \cong T \) where \( A \cong \mathbb{Z}_8 \) and since

\[
\text{ind}_{T_I \rightarrow T} f_x(S) = f_x(S) \otimes \text{reg}(A)
\]

we have that all the Chern classes of this are of degree \( c^2 - c^1 \).
Various details of the Adams conjecture in the real case.

First stage is computation of $H^*(BO_n(\mathbb{F}_p), \mathbb{Z}_2)$, $p$ an odd prime. The claim is that

$$H^*(BO_n(\mathbb{F}_p), \mathbb{Z}_2) = \mathbb{Z}_2[\omega_1, \ldots, \omega_n]$$

where $\omega_j$ is the $j$th Stiefel-Whitney class, or more precisely the image of the universal Stiefel-Whitney class $w_j$ under the map $BO_n(\mathbb{F}_p) \to BO_n$.

By $O_n$ we mean the algebraic group, which is the subgroup of $GL_n$ leaving the quadratic form $\sum_{i=1}^n x_i^2$ invariant. Since we are off 2 we have the canonical isomorphism $\mathbb{Z}_2 \cong \mu_2$ and hence the corresponding 2-torus $\mathbb{Z}_2^n \to O_n$ of diagonal matrices.

A more precise claim is that

$$H^*(BO_n(\mathbb{F}_p), \mathbb{Z}_2) \cong H^*(B\mathbb{Z}_2^n, \mathbb{Z}_2) = \mathbb{Z}_2[\Psi_1, \ldots, \Psi_n]$$

where $\omega_j \mapsto j$th elementary symmetric function of the $\Psi_i$.

We break up the proof of this isomorphism in 2 parts, the first being to show that the restriction homomorphism to this 2-torus is injective.

By Chevalley, if $4 | 8-1$, then the normalizer of the torus in $O_n$ contains the following 2-subgroups of $O_n(\mathbb{F}_p)$. More precisely let $T_n$ be the maximal torus of $O_n$. It appears so:
We have to describe the 2-torus in $\mathbb{R}^2$ which is $SO_2$ and consists of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A = (A^t)^{-1}$, $\det A = 1$.

That is $SO_2 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 = 1 \right\}$.  

(Note that once a fourth root of $1$, $i$, becomes available then this torus becomes isomorphic to $\mathbb{C}^*$ via the formulas $a + ib = z$ and $a - ib = \frac{1}{z}$).

The normalizer of the torus $T_n$ is clearly $\left( \Sigma^m_{i=1} O(2) \right) \times \mathbb{Z}_2^n$, $n = 2m+\varepsilon$, $\varepsilon = 0, 1$.

and its group of points over $\mathbb{F}_q$ has order $\left| N'_n \right| = m! \left( 2(q-1) \right)^m \cdot 2^\varepsilon$.

I wish now to compute the order of $O_n(\mathbb{F}_q)$. Suppose $i$ is a ground so that we assume the quadratic form is hyperbolic $V = V + V^*$ (case $n = 2m$ with $Q(w+x) = \langle a_0, \lambda \rangle$. Then the number of choices for
I wish now to compute the order of \( O_n(FF_q) \). Suppose for simplicity that \( i \in FF_q \) so I can write \( x^2 + y^2 = (x+iy)(x-iy) \). This implies that a form in 2-variables \( ax^2 + by^2 \) with discriminant 1 in \( FF_q(FF_q)^2 \) is isotropic. One knows that any form with \( \geq 3 \) variables represents zero by \( \sum_{i=1}^{2m} x_i^2 \). Warning so my hypothesis implies that the quadratic function \( \sum_{i=1}^{2m} x_i^2 \) with discriminant 1 is hyperbolic. Without assuming \( i \in FF_q \) one has to be slightly more careful.

I begin by counting the number of isotropic flags in the case \( n = 2m \). These are flags \( 0 = W_1 \subset \cdots \subset W_m \) where \( W_m \) is totally isotropic. I write \( V = W + W^* \) \( Q(w + \lambda) = \langle w, \lambda \rangle \). Then the number of non-zero isotropic vectors in \( V \) is

\[
\text{card} \left\{ w + \lambda \mid \langle w, \lambda \rangle = 0 \right\} = (2^m - 2)^{m-1} + 2^{m-1} = (2^m - 1)^{m-1},
\]

where \( w \neq 0 \). Let \( w_1, w_2, \ldots \).

Thus
\[
\text{no. of choices for } W_1 = \frac{2^m - 1}{2^m - 1}. \]

Then \( W_2 \) is an isotropic line in \( W_1/W_1 \) which is hyperbolic since it has determinant 1. Thus we can induct and we find

\[
\text{no. of isotropic flags} = \frac{m}{\prod_{j=1}^{m} (2^j - 1) (2^j + 1)} = 2 \frac{2^m - 1}{2^m - 1} \frac{m^2 - 1}{2^m - 1}.
\]

If \( W_m \) is a maximal isotropic subspace, then it has an isotropic complement \( Z \). Choose a complement \( Z \); then \( Q(w + z) = B(w, z) + Q(z) \) where \( B: W \times Z \to FF_q \) is a non-degenerate pairing, so \( Q(z) = B(Tz, z) \) for a unique transp \( T: Z \to W \), whereas \( \{ -Tz + z \} \) will be an isotropic complement to \( W \).
Moreover the other isotropic complements are given by skew-symmetric maps $T: W \to \mathbb{F}_q$, hence are in number $\frac{m(m-1)}{2}$. Thus

$$|O_{2m}(\mathbb{F}_q)| = 2 \cdot \prod_{j=1}^{m-1} \frac{q^{n-1}}{q-1} \cdot \prod_{j=1}^{m-1} \frac{q^2 - 1}{q-1} \cdot \frac{m(m-1)}{2} \cdot \frac{m(m-1)}{2} \cdot q^n.$$ 

no of isot. flags. no of bases splitting the flag no of complements to an isotropic subspace

$$= 2 \cdot q^{\frac{m(m-1)}{2}} \cdot (q-1)^{m-1} \cdot \prod_{j=1}^{m-1} (q^2 - 1)$$

2 components of $O(2m)$ Euler class $p_1 \ldots p_{m-1}$

For $n = 2m + 1$ we have $V = W + W^* + \mathbb{F}_q$ and the number of choices for the first vector of an orthonormal frame is

$$\text{Card } \{ (w + \alpha + z) | \langle w, \lambda \rangle + z^2 = 1 \}.$$ 

$$= (q-2) \left\{ \left( \frac{q^m - 1}{q-1} \right) \frac{q^m - 1}{q-1} \right\} + 2 \left\{ \left( \frac{q^m - 1}{q-1} \right) \frac{q^m - 1}{q-1} + 1 \cdot q^m \right\}$$

$$= q^m - 1 \left[ q^m - 2q^{m-2} + 2 + 2q^m - 2 + 2q^m \right]$$

$$= q^m (q^m + 1).$$

Once this vector is chosen the rest is in the complement which has dimension $2m$, so

$$|O_{2m+1}(\mathbb{F}_q)| = 2 \cdot q^{m^2} \cdot \prod_{j=1}^{m} (q^{2j} - 1)$$

2 components $p_{12} \ldots p_m$.
so therefore
\[
\frac{|O_{2m}(F_8)|}{|N_{2m}(F_8)|} = \frac{2 \cdot 8^m}{2} \cdot \frac{5^{m-1}}{m(8-1)} \cdot \frac{8^{2j-1}}{j=1 2j(8^j-1)}
\]
\[
\frac{|O_{2m+1}(F_8)|}{|N_{2m+1}(F_8)|} = \frac{2 \cdot 8^m}{2^4} \cdot \frac{5^m}{j=1 2j(8^j-1)} \cdot \frac{8^{2j-1}}{j=1 2j(8^j-1)}
\]
and this is prime to 2 provided \(4 \mid 8-1\).

(The above formulas hold even if \(4 \nmid 8-1\) but \(2 \mid 8-1\). In effect your counting argument works for a hyperbolic form and you can arrange this by selecting the first vector if \(n=2m+1\) or the first two vectors if \(n=2m\) so as to render the remainder hyperbolic.)

Conclusion: if \(4 \mid 8-1\), then \(N_n(F_8)\) contains a dyellow 2-subgroup of \(O_n(F_8)\). (This result occurs in Chevalley's Tohoku paper.)

The next point is to consider the structure of the dyellow 2-subgroup of \(N_n(F_8)\).

Assertion: The mod 2 cohomology of \(N_n(F_8)\) is detected by the family of elementary abelian 2-subgroups.

We know that
\[
N_n(F_8) = (\Sigma_m S O_{2m}(F_8))^\times \times \mathbb{Z}_2^e \quad n=2m+e
\]
By our lemma it suffices to prove the result for \( O_2 = N_2 \).

