Write up paper

Outline:

I. Review of cobordism theory

(i) geometric interpretation of $U^*(X)$, $U^*(X,U)$

Euler classes,

Riemann-Roch theorem

(ii) characteristic classes for a mult. theory / U

a) projective bundle theorem

b) formal group law.

(iii) Gysin homomorphism for a proj. bundle

its applications { blowing up / geometric Chern classes

Operations in cobordism theory:

affine categories and operations in nice GCT's

structure of the affine category of operations for U & N

~~simplica~~ how an algebraic theorem on ~~leads to~~ simplification of group law leads to a simplification of the theory.

1. Localization and typical group laws.

2. unoriented cobordism (laws of height ∞)

3. K theory (laws of height 1)

Stong - Hattori theorem

Proof of theorem that $L \xrightarrow{\sim} U^*(pt)$

Petrie, $H^*(PU(p))$
Milgram, Dickson

~~Thm~~ Given sphere ~~fibration~~ over X get right action of $\mathcal{A}$ on $H^*(X)$.

$$\int_X (xa)\cdot y = \int_X x\cdot a(y) \qquad \underline{X \text{ manifold}}.$$

$$\bigcap_{M^n} \operatorname{Ker}\{H^*(BO) \longrightarrow H^*(M^n)\} = I_n$$

$$I_n^q = \operatorname{Ker} H^q(BO) \longrightarrow \eta_*(K(\mathbb{Z}_2, n-q)^*)^*$$

Modular form Colloq. Lectures 1974

~~Lemma~~: Suppose G is a finite group acting on ~~a~~ a closed manifold M. Then M/G is a rational homology manifold hence has Pontryagin classes.

<u>Proof</u>: It's a local question & have to know only ~~that~~ what's the case ~~when~~ for a group acting on a vector space

More refined example: Suppose G preserves a weakly complex structure on M. Then M/G should have rational ~~Todd classes~~ Chern classes. It's a question of understanding the Atiyah-Segal paper.

<u>Problem</u>: Suppose that ~~___~~ V is a repn. of G say cx. and that we form $\mathbb{P}(V)/G$. Note that

$$H^*(\mathbb{P}V/G, \mathbb{Q}) \xrightarrow{\sim} H^*(\mathbb{P}V)$$

since G preserves the generator. Now $\mathbb{P}V/G$ has rational Chern classes. Find a formula.

<u>First reformulation</u>: The rational Chern classes ~~are~~ really come from rational Todd classes which are determined by duality. Thus have map

$$K(\mathbb{P}V/G) \longrightarrow \mathbb{Z}$$

given by $E \longmapsto \chi(H^*_G(\mathbb{P}V, \Omega(E)))$

Another variation of odd order business
G finite action on a manifold M
$\Rightarrow \partial(M/G) = 0$ if G is of odd order.

$$\begin{tikzcd}
K(X/G) \arrow[r] \arrow[d] \arrow[dr, phantom, "?"] & K_G(X) \arrow[r] \arrow[d] & K(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "?"] & H(X) \arrow[d] \\
K(Y/G) \arrow[r] & K_G(Y) \arrow[r] & K(Y) \arrow[r] & H(Y)
\end{tikzcd}$$

over the rationals we have

Take up rational cohomology first. Then can you define a map $H^*(X/G) \longrightarrow H^*(Y/G)$. Yes use Poincaré duality

$$\begin{tikzcd}
H^*(X/G) \arrow[r] \arrow[d, "\cong"] & H^*(Y/G) \arrow[d, "\cong"] \\
H^*(X)^G \arrow[r] & H^*(Y)^G
\end{tikzcd}$$

we should be a bit careful. Is it clear that if G acts on X compact oriented manifold then $H^*(X)^G$ sat. P.D?

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relativize Sullivan's construction with fundamental cycle? A bit less precise unless one can prove that Poincaré duality holds for K. Conjecture

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~~X weakly complex G-m~~

$X \longrightarrow Y$ proper oriented G-map where G acts trivially on Y then get for any space Z a map

$$\begin{tikzcd}
K(X/G;Z) \arrow[r] & K_G(X;Z) \arrow[r, "f^*"] & K_G(Y;Z) = R(G)\otimes K(Y;Z) \arrow[d, "\int_G"] \\
& & K(Y;Z)
\end{tikzcd}$$

thus get a fundamental class in $K(Y;X/G)$

X is a weakly complex G-manifold

<u>Atiyah-Segal version</u>: Define a map

$$K(X/G) \longrightarrow \mathbb{Z}$$

as follows. Given E over X/G, get f^*E G-bundle over X, integrate over X to obtain an element of R(G), then take inner product with 1. If you use cohomology then you get an element ~~an element~~ of $H_*(X/G)$ with rational coefficients.

Generalization: Use equivariant cobordism except no inner product with 1.

~~Compute in the case of projective space~~

Inner product with 1 = average over the group.

more generally suppose given ~~as~~ complex oriented proper G-map

$$X \xrightarrow{f} Y$$

$$X/G \qquad\qquad Y/G$$

define (an induced) ~~a~~ map

$$K(X/G) \longrightarrow K(Y/G)$$

and proves a Riemann-Roch formula in which the Todd class enters.

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{j} & \tilde{X} \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xhookrightarrow{i} & X \end{array}$$

$\nu = 2 + E'$

$f_* 1 = 1$ so

$$\begin{aligned} i_* y &= f_* f^* i_* y \\ &= f_*\{j_* s_* y\} \\ &= f_*\{j_*(g^* y \cdot s_* 1)\} \end{aligned}$$

~~Idea~~ Any hope that one can show that

$$K(X) \longrightarrow \mathbb{Z}_{todd} \otimes_{U(pt)} U(X)$$

is an isomorphism by power operations? idea would be to show that power operations can be defined on the left integrally. Problem of proving surjectivity, so start with an element on the left, and use the ~~filtration by powers~~ Atiyah tom Dieck expansion for it

ontoness is clear? enough to do for ~~skeleton~~ ~~of Grassmannians~~ Thom spaces over Grassmannians where the needed elements are already there? ?

~~next idea~~

How do you prove the exactness axiom?

$$U^*(Y) \longrightarrow U^*(X) \longrightarrow U^*(X-Y)$$

basically like the abelian case

V representation of G cyclic with generator g.

eigenvalues λ_i $i = 1, \dots, n = \dim V$ with multiplicities

Then want to consider action of ψ^k on $K_G(PV)$

where $k \equiv 1 \mod |G|$.

ψ^2

$R(G) \longrightarrow \mathbb{C}$

$\chi \longmapsto \chi(g)$

$$\frac{\mathbb{C}[T]}{\prod_i (T-\lambda_i)} = \bigoplus_{j=} \frac{\mathbb{C}[T]}{(T-\lambda_j)^{n_j}}$$

note that $(\lambda_i)^g = 1$ so $\lambda_i^k = \lambda_i$

and $\psi^k(T) = T^k$

$$2^N \equiv 1 \ (g)$$

$$\frac{\mathbb{C}[T]}{(T-\lambda)^n}$$

Its eigenvalues becomes $(2^N) \dots (2^N)^a$ where a = index of nilpotence

$$\psi^k(T-\lambda) = T^k - \lambda^k = (T-\lambda)(T^{k-1} + T^{k-2}\lambda + \dots + \lambda^{k-1})$$

$\mod (T-\lambda)^2$ $\quad k\lambda^{k-1} = k$.

$$\equiv (T-\lambda)\cdot k \quad \mod (T-\lambda)^2$$

so eigenvalues are $1, k, \dots, k^{n_\lambda}$ for each eigenvalue of mult. n_λ

to count

$$N_k = \text{card } X(\mathbb{F}_{q^k})$$

$$\zeta(s) = e^{\sum N_k \frac{z^k}{k}} \qquad\qquad z = q^{-s}$$

$$\sum q^k \frac{z^k}{k} = \log\left(\frac{1}{1-qz}\right).$$

$$\frac{1}{1-q^n z} = \zeta_{A^n_{\mathbb{F}_q}}(s)$$

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now of course $X(\mathbb{F}_{q^n} - 0) = \boxed{q^n - 1}$

$$\frac{1-z}{1-q^n z} = e^{(q-1)\sum \cancel{A} \frac{q^k-1}{q-1}\frac{z^k}{k}}$$

so take $\left(\dfrac{1-z}{1-qz}\right)^{1/q-1}$

exists as a rational function

$q-1$

G

1 2 3 4

1

X G group and G acts on X <u>over</u> $\mathbb{F}_q$ and G acts freely. Then G acts freely on each $X(\mathbb{F}_{q^n})$. And we find that

$$\zeta_X(s) = \left\{\zeta_{X/G}(s)\right\}^{|G|}$$

<u>example</u>: Let X be 2 space minus lines so that

$$\zeta_X(s) = \frac{(1-qz)^{q+1}}{(1-q^2z)(1-z)}$$

and let G be a group of linear transformations then $G \subset Gl(2,q)$ which has order $(q^2-1)(q^2-q)$

If A is a matrix over $\mathbb{F}_q$ and if $Av = v$ then A has eigenvalues 1, but the eigenspace of eigenvalue 1 is defined over $\mathbb{F}_q$ so has been removed for $A \neq 1$.

