Outline:

1. Review of cobordism theory
   (i) geometric interpretation of $U^*(X)$, $U^*(X,Y)
      Euler classes, Riemann-Roch theorem
   (ii) characteristic classes for a multi. theory $\mathcal{U}$
       a) projective bundle theorem
       b) formal group law
   (iii) Lefshetz homomorphism for a proj. bundle
      its applications following up geometric Chern classes

- Operations in cobordism theory:
  affine categories and operations in nice GCT\*$^2$
  structure of the affine category of operations for $\mathcal{U}$ & $\mathbb{N}$
  implies how an algebraic theorem leads to simplification of group law leads to a simplification of the theory.
  - 1. off-height localization and typical group laws
  - 2. unoriented cobordism (laws of height $\infty$)
  - 3. K theory (laws of height 1)

- Stone-Weierstrass theorem

Proof of theorem that $L \cong U^*(pt)$
H* \text{(PU}(p))

Milgram, Dickson

Given sphere fibration $\xi$ over $X$ get right action of $A$ on $H^*(X)$.

$$\int (xa) \cdot y = \int x \cdot a(y), \quad X \text{ manifold}$$

$$\bigcap M^* \ker \left\{ H^*(BO) \to H^*(M^*) \right\} = I_n$$

$$I_n^0 = \ker H^0(BO) \to \eta^* (K(\mathbb{Z}_2, n-8))$$

Modular form Colloq. Lectures 1974
Suppose $G$ is a finite group acting on a closed manifold $M$. Then $M/G$ is a rational homology manifold hence has Pontryagin classes.

Proof: It's a local question and we have to know only what's the case for a group acting on a vector space.

More refined example: Suppose $G$ preserves a weakly complex structure on $M$. Then $M/G$ should have rational Chern classes. It's a question of understanding the Atiyah-Segal paper.

Further variation of odd order invariants

$G$ finite action on a manifold $M$ is of odd order.

Problem: Suppose that $V$ is a rep. of $G$ say $\mathfrak{g}$ and that we form $\mathbb{P}(V)/G$. Note that

$$H^*(\mathbb{P}(V)/G, \mathbb{Q}) \to H^*(\mathbb{P}(V))$$

since $G$ preserves the generator. Now $\mathbb{P}(V)/G$ has rational Chern classes. Find a formula.

First reformulation: The rational Chern classes really come from rational Todd classes which are determined by duality. Thus have map

$$K(\mathbb{P}(V)/G) \to \mathbb{Z}$$

given by $E \to \chi(H^*_G(\mathbb{P}(V)), \Omega(E))$. 
\[
\begin{align*}
K(X/G) & \rightarrow K(X) \rightarrow K(X) \rightarrow H(X) \\
\downarrow \quad ? & \quad \downarrow \quad ? & \quad \downarrow \\
K(Y/G) & \rightarrow K(Y) \rightarrow K(Y) \rightarrow H(Y)
\end{align*}
\]

Over the rationals we have

Take up rational cohomology first. Then can you define a map

\[
\begin{align*}
\mathbb{H}^*(X/G) & \rightarrow \mathbb{H}^*(Y/G) \\
\downarrow \cong & \quad \downarrow \cong \\
\mathbb{H}^*(X)^G & \rightarrow \mathbb{H}^*(Y)^G
\end{align*}
\]

we should be a bit careful. Is it clear that if G acts on X compact oriented manifold then \(H^*(X)^G\) sat. P.D??

Relativize Sullivan's construction with fundamental cycle? A bit less precise unless one can prove that Poincaré duality holds for K. Conjecture

\[
\begin{align*}
x & \quad \text{weakly complex} \\
x & \rightarrow y \quad \text{proper oriented G-map where G acts trivially on y} \\
\text{then get for any space Z at map} \\
K(X/G;Z) & \rightarrow K_G(X;Z) \rightarrow K_G(Y;Z) = R_G \otimes K(Y;Z) \\
\downarrow f_! & \quad \downarrow f_! \\
K(Y;Z) & \rightarrow K(Y) \\
\end{align*}
\]

Thus get a fundamental class in \(K(Y;X/G)\)
A weakly complex $G$-manifold $X$ is defined as a weakly complex $G$-manifold.

Atiyah-Segal version: Define a map

$$K(X/G) \rightarrow \mathbb{Z}$$

as follows. Given $E$ over $X/G$, get $f^*E$ $G$-bundle over $X$, integrate over $X$ to obtain an element of $R(G)$, then take inner product with 1. If you use cohomology, then you get an element of $H^*_c(X/G)$ with rational coefficients.

Generalization: Use equivariant cobordism except no inner product with 1.

Compute in the case of projective space.

Inner product with 1 = average over the group.

More generally, suppose given as complex-oriented $G$-map

$$X \xrightarrow{f} Y$$

$$X/G \quad Y/G$$

Define an induced map

$$K(X/G) \rightarrow K(Y/G)$$

and prove a Riemann-Roch formula in which the Todd class enters.
With any hope that one can show that

\[ K(X) \longrightarrow \mathbb{Z} \otimes \mathbb{U}(X) \]

is an isomorphism by power operations? Idea would be to show that power operations can be defined on the left integrally. Problem of proving surjectivity, so start with an element on the left, and use filtration by power. Atiyah from Dieck expansion for it.

Exactness is clear? Enough to do for skeleton?

Isomorphism spaces over Grassmannians where the needed elements are already there?

Next idea:

How do you prove the exactness axiom?

\[ U^*(Y) \longrightarrow U^*(X) \longrightarrow U^*(X-Y) \]
basically like the abelian case

representation of G cyclic with generator g.

\[ \text{eigenvalues } \lambda_i, \quad i = 1, \ldots, n = \dim V \]

Then want to consider action of \( \psi^k \) on \( K_0(G, \mathbb{P}V) \)

where \( k \equiv 1 \mod \dim G \).

\[ \psi^2 \]

\[ \mathbb{C}[T] \]

\[ \frac{\mathbb{C}[T]}{\prod (T - \lambda_i)^{n_i}} = \bigoplus \frac{\mathbb{C}[T]}{(T - \lambda_i)^{n_i}} \]

Note that \( \psi (\Delta g) = 1 \) so \( \lambda_i^k = \lambda_i \)

and \( \psi^k (T) = T^k \)

\[ \mathbb{C}[T] \]

\[ \frac{\mathbb{C}[T]}{(T - \lambda)^n} \]

\[ \psi^k (T - \lambda) = T^k - \lambda^k = (T - \lambda)(T^{k-1} + \lambda T^{k-2} + \ldots + \lambda^{k-2} T + \lambda^{k-1}) \]

\[ \mod (T - \lambda)^2 \]

\[ k \lambda^k \]

\[ \equiv (T - \lambda)^k \mod (T - \lambda)^2 \]

so eigenvalues are \( 1, k, \ldots, k^{\nu_i} \) for each eigenvalue of mult. \( \nu_i \)
\[ N_k = \text{card} \ X(F_{q^k}) \]

\[ \zeta(s) = e^{\sum N_k \frac{z^k}{k}} \quad z = q^{-s} \]

\[ \sum q^k \frac{z^k}{k} = \log \left( \frac{1}{1-qz} \right) \]

\[ \frac{1}{1-q^n z} = \zeta(s) \]

Now of course \( X(F_{q^n} - 0) = [q^n - 1] \)

\[ \frac{1-z}{1-q^k z} = e^{(q-1) \sum \frac{q^k}{q^k - 1} \frac{z^k}{k}} \]

So take \[ \left( \frac{1-z}{1-q^n z} \right)^{\frac{1}{q^n-1}} \]

exists as a rational function.
$X$ is a group and $G$ acts on $X$ over $F_8$, and $G$ acts freely. Then $G$ acts freely on each $X(F_8^r)$. And we find that

$$\mathcal{F}_X(s) = \left\{ \frac{\mathcal{F}_{X/G}(s)}{|G|} \right\}$$

Example: Let $X$ be a space minus lines so that

$$\mathcal{F}_X(s) = \frac{(1-gz)^{g+1}}{(1-g^2z)(1-z)}$$

And let $G$ be a group of linear transformations, then $G \subseteq \text{Gl}(2, F_8)$ which has order $(q^2 - 1)(q^2 - q)$.