Lemma: Suppose that \( H \) is a family of subgroups of \( G \) with detect mod 2 cohomology. Then one can construct a detecting family for \( \Sigma m \in G \) as follows. For each dyadic partition

\[
\prod_{i=0}^{n} \left( \Sigma_{2^i} S O_2(\mathbb{F}_2) \right)^{a_i}
\]

where \( m = \sum_{i=0}^{n} a_i 2^i \) is the dyadic expansion of \( m \), and where \( \Sigma_{2^i} \) is an \( i \)-iterated wreath product, so

\[
\Sigma_{2^i} S O_2(\mathbb{F}_2) = \underbrace{\mathbb{Z}_2 S (\mathbb{Z}_2 S \cdots (\mathbb{Z}_2 S O_2(\mathbb{F}_2)) S \cdots)\mathbb{Z}_2}_{\text{\( i \) times}}
\]

If I know that \( \Sigma_{2^i} S O_2(\mathbb{F}_2) \) is detected by elementary abelian 2-groups, then by the known facts on wreath products, one knows that the family

\( A_j \times A_j \) and \( \mathbb{Z}_2 \times A_j \)

detects for \( \Sigma_{2^i} S O_2(\mathbb{F}_2) \). By induction there will be elementary abelian 2-groups.

So for \( O_2(\mathbb{F}_2) = \mathbb{Z}_{2^{e-1}} \times \mathbb{Z}_2 \) a dihedral group it is necessary to check that its mod 2 cohomology is detected by elementary abelian 2-groups. Can restrict to the Sylow subgroups \( \mathbb{Z}_2 \times \mathbb{Z}_2 \).

Take generators \( x, y, z \) with relations...
\(x^2 = 1, \ y^2 = 1, \ yxy^{-1} = x^1\). To compute the cohomology mod 2, I use the Leray spectral sequence of the exact sequence

\[0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow 0\]

Now note that the cohomology of the fibre

\[H^*(BP_2^{\mu+1}) = \begin{cases} \mathbb{Z}_2 [u, y]/(u^2, y) & \text{if } \nu = 2 \\ \mathbb{Z}_2 [u, y]/(u^2) & \text{if } \nu > 2 \end{cases}\]

\(u \in H^1(BP_2^{\mu+1})\) is represented by the map \(Z_2 \rightarrow Z_2\) and transgresses to the quadratic function \(t_1^2 + t_1 t_2\) where \(t_i : Z_2 \times Z_2 \rightarrow Z_2\) are the projection maps viewed as elements of \(H^1(BP_2\times Z_2)\). Now \(y\) is an infinite cycle in the spectral sequence, since the natural representation of \(Z_2 \times Z_2\) on \(R^2\) has an Euler class which restricts to \(y\). Thus only \(c_2\) is non-zero in the spectral sequence and so

\[H^*(BP_2^{\mu+1} \times Z_2) = \mathbb{Z}_2 [t_1, t_2, e]/(t_1^2 + t_1 t_2)\]

where the \(t_i\) are the elements of degree 1 represented by the maps \((x, y) \rightarrow (1, 0)\) and \((0, 1)\) respectively and where \(e\) is the Euler class of the natural representation \(E\).

Consider the two elementary abelian subgroups \(\{x^{2m}, y\}\) and \(\{x^{2n}, xy\}\). Then for the first the restriction map

\[H^*(BP_2^{\mu+1} \times Z_2) \rightarrow H^*(BP_2^{\mu+1}) = \mathbb{Z}_2 [V_1, V_2]\]

\[t_1 \rightarrow 0\]

\[t_2 \rightarrow V_2\]

\[e \rightarrow e_{V_1 \cdot (V_1 + V_2)}\]
and for the second the restriction map sends

\[ t_1 \mapsto V_2 \]
\[ t_2 \mapsto V_2 \]
\[ e \mapsto v_1 \cdot (v_1 + v_2) \]

\( x^{2^{n-1}} \) acts as -1
\( xy \) acts as a reflection

and we see that each restriction map picks up each of the minimal primes. (The first kills \( t_1 \), the second \( t_1 + t_2 \).

Thus the assertion on page 5 is proved.

Theorem: If \( 4 | q-1 \), then

\[ H^*(BO_n(F_q)) \to H^*(BZ^2) \]

is injective on mod 2 cohomology.

Now the image is obviously contained in the image of the invariants under \( \Sigma n \) and so we have the inclusion

\[ H^*(BO_n(F_q)) \to \mathbb{Z}_2[w_1, w_n] \]

Actually, this map is an isomorphism since the universal etale Stiefel-Whitney classes give rise to elements in \( H^*(BO_n(F_q)) \) with the correct restrictions to the 2-torus.
Thus we have proved

Proposition: If $4 \mid q-1$, then

$$H^*(BO_n(F_4)) \xrightarrow{TT} H^*(BA)$$

is injective, where $A$ runs over representatives for the conjugacy classes of maximal elementary abelian 2-subgroups.

Given such an $A$, we can decompose $V = F_4^n$ into eigenspaces under $A$. If $v, w$ belong to different eigenspaces then $\exists \alpha \in A$ such that $\alpha v = v$, $\alpha w = -w$, or possibly this holds with $v, w$ interchanged; hence $B(v, w) = B(\alpha v, \alpha w) = -B(v, w)$ so $B(v, w) = 0$ as $q$ is odd. Thus the eigenspaces are perpendicular and so if $A$ is a maximal elementary 2-group, then the eigenspaces give a decomposition of $V = L_1 + \ldots + L_n$ as an orthogonal sum of 1-dimensional quadratic spaces. A 1-dimensional quadratic space is classified by the image of $Q: L \rightarrow F_4^*$, which is a coset of $(F_4^*)^2$. Thus to $A$ we can associate the number of $L_i$ which belong to the $0$-coset of $F_4^*/(F_4^*)^2 \cong \mathbb{Z}_2$ and this gives an invariant of the conjugacy class of $A$. It's clear that we get any number $j \in \{0, 1, \ldots, L_n \}$ from some $A$. In effect, the discriminant is the only invariant of a quadratic space over $F_4$, and hence the problem if $E(L_j) \in \mathbb{F}_4^*$ is this invariant we must have $j = \sum E(L_i) \equiv 0 \mod 2^i$ (2). But taking $n=2$, we know that the form $x^2 + y^2$ is equivalent to $xx^2 + xy^2$, hence we can get in a 2-dimeions space two orthogonal lines with non-zero
E-invariant. Thus

**Proposition:** There are \( \left[ \frac{n}{2} \right] + 1 \) conjugacy classes of maximal elementary abelian 2-subgroups of \( O_n(F_2) \), \( n \geq 4 \).

\( N(A_j) \)

The normalizer of the \( A_j \) corresponding to \( V=L_1+\ldots+L_n \)

where \( \epsilon(L_i) = 1 \), \( 1 \leq i \leq 2j \) and \( \epsilon(L_i) = 0 \) for \( i > 2j \)

is clearly \( A_j \) semi-direct product \( \Sigma_{2j} \times \Sigma_{n-2j} \). Thus

\[
H^*(BA_j) \cong H^*(BO_{2j}) \otimes H^*(BO_{n-2j})
\]

is a free \( H^*(BO_n) \) module of rank \( \binom{n}{2j} \).

Indeed \( [H^*(BO_n): H^*(BO_{2j})] = n! \) and \( [H^*(BO_n): H^*(BO_{2j} \times BO_{n-2j})] = (2j)! \binom{n}{2j} \). Thus we have a diagram

\[
\begin{array}{ccc}
H^*(BO_n) & \xrightarrow{\Theta} & H^*(BO_n(F_2)) \\
\downarrow & & \downarrow \\
\prod_{j=0}^{\left[ \frac{n}{2} \right]} & & H^*(BO_{2j}) \otimes H^*(BO_{n-2j})
\end{array}
\]

where the composite is the product of the natural maps \( H^*(BO_n) \rightarrow H^*(BO_{2j} \times BO_{n-2j}) \). I want to see how— for the second map is from being an isomorphism.

From the spectral sequence of the fibration

\[
O_n/O_n(F_2) \rightarrow BO_n(F_2) \rightarrow BO_n
\]

in étale homotopy theory, I should be able to show that \( H^*(BO_n(F_2)) \cong H^*(BO_n) \otimes H^*(SO_n) \) as \( H_*(BO_n) \)-modules,
hence $H^*(BO_n(F_2))$ should be of rank $2^{n-1}$ over $H^*(BO_n)$.

Note that

$$\sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-1}{2j-1} + \binom{n-1}{2j} = \sum_{i=0}^{n-1} \binom{n-1}{i} = (1+1)^{n-1} = 2^{n-1}.$$

Thus we see that the map $\mathbb{H}$ of $(\ast)$ on page 10 should be an isomorphism after localizing at the zero ideal of $H^*(BO_n)$. (Well actually this has to be true for spectral reasons, so that our feeling that the etale spectral sequence degenerates is correct. In fact this can be used to prove the spectral sequence degenerates, since the first non-zero differential would have to change the rank over $H^*(BO_n)$ after localizing at 0-ideals.)

**Conclusion:** $H^*(BO_n(F_2)) = \mathbb{Z}_2[\omega_1, \ldots, \omega_n, e_1, \ldots, e_{n-1}] / J$

where $J$ is some ideal of relations which after a suitable filtration become like $\epsilon_i^2 = 0$. This ought to imply that $J = (r_1, r_2, \ldots, r_{n-1})$ where degree $r_i = 2i$ and the sequence $E^2_{i\mathbf{j}}$ is regular.