X 1 space $\mathbb{F}_q^*$ $q-1$ acts freely ie

$x \longmapsto \lambda x$ and

no fixed points $X = A^1 - 0$

$$\frac{1-z}{1-qz} = \left(\zeta_{X/G}\right)^{q-1}$$

$$X^p = 1$$

1, 1, ...

Suppose $A \in Gl(\mathbb{F}_q) \ni A^p = I$

get a ~~separable~~ ~~inseparable~~ extension.

$\exists f \in S(V) \ni$ $\quad Af = f + 1$

$$\boxed{\begin{aligned} Ae_0 &= e_0 + e_1 \\ Ae_1 &= e_1 + e_2 \\ &\vdots \\ Ae_{p} &= e_p \end{aligned}}$$

$$A\left(\frac{e_0}{e_1}\right) = \frac{e_0 + e_1}{e_1} = \frac{e_0}{e_1} + 1$$

$p = 2$.

$$A\left(\frac{e_{p-1}}{e_p}\right) = \frac{e_{p-1}}{e_p} + 1$$

$$f(x) = \sum' a_\alpha x^\alpha$$

$$\left(\frac{e_{p-1}}{e_p}\right)^p - \frac{e_{p-1}}{e_p}$$

$$= \frac{e_{p-1}^p e_p - e_{p-1} e_p^{p-1}}{e_p^p} = a$$

It says that

$$\sum_N (\log I_N) t^n =$$

$$f(x)\, f(y)$$

whole idea might

$$a_N = \sum_{r=0}^{N} \langle {N \atop r} \rangle b_r$$

$$\cancel{a(t) = \varphi(t)\, b(t)} \quad \text{~~where~~}$$

$$\cancel{\varphi(t) = \sum}$$

$$\sum_N \frac{a_N}{N!} t^N = \sum_{r=0}^{N} \frac{t^{N-r}}{(N-r)!} \cdot \frac{b_r t^r}{r!}$$

so you introduce a mystical function $\Phi(x) = \sum \frac{x^r}{r!}$

$$\sum_N \frac{a_N}{N!} x^N = \Phi(x) \sum \frac{b_r}{r!} x^r$$

and the inversion formula given

ζ function of a scheme of finite type over $\mathbb{F}_q$

$$\mathrm{Card}\left\{\mathbb{F}_q[T_1, \ldots, T_N]\,(\mathbb{F}_{q^n})\right\} = \mathrm{Card}(\mathbb{F}_{q^n})^N$$

$$\mathrm{Card}(\mathbb{F}_{q^n}) = q^n$$

$$\zeta_X(s) = \exp\left\{\sum_{n\geq 1} \frac{\mathrm{card}\, X(\mathbb{F}_{q^n})}{n} z^n\right\} \qquad z = q^{-s}$$

$$\sum_{\cancel{\mathrm{card}}\ n\geq 1} \frac{\mathrm{card}\ \mathbb{F}_q[T_1, \ldots, T_N]\,(\mathbb{F}_{q^n})}{n} z^n = \sum_{n\geq 1} \frac{q^{nN}}{n} z^n$$

$$= -\log(1-q^N z)$$

hence

$$\zeta_{A^N_{\mathbb{F}_q}}(s) = \frac{1}{1-q^N z}$$

Now let us try to remove the ~~finite~~ proper subspaces rational over $\mathbb{F}_q$

Let V be a vector space over $\mathbb{F}_q$ of dim N. I want to let $\zeta^m_V(s)$ to be the ζ of

$$V_\Omega - \bigcup_{W\subset V} W_\Omega$$

$N = 1$ this is

$$\zeta^m_V(s) = \frac{1-z}{1-qz}$$

If $N=2$ then all subspaces are 1-dimensional $+\,0$.

so
$$\zeta^m_V(s) = \frac{(1-qz)^{q+1}\,(1-z)}{(1-q^2z)(1-z)^{q+1}} = \frac{(1-qz)^{q+1}}{(1-q^2z)(1-z)^{q}}$$

$$\zeta_V(s)\cdot\left(\frac{1}{\zeta_{\mathbb{F}_q-0}(s)}\right)^{q+1}\cdot\left(\frac{1}{\zeta_0(s)}\right)$$

In general $\cancel{\zeta_V(s)}$

$$\prod_{\substack{W \text{ subspace} \\ \text{of } V}} \zeta^m_W(s) = \zeta_V(s)$$

so one can invert this ala Möbius I guess

$W^n \subset V^N$

$$\frac{Gl(N)}{\begin{pmatrix} Gl(n) & M(n,N-n) \\ & Gl(N-n) \end{pmatrix}} \qquad \text{card } Gl(N)$$

$$\text{card } Gl(N) = (q^N-1)(q^N-q)(q^N-q^2)\cdots(q^N-q^{N-1})$$
$$= q^{\frac{N(N-1)}{2}}\prod_{i=1}^{N}(q^i-1)$$

$$\text{card } \mathrm{Grass}_r(N) = \frac{q^{\frac{N(N-1)}{2}} \prod_{i=1}^{N} (q^i - 1)}{q^{\frac{r(r-1)}{2} + \frac{(N-r)(N-r-1)}{2}} \prod_{i=1}^{r} (q^i-1) \prod_{i=1}^{N-r} (q^i - 1) \; q^{r(N-r)}}$$

$$\frac{a(a-1)}{2} + \frac{b(b-1)}{2} + ab = \frac{a^2 - a + b^2 - b + 2ab}{2}$$

$$= \frac{(a+b)^2 - (a+b)}{2}.$$

$$\therefore \quad \text{card } \mathrm{Grass}_r(N) = \frac{\prod_{i=1}^{N} (q^i - 1)}{\prod_{i=1}^{r} (q^i-1) \prod_{i=1}^{N-r} (q^i-1)} = \left\langle {N \atop r} \right\rangle$$

$$\boxed{\log \zeta_N(s) = \sum_{r=0}^{N} \left\langle {N \atop r} \right\rangle \log \zeta'_r(s)}$$

$$\log \left\{ \frac{1}{1 - q^N z} \right\} = \sum$$

$$a_N = \sum_{r=0}^{N} \binom{N}{r} b_r$$

$$\sum a_N t^N = (1+t)^N \left\{ \sum b_r t^r \right\}.$$

$$\sum_{r=0}^{N} \left\langle {N \atop r} \right\rangle t^r = \prod_{i=1}^{N}(q^i-1) \sum \frac{t^r}{\prod_{i=1}^{r}(q^i-1)\prod_{i=1}^{N-r}(q^i-1)}$$

Take $\alpha_i = q^i - 1$

$$\prod_{i=1}^{r} \alpha_i \prod_{i=1}^{N-r} \alpha_i$$

$$\sum_{r=0}^{N} \binom{N}{r} t^r = N! \sum_{r=0}^{N} \frac{t^r}{r!} \cdot \frac{1^{N-r}}{(N-r)!}$$

$$\sum_{r,N} \frac{1}{N!} \sum \boxed{\left\langle {N \atop r} \right\rangle x^r y^{N-r}} = \left(\sum \frac{1}{r!} x^r\right)\left(\sum \frac{1}{s!} y^s\right)$$

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thus

$$\sum_{a,b} \frac{1}{\prod_{i=0}^{a+b} \alpha_i} \left\langle {a+b \atop a} \right\rangle x^a y^b = (\quad)(\quad)$$

so there is a basic function

$$f(x) = \sum \frac{x^n}{\prod_{i=1}^{n} \alpha_i} = \boxed{\sum_{n \geq 0} \frac{x^n}{(q^n-1)\cdots(q-1)}}$$

which I want to know

~~and~~

$$a_N = \log \frac{1}{1-q^N z}$$ or we might take

$$a_N = \frac{\cancel{+}q^N}{1-q^N z}$$ ess. derivative.

then we have

$$\sum_N \frac{a_N}{N!} x^N = \sum_N \frac{q^N x^N}{1-q^N z \cdot \prod_{i=1}^{N} (q^i - 1)}$$

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let $q \longrightarrow 0$ keeping

And it's almost ~~related~~ to the law for connected K-theory.