If $A$ is a matrix over $F_8$ and if $A0 = 0$, then $A$ has eigenvalue 1, but the eigenspace of eigenvalue 1 is defined over $F_8$ so has been removed for $A \neq 1$.

$X$ is a space $F_8^r$ acts freely, i.e.

$$x \mapsto \lambda x$$

No fixed points $X = A1 - 0$.

$$\frac{1-z}{1-gz} = \left( \mathcal{F}_{X/G} \right)^{g-1}$$
\[ X^p = 1 \]

Suppose \( A \in \text{GL}(\mathbb{F}_p) \Rightarrow A^p = I \)

get a \text{ inseparable extension }.

\[ \exists f \in S(V) \quad \text{Af} = f + 1 \]

\[
A(e_0) = e_0 + e_1 \\
A(e_1) = e_1 + e_2 \\
A(e_p) = e_p
\]

\[
A \left( \frac{e_0}{e_1} \right) = \frac{e_0 + e_1}{e_1} = \frac{e_0}{e_1} + 1
\]

\[ p = 2. \]

\[
A \left( \frac{e_p}{e_p} \right) = \frac{e_p}{e_p} + 1
\]

\[
\left( \frac{e_p}{e_p} \right)^p = \frac{e_p - 1}{e_p} \\
\left( \frac{e_p}{e_p} \right)^{p-1} = e_p - e_p - e_p - \ldots - e_p = a
\]

\[
f(x) = \sum a_x x^x
\]
It says that
\[ \sum_n (\log I_n) t^n = \]
\[ f(x) f(y) \]

whole idea might
\[ a_N = \sum_{n=0}^{N} \langle N \rangle b_n \]

\( \phi(t) \Rightarrow \phi(t) B(t) \) where
\[ \phi(t) = \sum \]

\[ \sum_N \frac{a_N t^N}{N!} = \sum_{n=0}^{N} \frac{t^{N-n}}{(N-n)!} \frac{b_n t^n}{n!} \]

so you introduce a mystical function
\[ \phi(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

\[ \sum_N \frac{a_N x^N}{N!} = \phi(x) \sum_{n=0}^{\infty} \frac{b_n x^n}{n!} \]

and the inversion formula given
A function of a scheme of finite type over $\mathbb{F}_q$

$$\text{Card } \left\{ \mathbb{F}_q \left[ T_1, \ldots, T_N \right] (\mathbb{F}_q^n) \right\} = \text{Card } (\mathbb{F}_q^n)^N$$

$$\text{Card } (\mathbb{F}_q^n) = q^n$$

$$I(s) = \exp \left\{ \sum_{n \geq 1} \frac{\text{card } \mathbb{F}_q \left[ T_1, \ldots, T_N \right] (\mathbb{F}_q^n)}{n} z^n \right\}$$

$$\sum_{n \geq 1} \frac{\text{card } \mathbb{F}_q \left[ T_1, \ldots, T_N \right] (\mathbb{F}_q^n)}{n} z^n = \sum_{n \geq 1} \frac{q^n N}{n} z^n$$

$$= -\log \left( 1 - q^N z \right)$$

hence

$$J_{\mathbb{F}_q}(s) = \frac{1}{1 - q^N z}$$

Now let us try to remove the finite subspaces rational over $\mathbb{F}_q$.

Let $V$ be a vector space over $\mathbb{F}_q$ of dim $N$. I want to let

$$J^m_V(s)$$

to be the $J$ of

$$V = \bigcup_{\omega \in V} \omega V$$
If \( N = 2 \) then all subspaces are 1-dimensional + 0.

So

\[
J^m_v(s) = \frac{(1 - g^2 z)^{q+1}}{(1 - g^2 z)(1 - z)^{g+1}} = \frac{(1 - g^2 z)^{q+1}}{(1 - g^2 z)(1 - z)^{g+1}}
\]

In general

\[
\bigoplus_{W \text{ subspace}} W \uparrow^V = J_v(s)
\]

So one can invert this a la Mobius, I guess

\[
W \cong V
\]

\[
\text{card } \text{GL}(N) = (g^N - 1)(g^N - g)(g^N - g^2) \cdots (g^N - g^{N-1})
\]

\[
= g^{\frac{N(N-1)}{2}} \prod_{i=1}^{N}(g^i - 1)
\]
\[ \text{card Grass}_{n} (N) = \frac{8 \frac{N(N-1)}{2}}{\prod_{i=1}^{N} (q^i - 1) \prod_{i=1}^{r} (q^i - 1)} \]

\[ \frac{a(a-1)}{2} + \frac{b(b-1)}{2} + ab = \frac{a^2 - a + b^2 - b + 2ab}{2} \]

\[ = \frac{(a+b)^2 - (a+b)}{2} \]

\[ \text{card Grass}_{n} (N) = \frac{\prod_{i=1}^{N} (q^i - 1)}{\prod_{i=1}^{r} (q^i - 1)} \prod_{i=1}^{\frac{N-n}{2}} (q^i - 1) = \langle N \rangle \]

\[ \log J_{n}(s) = \sum_{n=0}^{N} \langle N \rangle \log J_{n}(s) \]

\[ \log \left\{ \frac{1}{1 - q^{N+1}} \right\} = \sum \]

\[ a_{n} = \sum_{n=0}^{N} \left( \frac{N}{n} \right) b_{n} \]

\[ \sum_{n} a_{n} t^{n} = \frac{1}{1 + t} \sum_{n} b_{n} t^{n} \]
\[ \sum_{n=0}^{N} <N> x^n = \sum_{n=0}^{N} \frac{t^n}{\prod_{i=1}^{N} (8^i-1)} \cdot \prod_{i=1}^{N} \left( \frac{t^i}{(8^i-1)} \right) \]

Take \( \alpha_i = q^i - 1 \)

\[ \prod_{i=1}^{N} \frac{\alpha_i}{\alpha_i} = \prod_{i=1}^{N} \frac{\alpha_i}{\alpha_i} = \]

\[ \sum_{r=0}^{N} \binom{N}{r} x^r = N! \sum_{r=0}^{N} \frac{x^r}{r!} \cdot \frac{1}{(N-r)!} \]

\[ \sum_{r=0}^{N} \frac{1}{N!} \sum_{r=0}^{N} \binom{N}{r} x^n y^{N-n} = \left( \sum_{r=0}^{N} \frac{1}{r!} x^r \right) \left( \sum_{r=0}^{N} \frac{1}{r!} y^r \right) \]

Thus

\[ \sum_{i=0}^{\infty} \frac{q^i}{\prod_{i=0}^{\infty} (8^i-1)} x^i y^i = \left( \sum_{i=0}^{\infty} \frac{q^i}{\prod_{i=0}^{\infty} (8^i-1)} x^i \right) \left( \sum_{i=0}^{\infty} \frac{q^i}{\prod_{i=0}^{\infty} (8^i-1)} y^i \right) \]

So there is a basic function

\[ f(x) = \sum_{n=0}^{\infty} \frac{x^n}{\prod_{i=1}^{\infty} (8^i-1)} = \sum_{n=0}^{\infty} \frac{x^n}{(8^n-1) \cdots (8-1)} \]

which I want to know.
\( a_N = \log \frac{1}{1-q^Nz} \) or we might take

\[ a_N = \frac{q^N}{1-q^Nz} \]

ess. derivative.