If we form the quadratic extension $F_2 \subset F_2^2$, then all of the $A_j$ in $O_n(F_2)$ become conjugate to $A_0$ in $O_n(F_2^2)$. This is because the quadratic space $\mathbb{A}(F_2, \omega^2, \omega^2 \langle F_2^2 \rangle)$ becomes standard since $\omega^2$ exists in $F_2^2$. Therefore we obtain a commutative diagram.
where \( f_j : A_j(\mathbb{F}_q) \rightarrow A_0 \) is defined by conjugation with a matrix in \( O_n(\mathbb{F}_q) \) and where \( A_0 \) denotes the diagonal matrices.

This diagram shows that

\[
\text{Im} \left\{ H^\ast(BO_n(\mathbb{F}_q)) \rightarrow H^\ast(BO_n(\mathbb{F}_q)) \right\} \subset H^\ast(BA_0)^{N_0}
\]

and so we have proved

**Proposition:** \( H^\ast(BO_n(\mathbb{F}_p^{\infty})) \rightarrow H^\ast(BA_0)^{N_0} \cong \mathbb{Z}_2[\omega_1, \ldots, \omega_n] \)

for any odd prime \( p \).

Actually this map is an isomorphism, as we see from the étale Steenrod-Whitney classes. To produce these independently of étale cohomology I need a Brauer theorem for quadratic representations.

Analogues for the symplectic groups: \( \text{Sp}_{2n} = \text{Sp}(n) \) is the algebraic group of \( n \times n \) matrices preserving the form \( \sum_{i=1}^{n} x_i y_i \); although it's probably easier to work invariantly since the isomorphism problem is clear (i.e. no Witt groups).
\[ |\text{Sp}_{2n}(F_q)| = \prod_{j=1}^{n} \frac{q^{2j-1} - 1}{q - 1} \cdot (q-1)^n q^\left(\frac{n(n+1)}{2}\right) q^\left(\frac{n(n+1)}{2}\right) \]

iso. flags base per flag
iso. subspace + bases complements.

\[ = q^n \prod_{j=1}^{n} (q^{2j-1}) \]

\[ \text{eq}_{15}, \ldots, \text{eq}_n \quad \text{quaternionic Chern classes.} \]

For example, \( \text{Sp}_2 = \text{Sl}_2 \) has \( q(q^2-1) \) elements. Note that the Weyl group of \( \text{Sp}_{2n} \) is the same as that for \( O_{2n} \), namely \( \Sigma_n S \mathbb{Z}_2 \). Let \( N_{2n} \) be the normalizer of the torus. Then

\[ |N_{2n}(F_q)| = (q-1)^n \cdot 2^n \cdot n! \]

which again checks Chevalley's theorem.

Note that the normalizer \( N_2 \) has a different structure here than for \( O_2 \). Here

\[ N_2 = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{semi-direct product + amalg. with } \mathbb{Z}_4 \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

Thus

\[ N_2(F_q) \cong \mathbb{Z}_{q-1} \times \mathbb{Z}_4. \]

Restricting to the torus subgroup we get generalized quaternion gp.

\[ \mathbb{Z}_2 \times \mathbb{Z}_4 \quad \text{generators } x, y \quad x^{2n} = 1, \quad y^4 = 1, \quad y^2 = x^{2n} \]

\[ yxy^{-1} = x^{-1} \]

This is the subgroup of \( \mathbb{H} \) generated by \( x = \exp 2\pi i/2^n \) and \( y = j \).
Before computing the cohomology of this we do first that of $N_2 = S^1 \times Z_4$ by means of the spectral sequence for the extension

$$0 \rightarrow S^1 \rightarrow S^1 \times Z_4 \rightarrow Z_2 \rightarrow 0.$$

Note that $Z_2$ acts trivially on $H^*(BS^1) = Z_2[y]$, that $y^2 = 0$ comes from the Euler class of the representation of $S^1 \times Z_4$ on $H_1$, and that the group $Z_2$ acts freely on $S^3$ hence has periodic cohomology. This forces $T(y) = t^3$ where $t$ generates $H^1(BZ_2)$ as

$$H^*(B(S^1 \times Z_4)) = Z_2[t, e]/(t^3).$$

Now for the group $Z_2 \times Z_4$ we use the extension

$$0 \rightarrow Z_2^{n-1} \cdot x \rightarrow Z_2 \times Z_4 \rightarrow Z_2 \times Z_2 \rightarrow 0.$$

Now $H^*(BZ_2^{n-1}) = Z_2[u, y]/(u^2 = 2^n y)$ and by killed $x^4$ we get a map of $Z_2 \times Z_4$ to a familiar extension, showing that

$$T(u) = t_1^2 + t_1 t_2 + 2^{n-2} t_2^2$$

where $t_i \in H^1(BZ_2 \times Z_2)$ are characterized by

$$t_1(x) = 1, \quad t_1(y) = 0$$

$$t_2(x) = 0, \quad t_2(y) = 1$$

By use of the map $Z_2 \times Z_4 \rightarrow S^1 \times Z_4$ which takes $x$ to $t_2$, we see that $t_2^3 = 0$ in $H^3(B(Z_2 \times Z_4))$. As $t_2^3$ is relatively prime to $t_1^2 + t_1 t_2 + 2^{n-2} t_2^2$ we see that this is what $T(y)$ must be and also that the spectral sequence can be calculated to show that

$$H^*(B(Z_2 \times Z_4)) = Z_2[t_1, t_2, e]/(t_1^2 + t_1 t_2 + 2^{n-2} t_2^2, t_2^3).$$
I want to determine if there are proper subgroups of \( \mathbb{Z}_2^v \times \mathbb{Z}_4 \) which detect the non-zero element of \( H^3 \) which one computes is represented by \( t_i^2 \).

Suppose \( H \) is such a subgroup. If \( H = \mathbb{Z}_2^v \), as \( v \geq 2 \) the generator \( z \in H^1(\mathbb{Z}_2^v) \) has square zero; hence \( t_i \rightarrow \lambda_i z \), \( t_i^2 \lambda_1 \rightarrow 0 \). Thus \( H \not= \mathbb{Z}_2^v \) so \( H = \langle x \rangle \) with \( j \geq 1 \); as \( x \) \( x^a \) \( x^b \) \( y \), we may, by replacing \( H \) by a conjugate, assume that \( i = 0 \) or \( 1 \). In the former case \( t_i^2 H = 0 \), hence \( t_i^2 \lambda_1 \rightarrow 0 \). In the latter \( (t_i^2 + t_i^2) H = 0 \), hence \( t_i = 0 \) and \( t_i^3 = 0 \) have the same restriction to \( H \), so \( t_i^2 \lambda_1 \rightarrow 0 \).

Conclusion: If \( 4 \mid q - 1 \), then no proper subgroup of the Sylow of \( N_2(\mathbb{F}_q) = \mathbb{Z}_{q-1} \times \mathbb{Z}_4 \) detects cohomology in dimension 3.

Since \( N_2 = \sum S N_2 \), we find that there is a detecting family of subgroups for \( N_2(\mathbb{F}_q) \) which are of form \( (Q_{4+1})^i \times \mathbb{Z}_2^j \) where \( q - 1 = 2^k \) (odd) and \( Q_{4+1} \) is the generalized quaternion group \( \mathbb{Z}_2 \times \mathbb{Z}_4 \). So we conclude.

Proposition: If \( 4 \mid q - 1 \), then

\[
H^*(B\text{Span}(\mathbb{F}_q)) \overset{\cdot \leftarrow}{\longrightarrow} \prod_{K} H^*(BK)
\]

where \( K \) runs over representatives for the conjugacy classes of subgroups of \( \text{Span}(\mathbb{F}_q) \) of the form \( (Q_{4+1})^i \times \mathbb{Z}_2^j \).

Note that \( \mathbb{Z}_2^n \subset \text{Span} \) is the unique (up to conjugation) maximal elementary abelian 2-subgroup, where this inclusion...
comes from putting $\mathbb{Z}_2 \to \text{Sp}_2$ as $t \mathbf{1}$ and taking the direct sum $n$-times.

The proposition on the preceding page should be more precisely formulated as follows.

