January 25, 1970: \( H^*(B\Sigma_n, \mathbb{Z}_p) \) p odd.

Form the ring

\[
R = \bigoplus_{n \geq 0} H_\ast (B\Sigma_n) \mod p \text{ coefficients}
\]

with multiplication

\[
H_\ast (B\Sigma_i) \otimes H_\ast (B\Sigma_j) \xrightarrow{\sim} H_\ast (B\Sigma_i \times \Sigma_j) \xrightarrow{\partial} H_\ast (B\Sigma_{i+j})
\]

where the second map is restriction for the homomorphism \( \Sigma_i \times \Sigma_j \rightarrow \Sigma_{i+j} \).

It's useful to note that we can also form a ring

\[
R(X) = \bigoplus_{n \geq 0} H_\ast (E\Sigma_n \times \Sigma_n X^n)
\]

(\( X \) can also be a pair of spaces.)

The same way. (\( \Sigma_n = \{e\} \) if \( n = 0, 1 \); it thus is the automorph of a set with \( n \) elements. \( E\Sigma_n \times \Sigma_n X^n = * \) if \( n = 0 \).) (Alternative notations to be decided upon. Later):

\[
E\Sigma_n \times \Sigma_n X^n = E\Sigma_n \Sigma_n X
\]

\[
H_\ast (E\Sigma_n \times \Sigma_n X^n) \rightarrow H_\ast \Sigma_n (X^n).
\]

Note that if \( X = (S^1, pt) \), then \( R(X) \) gives the homology of \( \Sigma_n \) with twisted coefficients modulo \( p \) for all \( n \).
Let $B$ be a $\mathbb{Z}_2$-graded ring $B = B^+ \oplus B^-$ (alternative notation $B^{ev} \oplus B^{odd}$). Then a ring hom. $R(X) \to B$ respecting the grading is a collection of elements:

$$\alpha^+_h : H^*_h(X^h) \to B^+, \quad \alpha^-_h : H^*_h(X^h) \to B^-$$

or equivalently elements

$$\alpha^+_h \in \left( \frac{H^*}{\sum_h} (X^h) \otimes B \right)^{ev}$$

such that

$$\sum_{i+j} \frac{\alpha^+_i \times \alpha^-_j}{\sum_i \times \sum_j} \alpha^+_{i+j} = \alpha^+_i \otimes \alpha^-_j,$$

with the convention that $\alpha^+_0 = 1$.

We denote by

$$\prod_{h \geq 0} H^*_{\sum_h} (X^h; B)$$

the set of such families $(\alpha_h)_{h \geq 0}$. As we work always with elements of even degree there should be no problem with signs so this should form a ring natural in $B$. Thus $R(X)$ should be an affine ring scheme in the $\mathbb{Z}_2$-graded framework.

(This is all fine except that we would like the formula

$$\text{Hom}_{\text{args}}(R(X), B) \cong \text{Hom}_{\text{args}}(R,J,B) \otimes H^*(X)$$

where args is the category of skew rings. In the mod 2 case we proved this formula by means of the $\psi$ map.)
\[ \operatorname{Hom}_{\Sigma_p}(R(X), B) = \prod_k H_{\Sigma_k}(X^k, B) \xrightarrow{\cong} \prod_k H_{\Sigma_k}(pt, B) \otimes H(X) \]

\[ (d_k) \cdot (a_R^X) \xrightarrow{\cong} (d_k) \otimes a \]

and the fact that we had an isomorphism is because the hom.

fitted into

\[ \prod_k H_{\Sigma_k}(X^k, B) \]

\[ \xrightarrow{\cong} \prod_k H(B\Sigma_k, H(X^k) \otimes B) \]

\[ \cong \prod_k H(B\Sigma_k, B) \otimes H(X) \]

since \[ \prod_k H(B\Sigma_k, V^k \otimes B) \] additive in \( V \).

Somehow this all has to be generalized, which means ultimately
that \( R(X) \) has to be enlarged to include twisted coefficients.
Before doing this we should work out the operations

corresponding to \( Z_p \subset \Sigma_p \).

Start with

\[ P: H^{ev}(X) \longrightarrow H^{ev}(B\mathbb{Z}_p \times X)^N \]

where \( N \) is the normalizer of \( Z_p \) in \( \Sigma_p \), in this case \( N = Z_p^* = \mathbb{Z}_{p-1} \).

Now \( P \) is a ring homomorphism since after inverting \( v = e(N) \)

it is and since \( v \) is a non-zero divisor in \( \mathbb{Z}_p \).

\[ H^*(B\mathbb{Z}_p) \cong \mathbb{Z} \wedge [u] \otimes S[v] \]
Here $u$ denotes the generator of $H^1(B\mathbb{Z}_p)$ with $\beta u = v$.

Now if $i \in \mathbb{Z}_p^*$ then $i^*(u) = i^* u$ so
\[
H^*(B\mathbb{Z}_p) \cong \wedge [dw] \otimes S[w]
\]
where $w = c_{p-1}(\text{reg } \mathbb{Z}_p) = -vp^{-1}$ (Wilson's theorem $(p-1)! \equiv -1 \pmod{p}$) and $dw$ is an abusive notation for $-uv^2$, so that $\beta dw = w$. So set

\[
P_x = \sum_{i=0} (w^i \mathcal{S} x) + \sum_{i=1} dw^i w^{i-1} \mathcal{S} x
\]

where if $x \in H^Q(X)$, then
\[
\deg(\mathcal{S} x) = 2p_g - 2i(p-1) = 2p_g + 2(p-i)(g-i)
\]
\[
\deg(\mathcal{S} x) = 2p_g - 2i(p-1) + 1 = 2p_g + 2(p-i)(g-i) + 1
\]

Now these aren't stable operations and the correct stable operations...
\[ w^{-8} P_x = \sum_{i \geq 0} w^{-8+i} S_t^i x + \sum_{j \geq 1} dw \cdot w^{j-1} S_t^i x \]

total degree
\[ 2pq - q^2(p-1) = 2q \]

Not clear yet, but perhaps will become so when the Milnor generators are introduced.

Suppose fixed therefore elements \( dw, w \) such that

\[ H^*(B\mathbb{Z}_p)^N = \Lambda[ dw ] \otimes S[ w ] \]

Then I get a basis \( w^i, dw \cdot w^i \) for \( H^*(B\mathbb{Z}_p)^N \) and I let \( \delta_x^i, \varepsilon_x^i \in H_*(B\mathbb{Z}_p)_N \to H_*(B\mathbb{S}_p) \) be the dual basis.

Then
\[
\deg (\delta_x^i) = 2i (p-1) - 1 \quad i \geq 1 \\
\deg (\varepsilon_x^i) = 2i (p-1) \quad i > 0
\]

and we have the formula
\[ P_x = \sum_{i \geq 0} w^i \langle \delta_x^i, P_x \rangle + \sum_{j \geq 1} dw \cdot w^{j-1} \langle \varepsilon_x^i, P_x \rangle \\
S_t^i x \quad S_t^j x \]
Up to isogeny we have that for $x \in H^1_\ell(X)$

\[ S^x_i x = p^{g-i} x \]

\[ SE_x x = \beta p^{g-2} x \]
January 26, 1970

The following example arose in conversation with Scherlau and shows that Norm, for a double covering is a new operation not expressible in terms of $f^*$. Scherlau asked whether there was a simple formula for $w_1(f^*E)$, where $f: X \rightarrow Y$ is a double covering, $E$ a real vector bundle on $X$, in terms of the characteristic classes of $E$ and the covering and $f$.