Then we have

\[
\sum_{N} \frac{a_N}{N!} x^N = \sum_{N} \frac{q^N x^N}{1-q^Nz} \cdot \prod_{i=1}^{N} (q^i - 1)
\]

Let \( q \to 0 \) keeping almost and it's related to the law for connected K-theory.

\[
0 \quad \frac{1}{1-z}
\]

\[
1 \quad \frac{1-z}{1-q^1 z}
\]

\[
2 \quad \frac{(1-q^2 z)^{q+1}}{(1-q^2 z)(1-z)^{q}}
\]

\[
(\theta)(q+1) = \theta^{q+1}
\]

\[
3 \quad \int \left( \frac{1}{2} \right)^{q+1} \left( \frac{1}{1-q} \right)^{q+1} \left( \frac{1}{1-z} \right) \frac{1}{1-q^3 z}
\]

\[
\int \left( \frac{1-q^2 z}{1-q^2 z} \right)^{q+1} \left( \frac{1-q^3 z}{1-q^3 z} \right)^{q+1} \frac{1}{1-q^3 z}
\]

\[
\frac{(q^4-1)(q^3-1)}{(q^3-1)(q^4-1)}
\]
\[ S_4 = \frac{1}{1 - q^4} \left( \frac{1 - q^3 z}{1 - q^2 z} \right) \left( \frac{1}{1 - q^2 z} \right) \left( \frac{1 - q^2 z}{1 - q z} \right)^{q^3 + q + 1} \left( 1 - q z \right)^{q^4 - 1} \left( 1 - q \right) \left( 1 - q^3 \right)^{q^3 + q + 1} \]

\[ (q^4 + q^3 + q^2 + q + 1) = q^4 + q^3 + q^2 + q \]

\[ (q^2 + 1)(q^2 + q + 1) = q^4 + q^3 + q^2 \]

\[ (q^2 + q)(q^3 + \cdots + 1) = q^5 + \cdots + q^2 + q + \cdots + q \]

\[ (q^3 + q^2 + q + 1)(q^2 + q + 1) = q^5 q^4 q^3 \]

\[ q^4 q^3 + q^2 \]

\[ q^3 + q^2 + q \]

\[ q^2 + q + 1 \]

\[ q + 1 \]

\[ q^3 - q^2 - q - 1 \]
\[ \sum_{i=0}^{\infty} q^i - (q^3 + q)(q^2 + q + 1) \]

\[
\text{If } q^3 + q^2 + q + 1 = 0, \text{ then: }
\]

\[ -q^3 - q. \]

\[
\int_4^1 \frac{(1 - q^3 z)^{q^3 + q^2 + q + 1}}{(1 - q^4 z)(1 - q^5 z)^{q^3 + q^2 + q + 1}} \text{, (sketch)}
\]
January 9, 1970. (very groggy)

Basic question: Let $G$ be a finite group and let $\text{CG}_*(pt)$ be the cobordism ring of compact oriented $G$-manifolds. Let $Q(G)$ be the graded ring of period $4$ which is $0$ in odd dimensions and for $q=0, 4$ is the $\text{Grothendieck}$ group of symmetric non-degenerate quadratic $G$-spaces with $G$-action and for $q=2, 4$ is the anti-symmetric ones. Then the usual proof that the index is a homomorphism from $SO_*(pt) \to \mathbb{Z}[t^\pm_1]$ should generalize to show that one obtains a form $\alpha$ in the diagram

$$
\begin{array}{ccc}
\text{CG}_*(pt) & \xrightarrow{\alpha} & Q(G) \\
\downarrow & & \downarrow \\
\text{SO}_G^*(pt) & \xrightarrow{\beta} & R
\end{array}
$$

where $\alpha$ associates to $X$ the form on $H^*(X, \mathbb{Q})$. The question is whether there is a natural extension $\beta$ of $\alpha$?

Let's review the situation without the group. Here the existence of $\beta$ is immediate since the vertical arrow is an isomorphism. However one does more. Namely one shows how to endow $K\Omega^* \otimes \mathbb{Z}[\frac{1}{2}]$ with a Thom isomorphism for SO-bundles in such a way that

$$
\text{SO}_G^*(pt) \to K\Omega^*[pt][\frac{1}{2}] = \mathbb{Z}[\frac{1}{2}][t_1, t_1^{-1}]
$$
is the signature map. Can one do the same thing
equiariantly?

**Question:** Let $G$ be a compact Lie group. Does $KO_G\left[\frac{1}{2}\right]$ have an equivariant Thom isomorphism for oriented $4k$-$G$-bundles?

The idea intuitively is that if $E$ is an oriented $4k$-$G$-bundle over a $G$-space $X$, then one can construct from $\Lambda^* E$ with $\ast$ operators a class in $KO_G(E, E - X)$, which one knows at least has the augmentation $\sim 2^k$. So there's a chance at least that it is a Thom class after $\frac{1}{2}$ is adjoined.

By the usual argument, one can suppose $X$ is a point. Let's also assume that has an underlying complex structure, e.g. if $G$ is a finite group of odd order, and see what happens for $KU$ where we have a Thom class coming from the complex structure. What we have to do then is to compute the ratio of the signature class $\sigma(E)$ and $\Lambda_E^*$; it is an element of $R(G)$ with augmentation $\sim 2^k$, hence a unit in $R(G)[\frac{1}{2}]$ but maybe not in $R(G)[\frac{1}{2}]$. 
January 9, 1970 (still gruppy)

On symplectic cobordism: Let $h^8(X) = Sp^3(X \times \mathbb{Z})$ when $\mathbb{Z}$ is fixed. I recall the Gysin sequence

$$\tilde{h}^{8+3}(EZ_2 \times \mathbb{Z}, S^3) \xrightarrow{\tau \times \ast} \tilde{h}^8(B\mathbb{Z}) \xrightarrow{\omega} h^{8+4}(B\mathbb{Z})$$

$$\tilde{h}^{8+3}(\mathbb{RP}^3) \xrightarrow{\iota^*} \tilde{h}^8(B\mathbb{Z}) \xrightarrow{\text{reap}} h^8(\mathbb{RP}^3)$$

I wish to understand the composition

$$\tilde{h}^{8+3}(\mathbb{RP}^3) \xrightarrow{\iota} h^8(B\mathbb{Z}) \xrightarrow{\text{reap}} h^8(\mathbb{RP}^3)$$

Let $i : \text{pt} \to \mathbb{RP}^3$ be the origin of this Lie group and choose a framing. That is, a basis for the Lie algebra. Now $\mathbb{RP}^3$ is parallelizable since it is a Lie group. It seems that there are two possible trivializations of the tangent bundle, using left or right invariant vector fields. (These are probably different since there difference should be measured by the adjoint action homomorphism $\mathbb{RP}^3 \to SO(3)$, which should represent a non-trivial element of $\pi_4(BO) = \mathbb{Z}$.) Choosing one we get a map $f : \tilde{h}^{8+3}(\mathbb{RP}^3) \to h^8(\text{pt})$ satisfying $f \circ i = \text{id}$.

and so the exact sequence

$$\xrightarrow{f \circ i} h^8(\text{pt}) \xrightarrow{\tau \times \ast} \tilde{h}^{8+3}(\mathbb{RP}^3) \xrightarrow{f^*} \tilde{h}^{8+3}(\mathbb{RP}^3)$$

splits giving isomorphisms: for all $g$

$$\tilde{h}^{8+3}(\mathbb{RP}^3) = h^8(\text{pt}) \oplus \tilde{h}^{8+3}(\mathbb{RP}^3).$$
I want to clear up something that was slurred over before. I claim that

\[ E \mathbb{Z}_2 \times \mathbb{Z}_2 S^3 \xrightarrow{p_1} B \mathbb{Z}_2 \xrightarrow{p_2} \mathbb{R} \mathbb{P}^3 \xrightarrow{\text{incl.}} \]

is homotopy commutative. Indeed just recall that \( B \mathbb{Z}_2 = BO(1) \) and that it's enough to see what happens in \( H^4(\cdot; \mathbb{Z}_2) \). Thus in fact we know that there is a long exact sequence

\[ \tilde{h}^{8+3}(\mathbb{R} \mathbb{P}^3) \xrightarrow{j} h^8(B \mathbb{Z}_2) \xrightarrow{\delta} \tilde{h}^{8+4}(B \mathbb{Z}_2) \xrightarrow{\text{res}_2} \tilde{h}^{8+4}(\mathbb{R} \mathbb{P}^3) \]

which one perhaps also ought to view as a spectral sequence. We want to compute the first differential which is the composition

\[ h^{8+3}(\mathbb{R} \mathbb{P}^3) \xrightarrow{j} h^8(\mathbb{R} \mathbb{P}^3) \xrightarrow{\text{res}_3} h^8(\mathbb{R} \mathbb{P}^3) \]