**Proposition:** Let $Q^{n}_{2^{n+1}} \hookrightarrow \text{Sp}_{2^n}(\mathbb{F})$ be the subgroup obtained by taking the direct sum $n$-times of the inclusion $Q^{n}_{2^{n+1}} \to N_2(\mathbb{F})$ considered above. Then

$$H^*(B\text{Sp}_{2^n}(\mathbb{F})) \to H^*(BQ^{n}_{2^{n+1}}).$$

**Proof:** By induction on $n$. If $n$ is not a power of 2 then we can write $n = i + j$ with $i, j < n$ such that $\text{Sp}_{2^i}(\mathbb{F}) \times \text{Sp}_{2^j}(\mathbb{F}) \to \text{Sp}_{2^n}(\mathbb{F})$ contains a Sylow 2-subgroup. If $n = 1$ it's OKAY. If $n = 2m$, then $\mathbb{Z}_2 \times \text{Sp}_{2^n}(\mathbb{F})$ also contains a Sylow subgroup. Then by basic wreath theory we know that $\mathbb{Z}_2 \times Q_{2^{n+1}}^{m} \to \text{Sp}_{2^n}(\mathbb{F})$ and $Q_{2^{n+1}}^{m} \times Q_{2^{n+1}}^{m} \to \text{Sp}_{2^n}(\mathbb{F})$ detect cohomology. I have to produce an element of $\text{Sp}_{2^n}(\mathbb{F})$ which conjugates $\mathbb{Z}_2 \times Q_{2^{n+1}}^{m}$ into $Q_{2^{n+1}}^{m}$. Now we can decompose $W \times W$ under $\mathbb{Z}_2$, since the char. is odd, say

$$W \times W = \Delta W \oplus \{\omega_3 - \omega_2\}$$

and using the isomorphism

$$(\omega_1, \omega_2) \mapsto \frac{(\omega_1 + \omega_2, -\omega_1 + \omega_2)}{2}$$
which is symplectic, we can conjugate and suppose that \( Z_2 \times H \) acts on \( W \times W \) where \( Z_2 \) acts non-trivially on the first factor and trivially by the second. Now here \( H \) has an element \( \varepsilon \) in its center acting \( \varepsilon^{-1} \) on \( W \), so we see that the representation is the restriction of that of \( H \times H \) on \( W \times W \) by means of the homomorphism \( \Theta: Z_2 \times H \to H \times H \) such that \( \Theta(-1, h) = (\varepsilon h, h) \). This gives the desired conjugation of \( Z_2 \times Q_{2^{m+1}} \) into \( Q_{2^{2m+1}} \) and proves the proposition.

Just to be on the safe side we check the conjugation over \( \mathbb{C} \). Thus we are given the subgroup \( \{ C^* : C^2 \in \text{SL}_2(\mathbb{C}) \} \)

\[
Q = \{ \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix}, \begin{pmatrix} 0 & -z^{-1} \\ z & 0 \end{pmatrix} \mid z \in \mathbb{C}^* \}
\]

and we take the matrix

\[
\begin{pmatrix}
1 & -\frac{1}{2}i \\
\frac{1}{2}i & 1
\end{pmatrix}
\]

which is symplectic since \( \begin{pmatrix} 1 & -\frac{1}{2}i \\ 1 & \frac{1}{2}i \end{pmatrix} \) has determinant \(+1\) and the above matrix is the Kronecker product. Then conjugation

\[
\begin{pmatrix}
\frac{1}{2}i & +\frac{1}{2}i \\
-\frac{1}{2}i & \frac{1}{2}i
\end{pmatrix}
\begin{pmatrix}
0 & -z^{-1} \\
z & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2}i & -\frac{1}{2}i \\
\frac{1}{2}i & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & -z^{-1} \\
z & 0
\end{pmatrix}
\begin{pmatrix}
0 & -z^{-1} \\
z & 0
\end{pmatrix}
\]
\[
\begin{bmatrix}
\frac{1}{2} I & \frac{1}{2} I \\
-I & I
\end{bmatrix}
\begin{bmatrix}
0 & I \\
I & 0
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{2} I & \frac{1}{2} I \\
-I & I
\end{bmatrix}
\begin{bmatrix}
I & \frac{1}{2} I \\
0 & -I
\end{bmatrix}

Thus we get both the $\mathbb{Z}_2$ and $\Delta Q$ in $\mathbb{Q} \times \mathbb{Q}$ after conjugation.

Now we know from the étale cohomology spectral sequence that

\[ gr_n \{ H^*(\text{BSp}_2(\mathbb{F}_8)) \} \cong H^*(\text{BSp}_2) \otimes H^*(

\text{Sp}_2) \]

is at least additively an exterior algebra with generators of degrees $2g-1$, $j=0, \ldots, n$. For $n=1$ I know that

\[ H^*(\text{BSp}_2(\mathbb{F}_8)) \xrightarrow{c} H^*(\text{BQ}_2) \]

has for image the ring $\mathbb{Z}_2 \llbracket x \rrbracket \otimes S[y]$ where $y \otimes x$ have degrees $3+1$ respectively. This suggests the following conjecture:

Conjecture: The image of the map $H^*(\text{BSp}_2(\mathbb{F}_8)) \rightarrow H^*(\text{BQ}_2)$ is contained in the ring $\mathbb{Z}_2 \llbracket y \rrbracket \otimes S[c_1, \ldots, c_n]$ and

\[ H^*(\text{BSp}_2(\mathbb{F}_8)) \cong \mathbb{Z}_2 \llbracket d_1, \ldots, d_n \rrbracket \otimes S[c_1, \ldots, c_n] \]

where the $c_i$ are the quaternionic Chern classes and where $d$ is the derivation of degree $-1$ of $H^*(\text{BQ}_2)$ which carries the $x$ to $y$ and kills other elements of $H^*(\text{BQ}_2)$. 
Suppose $h(X) = [X, B]$ where $B$ is an infinitely commutative $H$-space. Then by Boardman we can embed within a connected cohomology theory $h^*$ with $h = h^0$. Now suppose that $f: Y \to X$ is a covering of degree $d$. In the suspension category we have maps $f_! : Y \to X$ (this is the map induced by $f$) and $f^! : X \to Y$ (this is the trace and is defined since $f$ is proper and framed.) The composition

$$X \overset{f_!}{\leftarrow} Y \overset{f^!}{\rightarrow} X$$

corresponds to the maps on any cohomology theory

$$\{x, z\} \overset{f^*}{\to} \{y, z\} \overset{f_*}{\to} \{x, z\}.$$

By the projection formula

$$f_*(f^*u) = f_! f^* u = f_! 1 \cdot u$$

the endomorphism $f_! f^!$ in the suspension category is

The idea is that if $k^*$ is a multiplicative cohomology theory, then $f_! f^* u = f_! 1 \cdot u$ for $u \in k^0(X)$ and $e(f_! 1) = d$ where $e: pt \to X$. Thus $f_! 1 = d \in k^0(X)$, so it is nilpotent. Thus if $f_!$ exists in $k^0(pt)$, $f_! 1$ will be a unit in $k^0(X)$. This should also hold for any theory which is a module over $k^*$. Thus taking $k^*(X) = \{x, s\}[\mathbb{Z}]$, the question should be true in general.
Lemma: Let $B$ be a homotopy symmetric $H$-space representing a trace theory $k$. Let $f: Y \to X$ be a finite covering of degree $d$ where $X$ is a finite complex of dimension $n$. Then if $x \in \text{Ker } f^*: k(X) \to k(Y)$, we have $d^n x = 0$.

Proof: We use induction on the dimension $n$ of $X$, the case $n=0$ being trivial since then $f$ has a section. Let $X^{(r-1)}$ be the $(r-1)$-skeleton of $X$; then there is a cofibration

$$
X^{(r-1)} \xrightarrow{i} X \xrightarrow{\pi} \bigvee_{i \in I} e_i/\partial e_i
$$

where $I$ is the set of $r$-cells of $X$. This gives rise to an exact sequence

$$
k(X^{(r-1)}) \xrightarrow{i^*} k(X) \xrightarrow{\sum_{i \in I} u_i} \bigoplus_{i \in I} k(e_i^*, \partial e_i^*).
$$

By induction hypothesis $d^{r-1} i^* x = i^* (d^{r-1} x) = 0$, hence there are elements $x_i \in k(e_i^*, \partial e_i^*)$ such that

$$
d^{r-1} x = \sum u_i x_i
$$

If we pull the covering $Y$ back to $e_i$, which we think of as a map $e_i \to Y$, then the covering is trivial of degree $d$ so that the composition

$$
k(e_i, \partial e_i) \xrightarrow{f^*} k(e_f^* e_i, \partial e_f^* e_i) \xrightarrow{f^*} k(e_i, \partial e_i)
$$
is multiplication by $d$. (More precisely we have this kind of commutative diagram)

$$
\begin{array}{c}
\begin{array}{c}
k(Y) \xleftarrow{V_i} k(f^{-1}e_i | \mathbb{E}_i) \cong k_c(Y_i | e_i) \\
\downarrow f^* \quad \uparrow f^* \\
k(X) \xleftarrow{u_i} k(\mathbb{E}_i | e_i) \cong k_c(e_i | \mathbb{E}_i | e_i)
\end{array}
\end{array}
$$

so

$$
d^n x = \sum u_i \, dx_i \\
= \sum u_i \, f^* f^* x_i \\
= f^* f^* (\sum u_i x_i) = f^* f^* (d^{n-1} x) = 0
$$

which was to be proved.

(The good way to rewrite the proof is to put $A = X^{(n-1)}$

$$
k(A) \leftarrow k(X) \leftarrow k_c(X-A)
$$

and then argue that $Y$ is trivial over $X-A$ so that $f^* f^* z = dz$ if $z \in k_c(X-A).$)
The point is that \( P_{x_{ij}} = (c_{p^{-1}}^{1/2})_{ij} \) up to sign since the Bockstein is zero by assumption and the other operations are of too high degree. Thus setting \( w = c_{p^{-1}}^{1/2} \) we get that

\[
P_{x_i} = \sum_{i=0}^{2k} [w\cdot i] u_i
\]

might be algebraic, whereas the \( u_i \) are algebraic!!

In the examples we know that the normalizer of the torus carries the cohomology by Sylow theory.