If $E$ is a line bundle, then we know that

$$w_1(f^*E) = f^*(w_1E) + r$$

$\ r = \text{char. class of the covering } f$

$$w_2(f^*E) = \text{Norm}_f w_1(E)$$

Suppose that $w_2(f^*E)$ could be expressed in terms of $r$, $f^*$, $w_1(E)$. The relevant monomials are

$$f^*(w_1E)^2, (f^*(w_1E))^2, \ r f^*w_1(E), r^2$$

and these are the only way to get elements of degree 2 in the base. Now apply $f^*$. Then

$$f^*w_2(f^*E) = w_2(f^*f^*E) = w_2(E + \sigma E) = w_1(E) \cdot \sigma w_1(E)$$

where $\sigma$ is the $\mathbb{Z}_2$-action on $X$.

$$f^* f^* w_1(E)^2 = w_1(E)^2 + (\sigma w_1(E))^2$$

$$f^* (f^*(w_1(E))^2 = (w_1E + \sigma w_1(E))^2 = (w_1E)^2 + (\sigma w_1E)^2$$

$$f^*(\ r f^*w_1(E)) = \ r (w_1E + \sigma w_1(E))$$

$$f^* r^2 = r^2$$
and you see that you don't get the cross term

$\omega_1(\mathcal{E}) \cdot \omega_2(\mathcal{E})$

from the expressions.
January 26, 1970

Let $p$ be an odd prime and work over $\mathbb{Z}_p$. I propose to complete the symmetric invariants in $\Lambda \left[ x_1, \ldots, x_n \right] \otimes S[y_1, \ldots, y_n]$. We regard this as the algebraic de Rham complex of affine $n$-space with $d y_i = x_i$. Let $c_i$ be the $i$th symmetric function of $y_1, \ldots, y_n$. I claim that the natural map

\[
\Lambda \left[ d c_1, \ldots, d c_n \right] \otimes S[c_1, \ldots, c_n] \xrightarrow{\cong} \Lambda \left[ x_1, \ldots, x_n \right] \otimes S[y_1, \ldots, y_n]^\Sigma_n
\]

if $\frac{1}{p}$ exists.

is an isomorphism. Now we know this map is injective since

\[
d c_1 \wedge \cdots \wedge d c_n = \text{Jac} \left( \frac{c_1 c_2 \cdots c_n}{y_1, \ldots, y_n} \right) \, d y_1 \wedge \cdots \wedge d y_n
\]

and the Jacobian is non-zero since the map $\mathbb{A}_n \to \mathbb{A}_n$, given by the $c_i$, is generically etale (invariants of a group acting on a field gives a separable extension). Actually

\[
\text{Jac} \left( \frac{c_1 c_2 \cdots c_n}{y_1, \ldots, y_n} \right) = \pm \prod_{i \neq j} (y_i - y_j)
\]

since the right side divides the lift and the degrees are equal $\left( \frac{n(n-1)}{2} \right)$. This we did before.

The new idea is to note that for a Galois covering $X \xrightarrow{f} Y$

\[
\Delta^* = f^*(\Delta^*)^G.
\]

In fact for any vector bundle $E$ on $Y$ one has

\[
(f^*E)^G = E \text{ and } f^*\Delta^* = \Delta^* \text{ as } f \text{ is etale. Now this means that}
\]

\[
\Delta = \prod_{i \neq j} (y_i - y_j)^2
\]

The map $\otimes$ becomes an isomorphism. Suppose that $\Lambda = a d y_1 \wedge \cdots \wedge d y_n$. 

is an invariant $n$-form. Then $a$ is skew-invariant, hence
divisible by $\mathcal{J}^a$ and so
\[ a = f(c_1, \ldots, c_n) \mathcal{J}^a, \text{ so } \lambda = f(c) dc_1 \wedge \cdots \wedge dc_n \]
Suppose $\lambda$ is an invariant $q$-form. By the above argument $J^N$
\[ \Delta^N : \lambda = \sum_{c_1 < \cdots < c_q} f_{c_1, \ldots, c_q} dc_1 \wedge \cdots \wedge dc_q \]
where $f_{c_1, \ldots, c_q}$ are elements of $S[c]$. Assume $N$ least such
that this holds. Let $I = (c_1, \ldots, c_q)$ be complementary to $(c_1, \ldots, c_q)$
(Hörmander's notation), then
\[ \Delta^N : \lambda \cdot dc_I = f_I dc_1 \wedge \cdots \wedge dc_n \]
but $\lambda \cdot dc_I$ is invariant hence of the form $q(c) dc_1 \wedge \cdots \wedge dc_n$, so
\[ \Delta^N : q = f_I \]
for all $I$ showing that $N = 0$ by minimality. Thus (x) is proved. (This argument generalizes to other Lie groups, see page 6)

Can the above argument be generalized to Dickson's theorem?

Let $V$ be a vector space of dimension $n$ over $F_q$. No set
\[ \Pi^I (x + \lambda) = x^n + c_{n-1}^{n-1} X^{n-1} + \cdots + c_0^{n-1} X \]
where the $c_i$'s are the analogues of the elementary symmetric functions
and we know that
Consider the mapping \( \pi : V \to \Omega^n \) given by the functions \( g_j \).
If the \( \pi \) action of \( \text{Gl}(V) \) at a geometric point \( \pi \) is not free, say \( g \pi = \pi \) where \( g \neq 1 \), then \( \ker (g-1) \) is defined over \( \mathbb{F}_p \); it follows that \( \pi \in U W_0 \) where \( W_0 \) runs over the codimension 1 subspaces of \( V \). Thus the bad set of \( \pi \) is this union of \( \pi \) where \( c_{g^{-1}} = 0 \). Therefore as \( \pi \) is a Galois covering, on the complement we have that

\[
\Lambda \left[ \frac{d c_{g^{-1}}}{g^{-1}} \cdots, \frac{d c_2}{c_2} \right] \otimes S[\frac{c_{g^{-1}}}{c_{g^{-1}}}, \cdots, \frac{c_{g^{-1}}}{c_{g^{-1}}}] \longrightarrow (\Lambda V^* \otimes S V^*)^{\text{Gl}(V)}
\]

is injective and \( \Lambda V^* \otimes S V^* \) becomes an isomorphism after \( c_{g^{-1}} \) is inverted.
Suppose that \( \lambda \in \Lambda^n V^* \otimes S V^* \) is an invariant \( n \)-form.
Choose a basis \( e_1, \ldots, e_n \) and let \( y_1, \ldots, y_n \) be the dual basis of \( V^* \).
Then

\[
\lambda = a \, dy_1 \cdots dy_n
\]

where since

\[
g^* (dy_1 \cdots dy_n) = (\det g)(dy_1 \cdots dy_n)
\]

we have

\[
g^* (a) = (\det g)^{-1} a.
\]

Let \( S \subset V^* - 0 \) be a set of representatives for the lines in \( V^* \)
and set

\[
\frac{c_{g^{-1}}}{c_{g^{-1}}} = \prod_{\lambda \in S} \lambda
\]

Note that

\[
c_{g^{-1}} = \prod_{\lambda \in S} \frac{c_{g^{-1}}}{c_{g^{-1}}} \prod_{z \in \mathbb{F}_p^*} \frac{z}{\lambda} = (-1)^{\mathbb{F}_p^*} c_{g^{-1}}
\]
and that \( Y \) divides any element of \( S(V^*) \) which vanishes on \( U \). It follows that \( Y \) is a semi-invariant so

\[
g^*Y = (\det g)^a Y \quad \text{for some } a \in \mathbb{Z}_{8-1}
\]