It is given by the correspondence

\[ S^n \times \mathbb{Z}_2 S^3 \xrightarrow{p_1} \mathbb{R} \mathbb{P}^n \xrightarrow{p_2} \mathbb{R} \mathbb{P}^3 \]

where we use the \( n \) to keep from being confused. Note that this can be factored.
and that orientations of $p_{1}$ and $p'_{1}$ ($\mathbb{RP}^3$ parallelizable) combine to give $\Omega$-orientation for $\alpha$. (Actually since $\alpha$ is a double covering it has a canonical orientation, however, as yet I see no reason why this state orientation coincides with the symplectic one.) So we have that

$$\nu_n (\Omega(z)) = p_{1}' \times (\alpha \cdot 1 \cdot p_{2}' \times z)$$

Now if $n = 3$ we have this diagram

$$\begin{array}{ccc}
S^3 \times_{Z_2} S^3 & \longrightarrow & S^3 \\
\downarrow \alpha & & \downarrow \gamma \\
\mathbb{RP}^3 \times \mathbb{RP}^3 & \longrightarrow & \mathbb{RP}^3 \\
\downarrow p_{1}' & & \downarrow f \\
\mathbb{RP}^3 & \longrightarrow & \mathbb{RP}^3 \\
\mu & & \mu \\
\end{array}$$

(frame the $T$ of $\mathbb{RP}^3$ via $\mu$-invariant fields) so that $p_{1}'$ is similarly $p_{1}$.)

where the big square comes from the quaternion bundle map

$$\begin{array}{ccc}
S^3 \times_{Z_2} H & \longrightarrow & H \\
\downarrow p_{1}' & & \downarrow \phi \\
\mathbb{RP}^3 & \longrightarrow & \mathbb{RP}^3 \\
\end{array}$$

($H$ acts on the right)
I wish to orient $\alpha$ and $p_1^\prime$ via $f$ and $g$. Then

$$\text{rez}_3(f \times g) = p_1^\prime \times (\mu^* \times 1 \cdot p_2^\prime \times z)$$

It therefore seems necessary to compute

$$\nu_1 \in \text{Sp}^o(\mathbb{R}P^3) \cong \text{Sp}^o(\mathbb{R}^4) \oplus \text{Sp}^{-2}(\mathbb{R}^2) \oplus \text{Sp}^o(\mathbb{R}P^2)$$

$$\nu_1 = (2 + \varrho + ?)$$

(since $\nu_1 = 0$)

Now

$$\text{Sp}^o(\mathbb{R}P^2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \text{ or } \mathbb{Z}_4 \text{ since}$$

$\text{Sp}^1(\mathbb{R}^4) \xrightarrow{2} \text{Sp}^{-2}(\mathbb{R}^2) \xrightarrow{2} \text{Sp}^o(\mathbb{R}P^2) \xrightarrow{2} \text{Sp}^o(\mathbb{R}P^1) \xrightarrow{2}$$

$\mathbb{Z}_2 \xrightarrow{2} \mathbb{Z}_2 \xrightarrow{2} \mathbb{Z}_2 \xrightarrow{2} \mathbb{Z}_2$

(I believe it is known that $\pi^o_1(\mathbb{R}P^2) = \mathbb{Z}_4$, hence and so it should be that $\text{Sp}^o(\mathbb{R}P^2) = \mathbb{Z}_4$. The conjecture then is that $\nu_1 - 2$ generates $\text{Sp}^o(\mathbb{R}P^3) = \mathbb{Z}_4$, and we can test this by showing that $\nu_1 - 2$ restricts to the non-zero element of $\text{Sp}^o_4(\mathbb{R}P^4) = \mathbb{Z}_2$).

I want to review the calculation of $\text{Sp}^{-2}(\mathbb{R}^4)$ for $q = 0, 1, 2, 3$. We start with

$$Q: \text{Sp}^0(S^q, *) \rightarrow \text{Sp}^0(B \mathbb{Z}_2 \times (S^q, *))$$

where $0 \leq q \leq 3$. Let $x \in \text{Sp}^0(S^q, *) = \text{Sp}^{-2}(\mathbb{R}^4)$. Then $s_x = 0$
for $x > 0$, so our localization formula gives

$$\omega^n \{ Qx - x \} = 0 \quad \text{for } n \gg 0.$$ 

Now the 8th order Lyapunov sequence

$$\text{Sp}^{3+e}(\text{RP}^3 \wedge (S^6, *)) \xrightarrow{\zeta} \text{Sp}^{2}(B\mathbb{Z}_2 \times (S^6, *)) \xrightarrow{\omega^e} \text{Sp}^{2+e}(B\mathbb{Z}_2 \times (S^6, *)) \quad \text{if } q \gg l \quad (\text{e.g. } q = 4)$$

shows that we must have

$$\omega^e \{ Qx - x \} = 0$$

and hence

$$\exists y \in \text{Sp}^{3+e}(\text{RP}^3 \wedge S^6) = \text{Sp}^{3+e}(\text{RP}^3) \quad \text{such that}$$

$$Qx - x = \zeta(y) \quad \text{in } \text{Sp}^{2}(B\mathbb{Z}_2 \times (S^6, *))$$

Now restricting from $B\mathbb{Z}_2$ to point and using that $\omega^0(Qx) = x^e = 0$ since $l > 0$ we have

$$-x = \text{res} \zeta(y) \quad \text{in } \text{Sp}^{0}(S^6) = \text{Sp}^{e}(pt).$$

Recalling the defn. of $\zeta$, this means we have a surjectivity arrow

$$\text{Sp}^{3-e}(\text{RP}^3) \xrightarrow{\delta^e} \text{Sp}^{3+e}(S^3)$$

onto

$$\text{Sp}^{e}(pt) \twoheadrightarrow 0$$
Recall that
\[ \widetilde{Sp}^{3,2} (RP^3) \cong Sp^{-l}(pt) \oplus \widetilde{Sp}^{3,2} (RP^2) \]
\[ \widetilde{Sp}^{3,2} (RP^1) = Sp^{2,2}(pt) \]

Now the composition
\[ \begin{array}{c}
Sp^{-l}(pt) \\
\downarrow \quad \downarrow \\
Sp^{3,2}(RP^3) \\
\downarrow \quad \downarrow \\
Sp^{3,2}(S^3) \\
\downarrow \quad \downarrow \\
Sp^{-l}(pt)
\end{array} \]

is multiplication by 2 since it sends
\[ [Z \to pt] \mapsto [Z \to pt \to RP^3] \mapsto [Z \times Z_2 \to Z_2 \to S^3] \mapsto [Z \times Z_2 \to pt] \]

Consequently we obtain a surjective homomorphism \( \phi \)
\[ \begin{array}{c}
\widetilde{Sp}^{3,2} (RP^2) \\
\downarrow \quad \downarrow \\
Sp^{-l}(pt) \oplus 2Sp^{-l}(pt) \to O, \quad 0 < l \leq 3
\end{array} \]

which we note is compatible with mult. by elements of \( Sp^*(pt) \), \( \phi \) for all \( l \). ((**) also surjective by Yamasaki.)

Now for \( l=0 \) we know \( Sp^0(pt) = \mathbb{Z} \). Hence we see that
\[ \widetilde{Sp}^2 (RP^2) = \mathbb{Z}_2 \Rightarrow Sp^{-1}(pt) \text{ is } 0 \text{ or } \mathbb{Z}_2 \]

(By use of the homomorphism \( Sp^*(X) \to KO^*(X) \), which we should be able to calculate, we know it is \( \mathbb{Z}_2 \).)

Call the generator of \( Sp^{-1}(pt) \) \( \eta \).
Next for $l = 2$ we get

$$\tilde{Sp}^1(\mathbb{RP}^2) = \mathbb{Z}_2 \oplus \eta \cdot (\text{gen of } \tilde{Sp}^2(\mathbb{RP}^2))$$

so

$$\tilde{Sp}^2(pt) = \mathbb{Z}_2 \cdot \eta^2$$

(or 0 which is out by KO).