Basic question: Injectivity of restriction to the normalizer for a general infinitely commutative \( H \)-space?

\[
\begin{align*}
BN_n, \overline{F}_p &\quad \rightarrow \quad BN_n, \overline{Z}_p^\ell &\quad \rightarrow \quad BN_n, \overline{C} \\
BGL_n, \overline{F}_p &\quad \rightarrow \quad BGL_n, \overline{Z}_p^\ell &\quad \rightarrow \quad BGL_n, \overline{C} \\
\text{Spec } (\overline{F}_p) &\quad \rightarrow \quad \text{Spec } \overline{Z}_p^\ell &\quad \rightarrow \quad \text{Spec } C
\end{align*}
\]

and now we know that by our cohomological calculation

\[
\lim_{\rightarrow} BGL_n(\overline{F}_p) \rightarrow \text{lim}_{\rightarrow} BGL_n, \overline{F}_p
\]

and similarly for \( N_n \). But the point is that \( N_n(\overline{F}_p) \rightarrow GL_n(\overline{F}_p) \) descent is always possible at \( k\) times \( q^{k-1} \).
February 23, 1970.

It remains to check that $RO_k(G) \oplus RSp_k(G)$ is a $\lambda$-ring in such a way that the forgetful homomorphism to $R_k(G)$ is a $\lambda$-ring map.

For this aspect of the theory it is probably simpler to use the unitary theory. So we suppose given a ring $A$ with an involution — and an element $A$ with $A^\lambda = 1$. A unitary module is a fin. gen. proj $A$-module $E$ together with a non-degenerate sesquilinear form $B(x,y)$, $E \otimes E \to A$

such that

$$B(y,x) = \lambda B(x,y)$$

We agree to identify two such $E$'s if the forms differ by a real unit of $A$. Call the resulting Grothendieck group $KU^\lambda(A)$. I want to define the tensor product map

$$KU^\lambda(A) \otimes KU^\lambda(A) \to KU^\lambda(A)$$

$$[E] \otimes [F] \to [E \otimes F]$$

For this it suffices to check that the relations go to zero. But if $E' \in E$ is isotropic, then so is $E \otimes F = E \otimes F$ and
\[
(E' \oplus E/E') \otimes F \cong E' \otimes F \oplus E \otimes F/(E' \otimes F)
\]

\[
(E'/E) \otimes F \cong (E' \otimes F)/E' \otimes F
\]

so it's clear.

Next I want to define the \(\lambda\)-operations

\[
KU^\lambda(A) \xrightarrow{\lambda=1} \bigoplus_{\lambda \geq 1} KU^\lambda(A)
\]

\[
[E] \mapsto \sum [\otimes E]
\]

has sesquilinear form

\[
\theta(x_i-x_j, y_i-y_j) = \det \{E(x_i, y_j)\}
\]

So again we must check that the basic relations hold. Thus suppose \(E \subset E'\) is isotropic direct summand. If you could find a complement \(Z\) to \(E'\), then we would have

\[
E \cong (E' \oplus Z) + (E' \oplus Z)^\perp \quad \text{o.d.s.}
\]

and there is no problem with additivity for o.d.s. case.

The basic problem is that if \(E\) is isotropic...
Cohomology mod 2 of $\text{BS}p_{2n}(\mathbb{F}_8)$ is $4i-1$.

Form $\bigoplus_{n \geq 0} H^*_{\mathbb{Z}}(\text{BS}p_{2n}(\mathbb{F}_8))$ ring, and there is a basic map

Let $N_{2n}$ be the normalizer of the torus $\cong \sum_{n \geq 1} S N_2$

$N_2(\mathbb{F}_8) = \mathbb{Z}_4 \times \mathbb{F}_8^*$

Then I get a surjection by Sylow theory

$$H^*_{\mathbb{Z}}(BN_{2n}(\mathbb{F}_8)) \longrightarrow H^*_{\mathbb{Z}}(\text{BS}p_{2n}(\mathbb{F}_8))$$

permitting me to define the additive diagonal in

$$\bigoplus_{n \geq 0} H^*_{\mathbb{Z}}(\text{BS}p_{2n}(\mathbb{F}_8))$$

Now we know that the composition

$$S \left\{ H^*_{\mathbb{Z}}(BN_2(\mathbb{F}_8)) \right\} \longrightarrow H^*_{\mathbb{Z}}(BN_n(\mathbb{F}_8)) \longrightarrow H^*_{\mathbb{Z}}(\text{BS}p_n(\mathbb{F}_8))$$

is surjective. $\mathbb{Z}_2[\beta_i]$ where $\beta_i$ is a basis for $H^*_{\mathbb{Z}}(BN_2(\mathbb{F}_8))$ now actually since we have a ring homomorphism, we can throw away the $\beta_i$ which go to zero in $\text{Sp}_2(\mathbb{F}_8)$. Thus we get two kernels

$$\mathfrak{f}_i \quad \text{dim} \quad 4^i \quad (i \geq 1)$$

$$\mathfrak{g}_i \quad \text{dim} \quad 4^{i-1} \quad (i > 1).$$

Given a trace theory \( k(X) = [X, B] \), I want to extend it to a generalized cohomology theory. The key step is to define a map \( QB \rightarrow B \), i.e., a family of maps \( \Omega^nS^nB \rightarrow B \).

If \( X \) is a \( \infty \)-manifold, then

\[
\lim_\gamma \left[ X, \Omega^nS^nB \right] = \lim_\gamma \left[ S^nX, S^nB \right]
\]

\[
= \left\{ \text{cobordism classes of maps } X \overset{f}{\leftarrow} Y \overset{g}{\rightarrow} B \right\} \\
\text{where } f \text{ is proper framed of rel. dim. 0}
\]

Thus we have to define the element \( f_+g^*(\text{Emin}) \) in \( k(X) \), where Emin is the canonical element of \( k(B) \). In other words we have to extend the trace to proper framed maps of rel. dim. 0.

Now I would like to use \( \pi_0 \) theorems in surgery to do this. Thus I wish to make \( f \) into a homotopy equivalence and apply the inverse of \( f \) to pull \( g \) down. Unfortunately this method supposes that \( f_*f^* = \text{id} \) in case \( f \) is a homotopy equivalence for any generalized coh. theory. But if I recall seattle this needn't be the case, although it might be so when the boundary is non-empty.

To put ourselves in good surgery position we replace \( X \), which is always assumed to be of the homotopy type of a finite complex, by a regular neighborhood in some high Euclidean space. Thus \( X \) is a compact \( \infty \)-manifold with boundary with no compact components and \( \pi_1(\partial X, x) \overset{\sim}{\rightarrow} \pi_1(X, x) \) for all \( x \in \partial X \). For simplicity suppose that \( X \) is connected. It suffices to define \( f_+ \).
only for f of degree 1 where the degree is the integer obtained by mapping pt → X and B → pt. According to Sullivan there is no problem in showing that Y may be surgered until f is a homotopy equivalence and that one may carry the map g: Y → B along with the surgery.

Now the problem becomes whether f*1 = 1. The idea is that if so, then there is a manifold W → X with ∂W = X ∪ Y and h|X = id, h|Y = f.

Actually W → X × I is given. Now one should be able to surger h keeping it fixed on h∗(X × [0, 1]) until it's a homotopy equivalence. But then X co-diffeomorphic to Y by the h-cobordism theorem and f is equivalent to a diffeomorphism which is extremely unlikely.

Conclusion: Surgery is no good.
February 23, 1970. On Mumford's conjecture

Let $k$ be an algebraically closed field of characteristic $p$, let $G$ be a reductive algebraic group over $k$, and let $E$ be a representation of $G$. Assume that $E$ has a 1-dimensional quotient $E \rightarrow L$ stable under $G$. Then for some integer $m \geq 1$, the map $S_m^E \rightarrow S_m^L = L \otimes_{F_m} F$ has an equivariant section.

1) Replacing $E$ by $E \otimes L^1$, we may suppose that $L$ is a trivial representation and prove that

$$(S_m^E)^G \rightarrow (S_m^L)^G$$

is surjective for some $m$.

2) Let $T$ be a maximal torus and let $N$ be its normalizer. It is clear that the conjecture is true for $N$ (Nagata). Let $m$ be such that

$$(S_m^E)^N \rightarrow (S_m^L)^N$$

is surjective. I would like to show that $m$ works for $G$.

3) Claim that

$$(S_m^E)^G = (S_m^E)^{G(F_q)}$$

for some large finite field $F_q$. This is because $G(F_p)$ is dense in $G$ for the Zariski topology.
4) By Chevalley

Question is equivalent to detecting cohomology in unipotent groups. Thus $G$ is a unipotent alg. group acting on $V$ and we have an element of $H^1(G, V)$, we have to have a criterion which enables us to decide when this is zero or better when the extension

$$
0 \longrightarrow S_p V \longrightarrow S_p 1 \longrightarrow 0
$$

is trivial. Note that for the Mumford conjecture if we want

$$S_p V \longrightarrow 1$$

to have a section for a given $V$, it suffices to have the element vanish on $H^1(G, V)$

Corollary: $H^1(G, V) \xleftarrow{\text{lim}} H^1(G(F_\ell), V)$

Proof: An element of $H^1(G, V)$ is an exact sequence

$$0 \rightarrow V \rightarrow \tilde{V} \rightarrow 1 \rightarrow 0$$

which is zero iff $\tilde{V}^G \rightarrow 1$. But by density $\tilde{V}^G \cong \tilde{V}^G(F)$
Proof of Mumford conjecture: $G$ alg. gp. reductive.  
representation of $G$. $L$ 1-dim. quotient of $V$. 
for some $n$ the map $S_n V \to S_n L$ has an inv. section.