~~dim~~

0 $\quad \frac{1}{1-z}$

1 $\quad \frac{1-z}{1-qz}$

2 $\quad \frac{(1-qz)^{q+1}}{(1-q^2 z)(1-z)^q}$

$(q-1)(q+1)^{+1} = q^2 \cancel{-1}$

3.

$$\varphi'_3 \left(\varphi'_2\right)^{q+1} \left(\varphi'_1\right)^{q+1} \varphi'_0 = \frac{1}{1-q^3 z}$$

$$\varphi'_3 \left(\frac{(1-qz)^{q+1}}{(1-q^2 z)\cancel{(1-z)^q}}\right)^{q+1} \left(\frac{\cancel{1-z}}{1-qz}\right)^{q+1} \frac{1}{1-z}$$

$$\boxed{\frac{(1-q^2 z)^{q+1} (1-z)^{q^2}}{(1-q^3 z)(1-qz)^{q(q+1)}}}$$

$$\frac{(q^4-1)(q^3-1)}{(q-1)(q^2-1)}$$

$$\varphi_4' = \frac{1}{1-q^4 z}\left(\frac{(1-q^3 z)(1-qz)^{q(q+1)}}{(1-q^2 z)^{q+1}(1-z)^{q^2}}\right)^{q^3+q^2+q+1}\left(\frac{(1-q^2 z)(1-z)^{q}}{(1-qz)^{q+1}}\right)^{q^2+1\,\frac{q^3-1}{q-1}}$$

$$\left(\frac{1-qz}{1-z}\right)^{\frac{q^4-1}{q-1}}(1-z)$$

$$\frac{(1-q^3 z)^{\frac{q^4-1}{q-1}}}{1-q^4 z}\;\frac{(1-qz)^{q(q+1)(q^3+\dots+1)-(q+1)(q^2+1)(q^2+\dots+1)}}{(1-q^2 z)^{\underbrace{\frac{q^2-1}{q-1}\frac{q^4-1}{q-1}-(q^2+1)\left(\frac{q^3-1}{q-1}\right)}_{q^3+q+1}}\;(1-z)^{q^2(q^3+\dots+1)-q(q^2+1)\frac{q^3-1}{q-1}}}$$

$$(q+1)(q^3+q^2+q+1) = \begin{array}{l} q^4+q^3+q^2+q \\ \quad\;\; q^3+q^2+q+1 \end{array}$$

$$(q^2+1)(q^2+q+1) = \begin{array}{l} q^4+q^3+q^2 \\ \qquad\quad\;\; q^2+q+1 \\ \hline \end{array}$$

$$\cancel{q}\quad q^3+q+1.$$

$$(q^2+q)(q^3+\dots 1) = \cancel{q^5}+\dots+q^2 + q^4+\dots+q$$

$$(q^3+q^2+q+1)(q^2+q+1) = \begin{array}{l} \cancel{q^5}\;\; q^4\;\; q^3 \\ \quad\;\; q^4\;\; q^3+q^2 \\ \qquad\quad\;\; q^3+q^2+q \\ \qquad\qquad\quad\; q^2+q+1 \end{array}$$

$$\cancel{q^4}\quad -q^3-q^2-q-1$$

$$\cancel{q^5} + \cdots + q^2 - (q^3+q)(q^2+q+1)$$

$$\cancel{q^5} + \cancel{q^4} + q^3$$

$$q^3 + q^2 + q$$

$$-q^3 - q.$$

$$\zeta_4' = \frac{(1-q^3 z)^{q^3+q^2+q+1}\,(1-z)^{q^3+q}}{(1-q^4 z)(1-q^2 z)^{q^3+q+1}(1-qz)^{q^3+q^2+q+1}\,\cancel{(1-z)^{q^3+q}}}$$

January 9, 1970. (very groggy)

Basic question: Let G be a finite group and let $CG_*(pt)$ be the cobordism ring of compact oriented G-manifolds. Let $Q_q(G)$ be the graded ring of period 4 which is 0 in odd dimensions and for $q \equiv 0\ (4)$ is the Grothendieck-Witt group of symmetric non-degenerate quadratic $\mathbb{Q}$-spaces with G-action and for $q \equiv 2\ (4)$ is the anti-symmetric ones. Then the usual proof that that the index is a homomorphism from $SO_*(pt) \longrightarrow \mathbb{Z}[t_4]$ should generalize to show that one obtains a ring hom. α in the diagram

$$\begin{tikzcd}
CG_*(pt) \arrow[r, "\alpha"] \arrow[d] & Q_*(G) \\
SO^*_G(pt) \arrow[ur, dashed, "\beta"'] &
\end{tikzcd}$$

where α associates to X the cup-product form on $H^*(X, \mathbb{Q})$. The question is whether there is a natural extension β of α?

Let's review the situation without the group. Here the existence of β is immediate since the vertical arrow is an isomorphism. However one does more. Namely one shows how to endow $KO^* \otimes \mathbb{Z}[\frac{1}{2}]$ with a Thom isomorphism for SO-bundles in such a way that

$$SO^*(pt) \longrightarrow KO^*(pt)[\tfrac{1}{2}] = \mathbb{Z}[\tfrac{1}{2}][t_4, t_4^{-1}]$$

is the signature map. Can one do the same thing equivariantly?

<u>Question:</u> Let G be a compact Lie group. Does $KO_G[\frac{1}{2}]$ have an equivariant Thom isomorphism for oriented $4k$-G-bundles?

The idea intuitively is that if E is an oriented $4k$-G-bundle over a G-space X, then one can construct from $\Lambda^{\cdot} E$ with $*$ operators a class in $KO_G(E, E-X)$, which one knows at least has the augmentation $\sim 2^k$. So there's a chance at least that it is a Thom class after $\frac{1}{2}$ is adjoined.

By the usual argument one can suppose X is a point. Let's also assume that has an underlying complex structure, e.g. if G is a finite group of odd order and see what happens for KU where we have a Thom class coming from the complex structure. What we have to do ~~use to compute~~ then is to compute the ratio of the signature class $\sigma(E)$ and U_E; it is an element of $R(G)$ with augmentation $\sim 2^k$, hence a unit in $\widehat{R(G)}[\frac{1}{2}]$ but maybe not in $R(G)[\frac{1}{2}]$.