Using this we see that for $l = 3$

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \tilde{Sp}^0(\mathbb{RP}^2) \rightarrow \tilde{Sp}^1(pt) \rightarrow 0$$

Now in fact from KO-theory we know that $\tilde{KO}^0(\mathbb{RP}^2) = \mathbb{Z}_4$ hence some will be true for $\tilde{Sp}^0(\mathbb{RP}^2)$. Therefore we have a surjection

$$\mathbb{Z}_4 \rightarrow \tilde{Sp}^3(pt) \rightarrow \tilde{KO}^3(pt) = 0$$

The really good thing would be for $\tilde{Sp}^3(pt) = 0$ for then and only then would $\tilde{Sp}^3(pt)$ be determined by KO char. nos. (I believe from stable homotopy theory one knows that $\pi_3^s(pt) = \mathbb{Z}_2$; it follows that $\eta^3$ must be zero in $\tilde{Sp}^3(pt)$ since it comes from an element of order 2 in the $\mathbb{Z}_2$. Consequently $\tilde{Sp}^3(pt)$ is at most $\mathbb{Z}_2$, generated by the image of the generator of $\pi_3^s(pt)$ which I believe people call $\sigma$. I also believe this element comes from the Hopf fibration $S^3 \rightarrow S^7 \rightarrow S^4$, hence is represented by the manifold $S^3$ framed in some way.)
January 11, 1970  (very groggy)

Let $h^*$ be a generalized cohomology theory. Then there is a canonical map

$$\alpha : h^0(RP^3) \to h^{8-3}(RP^3)$$

defined as the composition

$$h^0(RP^3) \xrightarrow{P_2^*} h^0(S^3 \times \mathbb{Z}_2 S^3) \xrightarrow{p_1^*} h^{8-3}(RP^3)$$

where the framed orientation for $p_1$ comes from the trivialization

$$S^3 \times \mathbb{Z}_2 S^3 \xrightarrow{\mu} S^3$$

In other words, we are using the fact that the (right) quaternionic line bundle $S^3 \times \mathbb{Z}_2 H$ over $RP^3$ is trivialized by $\mu : S^3 \times \mathbb{Z}_2 H \to H$.

Now, the map $\alpha$ is represented by an $S$-map $S^3 RP^3 \to RP^3$, which one can compute as follows:

$$h^0(RP^3) \xrightarrow{P_2^*} h^0(S^3 \times \mathbb{Z}_2 S^3) \xrightarrow{p_1^*} h^{8-3}(RP^3)$$

$$h^0(RP^3) \xrightarrow{P_2^*} h^0(S^3 \times \mathbb{Z}_2 H) \xrightarrow{p_1^*} h^{8-3}(RP^3 \times H)$$

where $p_1$ supports $(x, y) \mapsto (z_x, z_y)$.
Tracing this backwards, we see that the map

$$\mathbb{RP}^3 \times H^* \longrightarrow \mathbb{RP}^3 \times \mathbb{R}^+$$

$$\begin{pmatrix} z_2 x, y_n \end{pmatrix} \longrightarrow \begin{pmatrix} z_2 x^{-1} y, n \end{pmatrix}$$

g\in S^3, n > 0

extends to a map

$$\mathbb{RP}^3 \wedge (S^4, *) \longrightarrow \mathbb{RP}^3 \wedge (S^4, *)$$

which is the map we want.

Returning to \( x \) we recall that by using a framing of \( \mathbb{RP}^3 \)
we can construct a decomposition

$$h^8(\mathbb{RP}^3) = h^8(pt) \oplus h^{8-3}(pt) \oplus h^8(\mathbb{RP}^2)$$

Therefore the map \( x \) can be broken up into \( G \)-components which are maps:

- \( h^{8-3}(pt) \longrightarrow h^{8-3}(pt) \quad \Sigma^0 \longrightarrow \Sigma^0 \)
- \( h^{8-3}(pt) \longrightarrow h^{8-6}(pt) \quad \Sigma^3 \longrightarrow \Sigma^0 \)
- \( h^{8-3}(pt) \longrightarrow h^{8-3}(\mathbb{RP}^2) \quad (\mathbb{RP}^2, *) \longrightarrow \Sigma^0 \)
- \( h^8(\mathbb{RP}^2) \longrightarrow h^{8-3}(pt) \quad \Sigma^3 \longrightarrow (\mathbb{RP}^2, *) \)
- \( h^8(\mathbb{RP}^2) \longrightarrow h^{8-6}(pt) \quad \Sigma^6 \longrightarrow (\mathbb{RP}^2, *) \)
- \( h^8(\mathbb{RP}^2) \longrightarrow h^{8-3}(\mathbb{RP}^2) \quad \Sigma^{3}(\mathbb{RP}^2, *) \longrightarrow (\mathbb{RP}^2, *) \)
Now in fact I believe one knows that

\[{ Y, \Sigma^3(\mathbb{R}P^2_\times) \wedge Z} = \{(\mathbb{R}P^2_\times)^\wedge Y, Z\}\]

(To check dimensions, take \(Z = K(\mathbb{Z})\) and \(Y = \Sigma^i\). Then the left is

\[H^i_{\mathbb{Z}+3}(\mathbb{R}P^2_\times, Z_2) \neq 0 \text{ for } i = -5, -4, -3\]

The right is

\[H^i(\Sigma^3(\mathbb{R}P^2_\times), Z_2) = H^{-i}(\mathbb{R}P^2_\times, Z_2) \neq 0 \text{ for } i = -5, -4, -3\]

Put \(Z = \Sigma^3\).

\[{ Y, (\mathbb{R}P^2_\times)} = \{(\mathbb{R}P^2_\times)^\wedge Y, \Sigma^3\}\]

So therefore the six maps under consideration are

\[
\begin{align*}
\Sigma^0 & \rightarrow \Sigma^0 \\
\Sigma^3 & \rightarrow \Sigma^0 \\
(\mathbb{R}P^2_\times) & \rightarrow \Sigma^0 \\
(\mathbb{R}P^2_\times) & \rightarrow \Sigma^0 \\
\Sigma^3(\mathbb{R}P^2_\times) & \rightarrow \Sigma^0 \\
(\mathbb{R}P^2_\times)^\wedge (\mathbb{R}P^2_\times) & \rightarrow \Sigma^0 \\
\end{align*}
\]

Now we are interested in what happens after one goes from \(\Sigma^0 \rightarrow \mathbb{R}P^2\), so we get potential elements in

\[
\#
\begin{align*}
S^0(\mathbb{R}P^2) & = \mathbb{Z} \quad \text{(probably 2)} \\
S^3(\mathbb{R}P^2) & = 0 \\
S^0(\mathbb{R}P^2) & = \mathbb{Z}_4 \\
S^0(\mathbb{R}P^2) & = \mathbb{Z}_4 \\
S^3(\mathbb{R}P^2) & = ? \\
S^0(\mathbb{R}P^2)^\wedge (\mathbb{R}P^2_\times) & = ? \\
\end{align*}
\]
Here is a table of stems
\[ \pi_5^s(S) : \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_2, \mathbb{Z}_2 \] ignoring 3 torsion
\[ \eta^3 = 4 \nu \]

Let \( M_2 = (RP^2, *) \), then from Toda-Asakai paper we have that
\[ \{ \Sigma_i, M_2 \} = \{ \Sigma_i, \Sigma_3 \} \]

is given by the table

<table>
<thead>
<tr>
<th>i = 0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Z}_2 )</td>
<td>( \mathbb{Z}_2 )</td>
<td>( \mathbb{Z}_2 )</td>
<td>( \mathbb{Z}_2 + \mathbb{Z}_2 )</td>
<td>( \mathbb{Z}_2 + \mathbb{Z}_2 )</td>
</tr>
</tbody>
</table>

Here
\[ \Sigma_1 \to M_2 \xrightarrow{\pi} \Sigma^2 \]

are the basic maps from the third structure

and \( \bar{\eta} \in \{ M_2, \Sigma_3 \} \) satisfy
\[ \bar{\eta} \cdot \pi = \eta \]

and one has the relations
\[ 2 \bar{\eta} = \eta^2, \quad \bar{\eta} \pi = \bar{\eta} \pi \]

Finally \( \{ \Sigma_i M_2, M_2 \} \) is given by
\[ \begin{array}{cccccc}
  i < -1 & \bar{\eta} & 0 & -1 & 0 & 1 & 2 \\
  0 & \mathbb{Z}_2 & \mathbb{Z}_2^2 & \mathbb{Z}_2 + \mathbb{Z}_2 & \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \\
  \pi \bar{\eta} & 1 & \bar{\eta} \pi & (\bar{\eta} \pi)^2 & \cdots & \cdots
\end{array} \]
From the A-Toda tables one sees that

\[ \begin{array}{c c c c}
2 & \to & \{ \Sigma^5 M_2 \} & \to \{ \Sigma^3 M_2, M_2 \} \\
& & \to & \{ \Sigma^4 M_2 \} & \to \\
& & & \mathbb{Z}_2 \& \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\
\end{array} \]

so \[ \{ \Sigma^3 M_2, M_2 \} \] is either \( \mathbb{Z}_2 \) or \( \mathbb{Z}_2 + \mathbb{Z}_4 \).