It's enough to do for $k = \overline{F}_p$. One knows that True for the normalizer of a torus $T$. Assume $G = [G, G]$ so that just have to prove invariants are onto.

$$(S_k V)^G = \bigcap_{g \in G} (S_k V)$$

$$(S_k V)^{\text{G}(\overline{F}_p)} \to S_n L \quad \text{all } k \equiv 0 \ (N)$$

\begin{align*}
\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} & \quad \begin{pmatrix} a & y \\ b & 1 \end{pmatrix} \\
0 & 1 & \rightarrow & \begin{pmatrix} ax + by \\ y \\ z \end{pmatrix}
\end{align*}

Non-conj. since image of one is 2 dim.
4 here is 1-dim.
\[ \lim_{n \to \infty} R(\text{Gl}_n(F_0), \mathbb{C}) \text{ is the free monoid} \]

(with unit) with generators \((f_i)\) for irreducible monic polynomial of degree \(\geq 1\) with \(f(0)\) a unit (equivalently a Galois conjugacy class of elements of \(\overline{F}_0^\times\)) and \(i\) is an integer \(\geq 1\) of degree \(|f_i| = i \deg f\). Thus the generators correspond to divisors of \(\text{Gl}_n\) rational over \(\overline{F}_0\) with irreducible support, e.g., a closed point of \(\text{Gl}_n\) over \(\overline{F}_0\) with multiplicity \(\geq 1\).

Indecomposable modules: Let \(A\) be an elementary abelian \(p\) gp. and let \(M\) be an indecomp. 
\(S(A^*)\) -module such that \(a^P M = 0\) all \(a \in A\).

\(M\) indecomposable \(\iff\) \(\text{End}_{S(A)}(M)\) local ring.

Assume that in \(\begin{array}{c} A \cr \text{Aut}(M) \end{array} \)

is a maximal elementary abelian \(p\) gp. Then we have that if \(\Theta \in \text{End}_{S(A)}(M)\) and \(\Theta^P = 0\), then \(\Theta \in A\).

\[ \text{understand maximal commutative subalgs of } \text{Gl}_n \]

are they all of rank \(< l\).
February 24, 1970:

Can you compute $H^*(\text{BGL}_n(\mathbb{Z}))$? In particular, how far off is the map

$$H^*(\text{BGL}_n(\mathbb{Z})) \longrightarrow H^*(\text{B} \Sigma_n)$$

from being an isomorphism. (This map is the analogue of the canonical map of stable cohomotopy theory to any given cohomotopy theory.)

Idea: Consider the action of $\text{GL}_n(\mathbb{Z})$ on the symmetric space $X = \text{GL}_n(\mathbb{R})/O_n(\mathbb{R}) = \text{pos} \text{ def} \text{ rea} \text{ symm} \text{ matrices}$. I shall assume, until I can check with Borel, that $\text{GL}_n(\mathbb{Z})$ acts nicely enough so that there is no problem with equivariant cohomology. Then, since $X$ is contractible

$$E_2 = H^*(\text{BG}, H^*(X)) \longrightarrow H^*_G(X)$$

shows that the equivariant cohomology is what I want. Now form the standard bundle

$$E = \text{GL}_n(\mathbb{R}) \times_{O_n(\mathbb{R})} \mathbb{C}^n \longrightarrow X$$

associated to the standard representation of $O_n(\mathbb{R})$ on $\mathbb{C}^n$. Then I should have that

$$\text{Spec } H^*_G(X) = (\text{Spec } H^*_G(\text{Flag}(E)))_{\Sigma_n}.$$
steps:

\[ H^*(B\text{Gl}_n(\mathbb{Z})) \cong H^*(E\text{Gl}_n(\mathbb{Z}) \times \text{Gl}_n(\mathbb{Z})/\text{O}_n) \]

\[ \cong H^*[E(\text{Gl}_n(\mathbb{Z}) \times E(\text{O}_n)] \times \text{Gl}_n(\mathbb{Z})/\text{O}_n \text{Gl}_n(\mathbb{R}) \]

\[ \cong H^*_{\text{O}_n} \left( \text{Gl}_n(\mathbb{R})/\text{Gl}_n(\mathbb{Z}) \right) \]

\[ \text{space of lattices in } \mathbb{R}^n. \]

Let \( L \) be this space of lattices. I am interested in the elementary abelian \( \ell \)-subgroups of \( \text{O}_n \) fixing lattices in \( \mathbb{R}^n \).

To make things simpler use the similar isomorphism

\[ H^*(B\text{Gl}_n(\mathbb{Z})) \cong H^*_{\text{O}_n} \left( \text{Gl}_n(\mathbb{C})/\text{Gl}_n(\mathbb{Z}) \right) \]

\[ \text{space of lattices in } \mathbb{C}^n, \text{ ie } \]

\[ \text{Z-submodules } L \text{ of } \mathbb{C}^n \Rightarrow \mathbb{Z} \subset \mathbb{C} \Rightarrow \mathbb{C}^n. \]

(Embed \( \text{Gl}_n(\mathbb{Z}) \) into \( \text{Gl}_n(\mathbb{C}) \) descends to \( \text{U}_n \)) According to my theorems, the spectrum of \( \text{elementary abelian } \ell \)-

conjugacy classes of \( \ell \)-subgroups of \( \text{U}_n \) which fix points of \( L \).

Conclusion:

\[ H^*(B\text{Gl}_n(\mathbb{Z}), \mathbb{Z}) \rightarrow \lim_{A} S(A^*) \text{ Fiso}. \]

A runs over the \( \ell \)-subgroups of elementary abelian \( \ell \)-subgroups of \( \text{Gl}_n(\mathbb{Z}) \).
The embedding $GL_n \mathbb{Z} \to GL_n \mathbb{Z}$ is an example of an elementary abelian $d$-subgroup coming from $\mathbb{Z}^d$. Acting on $\mathbb{Z}^d$ in the standard way, we obtain $\mathbb{Z}$-modules $\mathbb{Z}/n\mathbb{Z}$, where $n$ is a finite group of rank $\mathbb{Z}/n\mathbb{Z}$, containing an automorphism which is non-isomorphic to $\mathbb{Z}/n\mathbb{Z}$. We consider the representation $\mathbb{Z}/n\mathbb{Z}$ over $\mathbb{Z}/n\mathbb{Z}$, and $L \otimes \mathbb{Z}/n\mathbb{Z}$. Since we cannot use the standard way of torsion, we consider the representation $\mathbb{Z}/n\mathbb{Z}$ over $\mathbb{Z}/n\mathbb{Z}$, and $L \otimes \mathbb{Z}/n\mathbb{Z}$. We can form $\bigoplus_{\mathbb{Z}/n\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$, where $\mathbb{Z}/n\mathbb{Z}$ is indecomposable. Thus we have an action of $\mathbb{Z}/n\mathbb{Z}$, which does not change the coefficients of $\mathbb{Z}/n\mathbb{Z}$. Therefore, the action of $\mathbb{Z}/n\mathbb{Z}$ is indecomposable.
while the maximal rank of an \([l]\)-subgroup of \(\Sigma_n\) is

\[
\begin{bmatrix} n \\ l \end{bmatrix}
\]

coming from the product \(\Sigma_l \begin{bmatrix} n \\ l \end{bmatrix} \subset \Sigma_n\). Clearly

\[
\frac{n}{l-1} > \begin{bmatrix} n \\ l \end{bmatrix}
\]

for suitable \(n\), e.g. \(n = l-1\).

---

Proof of assertion at bottom of page 2: To show that the category of \([l]\)-subgys in \(\text{Gl}_n(\mathbb{Z})\) is equivalent to the category of pairs \((A, l)\) where \(A\) is an \([l]\)-subgroup of \(\text{Un}\) and \(l \in \pi_0(L^A)\), \(L = \text{Gl}_n(\mathbb{C})/\text{Gl}_n(\mathbb{Z})\). First step is to go from

\[
\begin{cases}
\{ \text{[l]-subgys of } \text{Gl}_n(\mathbb{Z}) \} & \longrightarrow & \{ \text{transitive } \text{Gl}_n(\mathbb{Z}) \text{ sets} \}
\end{cases}
\]

\[\longrightarrow \begin{cases}
\text{transitive } \text{Gl}_n(\mathbb{C})\text{-spaces over } \text{Gl}_n(\mathbb{C})/\text{Gl}_n(\mathbb{Z}) \text{ with } [l]\text{-subgys for isot. groups} & \rightarrow & \{ \text{trans. } \text{Gl}_n(\mathbb{C}) \text{ spaces over } L \text{ with } [l]\text{-subgys for isot. gys} \}
\end{cases}
\]

\[
\longrightarrow \begin{cases}
\text{trans. } \text{Un}\text{ spaces over } L \text{ with } [l]\text{-subgys for isot. groups } \text{+ homotopy classes of maps} & \rightarrow & \{ \text{trans. } \text{Un}\text{ spaces over } L \text{ with } [l]\text{-subgys for isot. groups } \}
\end{cases}
\]

We only have to check that the last step is an equivalence of categories. The functor backwards is

\[
\begin{cases}
\{ X \rightarrow L \} & \longrightarrow & \{ \text{Gl}_n(\mathbb{C}) \times \text{Un} \times X \rightarrow \text{Un} \text{ Gl}_n(\mathbb{C}) \times \text{Un} \times L \rightarrow L \}
\end{cases}
\]
and the point to check is that $\text{GL}_n^0(\mathbb{C}) \rightarrow \mathbb{C}$ is an equivariant homotopy equivalence for any compact subgroup of $\text{GL}_n(\mathbb{C})$. In effect given $A \subset \text{GL}_n(\mathbb{C})$ and $\lambda \in \pi_0(L^A)$ one knows how to conjugate $A$ into $U_n$. It seems clear.