To determine a take \( g \) to be a scalar matrix, whence

\[
g^*Y = z^{\frac{n}{8-1}} Y = z^{na} Y
\]

for all \( z \in \mathbb{F}_8^* \), hence

\[
n = \frac{\frac{n}{8-1}}{8-1} \equiv na \pmod{8-1}
\]

which tends to suggest \( a = 1 \) but doesn't prove that \( a = 1 \) unless \( n \) is prime to \( 8-1 \). To do it correctly use embedding of \( GL_1 \) in \( GL_n \). In fact since we've chosen a basis \( y_1, \ldots, y_n \) for \( V^* \) it follows that we can take

\[
\mathcal{J} = \prod \left( \frac{y_1 + z y_2 + \cdots + z^a y_n}{(z, z^2, \ldots, z^a) \in \mathbb{F}_8^a} \right)
\]

where \( W \) is the subspace \( y_1 = 0 \). Thus if \( g = \begin{pmatrix} z & 1 \\ \end{pmatrix} \) we have

\[
g^*Y = \prod \left( \frac{z y_1 + \mu}{\mu \in \mathbb{F}_8^*} \right)
\]

\[
= \prod \left( \frac{z y_1 + z \mu}{\mu \in \mathbb{F}_8^*} \right)
\]

\[
= z^{\frac{n}{8-1}} Y = z Y \quad \text{since } z^{\frac{n}{8-1}} \equiv 1 \pmod{8-1}
\]
Thus

\[ g^* \gamma = (\det g) \gamma \quad \text{all } g \in \text{Gl}(V) \]

Let \( f \in S(V^*) \) be a semi-invariant not an invariant, i.e. \( g^* f = \delta g (\det g)^{\varepsilon} f \) where \( 0 < \varepsilon < g-1 \). I claim that \( f \) is divisible by \( \gamma \). It suffices to show that \( f \) vanishes on a hyperplane \( W \). Take \( W : y_1 = 0 \) and let \( g \) be \( \begin{pmatrix} z & 1 \end{pmatrix} \) matrix. Then \((g^* f)(w) = f(gw) = f(w)\) and \((g^* f)(w) = (\det g)^{\varepsilon} f(w) = z^{\varepsilon} f(w)\). Thus \( f(w) = 0 \). So we have proved

**Lemma:** Let \( f \in S(V^*) \) satisfy \( g^* f = (\det g)^{\varepsilon} f \) for all \( g \in \text{Gl}(V) \) where \( \varepsilon \) is an integer \( 0 < \varepsilon < g-1 \). Then

\[ f = \gamma^{\varepsilon} f_+ \]

where \( f_+ \in S(V^*)^{\text{Gl}(V)} \)

As an application consider

\[ dc_{g^n_{-n-1}} \cdots dc_{g_{-1}} = \text{Jac} \left[ \begin{array}{c} \frac{g^n_{-n-1}}{y_1} \cdots \frac{g_{-1}}{y_{n-1}} \\ y_1 \cdots y_{n-1} \end{array} \right] dy_1 \cdots dy_n \]

Then the Jacobian \( J \) has degree

\[ (g^n_{-n-1-1}) + \cdots + (g_{-1-1}) = n(g_{-1}) - \frac{g_{-1}}{g-1} \]

so

\[ \text{Jac} \left[ \begin{array}{c} \frac{g^n_{-n-1}}{y_1} \cdots \frac{g_{-1}}{y_{n-1}} \\ y_1 \cdots y_{n-1} \end{array} \right] = d \cdot \gamma^{n-2} \cdot c_{g_{-1}}^{n-1} = d' \cdot \gamma^{n^2-n-1} \]

where \( d, d' \) are non-zero constants.
Consequently the map (\(*\)) is not an isomorphism on n-forms since \(\gamma^{n-2} dy_1 \wedge \cdots \wedge dy_n\) is invariant yet not a multiple of \(\text{Jac.} dy_1 \wedge \cdots \wedge dy_n\).

I yet don't have a conjecture as to what the invariant forms are and propose now to compute for \(n=2\).

Let \(V^e = F^e_y \oplus F^e_{y_2}^e\). Then

\[
\prod_{y_2} (x + z_1, y_1 + z_2, y_2) = \prod_{y_1} \left[(x + z_1)_{y_1}^{y_2} - (x + z_2)_{y_2}^{y_1}\right]^{-1}
\]

\[
= \prod_{y_1} \left(x^{\delta^0} - x y_2^{\delta^1} + 2 (y_1^{\delta^0} - y_1 y_2^{\delta^1})\right)
\]

\[
= (x^{\delta^0} - x y_2^{\delta^1})^{\delta^0} + 2 (y_1^{\delta^0} - y_1 y_2^{\delta^1})^{\delta^1}
\]

\[
= x^{\delta^0} - x y_2^{\delta^1} + 2 (y_1^{\delta^0} - y_1 y_2^{\delta^1})^{\delta^1}
\]

Borel has pointed out for a semi-simple Lie algebra (maybe even reductive) that the Jacobian \(\text{Jac} \begin{bmatrix} \frac{e_1}{y_1}, \cdots, \frac{e_n}{y_2} \end{bmatrix}\) has the same degree as \(\Delta^2\) where \(\Delta\) is the basic anti-invariant (product of the positive roots) at least in char. 0. Thus if \(\text{mod } \rho \neq 0\), \(H(\mathcal{B}/T) \rightarrow H(\mathcal{G}/T)\) is onto, then

\(H(\mathcal{B}/G) = H(\mathcal{B}/T)^W \quad \rho \neq 2 \quad \Rightarrow \quad H(\mathcal{B}/G) \otimes H(\mathcal{G}) \sim (H(\mathcal{B}/T) \otimes H(\mathcal{T}))^W\)

because the argument given on pages 1+2 generalizes.
there is a spectral sequence associated with $A$, $H^*(G) \otimes H^*(G) \rightarrow H^*(G \times G) \rightarrow H^*(G)$.

As noted, it is clear that $H^*(G)$ is connected, hence $Z$ is connected.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra, and by the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

Hence, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.

By the classification theorem, $H^*(G)$ is a finite-dimensional algebra.

The claim is that $H^*(G)$ is a finite-dimensional algebra.
Proof: Consider $H^*_A(G)$ where $A$ acts on $G$ by conjugation. Since $l$ is odd for $G$, we know that the fibre is thick in the spectral sequence

$$H^*_A \otimes H^*(G) \Rightarrow H^*_A(G)$$

so $H^*_A(G)$ is a free $H^*_A$ module with

$$H^*_A(G) \otimes_{H^*_A} \mathbb{Z}_2 \cong H^*(G).$$

Let $e_i$ be a basis for $PH^*(G)$ and lift them to elements $\tilde{e}_i$ of $H^*_A(G)$. Since $l$ is odd the $\tilde{e}_i$ generate an exterior subalgebras and so there is an algebra isomorphism

$$H^*_A(G) \cong S(A^*) \otimes \Lambda^{\ast} \otimes H^*(G)$$

which shows that $H^*_A(G)$ is an exterior algebra over $S(A^*)$ with odd degree generators.

By the localization thm, there is an isomorphism

$$H^*_A(G)[w^{-1}] \cong H^*_A(Z)[w^{-1}],$$

and as the latter is isom. to $S(A^*)[w^{-1}] \otimes H^*(Z)$, it follows

by looking at the odd-even grading of $Z$ that $H^*_A(Z)$ is generated by its elements of odd degree. Since $l$ is odd and
$H^*(Z)$ is a Hopf algebra. $H^*(Z)$ is an exterior algebra by Borel's theorem. Precisely:

Thus $Z$ is connected and $k$ is good for $Z$. 
January 27, 1970

$H^*(B\Sigma_n, \mathbb{Z}_p)$ where $p$ is odd (cont.)