But also we have

\[ \begin{array}{c c c c}
\{ \Sigma^5, \Sigma^0 \} & \to & \{ \Sigma^5, M_2 \} & \to \{ \Sigma^4, \Sigma^0 \} \\
\to & & \to & \to \\
0 & & 0 & \mathbb{Z}_2 \& \mathbb{Z}_2 \\
\end{array} \]

and therefore I see that the composition

\[ h^8(\mathbb{RP}^3) \to h^{8-3}(\mathbb{RP}^3) \to h^{8-6}(\mathbb{RP}^3) \]

for a theory over \( \text{Sp}^* \) is zero.
\[ \text{Sp}^{*}(pt) \rightarrow \text{KO}^{*}(pt) \text{ hom.} \]

\[ \text{assume, then } \text{det. by } \text{KO}^{*} \text{ nos.} \]

\[ \text{Sp}^{-1k-3}(pt) = 0 ! \]

\[ \text{Conclusion: } \mathcal{V} \in \widetilde{\text{Sp}}^{0}(\mathbb{RP}^{2}) \text{ and } w \in \widetilde{\text{Sp}}^{2}(\mathbb{RP}^{2}) \text{ generated } \widetilde{\text{Sp}}^{*}(\mathbb{RP}^{2}) \text{ and all torsion is of order 2.} \]

\[ \text{Sp}^{2k}(pt) \rightarrow \widetilde{\text{Sp}}^{2}(\mathbb{RP}^{2}) \rightarrow \text{Sp}^{0}(pt) \rightarrow \eta \]

\[ \text{Conclusion is that all torsion in } \text{Sp}^{*}(pt) \text{ is generated by } \eta. \text{ (unreasonable)} \]

\[ \text{Sp}(pt) \rightarrow \mathbb{Z}[c^{2}] \text{ } \text{deg } c = 2. \]

I have Adams operations on \( k(pt) \). I have \( \psi^{k} \), \( k \geq 1 \) because we have \( \lambda \). How about \( \psi^{-1} \)?

\[ E \quad E^{*} = \text{Hom}_{c}(E, c) = \text{Hom}_{R}(E, c) \]

\[ (\sigma f)(e) = f(e) \]

\[ (\sigma f)(e) = \frac{f(e)}{f(e)} \]

Question: Is \( E \cong E^{*} \)? Suppose that \( E \) endowed with a hermitian inner product \( \langle \cdot, \cdot \rangle \). Suppose \( \langle x, y \rangle \)

\[ \langle x, y \rangle = \langle y, x \rangle \]

\[ E = \mathbb{C}^{\mathbb{C}} \]

\[ \langle x, y \rangle = \lambda \mu \langle x, y \rangle \]

\[ \langle \sigma(x), \sigma(y) \rangle = \mu \lambda \langle x, y \rangle \]

I assume known that \( \psi^{-1} = \text{identity} \)
L-Novikov operations: $\Psi^{-1} = \text{id}$

alg. of Adams operations with $\Psi^{-1} = \text{id}$

alg. of Adams ops. $\mathbb{Z}[\tilde{T}^{\pm}] \oplus \mathbb{Z}(\tilde{T}) \subset \mathbb{Q}[\tilde{T}, \tilde{T}']$

$\tilde{T} \to \mathbb{Q}$

$T \to \mathbb{Q}$

$T \to k \quad \binom{k}{n} \in \mathbb{Z} \quad \text{all } n$

Want $\Psi^{-1} = \text{id}$

Now the problem is to make $\Psi^k$ act on

$\beta \in \mathcal{K}^{-2}(pt)$

$\Psi^k(\beta) = \frac{1}{k^*} \beta$. Definitely denominators needed for

Idea is to construct a subalgebra of Novikov operations corresponding to $\tilde{T}$. E.g. in complex $K$-theory have

$$(1+x)^{-1} = \sum_{n \geq 1} \binom{a}{n} x^n$$

which $\Psi^k$ preserves

We agree that by C.F.

$$\mathcal{K}^0(pt) \otimes \text{Sp}^*(X) \xrightarrow{\sim} \mathcal{K}^0(X).$$

Hence for any ring $R$

$$k(pt) \otimes \text{Sp}(pt) \xrightarrow{\text{Sp}(pt)} k(X)$$
January 14, 1970 (still groggy but getting better)

On symplectic cobordisms again. I recall how the calculations of $\text{Sp}^l(\mathbb{F}_t)$ for $0 \leq l < 4$ go. So let $x = \text{Sp}^{-l}(\mathbb{F}_t) \cong \text{Sp}^0(S_{s}^2, x)$. Then

$$w^j(Qx - x) = 0$$

for $j$ large

in $\text{Sp}^*(BZ_2 \times (S_{s}^3, *))$. By Gysin sequence,

$$\tilde{\text{Sp}}^3(R^3 \times (S_{s}^3, *)) \xrightarrow{j} \text{Sp}^0(BZ_2 \times (S_{s}^3, *)) \xrightarrow{w^l} \text{Sp}^{l+4}(BZ_2 \times (S_{s}^3, *))$$

Thus 

$$w(Qx - x) = 0$$

and

$$Qx - x = \tilde{f}(a) \quad a \in \tilde{\text{Sp}}^{2-4}(R^3)$$

Now restricting to a point of $BZ_2$ we get

$$\phi(f) = \tilde{f}(a) \quad 0 < l, \quad w_x = x = 0$$

where in general $w_x \circ \tilde{f}$ is the composition

$$\tilde{\text{Sp}}^3(R^3) \xrightarrow{P_2^*} \tilde{\text{Sp}}^3(S_2 \times (S^3, *)) \xrightarrow{P_4^*} \text{Sp}^{2, 3}(R^3)$$

Here I am interested in the maps $w \circ \tilde{f} = \tilde{f}$

$$\phi: \tilde{\text{Sp}}^3(R^3) \xrightarrow{\pi^*} \tilde{\text{Sp}}^3(S^3) \xrightarrow{\pi^*} \text{Sp}^{3, 3}(\mathbb{F}_t)$$

Now we know that

$$\tilde{\text{Sp}}^3(R^3) \xrightarrow{(i^*, f_\pi)} \tilde{\text{Sp}}^3(R^3) \times \text{Sp}^{3, 3}(\mathbb{F}_t)$$
If \( i: pt \to \mathbb{R}P^3 \) is framed so that \( f_*i_* = id \) we can compute \( \pi_0(i_*1) \) and we find it is
\[
\begin{array}{ccc}
\mathbb{Z}_2 & \to & S^3 \\
& \searrow & \downarrow \pi \\
& pt & \to \mathbb{R}P^3
\end{array}
\]

2. Thus we know from (4) that
\[
\varphi: Sp^3(\mathbb{R}P^3) \to Sp^{-l}(pt)
\]
and that on the \( Sp^{-l}(pt) \) factor it is multiplication by 2. Hence we get a surjection
\[
\tilde{Sp}^{3-l}(\mathbb{R}P^3) \to Sp^{-l}(pt)/2Sp^{-l}(pt)
\]
which by Nakayama means that
\[
(2) \quad \tilde{Sp}^{3-l}(\mathbb{R}P^2) \to Sp^{-l}(pt) \quad 0 < l \leq 3
\]
is surjective mod odd torsion. On the other hand
\[
2 \Rightarrow Sp^{-l}(pt) \to Sp^{3-l}(\mathbb{R}P^2) \to Sp^{2-l}(pt) \Rightarrow
\]
so using this and the surjectivity of (2) \( \mathbb{Z}_2 \) and \( Sp^0(pt) = \mathbb{Z} \),
we get
\[
(3) \quad \mathbb{Z}_2 \to Sp^{-1}(pt) \to Sp^{-2}(pt)
\]
0 \to \text{Sp}^{-2}(pt) \to \widetilde{\text{Sp}}^0(\mathbb{R}P^2) \to \text{Sp}^{-1}(pt) \to 0

Let $S^3 \xrightarrow{\pi} \mathbb{R}P^3$ be framed in the natural way. Considering the natural map $\mu: \text{Sp}^*(X) \to \text{KO}^*(X)$ furnished by the Thom isomorphism in $\text{KO}$-theory we see that $\mu(\pi_*1) = \pi_*1$. I will assume that $\pi_*1$ is $K$-theory in the direct image bundle which is clear $1+O(1)$. Thus

$$\pi_*1 = 2 + x \quad \forall \in \widetilde{\text{Sp}}^0(\mathbb{R}P^2) \quad \mu(\forall) = O(1) - 1$$