We know that $\text{GL}_n(\mathbb{Z})$ contains $\mathbb{Z}_2^n$ as diagonal matrices. Suppose $A \subset \text{GL}_n(\mathbb{Z})$ is an elementary $\mathbb{Z}_2$-subgroup of rank $n$. Then if I think of $A$ as acting on a free abelian group $E$ I can break up $E[1/2]$ into a sum of eigenspaces

$$E[1/2] = \bigoplus_{i=1}^n L_i[1/2],$$

where $L_i \subset E$ is the corresponding invariant subspace of $E$. I claim that

$$2E \subset \bigoplus_{i=1}^n L_i \subset E.$$

Only the first has to be proved, so let $e = \sum x_i$, $x_i \in L_i[1/2]$. Now choose an element $\sigma \in A \ni \sigma = +1$ on $L_i$ and $-1$ on the others. Then

$$e + \sigma e = \bigoplus_{i=1}^n 2x_i \in E \cap L_i[1/2] = L_i$$

showing that $x_i \in 1/2 L_i$ for all $i$, hence that $2E \subset \bigoplus L_i$ as claimed.

Note that $A$ acts trivially on $\bigoplus L_i \otimes \mathbb{Z}_2$. Here's how to construct non-conjugate $[2]$-subgroups of $\text{GL}_n(\mathbb{Z})$. 
rank n. Start with with the standard representation of $\mathbb{Z}_2^n$ or $\mathbb{Z}_2^n = M$. Then any lattice $2M \subseteq L \subseteq M$ is stable under $\mathbb{Z}_2^n$. Choosing a basis for $L$ i.e. an isomorphism $\mathbb{Z}_2^n \cong L$ as $\mathbb{Z}$ modules we obtain a subgroup of $\text{GL}_n(\mathbb{Z})$ isomorphic to $\mathbb{Z}_2^n$. In this way I get a map

$$L \rightarrow \{\text{conjugacy class of } \mathbb{Z}_2^n \text{ in } \text{GL}_n(\mathbb{Z})\}$$

which is in fact surjective by my previous argument. It seems that the only indeterminacy in going backwards is the ordering of the eigenspaces. This leads to the following.

**Conclusion:** Conjugacy classes of $\mathbb{Z}_2^n$ in $\text{GL}_n(\mathbb{Z})$ are in 1-1 correspondence with orbits of non-zero subspaces $V$ of $\mathbb{Z}_2^n$ such none of the coordinate vectors $(0, ..., 1, ..., 0)$ belong to $V$.

**Conclusion:** $\ell$-rank of $\mathbb{Z}_2^n \text{ in } \text{GL}_n(\mathbb{Z})$ equals $\left\lfloor \frac{n}{\ell-1} \right\rfloor$.

**Remark:** A similar analysis of a maximal $\mathbb{Z}_2$-subgroup $A$ of $\text{GL}_n(\mathbb{Z})$ of lower rank runs into trouble because we can't show that $2E \subseteq \text{sum of eigenspaces}$.
Green's paper and the cohomology mod p of $H^*(B\text{Gl}_n(\mathbb{F}_q))$.

Character theory:

$\bigoplus_n R_+(\text{Gl}_n(\mathbb{F}_q))$ becomes a ring.

Conjugacy classes: $\text{Hom}_{k[\mathbb{F}_q]}(\prod_n \text{Spec } R(\text{Gl}_n(\mathbb{F}_q)), \text{graded monoid})$

A point in this is a conjugacy class and the generators are the irreducible Jordan blocks.

$\text{Field contributed}$

Suppose $A$ is a semi-simple indecomposable matrix with Then the eigenvalues of $A$ are irreducible equation of degree $d$.

So let $N_d$ be the number of irreducible equations of degree $d$. Then you get one semi-simple irreducible class. Thus conj. classes are indexed by pairs $d, j$ where $d$ stands for eigenvalue field and $j$ for index of nilpotency.

Conj. $d_j \text{Gl}_n(\mathbb{F}_q)$ are sequences $(f_1, \xi_1), \ldots, (f_n, \xi_n)$ monic irreducible equation and

$$n = \frac{d}{j} \sum \xi_i \deg f_i$$
February 24, 1970: Projects:

1. Spectrum of an equivariant cohomology ring: It is necessary to add p=0, the localization + fixed-point theorems. I propose to rewrite this paper along the following lines: First separate out the dimension theorem + examples (Σn, G_n(Z), G_n(F_p)) as part I. Then organize part II by first computing what happens for H^*(X) and then reducing to this. Need:

\[ \text{Spec} \left[ \frac{A[x_1, \ldots, x_n]}{(\sigma_i(x_i) = a_i)} \right] \rightarrow \text{Spec} \ A \]

is a universal homeomorphism (and an F-iso. in char. p.) It might be nice to write this up in such a way that the analogues for K(G(X)) are clear.

Adams' conjecture:

2. Cohomology of finite classical groups with applications to and the Adams conjecture. I propose to write a complete proof of the Adams' conjecture without using stable cohomology. At the moment I need to check that the fact that BG(\mathbb{F}_p) \rightarrow BU induces a cohomology model + p implies that the Adams conjecture is true. The problem is knowing how to deal with these infinite complexes. I also need the orthogonal Brauer theory — some minor technical lemmas remain.

3. Cohomology of BG(\mathbb{F}_p). At the moment you have a general theorem which gives good information model for odd. For Sp = 0 some problems remain at 2. General question about restriction to T is unclear.
4. Symmetric groups. Nowhere near in verifiable form. The main theorem is the computation of \( \bigoplus_{n \in \mathbb{N}} H_\ast (E^n \times S^n, X^n) \) for any space \( X \), possibly also with twisted coefficients. The idea was to construct Steenrod ops.

\[
\begin{align*}
H(X) & \rightarrow \bigoplus_{n \in \mathbb{N}} \text{Hom}(R, H(X)) \\
\end{align*}
\]

and then use the Klein groups to put elements in \( R \).

I wanted to produce by Steenrod a multiplicative operation

\[
\begin{align*}
H(X) & \xrightarrow{\gamma} \bigoplus_{n \in \mathbb{N}} \mathbb{Z}_p [\ell_n, \tau_n] \otimes H(X) \\
\end{align*}
\]

\[
\begin{align*}
\gamma(x) & = \sum_{n \in \mathbb{N}} \ell_n(\beta x)^n + \tau_0 x \\
\gamma(\beta x) & = \sum_{n \in \mathbb{N}} \ell_n(\beta x)^n \\
& \quad \text{where } x \in H^1(X) \\
\end{align*}
\]

which would prove that \( \bigoplus_{n \in \mathbb{N}} \mathbb{Z}_p [\ell_n, \tau_n] \rightarrow R_{\text{min}} \) was free. Then I hoped to find an argument showing this map had to be an isomorphism. Finally I hoped to go back and put the symmetric groups into shape.

5. The Boardman-Vogt theorem and KADL-theory. I have now a formulation of BV thm. and what I think might be a buoyant proof using Milgram + Nakaoaka, to show \( \bigoplus_{n \in \mathbb{N}} E^n \times X^n \rightarrow QX \) induces an isomorphism on classifying space cohomology.

6. Higher K-theory. I have a definition of \( K_i(R) \), hopefully compute for \( R = \mathbb{F}_2 \), at least the odd char. part. And I have supporting evidence for symmetric groups.
February 7, 1975 (still going).

I want to compute the K(ADL) operations for the additive structure of BU.

Following are ideas of last night:
1. Look again at Mumford conjecture
2. symplectic cob.
3. Is Chern subring of $H^*(G, \mathbb{Z}_p)$ $F$-equivalent?
4. If $X$ is a $G$-manifold oriented, then $X/G$ is a rational homology manifold.
   Following this, if the quadratic form doesn't have to have det +1 so there are more invariants of the signature type.

5. $\mathbb{Z}_p \oplus \bigoplus_{k=1}^{n} H^*_p(GL_k(\mathbb{Z}_p), \mathbb{Z}_p)$ $l$ prime $l 
   \text{prime } l+g.$
   odd cohomology of the symmetric groups

6. Infinite loop space and the map
   $\lim_{k \to 1} X \overset{k}{\longrightarrow} \Sigma_\infty X$
   for a pointed space. Is this a homotopy equivalence?
   (Look at semi-simplicially to get formula for $QX$)

7. Can one construct interesting classes in $H^*(BPL)$ using the KADL method.

8. Higher order $K$-groups as homotopy of general linear group.
Take $U^*_{g, \text{mod}}(X)$ modified so as to have a f.g.r.p. law.