A review of the Milnor description of the dual of the Steenrod algebra:

Following Grothendieck, for any (anti-)commutative graded $\mathbb{Z}_p$-algebra $S$, introduce the base extension $H^*(X) \otimes S$ and consider the group of $S$-automorphisms of $H^*(X) \otimes S$. This gives a functor from $S$ to groups which is represented by the dual of the Steenrod algebra, $H^*_k(\mathbb{K}(\mathbb{Z}_p, \infty)) = A_\ast$. Thus any multiplicative stable operation

$$H^*(X) \longrightarrow S \otimes H^*(X)$$

is obtained from the universal one

$$\Theta: H^*(X) \longrightarrow A_\ast \otimes H^*(X)$$

by composition with a ring homomorphism $A_\ast \longrightarrow S$.

According to Milnor if $\eta \in H^1(X)$, then

$$\Theta(\eta) = 1 \otimes \eta + \sum_{i>0} \tau_i \otimes (\beta_\eta)_{p^i} \quad \tau_i \in A_{p^{i-1}}$$

$$\Theta(\beta \eta) = \sum_{i>0} \tau_i \otimes (\beta \eta)_{p^i} \quad \tau_i \in A_{p^{i-2}}, \, \tau_0 = 1$$

and moreover

$$\Lambda[\tau_1, \tau_2, \ldots] \otimes S[\beta_1, \ldots] \longrightarrow A_\ast$$

Now I want to apply these results to analyze the
cohomology operations obtained from the Klein group.

Fix an integer \( r \) and consider the operation

\[
H^e(X) \xrightarrow{P} H^e(BZ_p^r \times X).
\]

Then we know geometrically that

\[
P \cdot e(L) = e(p \otimes L)
\]

where \( p = \text{reg } Z_p^r \)

\[
= \sum_{i=0}^{n} c_{p^r-p_i} \cdot e(L)^{p_i}
\]

If \( \sigma \) is the generator of \( \mathbb{Z}_2 \), then

\[ P \sigma = c_{p^r-1} \sigma \]

showing that if we invert \( c_{p^r-1} \), we obtain a stable operation. As

\[ c_{p^r-1} \]

is a non-zero divisor in \( H^*(BZ_p^r) \), it follows that \( P \)

must be additive before inverting \( c_{p^r-1} \). Let \( R \) be the stable operation obtained from \( P \):

\[
R \cdot x = (c_{p^r-1})^{-1} \cdot P \cdot x \quad \text{if } x \in H^2(X)
\]

Then \( R \) extends to all degrees

\[
R: H^*(X) \longrightarrow H^*(BZ_p^r)[c_{p^r-1}] \otimes H^*(X)
\]

and hence there is a canonical ring homomorphism

\[
\Xi: A^* \longrightarrow H^*(BZ_p^r)[c_{p^r-1}]
\]

which I will now very carefully calculate. According to Milnor we must see what happens to the \( \xi_2, \xi_3 \), or equivalently...
what happens to one-dimensional classes and their Bockstein.

As \( \beta \eta \) is an Euler class I know that

\[
R(\beta \eta) = c_{p-1}^{-1} \sum_{i=0}^{n} c_{p-p^i}(\beta \eta)^{p^i}
\]

So

\[
\Psi(\xi_i) = \frac{c_{p-p^i}}{c_{p-1}} \quad i \geq 0
\]
we go back to operations associated to $\mathbb{Z}_p^*$.
Thus we get a map

$$H^e_0(X) \rightarrow H^e_0(B\mathbb{Z}_p^* \times X) \xrightarrow{Gl} (A^* \otimes S^*v)^* \otimes H^*(X)$$

conjecture

$$A[d_{\alpha_0, ..., \alpha_n}] \otimes S[\alpha_0, ..., \alpha_n] \otimes H^*(X)$$

Idea is that you have basic operation

$$H^*(X) \xrightarrow{\otimes} A \otimes H^*(X)$$

universal for stable operations, so add in $A[t, t^{-1}] \otimes H^*(X)$

Thus when you construct your map

$$H^e_0(X) \rightarrow \{R \otimes H^*(X)\}^{w, *}_{\otimes}$$

where

$$R = (H(B\mathbb{Z}_p^*)[c_{p^{-1}}^{-1}])^{Gl_n(\mathbb{Z}_p)}$$

you get a homomorphism

$$A[t, t^{-1}] \rightarrow R$$

So here's the situation:

$$H^e_0(X) \xrightarrow{p} H(B\mathbb{Z}_p^* \times X) \xrightarrow{c_{p^{-1}}}$$

gives a basic homomorphism from $A[t, t^{-1}] \xrightarrow{\alpha} H^*(B\mathbb{Z}_p^*)$

which I want to understand. According to Minhye, there is a nice way of describing $A$. 

$$\alpha$$
The idea is to introduce maps as a diagonal of sorts mimicking the Bott-Samelson pairing.

More significant is

\[
\begin{align*}
E_2^{pq} &= H_p^*(BG) \otimes H_q^*(G) \\
E_2^{pq} &= H_p^*(BG) \otimes H_q^*(G) \\
\end{align*}
\]

in the spectral sequence

\[
E_2^{pq} = H_p^*(BG) \otimes H_q^*(G) \Rightarrow H_*(kt)
\]

One knows that \(E_2^{pq}\) admits a module structure over \(E_2^{pq}\).

One knows that \(E_2^{pq}\) admits a module structure over \(E_2^{pq}\).

\[
d_2(x, y) = d_2(x, y) + (-1)^{deg x} x \cdot d_2 y.
\]

What is the nature of this pairing?

Now intuitively we have a map

\[
H_*(BG) \otimes H_*(G)
\]

1) Why the "norm" is equivalent to $\text{Pext}$:

Recall how we define the norm map for a finite covering of degree $k$, $f: X \to Y$, in terms of $\text{Pext}$.

$$
\begin{align*}
U^2g(X) \xrightarrow{\text{Pext}} U^2_{\Sigma k} (X^k) \xrightarrow{\text{Norm}} U^2_{\Sigma k} (X^k)_{\text{reg}} \simeq U^2_{\Sigma k} (Y)
\end{align*}
$$

where $(X/Y)^k_{\text{reg}}$ is the subset of $(X/Y)^k$ consisting of $k$-tuples $(x_1, \ldots, x_k)$ where the $x_i$ are distinct. (Alternatively, if we think of $\Sigma k$ as the automorphism of a set $S$, then

$$(X/Y)^k_{\text{reg}} = \text{Iso}_y (y \times S, X)$$

is the principal bundle describing the covering. Call this $\text{P}$. Then on lifting to $\text{P}$ we get a tautological isomorphism of $y \times S$ with $X$ but not equivariant for $\Sigma k$.)

Here's how to define $\text{Pext}$ in terms of the norm.