Now $\mu(\forall)$ has order 4. In fact the Whitney class of $O(1) - 1 = \eta$ is $1 + x$ and $(1 + x)^4 = \text{Ann} \mathbb{R}P^2$, but $(1 + x)^3 \neq 0, 1 \leq j < 4$. This shows the order is at least 4, hence card $\widetilde{\text{Sp}}^0(\mathbb{R}P^2) > 4$ and we already know this card is $\leq 4$ by (3)+ (4) above. Thus we find that

$$\text{Sp}^{-1}(pt) = \mathbb{Z}_2 \eta$$
$$\text{Sp}^{-2}(pt) = \mathbb{Z}_2 \eta^2$$
$$\widetilde{\text{Sp}}^0(\mathbb{R}P^2) = \mathbb{Z}_4 \forall \quad \text{where } \eta^2 \mapsto \forall.$$

Now $\varphi(\pi_*1)$ is represented by

$$S^3 \xrightarrow{\eta} S^3 \xrightarrow{\mu} \text{pt.}$$

hence $\varphi(\pi_*1) = 2 \cdot [S^3 \to \text{pt}] = 0$. Therefore for $k = 1$ in (2) since $\forall$ generates $\widetilde{\text{Sp}}^0(\mathbb{R}P^2)$. We conclude that

$$\text{Sp}^{-3}(pt) = 0.$$
Try $l = 4$: Take $x \in \text{Sp}^{-4}(pt)$, suppose $s_{\lambda} x = 0$, $\lambda > 0$. Then we get

$$\omega^2 (w Q x - x) = 0 \Rightarrow \omega (w Q x - x) = \eta(a) \quad a \in \tilde{\text{Sp}}^{2}(\mathbb{R}P^3)$$

Restricting to a point in $\mathbb{R}P^3$ we get $2a = 0$, so $a = 0$, so

$$w Q x - x = \eta(a) \quad a \in \tilde{\text{Sp}}^{-1}(\mathbb{R}P^3).$$

So restricting to a point again we get

$$-\nu = \eta(a)$$

But we have

$$0 \rightarrow \text{Sp}^{2}(pt) \rightarrow \tilde{\text{Sp}}^{0}(\mathbb{R}P^3) \rightarrow \text{Sp}^{-1}(pt) \rightarrow 0$$

$$0 \rightarrow 0 \rightarrow \tilde{\text{Sp}}^{-1}(\mathbb{R}P^3) \rightarrow \text{Sp}^{-2}(pt) \rightarrow 0$$

$$\therefore \tilde{\text{Sp}}^{-1}(\mathbb{R}P^3) = \mathbb{Z} \cdot \eta$$

since $\eta$ is a $\text{Sp}^{2}(pt)$-module homomorphism and since $\eta(\nu) = 0$. Conclude that

$$\text{Sp}^{-1}(pt) = \mathbb{Z} \quad \tilde{\text{Sp}}^{-1}(\mathbb{R}P^3) = \mathbb{Z} \cdot \eta$$

$\nu$ and $\eta$ commute with $w Q x - x$.

Then get

$$w (w Q x - x) = \eta(a) \quad a \in \tilde{\text{Sp}}^{2}(\mathbb{R}P^3)$$
January 19, 1990 (still groggy)

Atiyah's lectures:

The basic representation theory. Let $n = 4q$. In the Clifford algebra of $R^n$ let $\omega = e_1 \cdots e_n$ be the volume element. Identify $C(R^n)$ with $\Lambda(R^n)$ in the natural way. Consider the operator

$$(-1)^q L(\omega)$$

which has square $+ \text{id}$ and up to sign is the star operator. Then this gives the decomposition $\Lambda = \Lambda_+ \oplus \Lambda_-$.
Assume you can define partial Steenrod operations in $H^*(X \times Y)$ and that they preserve algebraic cycles.

Method: (mod 2)

\[
H^*(X \times Y) \rightarrow \prod' H^*_{\Sigma_k} \left( X \times Y \right) \xrightarrow{\Delta_X^*} \prod' H^*_{\Sigma_k} \left( X \times Y \right)
\]

\[
= \prod' H^*_{\Sigma_k} \left( Y^k \right) \otimes H^*(X)
\]

\[
= \prod' \left[ H^*_{\Sigma_k} \left( X \otimes (Y \otimes k) \right) \otimes H^*(X) \right]
\]

\[
= \left[ \prod' H^*_{\Sigma_k} (X) \right] \otimes H^*(Y)
\]

Example: (mod 2)

Let $u \in H^*(X \times Y)$, $Qu \in H^*_{\Sigma_2} (X^2 \times Y^2)$, res $\otimes (Qu) = u \otimes 2$

\[
V = (\Delta_X \otimes \text{id}_Y)^* Qu \in H^*_{\Sigma_2} (X \times Y^2), \quad \text{res } V = (\Delta_X \otimes \text{id}_Y)^* u \otimes 2
\]

Write $V = \sum_i x_i \otimes z_i$ with $x_i \in H^*(X)$ and $z_i \in H^*_{\Sigma_2}(Y^2)$

Example: $u = x \otimes y$, $Qu = Qx \otimes Qy$, $V = Px \otimes Qy$

so the end result should be

$x \otimes y \rightarrow Px \otimes Qy$. 
We know that \( P_x = (S^0 x) \omega^n + \ldots + (S^k x) \),
so we conclude that if \( u \in H^e(X \times Y) \) is algebraic and \( u = \sum x_i \otimes y_i \)
then

Suppose that \( u \in H_{2g}^e(X \times Y) \) and let
\[
  u = \sum_{i=0}^{2g} u_i \quad \text{with} \quad u_i \in H^i(X) \otimes H^{2g-i}(Y).
\]
Then write
\[
  u_i = \sum_{j=1}^{r_i} x_{ij} \otimes y_{ij}
\]
\( \{y_{ij}\}_{j=1}^{r_i} \) basis for \( H^{2g-i}(Y) \)
and we have

\[
P_x u = \sum_{i=0}^{2g} \sum_{j=1}^{r_i} P_{x_{ij}} \otimes y_{ij}
\]

\[
= \sum_{i=0}^{2g} \sum_{j=1}^{r_i} \sum_{\nu=0}^{i} \omega^{-\nu} (S^\nu y_{ij}) \otimes y_{ij}
\]

\[
= \sum_{i=0}^{2g} \sum_{\nu=0}^{i} \omega^{-\nu} \left( \sum_{j=1}^{r_i} (S^\nu y_{ij}) \otimes y_{ij} \right)
\]

\[
= \sum_{i=0}^{2g} \sum_{\nu=0}^{i} \omega^{-\nu} (S^\nu \otimes \text{id}) u_i
\]
work out the formulas for odd primes. Suppose \( p \)
the primes occurring in the torsion of \( H^*(X), H^*(Y) \) and that
\( p \gg \dim X \). In this case we should get a similar
formula with
\[
P_x u = \sum_{i=0}^{2g} \sum_{\nu=0}^{i} \omega^{-\nu} (p^\nu \otimes \text{id}) u_i
\]
\( \deg u = 2p+1 \)
\( p^\nu = 2(p+1) \nu \)
Consider the cohomology theory $h^0(X) = H^0(X \times \mathbb{Z})$ where $\mathbb{Z}$ is fixed. Can you define steered operations in $h^0$? In general if

$$h^0(X) = \{X, M\}$$

one has a product

$$M \times M \to M$$

and steered operations consist of maps

$$M \to \ldots \to M$$

Instead of $H$ take $N$ so we have good geometric representatives. Thus

$$h^*(X) = N^*(X \times \mathbb{Z})$$

given $N^k$, form product $W^k$

$X \times \mathbb{Z}$

however one can't directly pull-back to $\mathbb{Z}$. Therefore the natural theory goes from

$$N^\bullet_h(X \times \mathbb{Z}) \to N^\bullet_h(X \times (E \Sigma_r \times \Sigma_{lr} \mathbb{Z}^k))$$

even from the internal point of view in $X$. 
January 16, 1970 (feeling better!)