Does $U^*_{g, \text{mod}}(X) \otimes \mathbb{H}^*_{L^{2,6}}$ give the same spectral results?

Let \( G \) be a group and let \( K \) be a field. By an orthogonal (resp. symplectic) representation of \( G \) over \( K \) we mean a representation \( \rho \) of \( G \) in a finite dimensional vector space \( V \) endowed with an invariant non-degenerate symmetric (resp. skew-symmetric) bilinear form. Thus, the forms \( B \) and \( AB \), \( A \in K^* \), define the same structure on \( V \).

Introduce Grothendieck groups \( R_{K}(G) \) and \( R_{Sp_{K}}(G) \) by the relations:

(i) \( V = V' \oplus V'' \) orth. direct sum \( \Rightarrow [V] = [V'] + [V''] \)

(ii) \( W \subseteq W^\perp \subseteq V \) \( \Rightarrow [V] = [W^\perp/W] + [W/W^\perp] \)

and we check that there are maps

\[
\begin{array}{ccc}
R_{K}(G) & \overset{\text{forget}}{\rightarrow} & \overset{\text{forget}}{R_{K}(G)} \\
\downarrow \quad \text{hyperbolic} & & \downarrow \\
R_{Sp_{K}}(G) & \overset{\text{forget}}{\rightarrow} & \overset{\text{forget}}{R_{K}(G)}
\end{array}
\]

\( (\ast) \)

All the above should work for a commutative ringed topos and also in the unitary theory.

Then there should exist two further aspects:

1) \( A \)-ring structure on \( R_{K}(G) \oplus R_{Sp_{K}}(G) \).

2) Decomposition homomorphism

\[ d : R_{O_{\mathcal{E}^{1}}}^{A}(G) \rightarrow R_{O_{\mathcal{E}^{1}}^{A}}(G) \]
defined when $t$ is a regular element of $A$ (and maybe $A$ regular?)

Check compatibility of this structure with $(\ast)$.  

*Trick:* How to use the decomposition homomorphism to define the $\Lambda$-operations. The point is to check that the $\Lambda^g$ are compatible with the equivalence relations. Thus I am given $0 < W < W^* < V$ and I want to show that

$$[\Lambda^n V] = [\Lambda^n ((W + V/W) \oplus W/W)]$$

But consider the polynomial ring $K[T]$ and $t = T$. Then in $V \otimes K[T, T^{-1}]$ I have two "lattices"

$$L_1 = V \otimes K[T], \quad (W \cdot T^{-1} + W^\perp + V \cdot T) \cdot K[T] = L_2$$

and

$$L_1 \otimes_{K[T]} K = V$$

$$L_2 \otimes_{K[T]} K = W \oplus W^\perp \oplus \frac{V}{W^\perp}$$

To see the last equation use the exact sequence

$$\begin{align*}
W^+ & \rightarrow V[T] \xrightarrow{\text{inc.}} \quad \oplus \rightarrow L_2 \rightarrow 0 \\
\oplus & \quad T \quad T \\
W[T] & \quad \text{inc.} \quad \text{inc.} \quad W[T]
\end{align*}$$

which on killing $T$ yields the desired result. Note that $L_2$ has a non-degenerate form (split the flag $+$ check).

Now since there is a decomposition theorem we know that $\Lambda^g L_1$, $\Lambda^g L_2 < \Lambda^n V[T, T^{-1}]$ give the same element in $R_{\mathcal{O}_K}(G)$ which is what we want to have proved.
Question: Is $RO_A(G) \rightarrow RO_A[H^*(G)](G)$ onto?

For this, you probably need $A$ regular since this is required for $R$.

Outline of simplest path: Assume $K$ field char $\neq 2$, large.

1) definition of $RO_K(G) \oplus RSP_K(G)$
   + maps from and to $R_K(G)$

2) By induction on the length of a quadratic rep, get explicit description of $RO_K(G)$ and $RSP_K(G)$ as submodules of $R_K(G)$ and we get basic exact sequence.

3) $\kappa \to A \to K$ adic, then you check that

4) now introduce $\lambda$-ring structure on $RO_K(G) \oplus RSP_K(G)$ and check well-defined, actually you know this because to show that

\[
\begin{bmatrix}
\lambda^3 E
\end{bmatrix} = \begin{bmatrix}
\lambda^3 (W + \overline{W}/W) + W/W
\end{bmatrix}
\]

it is enough to know same in $R_K(G)$. To show $\lambda$-ring identities hold enough to use

$RO_K(G) \oplus RSP_K(G) \rightarrow R_K(G \times \mathbb{Z}_2)$
The only thing that remains is 3) namely to prove the dotted arrow exists. So I take an orthogonal representation $E$ of $G$ over $K$ and choose an invariant lattice $L$. It is necessary to show that the semi-simple $k[G]$-module associated to $L \otimes k$ carries a non-degenerate symmetric bilinear form.

Example: 1) suppose we can find an invariant lattice $L$ such that $\text{nil}^2 \subset L \subset \text{nil}^1$, where $\text{nil}^i = \{ x \in E \mid B(x, L) \subset A^i \}$. Then

$$0 \rightarrow \text{nil}^1 / L \rightarrow L \otimes k \rightarrow L \otimes k \rightarrow \text{nil}^1 / L \rightarrow 0$$

is exact. Now $L \otimes k \rightarrow L \otimes k \approx (L \otimes k)^*$ is the restriction of $B$ to $L$ so the image of this map carries a non-degenerate symmetric form. It remains to produce one for $\text{nil}^1 / L$. But $B : L \otimes L \rightarrow A^{i-1}$ is injective (since $L \subset \pi^{-1} \text{nil}^1$) and the reduction of this has $L$ for kernel so it induces a non-degenerate symmetric form on $\text{nil}^1 / L$.

2) suppose there exists an invariant lattice $L$ with $\pi^2 \text{nil}^1 \subset L \subset \text{nil}^1$. This satisfies the filtration

$$L = \pi^2 \text{nil}^1 \subset \pi L \subset \pi^2 \text{nil}^2$$

and

$$\{ x \in L \mid B(x, L^*) \subset \pi A^2 \} = L \cap \pi^2 \text{nil}^2$$

$$\{ x \in \pi \text{nil}^1 \mid B(x, \pi^2 \text{nil}^1) \subset \pi^2 A^2 \} = \pi \text{nil}^1 \cap \pi^2 \text{nil}^2$$
2) Suppose there is an invariant lattice \( L \) with 
\[ \pi^n L^\vee \subseteq L \subseteq \pi^{n-1} L^\vee \]
for some integer \( n \). Then setting \( L = \pi^{iL} \) we have 
\[ \pi^{n-i} L^\vee \subseteq \pi^i L \subseteq \pi^{n-1-i} L^\vee \]
so if we set \( n = 2i + \varepsilon \) \( \varepsilon = 0, 1 \) we have 
\[ \pi^i L_1^\vee \subseteq L_1 \subseteq \pi^{i+1} L_1^\vee \]
and hence \( L_{i+1} \) is an invariant lattice with 
\[ \pi^{i+1} L \subseteq L_{i+1} \subseteq \pi^{i+2} L \]
setting \( L_2 = L_1 \) if \( \varepsilon = 1 \) and \( L_2 = L_1^\vee \) if \( \varepsilon = 0 \) we have an invariant lattice with 
\[ \pi L_2 \subseteq L_2 \subseteq \pi^2 L_2 \]
and we are reduced to the preceding case.

3) Next suppose there is an invariant lattice \( L \) such that 
\[ \pi^2 L^\vee \subseteq L \subseteq \pi^0 L^\vee \]
Then 
\[ \pi \left( L + \pi L^\vee \right)^\vee = \pi \left( L^\vee \cap \pi^{-1} L \right) = \pi L^\vee \cap L \]

Thus if \( L_1 = L + \pi L \) we have that 
\[ \pi L_1^\vee \subseteq L_1 \subseteq \pi^2 L \]
and so are reduced to case 1).

4) Now we can always produce an invariant lattice $L$ satisfying $\pi^{8} L \subseteq L \subseteq L$ for some integer $g$. Assume $g$ least and let $g = 2j - e$ with $e = 0, 1, 2$ so that $\pi^{3} L \subseteq L \subseteq L$. Repeat preceding argument with $\pi^{3}$ instead of $\pi$. Thus

$$(L + \pi^{3} L)^{\vee} = L^{\vee} \cap \pi^{-3} L \subseteq L + \pi^{3} L$$

$$\pi^{3}(L + \pi^{3} L)^{\vee} = \pi^{3} L^{\vee} \cap L \subseteq L + \pi^{3} L$$

and we get an invariant lattice with $g$ replaced by \( j \). Now $j < g$ impossible so $j = g = e$ so $j = g = 1$.

Conclusion: It is always possible to find a lattice $L$ invariant under $G$ such that $\pi L \subseteq L \subseteq L$. 