Let $S$ have $k$ elements and let $\Sigma'$ be Aut $S$. Then have maps

$$
\begin{align*}
X & \xleftarrow{\text{ev}} X^S \times S \xrightarrow{pr_1} X^S \\
(f) & \leftarrow \circ (f, s)
\end{align*}
$$

and $\text{Pext}$ is the composition

$$
\begin{align*}
U^2g(X) \xrightarrow{\text{ind}^k} U^2_{\Sigma} (X) \xrightarrow{(\text{ev})^k} U^2_{\Sigma} (X^S \times S) \xrightarrow{\text{Norm}_{\Sigma'}} U^2_{\Sigma'} (X^S).
\end{align*}
$$
This definition of $P$ yields the same $P$ in virtue of the commutative diagram

$\xymatrix{ U_{2^k}(X) \ar[r]^{(ev)} & U_{2^k}(X^0 \times S) \ar[r]^{\text{Norm}_{p_1}} & U_{2^k}(X^0) \ar[r]^(0.4){\Delta^k} & U_{2^k}(X) }$

2) Steenrod operations for sheaves:

Given a finite covering $f: X \to Y$ and a sheaf $F$ of abelian groups on $X$ we have the sheaf

$$(f_* F)_y = \bigoplus_{x \in f^{-1}(y)} F_x$$

and also the analogues of the other elementary symmetric functions

$$(\text{Norm}_{f,0} F)_y = \bigotimes_{x \in f^{-1}(y)} F_x$$

$$(\sigma_j F)_y = \bigoplus_{I \subseteq f^{-1}(y)} \bigotimes_{x \in I} F_x$$

Thus,

$$\bigoplus \sigma_j F = \text{Norm}_{f}(\mathbb{Z} \oplus F).$$
Corresponding to the map (in fact isomorphism)

\[ H^0(X, F) \longrightarrow H^0(Y, f_* F) \]

there is the map

\[ H^0(X, F) \longrightarrow H^0(Y, \text{Norm}_f F) \]

\[ s \mapsto \left( y \mapsto \bigotimes_{x \in f^{-1}(y)} s_x \right) = (\text{Norm}_f s)_y. \]

The question now is to extend the norm of a section to higher cohomology classes.

The first step is to note that the norm map extends to complexes of sheaves. Thus if \( F^\cdot \) is a complex of sheaves (bounded below?) I can define

\[ (\text{Norm}_f F^\cdot)_y = \bigotimes_{x \in f^{-1}(y)} F^\cdot_x. \]

Now to write this up precisely will be a real mess, however the simplest approach appears to take \( P = (X/Y)_{\text{reg}} \) where \( k \) is the degree of \( f \) so that

\[ \begin{array}{c}
\text{P} \\
\downarrow \\
\text{P} \end{array} \xrightarrow{\text{P} \{b \mapsto k\}} \xrightarrow{X} \xrightarrow{Y} \]

where the horizontal arrows are principal \( \Sigma_k \) bundles. Then I have maps \( p_i : P \rightarrow X \) for \( i = 1, \ldots, k \) and I form the tensor product

\[ \bigotimes_{1 \leq i \leq k} p_i^* F^\cdot. \]
Now using the natural associativity and commutativity isomorphisms for the tensor product of sheaves, I see this tensor product has a natural $\Sigma^k$ action and so descends to a complex of sheaves on $Y$ which is denoted $\text{Norm}_f F^*$.

Suppose for simplicity that I am working with sheaves of $\mathbb{Z}_p$-modules. Then $F^* \to \text{Norm}_f F^*$ carries quasi-isos to quasi-isos and hence passes to the derived category. Next given a class $u \in H^8(X, F^*)$ where $F^*$ is a flat complex we may identify $u$ with a homomorphism $u: \Sigma^0 \to F^*$ whence we get a map

$$\text{Norm}_f (\Sigma^0) \to \text{Norm}_f F^*.$$

Now $\Sigma^0 = \mathbb{Z}_p[0]$ and one computes easily that

$$\text{Norm}_f (\Sigma^0) = \mathbb{Z}_p[0] \otimes \mathbb{Z}_p[0],$$

where $\mathcal{O}_f$ is the orientation bundle of the covering, that is the sheaf $P \times_{\Sigma^d} \mathbb{Z}_p[0]$ where $\Sigma^d$ acts by the sign representation on $\mathbb{Z}_p[0]$.

Therefore finally we get a map

$$\text{Norm}_f : H^8(X, F^*) \to H^8(Y, \text{Norm}_f F^* \otimes \mathcal{O}_f^8),$$

well-defined on $D^+(X, \mathbb{Z}_p)$.
Multiplicative property of the norm: Suppose given \( f : X \to Y \) a covering of degree \( d \) and \( f_!^* G' \in D^+(X, \mathbb{Z}_p) \) and \( u \in H^p(X, F) \) and \( v \in H^q(X, G) \). Then I have the element \( u \cdot v \in H^{p+q}(X, F \otimes G) \) defined as follows. Identify \( u \) with a map \( \mathbb{Z}_p[p] \to F \) and \( v \) with a map \( \mathbb{Z}_p[q] \to G \), then \( u \cdot v \) is the composition

\[
\mathbb{Z}_p[p+q] \cong \mathbb{Z}_p[p] \otimes \mathbb{Z}_p[q] \xrightarrow{u \otimes v} F \otimes G.
\]

Now I want to show why

\[
\text{Norm}_f(u \cdot v) = \text{Norm}_f u \cdot \text{Norm}_f v \quad (-1)
\]

and keep things sign-consistent.

First to understand multiplicativity of \( \text{Norm} \) on sheaves:

\[
\text{Norm}_f(F \otimes G) = \bigotimes_{x \in Y} (F \otimes G)_x \cong \bigotimes_{x \in Y} F_x \otimes G_x
\]

\[
\cong \bigotimes_{x \in Y} F_x \otimes \bigotimes_{x \in Y} G_x
\]

\[
\cong \text{Norm}_f F \otimes \text{Norm}_f G.
\]

Thus there is a canonical isomorphism

\[
\text{Norm}_f(F \otimes G) \cong (\text{Norm}_f F) \otimes (\text{Norm}_f G)
\]

such that if \( s \in H^p(X, F) \) and \( t \in H^q(X, G) \), then under this isom
\[ \text{Norm}(s \otimes t) = \text{Norm}(s) \otimes \text{Norm}(t). \]

The next stage is to understand the norm for complexes but again there is no problem. There is a canonical isomorphism:

\[ \text{Norm}_f (F \otimes G) \cong (\text{Norm}_f F') \otimes (\text{Norm}_f G'). \]

(What we are using here is that we have a fibered category over spaces with a good tensor product and Galois descent.) Thus the Norm map can be defined for vector bundles, fiber spaces, etc.

Now suppose given a cohomology class \( u \in H^q(X, F') \)

First for \( q=0 \) we note that we have a canonical map:

\[ \text{Norm}_f : H^0(X, F') \to H^0(Y, \text{Norm}(F')). \]

Which is compatible with the tensor product isomorphism (\( \ast \)). This because \( F' \mapsto \text{Norm}_f F' \) is a functor and because \( \text{Norm}_f \mathbb{Z} \) is canonically isomorphic to \( \mathbb{Z} \).