~~[illegible]~~ January 9, 1970 (still groggy)

On symplectic cobordism: Let $h^q(X) = Sp^q(X \times Z)$ where Z is fixed. I recall the Gysin sequence

$$\begin{tikzcd} \tilde{h}^{q+3}(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} S^3) \arrow[r, "\pi_*"] & h^q(B\mathbb{Z}_2) \arrow[r, "\omega"] & h^{q+4}(B\mathbb{Z}_2) \\ \tilde{h}^{q+3}(\mathbb{R}P^3) \arrow[u, "pr_2^*", "\cong"'] \arrow[ur, "\xi"'] & & \end{tikzcd}$$

I wish to understand the composition

$$\tilde{h}^{q+3}(\mathbb{R}P^3) \xrightarrow{\xi} h^q(B\mathbb{Z}_2) \xrightarrow{res_3} h^q(\mathbb{R}P^3)$$

Let $i: pt \longrightarrow \mathbb{R}P^3$ be the origin of this Lie group and choose a framing, That is, a basis for the Lie algebra. Now $\mathbb{R}P^3$ is parallelizable since it is a Lie group. It seems that there are two possible trivializations of the tangent bundle using left or right invariant vector fields. (These are probably different since there difference should be measured by the adjoint action homomorphism $\mathbb{R}P^3 \longrightarrow SO(3)$, which should represent a non-trivial element of $\pi_4(BO) = \mathbb{Z}$?) Choosing one we get a map $f: h^{q+3}(\mathbb{R}P^3) \longrightarrow h^q(pt)$ satisfying $f_* i_* = id$ and so the exact sequence

$$\longrightarrow h^q(pt) \underset{f_*}{\overset{i_*}{\rightleftarrows}} \tilde{h}^{q+3}(\mathbb{R}P^3) \xrightarrow{j^*} \tilde{h}^{q+3}(\mathbb{R}P^2) \xrightarrow{\delta} \cdots$$

splits giving isomorphisms for all q

$$\tilde{h}^{q+3}(\mathbb{R}P^3) = h^q(pt) \oplus \tilde{h}^{q+3}(\mathbb{R}P^2).$$

I want to clear up something that was slurred over before. I claim that

$$\begin{tikzcd} E\mathbb{Z}_2 \times_{\mathbb{Z}_2} S^3 \arrow[r, "pr_1"] \arrow[d, "pr_2"] & B\mathbb{Z}_2 \\ \mathbb{R}P^3 \arrow[ur, "incl."'] & \end{tikzcd}$$

is homotopy commutative. Indeed just recall that $B\mathbb{Z}_2 = BO(1)$ and that its enough to see what happens in $H^*(?;\mathbb{Z}_2)$. Thus in fact we know that there is a long exact sequence

$$\tilde{h}^{q+3}(\mathbb{R}P^3) \xrightarrow{\xi} h^q(B\mathbb{Z}_2) \xrightarrow{\omega} \tilde{h}^{q+4}(B\mathbb{Z}_2) \xrightarrow{res_3} \tilde{h}^{q+4}(\mathbb{R}P^3)$$

which one perhaps also ought to view as a spectral sequence (without the ~'s). We want to compute the first differential which is the composition

$$h^{q+3}(\mathbb{R}P^3) \xrightarrow{\xi} h^q(\mathbb{R}P^\infty) \xrightarrow{res_3} h^q(\mathbb{R}P^3)$$

It is given by the correspondence

$$\begin{tikzcd} & S^n \times_{\mathbb{Z}_2} S^3 \arrow[dl, "pr_2"'] \arrow[dr, "pr_1"] & \\ \mathbb{R}P^3 & & \mathbb{R}P^n \end{tikzcd}$$

where we use the n to keep from being confused. Note that this can be factored

$$\begin{tikzcd}
S^n \times_{\mathbb{Z}_2} S^3 \arrow[d, "\alpha"] \arrow[drr, "pr_1"] & & \\
\mathbb{R}P^n \times \mathbb{R}P^3 \arrow[d, "pr_2'"'] \arrow[rr, "pr_1'"] & & \mathbb{R}P^n \\
\mathbb{R}P^3 & &
\end{tikzcd}$$

and that the symplectic orientations of pr_1 and pr_1' ($\mathbb{R}P^3$ parallelizable) combine to give ~~a~~ symplectic orientation for α. (Actually since α is a double covering it has a canonical framed orientation, however, as yet I see no reason why this etale orientation coincides with the symplectic one.) So we have that

$$res_n(\xi(z)) = pr'_{1*}(\alpha_* 1 \cdot pr_2'^{*} z)$$

Now if $n=3$ we have this diagram

$$\begin{tikzcd}
 & S^3 \times_{\mathbb{Z}_2} S^3 \arrow[r, "\mu"] \arrow[d, "\alpha"] & S^3 \arrow[d, "\gamma"] \\
 & \mathbb{R}P^3 \times \mathbb{R}P^3 \arrow[r, "\mu"] \arrow[dl, "pr_2'"'] \arrow[d, "pr_1'"] & \mathbb{R}P^3 \arrow[d, "f"] \\
\mathbb{R}P^3 & \mathbb{R}P^3 \arrow[r] & pt
\end{tikzcd} \qquad (*)$$

$\mu(x,y) = x \cdot y$

(frame the τ of $\mathbb{R}P^3$ via right invariant fields so that pr_1' is similarly fr.)

where the big ~~whole~~ square comes from the quaternion bundle maps

$$\begin{tikzcd}
S^3 \times_{\mathbb{Z}_2} \mathbb{H} \arrow[r, "\mu"] \arrow[d, "pr_1"] & \mathbb{H} \arrow[d] \\
\mathbb{R}P^3 \arrow[r] & pt
\end{tikzcd}$$

($\mathbb{H}$ acts on the right)

I wish to orient α and pr_1' via f and γ. Then I get the formula

$$res_3(\xi z) = pr'_{1*}\left(\mu^* \gamma_* 1 \cdot pr_2'^* z\right)$$

It therefore seems necessary to compute

$$\gamma_* 1 \in Sp^0(\mathbb{R}P^3) \cong Sp^0(pt) \oplus Sp^{-3}(pt) \oplus \widetilde{Sp}^0(\mathbb{R}P^2)$$

$$\gamma_* 1 = (\ 2 \ + \ \underset{\substack{\uparrow \\ (\text{since } f_*\gamma_* 1 = 0)}}{0} \ + \ ?\ )$$

Now $\qquad \widetilde{Sp}^0(\mathbb{R}P^2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ ~~...~~ or $\mathbb{Z}_4$ since

$$\begin{array}{ccccccccc}
\widetilde{Sp}^{-1}(\mathbb{R}P^1) & \xrightarrow{2} & Sp^{-2}(pt) & \longrightarrow & \widetilde{Sp}^0(\mathbb{R}P^2) & \longrightarrow & \widetilde{Sp}^0(\mathbb{R}P^1) & \xrightarrow{2} & \\
\| & & \| & & & & \| & & \\
\mathbb{Z}_2 & \xrightarrow{2} & \mathbb{Z}_2 & & & & Sp^{-1}(pt) & & \\
 & & & & & & \| & & \\
 & & & & & & \mathbb{Z}_2 & &
\end{array}$$

(I believe it is known that $\tilde{\pi}_s^0(\mathbb{R}P^2) = \mathbb{Z}_4$, ~~...~~ and so it should be that $\widetilde{Sp}^0(\mathbb{R}P^2) = \mathbb{Z}_4$. The conjecture then is that $\gamma_* 1 - 2$ generates $\widetilde{Sp}^0(\mathbb{R}P^2) = \mathbb{Z}_4$, and we can test this by showing that $\gamma_* 1 - 2$ ~~...~~ restricts to the non-zero element of $\widetilde{Sp}^0(\mathbb{R}P^1) = \mathbb{Z}_2$.)

I want to review ~~...~~ the calculation of $Sp^{-q}(pt)$ for $q = 0, 1, 2, 3$. We start with

$$Q: \ Sp^0(S^\ell, *) \longrightarrow Sp^0(B\mathbb{Z}_2 \times (S^\ell, *))$$

where $0 < \ell \leq 3$. ~~...~~ Let $x \in Sp^0(S^\ell, *) = Sp^{-\ell}(pt)$. Then $S_\alpha x = 0$

for $\alpha > 0$, so our localization formula gives

$$w^n\{Qx - x\} = 0 \qquad \text{for } n \gg 0.$$

Now the Gysin sequence

$$\widetilde{Sp}^{q+3}(\mathbb{RP}^3 \wedge (S^\ell,*)) \xrightarrow{\xi} Sp^q(B\mathbb{Z}_2 \times (S^\ell,*)) \xrightarrow{w} Sp^{q+4}(B\mathbb{Z}_2 \times (S^\ell,*))$$

$\overset{\|}{0}$ if $q > \ell$ (e.g. $q = 4$)

show that we must have

$$w\{Qx - x\} = 0$$

and hence $\exists\, y \in \widetilde{Sp}^{3}(\mathbb{RP}^3 \wedge \Sigma^\ell) = \widetilde{Sp}^{3-\ell}(\mathbb{RP}^3)$ such that

$$Qx - x = \xi(y) \qquad \text{in } Sp^0(B\mathbb{Z}_2 \times (S^\ell,*)).$$

Now restricting from $B\mathbb{Z}_2$ to point and using that $\operatorname{res}(Qx) = x^2 = 0$ since $\ell > 0$ we have

$$-x = \operatorname{res} \xi(y) \quad \text{in } \widetilde{Sp}^0(S^\ell) = Sp^{-\ell}(pt).$$