I want to determine whether or not my method will permit one to show that elements of \( Sp^*(pt) \) are determined by their KO*-characteristic numbers. So I start with the pivotal map \( \Phi \)

\[
\begin{align*}
\xymatrix{
Sp^8(\mathbb{RP}^3) \ar[r]^{p_2^*} & Sp^8(S^3 \times S^3) \ar[r]^{p_1^*} & Sp^{6-3}(\mathbb{RP}^3).
}
\end{align*}
\]

Using the isomorphism

\[
\begin{align*}
\nu: \mathbb{RP}^3 \times S^3 & \longrightarrow \mathbb{RP}^3 \\
(z, x, y) & \longrightarrow z^2 x \cdot y
\end{align*}
\]

we see \( \Phi \) is also the map

\[
\begin{align*}
\xymatrix{
Sp^8(\mathbb{RP}^3) \ar[r]^{\nu^*} & Sp^8(\mathbb{RP}^3 \times S^3) \ar[r]^{p_2^*} & Sp^{6-3}(\mathbb{RP}^3).
}
\end{align*}
\]

I must check the following:

**Conjecture:** The map

\[
\begin{align*}
Sp^*(pt) & \longrightarrow KO^*(pt) [t_1, t_2, \ldots] \\
\chi & \longrightarrow \mu(\xi(x)) \quad \mu = Thom hom.
\end{align*}
\]

\( \deg_{h^*} = -4i \)
I'd like to prove this using the method I worked out for \( U^* \). Suppose \( g \) is the first dimension for which is fails to be injective, \( g = 4k + l \), \( 0 \leq l \leq 3 \), and let \( x \in \text{Sp}^{-8}(pt) \) be in the kernel. Then \( S^2 x = 0 \) for \( x > 0 \), so we get a relation

\[
\text{w}^j (w^*)Qx = 0 \quad \text{in} \quad \text{Sp}(B(Z_2 \times (S^2)^n)) = \text{Sp}^{-8}(Z_2)
\]

for \( j \) large. So if I expect to do as for \( U^* \) it is necessary to bring \( j \) down to 1. So I need to know that

**Conjecture:** 1) Assume \( K^0^* \) numbers determine \( x \) in \( \text{dim} < g \). Then if \( a \in \text{Sp}^{-8+4j+3}(RP^3) \) satisfies \( \text{E}(a) = 0 \) and \( j > 0 \), we have \( a = 0 \).

2) Hypotheses as in 1), if \( -x = \text{res}_z \gamma^o (z) = 2y + \langle x, z \rangle \) has zero \( K^0^* - \text{numbers} \), \( z \in \text{Sp}^{-8+3}(RP^2) \), then \( \langle x, z \rangle = 0 \).

The problem is to calculate what \( \gamma^o \) does to \( K^0^*(pt)[2j, \ldots] \). The t's don't enter because the integration is over the framed sphere, or if you prefer, the map comes from the 5-category.

**Problem:** What is the map

\[
K^0^*(RP^3) \xrightarrow{V^*} K^0^*(RP^3 \times S^3) ?
\]
\[ \begin{array}{cccc}
q = 2 & \mathbb{Z} & \mathbb{Z}_2 \omega & 0 \\
1 & \mathbb{Z}_2 \eta & \mathbb{Z}_2 \eta \omega & \mathbb{Z} \\
0 & \mathbb{Z}_2 \eta^2 & \mathbb{Z}_4 \gamma & \mathbb{Z}_2 \eta \\
-1 & 0 & \mathbb{Z}_2 \eta \gamma & \mathbb{Z}_2 \eta^2 \\
-2 & \mathbb{Z}_4 \chi & \mathbb{Z}_2 \tau \omega & 0 \\
-3 & 0 & 0 & \mathbb{Z}_4 \\
-4 & 0 & 0 & 0 \\
-5 & 0 & 0 & 0 \\
-6 & \mathbb{Z}_\beta & \mathbb{Z}_2 \beta \omega & 0 \\
\end{array} \]

\[ t^2 = 4 \beta \quad \eta^2 \omega = 2 \gamma \quad (\text{since } \eta \omega \neq 0) \]
\[ \eta^2 \gamma = ? \]

\[ \tilde{\text{Ko}}^3(\mathbb{R}P^3) \xrightarrow{\Phi} \tilde{\text{Ko}}^{3-3}(\mathbb{R}P^2) \]
(assume $f_{_\Phi} = 0$)

\[ \begin{array}{cccc}
\overline{q} = 2 & \mathbb{Z}_2 \eta + \mathbb{Z}_2 \omega & \mathbb{Z}_2 \eta + \mathbb{Z}_2 \\
1 & \mathbb{Z}_2 \eta^2 + \mathbb{Z}_2 \eta \omega & \mathbb{Z}_2 \eta^2 + \mathbb{Z}_2 \\
0 & 0 + \mathbb{Z}_4 \gamma & 0 + 0 \\
-1 & \mathbb{Z}_4 \chi + \mathbb{Z}_2 \eta \gamma & \mathbb{Z}_4 \chi + 0 \\
0 + \mathbb{Z}_2 \tau \omega & 0 + 0 \\
0 + 0 & 0 + \mathbb{Z}_2 \\
0 + 0 & 0 + \mathbb{Z}_2 \\
\mathbb{Z}_\beta + 0 & \mathbb{Z}_\beta + \mathbb{Z}_4 \\
\end{array} \]

This shows that the map $\Phi$ cannot be injective on the Ko level.
Notes on KO: \( k(X) = KO(X) \oplus K\text{Sp}(X) \).
So we study bundles endowed with a conjugate linear automorphism \( \sigma : \sigma^* = 1 \). Observe that \( k(X) \) is a \( \Lambda \)-ring.

Then isomorphism for \( KO \) says that if \( L \) is an \( S^3 \) bundle over \( X \), then the relative element of \( K\text{Sp}(L; L-X) \) defined by the complex \( 1 \to i_L \) is a Thom class, i.e.

\[
\begin{align*}
k(X) & \cong k(L; L-X) \\
\lambda_L &
\end{align*}
\]

Hence there is a Thom homomorphism

\[
\begin{align*}
\text{Sp}(X) & \to k(X) \\
e(L) & \mapsto L - 1
\end{align*}
\]

\( \text{Sp}(X) = \text{Sp}^{\infty}(X) \)

This is \( \iota \)

\[
k(pt) = \mathbb{Z}[\tau]/\tau^2 = 4
\]

Question: Take \( X = \mathbb{H}P^\infty \times \mathbb{H}P^\infty \) and \( L_i \) the two canonical line bundles. We can form \( L_1 \cdot L_2 \) in \( k(X) \). Does \( \sum t^j (L_1 \cdot L_2 - 1)^{j+1} \) lie in the image of

\[
\begin{align*}
\text{Sp}(X) & \to k(X)[\mathbb{L}] \\
\end{align*}
\]
Why your technique fails to determine the image of $\text{Sp}^*(\text{pt})$ in $N^*(\text{pt})$:

The idea would have been to use the formula:

$$w^n (w^n Qx - x) = \sum_{0 \leq b \leq n} w^n Qx - x \cdot a^b \cdot s_x(x)$$

or rather its image in $N^*(\text{BZ}_2)$, and to argue that the image is at most the set of $y \in N^*(\text{pt})$ satisfying this formula. Then the hope would be that this would be the image. However this set is the subring of $4$th powers in $N^*(\text{pt})$. This one can see more or less in principle: on one hand, taking the $4$th power of the squaring equation for an element $y$ of $N^*(\text{pt})$ shows that $y^4$ satisfies $(*)$ and on the other hand, the image of $\text{Sp}^*(\text{pt})$ is contained in the ring of $4$th powers.

If $x \in \text{Sp}^4(\text{pt})$, then $s_4x \equiv 0 \ (2)$.

$$w(w^n Qx - x) = a_1(s_4x) + f(b) \quad \text{be } \text{Sp}^3(\text{RP}^3)$$

Now restrict to a point of $\text{BZ}_2$. Since

$$e(\eta \otimes L) = w + a_1 e(L) + \cdots$$

on restricting to a point $a_1 \mapsto 1$. Also $f(b) \mapsto 2f(b)$ so

$$s_4x + 2f(b) = 0$$

$$\Rightarrow s_4x \equiv 0 \ (2).$$