Next point is to ask whether the functor:

\[ \begin{array}{ccc}
C^+(X) & \xrightarrow{\text{Norm}} & C^+(Y) \\
\downarrow & & \downarrow \\
D^+(X) & & D^+(Y)
\end{array} \]

has a derived functor of some sort. Now my instincts tell me...
that I want the left-derived functor because the norm somehow is like an \( f_! \). But to keep things simple suppose we are over a field \( K \). Then it is clear that \( F^* \to \text{Norm } F^* \) preserves quasi-isomorphisms in fact since

\[
H^*(\text{Norm } F^*) = \bigotimes_{x \in \text{f}y} H^*(F^*) = (\text{Norm } H(F))^y
\]

where \( H^* \) denotes the homology of a complex. Thus \( \text{Norm} \) extends to the derived categories and we get a map

\[
\text{Norm}_f : H^0(X, F^*) \to H^0(X, \text{Norm}(F^*)
\]

For higher cohomology we use the suspension isomorphism

\[
H^0(X, F^*) \cong H^0(X, \Sigma^g F^*)
\]

where

\[
\Sigma^g F^* = \Sigma^g \otimes F^*
\]

and \( \Sigma^g \) is the complex with generator \( g \) and \( dg = 0 \). Then we have

\[
H^0(X, F^*) \cong H^0(X, \Sigma^g F^*)
\]

\[
\downarrow \text{Norm}_f
\]

\[
H^0(Y, \text{Norm}_f \Sigma^g F^*)
\]

\[
\downarrow \text{Norm}_f
\]

\[
H^0(Y, \Sigma^g \otimes \text{Norm}_f F^*)
\]

where we have used an isomorphism
whose specific properties will determine the sign behavior. The formula that we want is

\[ \text{Norm}_f(u \cdot v) = (-1)^{\text{Norm}_f(u) \cdot \text{Norm}_f(v)} \]

To see that this is correct we note that for the trivial covering \( Y \times \{1, \ldots, k\} \) we want \( \text{Norm}_f^Y = y^d \) and that the sign difference between \( (y, y_2) \) and \( y_1 y_2 \) is as above. This formula forces us to use the isomorphism

\[ \text{Norm}_f \Sigma^\delta = P \prod_{\sigma}^{d(d-1)/2} (\sigma_0 \otimes \ldots \otimes \sigma_d) \]

which identifies

\[ \sigma_0 \otimes \ldots \otimes \sigma_d \leftrightarrow (\sigma_{d0} \otimes 1_g, (-1)) \]

The way to see the sign is to use that \( \sigma = \sigma_0^d \) and that

\[ \sigma \odot d = (\sigma \odot \delta^{-1}) \odot d = (-1)^{\frac{d(d-1)(g-1)}{2}} \sigma \odot (\delta^{-1}) \odot d \]

\[ = (-1)^{\frac{d(d-1)}{2} \frac{1}{g-1} + 1} \sigma \odot d \]
\[
\frac{d(d-1)}{2} \cdot \frac{g-1}{2} = \left( \frac{\sigma_g \otimes 1^g}{2} \right)^2 \\
= \left( -1 \right)^\frac{d(d-1)}{2} \cdot \frac{g(g-1)}{2} = \left( \sigma_g \otimes 1^g \right)^2
\]

**Conclusion:** If \( u \in H^g(X; F) \), then we can define
\[
\text{Norm}_f(u) \in H^{d,g}(X; \text{Norm}_f(F))
\]

by
\[
\begin{align*}
H^g(X; F) & \xrightarrow{\forall} H^0(X; \sigma_g \otimes F) \\
H^0(Y, \text{Norm}_f(\sigma_g \otimes F)) & \xrightarrow{\forall} H^0(Y, \sigma_g \otimes \text{Norm}_f F)
\end{align*}
\]

where \( \alpha(\sigma_g \otimes 1^g) = (-1)^\frac{d(d-1)}{2} \cdot \frac{g(g-1)}{2} \sigma_g \otimes 1^g \).

Now consider the case where the covering is orientable, by which I mean that the action of \( T_1 \) on the fibre \( f^{-1}y \) has a trivial sign representation. Then the sign representation is trivial. Choosing an orientation...
we get an oriented covering, i.e. $f_*(\mathbb{R})$ is an oriented bundle. Thus for an oriented covering we have an isomorphism $\mathbb{C}^d \otimes \mathbb{Z}$ and so the norm map

$$\text{Norm}_f : H^i_\mathbb{C}(X, F') \to H^i_\mathbb{C}(X, \text{Norm}_f F')$$

is defined. 

Now we are interested in the case where $f$ is the trivial covering $\text{pr}_1 : X \times A \to X$ where $A$ is an elementary abelian $p$-group acting trivially on $X$. Then for any sheaf $\mathcal{F}$ on $X$ we have

$$\text{Norm}_f f^*(\mathcal{F}) = \bigotimes_{a \in A} \mathcal{F}$$

and there is the Steenrod operation

$$u \mapsto \text{Norm}_f (f^*u) : H^i(X, \mathcal{F}) \to H^i_\mathbb{C}(X, \otimes \mathcal{F}), \quad \deg = |A|$$

So if $\mathcal{F}$ is a commutative ring we can compose this with $\otimes \mathcal{F} \to \mathcal{F}$ to get the usual Steenrod map.

to take $\mathcal{F} = \mathbb{Z}_p$ and we have defined

$$P : H^i_\mathbb{C}(X) \to H^i_\mathbb{C}(X)$$

for all $g$

(depending on a choice of orientation for $\mathcal{F}$)

satisfying

$$P(xy) = (-1)^{\frac{\deg(x)\deg(y)}{2}} P_x \cdot P_y$$

$$P(x+y) = P_x + P_y$$

(this requires proof).

Here's how to do things for $p$-odd. Recall that we have defined Steenrod operations

$$P : H^0(X) \rightarrow H^0_G(X)$$

where $G \rightarrow \Sigma_p$ is an oriented representation. The problem now is to compute the effect of $P$ on $x, \beta x$ where $x$ is a 1-dimensional class. First we consider the case where $G = \mathbb{Z}_p \rightarrow \text{Aut}_{\text{sets}}(\mathbb{Z}_p)$ is oriented by the ordering $0, \ldots, p-1$ of $\mathbb{Z}_p$.

Recall that $H^*(\mathbb{Z}_p, \mathbb{Z}_p)$ has a canonical generator $x$ corresponding to the covering $\mathbb{Z}_p \rightarrow \mathbb{Z}$ in the topos of (left) $\mathbb{Z}_p$-sets where $\mathbb{Z}_p$ acts on the right. (More generally in $H^*(\mathbb{Z}_p, G)$ there is a canonical element.) I claim that $\beta x = c(\eta)$ where $\eta$ is representation of $\mathbb{Z}_p$ which sends 1 to $e^{2\pi i / p}$.

Indeed, we have a map of exact sequences

$$
\begin{array}{cccccccc}
0 & \rightarrow & \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} & \rightarrow & \mathbb{Z}_p & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathbb{Z} & \xrightarrow{1} & \mathbb{C} & \xrightarrow{2\pi i p} & \mathbb{C}^* & \rightarrow & 0 \\
\end{array}
$$

which gives rise to a commutative diagram

$$
\begin{array}{ccc}
H^1(X, \mathbb{Z}_p) & \xrightarrow{\delta} & H^1(X, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^1(X, \mathbb{C}^*) & \xrightarrow{\delta'} & H^1(X, \mathbb{C})
\end{array}
$$

for any space $X$. 

By definition \( Sx = \beta x \) and \( S^*Q = c_1(L) \) if \( Q \) is a principal \( E \)-bundle over \( X \) with associated line bundle \( L \). The vertical arrow associates to a principal \( \mathbb{Z}_p \)-bundle \( Y \rightarrow X \) the \( \mathbb{Z}_p \)-principal \( E \)-bundle \( Y \times \mathbb{Z}_p \rightarrow \mathbb{Z}_p \), so the formula \( \beta x = c_1(L) \) is clear. Note this formula is independent of how \( S \) is defined.

Thus we have canonical generators

\[
H^*_Z(\mathbb{Z}_p) = \mathbb{Z}_p[x, \beta x] \quad \text{(p-odd)}
\]

Moreover if we let \( \mathbb{Z}_p^\times \) act on \( \mathbb{Z}_p \) by multiplication, then we have

\[
i^*(x) = ix \quad i \in \mathbb{Z}_p^\times
\]

where \( i^* \) is the map on cohomology induced by \( i: \mathbb{Z}_p \rightarrow \mathbb{Z}_p \).