Recalling the defn. of ξ this means we have a surjectivity arrow

$$\begin{tikzcd}
\widetilde{Sp}^{3-\ell}(\mathbb{RP}^3) \arrow[r, "\gamma^*"] \arrow[dr, "\text{onto}"'] & \widetilde{Sp}^{3-\ell}(S^3) \arrow[d, "\cong"', "\text{int. over } S^3"] & \\
& Sp^{-\ell}(pt) \arrow[r] & 0
\end{tikzcd}$$

Recall that

$$\begin{array}{ccc}
 & & Sp^{1-\ell}(pt) \\
 & & \big\downarrow 2 \\
 & & Sp^{1-\ell}(pt) \\
 & & \big\downarrow \\
\widetilde{Sp}^{3-\ell}(\mathbb{R}P^3) \cong & Sp^{-\ell}(pt) \oplus & \widetilde{Sp}^{3-\ell}(\mathbb{R}P^2) \\
 & & \big\downarrow \\
 & & \widetilde{Sp}^{3-\ell}(\mathbb{R}P^1) \cong Sp^{2-\ell}(pt) \\
 & & \big\downarrow 2
\end{array}$$

Now the composition

$$Sp^{-\ell}(pt) \xrightarrow{t_*^*} Sp^{3-\ell}(\mathbb{R}P^3) \xrightarrow{g^*} Sp^{3-\ell}(S^3) \longrightarrow Sp^{-\ell}(pt)$$

is multiplication by 2 since it sends

$$[Z \to pt] \longmapsto [Z \to pt \to \mathbb{R}P^3] \mapsto [Z \times \mathbb{Z}_2 \to \mathbb{Z}_2 \to S^3] \mapsto [Z \times \mathbb{Z}_2 \to pt]$$

Consequently we obtain a surjective homomorphism (*)

$$\widetilde{Sp}^{3-\ell}(\mathbb{R}P^2) \xrightarrow{\ *\ } Sp^{-\ell}(pt)/2\,Sp^{-\ell}(pt) \longrightarrow 0, \quad 0 < \ell \leq 3$$

$$\widetilde{Sp}^{3-\ell}(\mathbb{R}P^2) \xrightarrow{(**)} Sp^{-\ell}(pt) \longrightarrow Sp^{-\ell}(pt)/2\,Sp^{-\ell}(pt)$$

which we note is compatible with mult. by elements of $Sp^*(pt)$, for all ℓ. ((**) also surjective by Nakyama.)

Now for $\ell = 0$ we know $Sp^0(pt) = \mathbb{Z}$. Hence we see that

$$\widetilde{Sp}^2(\mathbb{R}P^2) = \mathbb{Z}_2 \implies Sp^{-1}(pt) \text{ is } 0 \text{ or } \mathbb{Z}_2$$

(By use of the homomorphism $Sp^*(X) \longrightarrow KO^*(X)$, which we should ~~know~~ be able to calculate, we know it is $\mathbb{Z}_2$.)
Call the generator of $Sp^{-1}(pt)$ η.

Next for $l=2$ we get

$$\tilde{Sp}^{1}(\mathbb{RP}^2) = \mathbb{Z}_2\, \eta\cdot(\text{gen. of } \tilde{Sp}^2(\mathbb{RP}^2))$$

so

$$Sp^{-2}(pt) = \mathbb{Z}_2 \cdot \eta^2 \qquad (\text{or } 0 \text{ which is out by } KO^*).$$

Using this we see that for $l=3$

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\eta^2\cdot(\text{gen. of } \tilde{Sp}^2(\mathbb{RP}^2))} \tilde{Sp}^{0}(\mathbb{RP}^2) \longrightarrow \underset{\mathbb{Z}_2}{\underset{\parallel}{Sp^{-1}(pt)}} \longrightarrow 0$$

Now in fact from KO-theory we know that $\tilde{KO}^0(\mathbb{RP}^2) = \mathbb{Z}_4$ hence same will be true for $\tilde{Sp}^0(\mathbb{RP}^2)$. Therefore we have a surjection

$$\begin{array}{ccccc} \mathbb{Z}_4 & \twoheadrightarrow & Sp^{-3}(pt) & \longrightarrow & KO^{-3}(pt) = 0 \\ \cup & & & & \\ 2 & \longmapsto & \eta^3 & & \end{array}$$

~~So we conclude that either 2η³ ... and Sp⁻³(pt) ...~~ The really good thing would be for $Sp^{-3}(pt) = 0$ for then and only then would $Sp^{-3}(pt)$ be determined by KO char. nos. (I believe from stable homotopy theory one knows that $\pi_s^{-3}(pt) = \mathbb{Z}_{24}$; it follows that η^3 must be zero in $Sp^{-3}(pt)$ since it comes from an element of order 2 in the $\mathbb{Z}_{24}$. Consequently $Sp^{-3}(pt)$ is at most $\mathbb{Z}_2$, generated by the image of the generators of $\pi_s^{-3}(pt)$ which I believe people call σ. I also believe this element comes from the Hopf fibration $S^3 \to S^7 \to S^4$, hence is represented by the manifold S^3 framed in some way.)

January 11, 1970 (very groggy)

Let h^* be a generalized cohomology theory. Then there is a canonical map

$$\alpha : \tilde{h}^q(\mathbb{R}P^3) \longrightarrow h^{q-3}(\mathbb{R}P^3)$$

defined as the composition

$$\tilde{h}^q(\mathbb{R}P^3) \xrightarrow{p_2^*} \tilde{h}^q(S^3 \times_{\mathbb{Z}_2} S^3) \xrightarrow{p_{1*}} h^{q-3}(\mathbb{R}P^3)$$

where the framed orientation for p_1 comes from the trivialization

$$\begin{tikzcd} S^3 \times_{\mathbb{Z}_2} S^3 \arrow[r, "\mu"] \arrow[d, "p_1"] & S^3 \arrow[d] \\ \mathbb{R}P^3 \arrow[r] & pt \end{tikzcd}$$

In other words we are using the fact that the (right) quaternionic line bundle $S^3 \times_{\mathbb{Z}_2} \mathbb{H}$ over $\mathbb{R}P^3$ is trivialized by $\mu : S^3 \times_{\mathbb{Z}_2} \mathbb{H} \to \mathbb{H}$.

Now the map α is represented by an S-map $\Sigma^3 \mathbb{R}P^3 \to \mathbb{R}P^3$, which one can compute as follows

$$\begin{tikzcd} h^q(\mathbb{R}P^3) \arrow[r, leftrightarrow, "p_1^*"] \arrow[d, "\cong\ \text{susp.}"'] & h^q(S^3 \times_{\mathbb{Z}_2} S^3) \arrow[rr, "p_2^*"] \arrow[d, "\cong\ \text{susp.}"'] & & h^{q-3}(\mathbb{R}P^3) \\ h_c^{q+1}(\mathbb{H}^*/\mathbb{Z}_2) \arrow[r, leftrightarrow, "p_1^*"] & h_c^{q+1}(S^3 \times_{\mathbb{Z}_2} \mathbb{H}^*) \arrow[r, "\text{ext. supports}"] & h_c^{q+1}(S^3 \times_{\mathbb{Z}_2} \mathbb{H}) \arrow[r, phantom, "\cong"] & h_c^{q+1}(\mathbb{R}P^3 \times \mathbb{H}) \arrow[u, "\text{inv. Thom iso}"'] \end{tikzcd}$$

$$(x, q) \longleftrightarrow (\mathbb{Z}_2 x, xq)$$

Tracing this backwards, we see that the map

$$\mathbb{R}P^3 \times \mathbb{H}^* \longrightarrow \mathbb{R}P^3 \times \mathbb{R}^+$$

$$(\mathbb{Z}_2 x, yr) \longmapsto (\mathbb{Z}_2 x^{-1} y, r) \qquad y \in S^3, r > 0$$

extends to a map

$$\mathbb{R}P^3 \wedge (S^4, *) \longrightarrow \mathbb{R}P^3 \wedge (S^1, *)$$

which is the map we want.