Now I want to determine the map

\[
P: H^*_Z(\mathbb{Z}_p) \rightarrow H^*_Z(\mathbb{Z}_p \times \mathbb{Z}_p)
\]

as \( P \) is almost a ring homomorphism, it's enough to know \( P \) and \( P(\beta x) \). Since \( \beta x = c_1(L) \) we can use geometry. Thus let \( i^* \) be the zero section with Thom class \( i^*1 \in H^2(L, L-X) \). Then \( P(i^*1) \in H^2(\mathbb{Z}_p \mathbb{Z}_p, \mathbb{Z}_p, L-X) \) must be the Thom class of \( L \) since it has the correct restriction and so taking \( i^* \) we find

\[
P(e(L)) = e(p \otimes L) = \prod_{i=0}^p e(\eta \otimes L) = \prod_{i=0}^p (e(L) + i e(\eta))
\]

as

\[
e(L) = (\beta x)^{p-1} e(L) = (\beta x)^{p-1} e(L)
\]
where to keep from getting lost we let \( y \) (resp. \( x \)) be the generator of the first (resp. 2nd) factor of \( H^1_{\mathbb{Z}_p} \). Thus for an arbitrary \( x \in H^1(x) \) we have the formula

\[
P(\beta x) = (\beta x)^p - (\beta y)^{p-1} \cdot \beta x.
\]

Now \( P x \in H^p_{\mathbb{Z}_p}(X) \) is well-defined after the orientation is chosen, hence is invariant under the action of \( (\mathbb{Z}_p^*)^2 \subset \mathbb{Z}_p^* \). Thus we see there are constants \( \alpha_1, \alpha_2 \in \mathbb{Z}_p \) such that

\[
Px = \alpha_1 (\beta y)^{p-1} x + \alpha_2 (\beta y)^{p-1} \beta x.
\]

To evaluate \( \alpha_1 \), take \( \sigma \) to be the generator \( \sigma \) of \( H^1_c(R) \). Then \( P x \) is the Thom class of the representation of \( \mathbb{Z}_p \) on \( R^p \). We want to restrict to the diagonal subbundle \( R \subset \bigtriangleup R^p \). Now write

\[
R^p = \bigtriangleup(R) + \overline{R^p}
\]

whence the Thom class of the representation of \( R^p \) is the product of the two Thom classes.

\[
\begin{align*}
P^\dagger : R^p &\to \overline{R^p} \\
\Delta &\uparrow \\
\downarrow P^\dagger & \\
R &\to \text{pt}
\end{align*}
\]

\[
U_{R^p} = p_1^* U_R \cdot p_2^* U_{\overline{R^p}}
\]

\[
P \varphi = \Delta^*(U_{R^p}) = \varphi \cdot e(R^p)
\]

where the last has to be
computed. Now \( \overline{R^p} = \eta \oplus \cdots \oplus \eta^{p-1} \) as real representations.

So orienting \( \eta^i \) by means of its complex structure we have

\[
\pm e(\overline{R^p}) = \prod_{i=1}^{p-1} e(\eta^i) = \left( \frac{p-1}{2} ! \right) \left( \beta y \right)^{p-1/2}.
\]

The sign comes from whether or not the orientations agree. If the minus sign appears, choose the other orientation for the covering \( \mathbb{Z}_p \to \mathbb{F}_p \); this choice can be made to define \( P \) in odd dimensions at the beginning so as to make the formulas simpler. With this convention we see that

\[
P^\sigma = \left( \frac{p-1}{2} ! \right) \left( \beta y \right)^{p-1/2} \sigma \quad \text{if} \quad \beta y = 0
\]

Next note that

\[
\beta(Pu) = 0.
\]

This is the same kind of result as the additivity of \( P \) and is proved in Steenrod's book by noting that \( \beta(P \circ t) = \text{ind}_{1 \to \mathbb{Z}_p} \left( \frac{Pu \circ t^0 \cdots t^{p-1}}{p-1} \right) \) restricts to zero on the diagonal. A good proof would involve producing the Pontryagin–Thomas operations

\[
H^*(X, \mathbb{Z}_p) \to H^*(X, \mathbb{Z}_{p^k})
\]

which I don't yet understand.

Assuming this we see from * page 13 that \( \alpha_1 + \alpha_2 = 0 \)

whence
\[ x \in H^*(X) \implies Px = \left( \frac{p-1}{2} \right)! (\beta y)^{\frac{p-1}{2}} x \oplus -\left( \frac{p-1}{2} \right)! (\beta y)^{\frac{p-1}{2} - 1} \beta x \]

Now we can begin to use induction to determine the map

\[ p^{(r)} : H^*(X) \longrightarrow \frac{H^*(X)}{Z_p^r} \]

which is the iterate

\[ H^*(X) \xrightarrow{p^{(1)}} H^*(BZ_p \times X) \xrightarrow{p^{(2)}} H^*(BZ_p^2 \times X) \longrightarrow \cdots \]

To see this note that \( p^{(r)} \mathcal{U} = \mathcal{U} \circ p^r \) as an equivariant class under \( \mathbb{Z}_p^r \) acting on itself by translations and that \( (\mathcal{U} \circ p^r) \circ p^{(r-1)} = p^{(r)} \mathcal{U} \) with \( \mathbb{Z}_p \times \mathbb{Z}_p^{r-1} \) action. These are the same clearly.
February 2, 1970. Stiefel-Whitney operations for $p$ odd. (cont.)

If $u \in H^2(X)$, then $p^{(1)} u \in H^2(\mathbb{Z}_p \times X)^{\mathbb{Z}_p}$ and hence there is an expansion

$$p^{(1)} u = 1 \otimes \alpha(u)_2 + d_{p-1} \otimes \alpha(u)_3 + c_{p-1} \otimes \alpha(u)_2$$

where $\alpha(u)_i$ represents a class in $H^i(X)$ depending naturally on $u$. Forgetting the $\mathbb{Z}_p$ action we see that $\alpha(u)_2 = u^p$. Now we know that

$$p^{(1)} u = 1 \otimes u^p + c_{p-1} \otimes u$$

if $u \in H^2(X, \mathbb{Z})$ since then $u = e(L)$ for some complex line bundle $L$. Apply $\beta$ to both sides

$$0 = \beta p^{(1)} u = 0 + \beta d_{p-1} \otimes \alpha(u)_3 + d_{p-1} \otimes \beta \alpha(u)_3 + c_{p-1} \otimes \beta \alpha(u)_2$$

Recall $d$ is the derivation of degree $-1$ of $H^*(\mathbb{Z}_p)$ induced to $\beta$ on elements of degree $1$, so

$$(d\beta + \beta d) c_{p-1} = \beta dc_{p-1} = (p-1) c_{p-1} = -c_{p-1}$$

Thus we find that

$$\alpha(u)_3 = \beta \alpha(u)_2$$

$$\beta \alpha(u)_3 = 0. \quad \text{(clear from } \beta^2 = 0)$$
I claim that \( \alpha(u)_2 = u \). Indeed, the only zero-
degree cohomology operations are multiplications by elements of \( \mathbb{Z}_p \)
(this follows from \( H^n(\mathbb{K}(\mathbb{Z}_p^n), \mathbb{Z}_p) = \mathbb{P} \) (by Hurewicz); alternatively \( H^b(X) \hookrightarrow H^b(\text{coh}_n(X)) \) and any class of \( H^n(\text{coh}_n X) \) is induced by
map to a sphere. Thus can assume \( u = \text{can}. \text{elt of } H^0(\mathbb{S}^2) \)
and so \( \alpha(u)_2 = u \). Note similarity between this and the proof
that \( p^0 = \text{id} \).