Returning to α we recall that by using a framing of $\mathbb{R}P^3$ we can construct a decomposition

$$h^8(\mathbb{R}P^3) = h^8(pt) \oplus h^{8-3}(pt) \oplus \tilde{h}^8(\mathbb{R}P^2)$$

Therefore the map α can be broken up into 6-components which are maps: and S-maps:

$$h^{8-3}(pt) \longrightarrow h^{8-3}(pt) \qquad \Sigma^0 \to \Sigma^0$$

$$h^{8-3}(pt) \longrightarrow h^{8-6}(pt) \qquad \Sigma^3 \to \Sigma^0$$

$$h^{8-3}(pt) \longrightarrow \tilde{h}^{8-3}(\mathbb{R}P^2) \qquad (\mathbb{R}P^2_{,*}) \to \Sigma^0$$

$$\tilde{h}^8(\mathbb{R}P^2) \longrightarrow h^{8-3}(pt) \qquad \Sigma^3 \to (\mathbb{R}P^2, *)$$

$$\tilde{h}^8(\mathbb{R}P^2) \longrightarrow h^{8-6}(pt) \qquad \Sigma^6 \to (\mathbb{R}P^2, *)$$

$$\tilde{h}^8(\mathbb{R}P^2) \longrightarrow \tilde{h}^{8-3}(\mathbb{R}P^2) \qquad \Sigma^3(\mathbb{R}P^2_{,*}) \to (\mathbb{R}P^2, *)$$

Now in fact I believe one knows that

$$\{Y, \Sigma^{-3}(\mathbb{R}P^2,*)\wedge Z\} = \{(\mathbb{R}P^2,*)\wedge Y, Z\}$$

(To check dimensions, take $Z = K(\mathbb{Z}_2)$ and $Y = \Sigma^i$. Then the left is

$$H_{i+3}(\mathbb{R}P^2,*;\mathbb{Z}_2) \quad \text{is } \neq 0 \quad \text{for} \quad i = \cancel{3,3}\ -1,-2$$

The right is

$$H^0(\Sigma^i(\mathbb{R}P^2,*);\mathbb{Z}_2) = H^{-i}((\mathbb{R}P^2,*);\mathbb{Z}_2) \neq 0 \qquad i=-1,-2.)$$

~~[illegible]~~ Put $Z = \Sigma^3$.

$$\cancel{\{\Sigma^3, (\mathbb{R}P^2,*)\}} \qquad \{Y, (\mathbb{R}P^2,*)\} = \{(\mathbb{R}P^2,*)\wedge Y, \Sigma^3\}$$

So therefore the six maps under consideration are

$$\Sigma^0 \to \Sigma^0$$
$$\Sigma^3 \to \Sigma^0$$
$$(\mathbb{R}P^2,*) \to \Sigma^0$$
$$(\mathbb{R}P^2,*) \to \Sigma^0$$
$$\Sigma^3(\mathbb{R}P^2,*) \to \Sigma^0$$
$$(\mathbb{R}P^2,*)\wedge(\mathbb{R}P^2,*) \to \Sigma^0$$

Now we are interested in what happens after one goes from $\Sigma^0 \to MSp$ so we get <u>potential</u> elements in

$$Sp^0(pt) = \mathbb{Z} \qquad \text{(probably 2)}$$
$$Sp^{-3}(pt) = 0$$
$$\widetilde{Sp}^0(\mathbb{R}P^2) = \mathbb{Z}_4$$
$$\widetilde{Sp}^0(\mathbb{R}P^2) = \mathbb{Z}_4$$
$$\widetilde{Sp}^{-3}(\mathbb{R}P^2) = ?$$
$$\widetilde{Sp}^0((\mathbb{R}P^2,*)\wedge(\mathbb{R}P^2,*)) = ?$$

Here is a table of stems

$$\begin{array}{lccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \pi_*^s(S): & \mathbb{Z}, & \mathbb{Z}_2\eta, & \mathbb{Z}_2\eta^2, & \mathbb{Z}_8\nu, & 0, & 0, & \mathbb{Z}_2\nu^2 \end{array}$$

ignoring 3 torsion

$$\eta^3 = 4\nu$$

Let $M_2 = (\mathbb{R}P^2, *)$, then from Toda-Araki paper we have that

$\{\Sigma^i, M_2\} = \{\Sigma^i M_2, \Sigma^3\}$ is given by the table

$i =$	0	1	2	3	4
	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2+\mathbb{Z}_2$
gen. of $\{\Sigma^{i-3}M_2, \Sigma^0\}$		π	$\eta\pi$	$\bar{\eta}$, $\cancel{\eta^2\pi}$	$\eta\bar{\eta}$, $\nu\pi$
gen. of $\{\Sigma^i, M_2\}$		i	$i\eta$	$\tilde{\eta}$, $\cancel{i\eta^2}$	$\tilde{\eta}\eta$, $i\nu$

here $\Sigma^1 \xrightarrow{i} M_2 \xrightarrow{\pi} \Sigma^2$ are the basic maps from the cell structure

and $\bar{\eta} \in \{M_2, \Sigma^0\}$, $\tilde{\eta} \in \{\Sigma^3, M_2\}$ satisfy $\bar{\eta} i = \eta$, $\pi\tilde{\eta} = \eta$

and one has the relations $2\bar{\eta} = \eta^2\pi$, $2\tilde{\eta} = i\eta^2$.

Finally $\{\Sigma^i M_2, M_2\}$ is given by

$i < -1$	-1	0	1	2
0	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2+\mathbb{Z}_2$	$\mathbb{Z}_2+\mathbb{Z}_2+\mathbb{Z}_2$
	$i\pi$	1	$i\bar{\eta}$ $\bar{\eta}\pi$	$(i\bar{\eta})^2$ $(\bar{\eta}\pi)^2$ $i\nu\pi$

From the A-Toda tables one sees that

$$\begin{array}{ccccccc} \xrightarrow{2} & \{\Sigma^5, M_2\} & \xrightarrow{\pi^*} & \{\Sigma^3 M_2, M_2\} & \xrightarrow{\iota^*} & \{\Sigma^4, M_2\} & \xrightarrow{2} \\ & \| & & & & \| & \\ & \mathbb{Z}_2 & & & & \mathbb{Z}_2 \tilde{\eta}\eta + \mathbb{Z}_2 \iota\nu & \end{array}$$

so $\{\Sigma^3 M_2, M_2\}$ is either $\mathbb{Z}_2{}^3$ or $\mathbb{Z}_2 + \mathbb{Z}_4$.

But also we have

$$\begin{array}{ccccc} \{\Sigma^5, \Sigma^0\} & \xrightarrow{\iota_*} & \{\Sigma^6, M_2\} & \xrightarrow{\pi_*} & \{\Sigma^4, \Sigma^0\} \\ \| & & & & \| \\ 0 & & & & 0 \end{array}$$

and therefore I see that the composition

$$h^8(\mathbb{R}P^3) \xrightarrow{\alpha} h^{8-3}(\mathbb{R}P^3) \xrightarrow{f_*} h^{8-6}(pt)$$

for a theory over Sp^* is zero!

$Sp^*(pt) \longrightarrow KO^*(pt)$ hom.

~~consider~~ assume ~~~~ det. by KO^* nos.

then ~~KO~~ $Sp^{-4k-3}(pt) = 0$!

$$\Downarrow$$

$\boxed{\text{Conjecture: } \gamma \in \widetilde{Sp}^0(\mathbb{R}P^2) \text{ and } \omega \in \widetilde{Sp}^2(\mathbb{R}P^2) \text{ generated } \widetilde{Sp}^*(\mathbb{R}P^2) \text{ and all torsion in } Sp^*(pt) \text{ is of order 2.}}$

$$Sp^{8-2}(pt) \xrightarrow{\cdot\omega} \widetilde{Sp}^8(\mathbb{R}P^2) \longrightarrow Sp^{8-1}(pt) \longrightarrow$$
$$\gamma \longmapsto \eta$$

conclusion is that ~~that~~ all torsion in $Sp^*(pt)$ is generated by η. (unreasonable)

~~$Sp(pt)$~~ $\quad Sp(pt) \longrightarrow \mathbb{Z}[2c^2; c^4] \quad$ deg $c = 2$.