Thus we obtain the formula

\[
P^1(u) = u^0 + dc_{p-1} \cdot \beta u + c_{p-2} \cdot u \quad \text{if } u \in H^2(X)
\]

which for generalization to higher rank elementary abelian
\( p \)-groups should be written

\[
P^1(\beta x) = (\beta x)^0 + c_{p-1}(\beta x)
\]

\[
P^1(\sigma x) = \sigma - dc_{p-1} \cdot \sigma(\beta x) + c_{p-1} \sigma x
\]

\[
= \sigma \left[ dc_{p-1} \cdot \beta x + c_{p-1} \cdot x \right]
\]

The result is that these formulas generalize to

1) \( P^A(\beta x) = \sum_{i=0}^{n-1} c_{p-p^i}(\text{reg} A) \cdot \beta x^{p^i} \)

2) \( P^A(\sigma x) = \sigma \left[ \sum_{i=0}^{n-1} dc_{p-p^i}(\text{reg} A) \cdot \beta x^{p^i} + c_{p-1}(\text{reg} A) \cdot x \right] \)
To prove these formulas we use induction on the rank of $A$, writing $A = B \times \mathbb{Z}_p$ where $y$ is the canonical generator of $\mathbb{H} / \mathbb{Z}_p).$ Then (recall $e(y) = e_y$)

$$e_t(\text{reg } A) = \prod_{i=0}^{p-1} e_t(\text{reg } B \circ y^i) = \prod_{i=0}^{p-1} e_{t+\frac{i}{p} y}(\text{reg } B).$$

By induction hypothesis

$$e_t(\text{reg } B) = \sum_{i=0}^{r-1} c_{p^{-i} p^i}(\text{reg } B) t^p^i$$

is an additive function of $t$, hence

$$e_t(\text{reg } A) = \prod_{i=0}^{p-1} \left\{ e_t(\text{reg } B) + i \frac{e_y(\text{reg } B)}{e_{p^{-1} y}(\text{reg } B)} \right\}$$

$$= e_t(\text{reg } B)^p - e_{p^{-1} y}(\text{reg } B)^{p^{-1} e_t(\text{reg } B)},$$

proving $e_t(\text{reg } A)$ is additive in $t$.

We now prove formula 1) by induction assuming true for $B$. Then

$$P^A(\beta x) = P^B(P^A(\beta x)) = P^B \left\{ (\beta x)^p - (\beta y)^{p^{-1} \beta x} \right\}$$

$$= (P^B(\beta x))^p - (P^B(\beta y))^{p^{-1} P^B(\beta x)}$$

$$= e_{\beta x}(\text{reg } B)^p - e_{\beta y}(\text{reg } B)^{p^{-1} e_{\beta x}(\text{reg } B)}$$

$$= e_{\beta x}(\text{reg } A).$$

by the above. Now we prove formula 2) starting with
\[ P'(\sigma x) = \sigma \{ y (\beta y)^{\sigma - 2} \beta x + (\beta y)^{\sigma - 1} x \} = (\beta y)^{\sigma - 2} \{ \sigma y \cdot \beta x - \beta y \} \]

Then

\[ P^A(\sigma x) = (P^B(\beta y))^\sigma \{ P^B(\sigma y), P^B(\beta x) - P^B(\sigma x) P^B(\beta y) \} \]

\[ = e_{\beta y} (\text{reg } B)^{\sigma - 2} \left\{ \sum_{i=0}^{\rho - 1} \sum_{j=0}^{\rho - 1} d_{p_{i, p_{i-j}}} (\text{reg } B) c_{p_{i-j}} (\text{reg } B) \left[ (\beta y)^{\rho - 1} \cdot \beta x - (\beta y)^{\rho - 1} \cdot \beta y \right] \right\} + c_{p_{\rho - 1}} (\text{reg } B) \left[ y e_{\beta x} (\text{reg } B) - x e_{\beta y} (\text{reg } B) \right] \]

using induction hypothesis. But

\[ \sum_{i=0}^{\rho} d e_{\beta x} (\text{reg } A) (\beta x)^{\rho - i} + c_{p_{\rho - 1}} (\text{reg } A) x = d e_{\beta x} (\text{reg } A) + c_{p_{\rho - 1}} (\text{reg } A) x \]

\[ = d \left\{ e_{\beta x} (\text{reg } B)^{\rho - 2} e_{\beta y} (\text{reg } B) \right\} + c_{p_{\rho - 1}} (\text{reg } A) x \]

\[ = e_{\beta y} (\text{reg } B)^{\rho - 2} \left\{ \sum_{i=0}^{\rho - 1} d e_{\beta x} (\text{reg } B) \cdot e_{\beta x} (\text{reg } B) - e_{\beta y} (\text{reg } B) \cdot d e_{\beta x} (\text{reg } B) \right\} + c_{p_{\rho - 1}} (\text{reg } A) x \]

\[ = e_{\beta y} (\text{reg } B)^{\rho - 2} \left\{ \sum_{i=0}^{\rho - 1} d c_{p_{i-j}} (\text{reg } B) \cdot \beta y^i c_{p_{i-j}} (\text{reg } B) \cdot \beta x^j + c_{p_{\rho - 1}} (\text{reg } B) \right\} \]

so comparing we get 2) for A.
February 3, 1970:

From char. 2 we are led to the following conjectures:

1) \( \text{Im} \left\{ H^*(B\Sigma_p^r) \to H^*(B\Sigma_p^r) \right\} = \Lambda [d\xi, d\zeta] \otimes S[c_{r-1}] \mid_{i=0}^{r-1} \)

2) \( \text{Im} \left\{ \Phi H^*(B\Sigma_p^r) \to H^*(B\Sigma_p^r) \right\} = \left( \Lambda [d\xi] \otimes S[c_{r-1}] \right) \mid_{i=0}^{r-1} \)

We know that the inclusions are valid because the elements \( d\xi, c_{r-1} \) come from the standard representation of \( \Sigma_p^r \) on \((F_q)^r\) where \( q \) is a finite field and \( v_p(q-1) = 1 \). The point is that we have a commutative diagram:

\[
\begin{array}{ccc}
H(\mathbb{Z}_p^r) & \to & H(B\Sigma_p^r) \\
\phi \uparrow & & \uparrow \\
H(B\Sigma_p^r) & \to & H(BGL_p(F_q)) = \Lambda [d\xi, d\zeta] \otimes S[c_{r-1}, q_i]
\end{array}
\]

and the arrow \( \phi \) commutes with \( d \). Unlike char. 2 we cannot conclude equality because \( \Lambda [d\xi] \otimes S[c_{r-1}] \) is not the subgroup of invariants in \( H^*(B\Sigma_p^r) \), so we have to find another method.

Idea: First prove conjecture

3) \( \exists \) a universal multiplicative (unstable) natural transformation

\[ R \colon H(X)^{ev} \to \left\{ R \otimes H(X) \right\}^{ev} \]

Moreover \( R = \mathbb{Z}_p[\xi_i, \zeta_i]_{i>0} \) where
\[ \nu(\beta x) = \sum_{i \geq 0} \xi_i(\beta x) \beta^i \]

\[ \gamma(\sigma x) = \sigma \left[ \sum_{i \geq 0} \tau_i(\beta x) \rho^i + \xi_0 x \right] \]

Here's how to construct such an operation \( \nu \). I claim that by induction on the rank of \( A \), we can show that

\[ \rho^A : H^e(\Lambda) \to \left[ \Lambda \left[ \delta_{p-p_{i-1}}(r \rho^A) \right] \otimes S[\delta_{p-p_{i}}(r \rho^A)] \right] \otimes H(X) \]

For \( n = 1 \) it's true as the image is contained in the \( G^e(A) \)-invariants.