I have Adams operations on $k(pt)$, have ψ^k $k \geq 1$ because we have λ. How about ψ^{-1}

$$E \qquad E^* = \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C}) \qquad = \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{C})$$

$$(\sigma f)(\sigma e) = f(e)$$

$$(\sigma f)(e) = \overline{f(\sigma^{-1} e)}$$

Question: Is $E \cong E^*$. suppose that E endowed with a hermitian inner product $\ni$ $\langle x, y\rangle$

$$\underline{\langle \sigma x, \sigma y\rangle = \langle y, x\rangle} \qquad \text{symm. if } x, y \text{ real}$$

$E = V \otimes_{\mathbb{R}} \mathbb{C} \qquad \langle \lambda x, \mu y\rangle = \lambda\bar{\mu}\langle x, y\rangle$

$$\langle \sigma(\lambda x), \sigma(\mu y)\rangle = \mu\bar{\lambda}\langle x, y\rangle$$

I assume known that $\psi^{-1} = $ identity.

L-Novikov operations: ~~[illegible]~~

alg of Adams operations with $\psi^{-1} = id$

alg. of Adams op. $\underbrace{\mathbb{Z}[T, T^{-1}] \otimes \bigoplus_{n \geq 0} \mathbb{Z}\binom{T}{n}}_{\Gamma} \subset \mathbb{Q}[T, T^{-1}]$

$$\Gamma \longrightarrow \mathbb{Q}$$

$$T \qquad q$$

$T \longrightarrow k$ $\qquad \binom{k}{n} \in \mathbb{Z}$ all n

---

want $\psi^{-1} = id$

now the problem is to make ψ^k act on

$\beta \in K^{-2}(pt)$ ~~[illegible]~~ $\qquad \beta \in KO^{-8}(pt)$

$\psi^k(\beta) = \cancel{k} \; \frac{1}{k}\beta.$ Definitely denominators needed for ~~[illegible]~~

Idea is to construct a subalgebra of Novikov operations corresp. to Γ. e.g. in complex K-theory have

$(1+X)^a - 1 = \sum_{n \geq 1}' \binom{a}{n} X^n$ which ~~[illegible]~~ preserves

---

we agree that by C.F.

$$KO^*(pt) \otimes_{Sp^*(pt)} Sp^*(X) \xrightarrow{\sim} KO^*(X).$$

hence for any ring R

$$k(pt) \otimes_{Sp(pt)} Sp(X) \longrightarrow k(X)$$

January 14, 1970 (still groggy but getting better)

On symplectic cobordism again. I recall how the calculations of $Sp^{-\ell}(pt)$ for $0 \le \ell < 4$ go. So let $x \in Sp^{-\ell}(pt) \cong Sp^0(S^\ell, *)$. Then

$$w^j(Qx - x) = 0 \qquad \text{for } j \text{ large}$$

in $Sp^*(B\mathbb{Z}_2 \times (S^\ell, *))$. By Gysin sequence

$$\widetilde{Sp}^{q+3}(\mathbb{R}P^3 \times (S^\ell,*)) \xrightarrow{\xi} Sp^q(B\mathbb{Z}_2 \times (S^\ell,*)) \xrightarrow{w} Sp^{q+4}(B\mathbb{Z}_2 \times (S^\ell,*))$$

$$\wr\!\wr$$

$$Sp^{q-\ell+3}(\mathbb{R}P^3) = 0 \quad \text{if } q > \ell \qquad \text{e.g. } q = 4$$

Thus

$$w(Qx - x) = 0 \quad \text{and}$$

$$Qx - x = \xi(a) \qquad a \in \widetilde{Sp}^{3-\ell}(\mathbb{R}P^3)$$

Now restricting to a point of $B\mathbb{Z}_2$ we get

(1) $$-x = res_0\, \xi(a) \qquad 0 < \ell,\ res_0 Qx = x^2 = 0$$

where in general $res_n \circ \xi$ is the composition

$$\widetilde{Sp}^q(\mathbb{R}P^3) \xrightarrow{p_2^*} \widetilde{Sp}^q(S^n \times_{\mathbb{Z}_2} S^3) \xrightarrow{p_{1*}} Sp^{q-3}(\mathbb{R}P^n).$$

Here I am interested in the map $res_0 \circ \xi = \varphi$

$$\varphi: \widetilde{Sp}^q(\mathbb{R}P^3) \xrightarrow{\pi^*} \widetilde{Sp}^q(S^3) \xrightarrow{\cong} \widetilde{Sp}^{q-3}(pt)$$

Now we know that

$$\widetilde{Sp}^q(\mathbb{R}P^3) \xrightarrow[(j^*, f_*)]{\sim} \widetilde{Sp}^q(\mathbb{R}P^2) \times Sp^{q-3}(pt)$$

If $i : pt \longrightarrow \mathbb{R}P^3$ is framed so that $f_* i_* = id$ we can compute $\xi \circ (i_* 1)$ and we find it is

$$\begin{tikzcd} \mathbb{Z}_2 \arrow[r] \arrow[d] & S^3 \arrow[r] \arrow[d, "\pi"] & pt \\ pt \arrow[r] & \mathbb{R}P^3 & \end{tikzcd}$$

2. Thus we know from (1) that

$$\varphi : \widetilde{Sp}^{3-\ell}(\mathbb{R}P^3) \twoheadrightarrow Sp^{-\ell}(pt) \qquad \ell \leq 3$$

and that on the $Sp^{-\ell}(pt)$ factor it is multiplication by 2. Hence we get a surjection

$$\widetilde{Sp}^{3-\ell}(\mathbb{R}P^2) \longrightarrow Sp^{-\ell}(pt)/2\,Sp^{-\ell}(pt)$$

which by Nakyama means that

(2) $$\widetilde{Sp}^{3-\ell}(\mathbb{R}P^2) \longrightarrow Sp^{-\ell}(pt) \qquad 0 < \ell \leq 3$$

is surjective mod odd torsion. On the other hand

$$\xrightarrow{2} Sp^{1-\ell}(pt) \longrightarrow \widetilde{Sp}^{3-\ell}(\mathbb{R}P^2) \longrightarrow Sp^{2-\ell}(pt) \xrightarrow{2}$$

so using this and the surjectivity of (2) ~~that~~ and $Sp^0(pt) = \mathbb{Z}$,
we get

(3) $$\mathbb{Z}_2 \twoheadrightarrow Sp^{-1}(pt)$$
$$\mathbb{Z}_2 \twoheadrightarrow Sp^{-2}(pt)$$

(4)

$$0 \longrightarrow Sp^{-2}(pt) \longrightarrow \widetilde{Sp}^0(\mathbb{R}P^2) \longrightarrow Sp^{-1}(pt) \longrightarrow 0$$

Let $S^3 \xrightarrow{\pi} \mathbb{R}P^3$ be framed in the natural way. Considering the natural map $\mu : Sp^*(X) \longrightarrow KO^*(X)$ furnished by the Thom isomorphism in KO-theory we see that $\mu(\pi_* 1) = \pi_! 1$. I will assume that $\pi_! 1$ is K-theory is the direct image bundle which is clear $1 + \mathcal{O}(1)$. Thus

$$\pi_* 1 = 2 + \gamma \qquad \gamma \in \widetilde{Sp}^0(\mathbb{R}P^3) \qquad \mu(\gamma) = \mathcal{O}(1) - 1$$

Now $\mu(\gamma)$ has order 4. In fact the Whitney class of $\mathcal{O}(1) - 1 = \eta$ is $1 + x$ and $(1+x)^4 = 1$ on $\mathbb{R}P^2$, but $(1+x)^j \neq 0$, $1 \leq j < 4$. This shows the order is at least 4, hence card $\widetilde{Sp}^0(\mathbb{R}P^2) \geq 4$ and we already know this card is ≤ 4 by (3) + (4) above. Thus we find that

$$Sp^{-1}(pt) = \mathbb{Z}_2 \eta$$
$$Sp^{-2}(pt) = \mathbb{Z}_2 \eta^2$$
$$\widetilde{Sp}^0(\mathbb{R}P^2) = \mathbb{Z}_4 \gamma \qquad \text{where } \eta^2 \longmapsto \gamma.$$

Now $\varphi(\pi_* 1)$ is represented by

$$\begin{tikzcd} S^3 \times \mathbb{Z}_2 \arrow[r] \arrow[d, "\text{antipodal action}"'] & S^3 \arrow[r] \arrow[d, "\pi"] & pt. \\ S^3 \arrow[r, "\pi"] & \mathbb{R}P^3 & \end{tikzcd}$$

hence $\varphi(\pi_* 1) = 2 \cdot [S^3 \to pt] = 0$. Therefore for $l = 1$ in (2) since γ generates $\widetilde{Sp}^0(\mathbb{R}P^2)$. We conclude that

$$Sp^{-3}(pt) = 0.$$

Try $\ell = 4$: Take $x \in Sp^{-4}(pt)$, suppose $s_\alpha x = 0$, $\alpha > 0$.
Then we get

$$w^2(wQx - x) = 0 \Rightarrow w(wQx - x) = \xi(a) \qquad a \in \widetilde{Sp}^3(\mathbb{R}P^3)$$
$$= 2a \qquad \qquad \overset{\shortparallel}{\mathbb{Z}}$$

~~restricting~~ Restricting to a point in $\mathbb{R}P^3$ we get $2a = 0$, so $a = 0$,
so

$$wQx - x = \xi(a) \qquad a \in \widetilde{Sp}^{-1}(\mathbb{R}P^3).$$

So restricting to a point again we get

$$-x = \varphi(a)$$

But we have

$$\begin{array}{ccccccccc} & & & & \mathbb{Z}_4 \cdot \gamma & & & & \\ 0 & \longrightarrow & \cancel{Sp^2(pt)} & \longrightarrow & \widetilde{Sp}^0(\mathbb{R}P^2) & \longrightarrow & Sp^{-1}(pt) & \longrightarrow & 0 \\ & & \downarrow \eta & & \downarrow \cdot\eta & & \downarrow \cdot\eta & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \widetilde{Sp}^{-1}(\mathbb{R}P^2) & \longrightarrow & Sp^{-2}(pt) & \longrightarrow & 0 \\ & & & & & & \mathbb{Z}_2\eta^2 & & \end{array}$$

$$\therefore \quad \widetilde{Sp}^{-1}(\mathbb{R}P^2) = \mathbb{Z}_2 \cdot \gamma\eta \xrightarrow{\varphi} 0 \qquad \text{since}$$

φ is a $Sp^*(pt)$-module homomorphism and since $\varphi(\gamma) = 0$. Conclude that

$$Sp^{-4}(pt) = \mathbb{Z} \qquad \widetilde{Sp}^{-1}(\mathbb{R}P^2) = \mathbb{Z}_2\eta\gamma$$

~~$\ell = 5$: Take $x \in Sp^{-5}(pt)$ with $s_\alpha x = 0$ for $\alpha > 0$.~~
~~Then get~~

~~$w(wQx - x) = \xi(a) \qquad a \in \widetilde{Sp}^2(\mathbb{R}P^3)$~~

January 14, 1970 (still groggy)

<u>Atiyah's lectures:</u>

The basic representation theory. Let $n = 4q$. In the Clifford algebra of $\mathbb{R}^n$ let $\omega = e_1 \cdots e_n$ be the volume element. Identify $C(\mathbb{R}^n)$ with $\Lambda(\mathbb{R}^n)$ in the natural way. Consider the operator

$$(-1)^q L(\omega)$$

which has square $+ \mathrm{id}$ and up to sign is the star operator. Then this gives the decomposition $\Lambda = \Lambda_+ \oplus \Lambda_-$

Assume you can define partial Steenrod operations in $H^*(X\times Y)$ and that they preserve algebraic cycles.

Method: (mod 2)

$$H^*(X\times Y) \longrightarrow \prod_{k\geq 0}{}' H^*_{\Sigma_k}(X^k \times Y^k) \xrightarrow{\Delta_X^*} \prod_{k\geq 0}{}' H^*_{\Sigma_k}(X\times Y^k)$$

$$\cong \prod_{k\geq 0}{}' H^*_{\Sigma_k}(Y^k)\otimes H^*(X)$$

$$\cong \prod_{k\geq 0}{}' \left[H^*_{\Sigma_k}(pt, H^*(Y)^{\otimes k})\otimes H^*(X)\right]$$

$$\cong \left[\prod_{k\geq 0}{}' H^*_{\Sigma_k}(X)\right]\otimes H^*(Y)$$

example: (mod 2)

let $u\in H^*(X\times Y)$ $\qquad Qu\in H^*_{\Sigma_2}(X^2\times Y^2)$, $\quad \mathrm{res}\,(Qu) = u^{\otimes 2}$

$$V = (\Delta_X\times id_{Y^2})^* Qu \in H^*_{\Sigma_2}(X\times Y^2), \quad \mathrm{res}\ V = (\Delta_X\times id_{Y^2})^* u\otimes u$$

write $V = \sum_{i=1}^{r} x_i\otimes z_i$ with $x_i\in H^*(X)$ and $z_i\in H_{\Sigma_2}(Y^2)$

Example: ~~[illegible]~~ $u = x\otimes y$ $\qquad Qu = Qx\otimes Qy$

$$V = Px\otimes Qy$$

so the end the result should be

$$x\otimes y \longmapsto Px\otimes y.$$

We know that $\quad P x = (Sq^0 x) w^n + \dots + (Sq^n x)$.

so we conclude that if

If $u \in H^{ev}(X \times Y)$ is algebraic ~~so is~~ and $u = \sum x_i \otimes y_i$

then

Suppose that $u \in H^{2q}(X \times Y)$ and let

$$u = \sum_{i=0}^{2q} u_i \qquad \text{with} \qquad u_i \in H^i(X) \otimes H^{2q-i}(Y).$$

Then write

$$u_i = \sum_{j=1}^{r_i} x_{ij} \otimes y_{ij} \qquad \{y_{ij}\}_{j=1}^{r_i} \text{ basis for } H^{2q-i}(Y)$$

and we have

$$P_X u = \sum_{i=0}^{2q} \sum_j P x_{ij} \otimes y_{ij}$$

$$= \sum_{i=0}^{2q} \sum_{j=1}^{r_i} \sum_{\nu=0}^{i} w^{i-\nu} (Sq^\nu x_{ij}) \otimes y_{ij}$$

$$= \sum_{i=0}^{2q} \sum_{\nu=0}^{i} w^{i-\nu} \boxed{\sum_{j=1}^{r_i} (Sq^\nu x_{ij}) \otimes y_{ij}}$$

$$= \sum_{i=0}^{2q} \sum_{\nu=0}^{i} w^{i-\nu} (Sq^\nu \otimes id)\, u_i$$

work out the formulas for odd primes. Suppose $p >$ the primes occurring in the torsion of $H^*(X)$, $H^*(Y)$ and that $p \gg \dim X$. In this case we should get a similar formula with

$$P_X u = \sum_{i=0}^{2q} \sum_{\nu=0}^{i} w^{i-\nu} (P^\nu \otimes id)\, u_i$$

$\deg w = 2(p-1)$

$P^\nu = 2(p-1)\nu$

Consider the cohomology theory $h^q(X) = H^q(X \times Z)$ where Z is fixed. Can you define Steenrod operations in h. In general if

$$h^q(X) = \{X, M\} \qquad M \text{ a spectrum}$$

one has a product

$$M \wedge M \longrightarrow M$$

and Steenrod operations consist of maps

$$\underbrace{M \wedge \cdots \wedge M} \longrightarrow$$

Instead of H take N so we have good geometric representatives. Thus

$$\cancel{N^*(X)} \qquad h^*(X) = N^*(X \times Z)$$

BGl_n

given

$$\begin{array}{ccc} W & & W^k \\ \downarrow f & \text{form product} & \downarrow \\ X \times Z & & X^k \times Z^k \end{array}$$

however one can't directly pull-back to Z. Therefore the natural theory goes from

$$\boxed{N^q(X \times Z) \longrightarrow N^{kq}(X \times (E\Sigma_k \times_{\Sigma_k} Z^k))}$$

even from the internal point of view in X.

January 16, 1970 (feeling better!)

I want to determine whether or not my method will permit one to show that elements of $Sp^*(pt)$ are determined by their KO^*-characteristic numbers. So I start with the pivotal map Φ

$$Sp^{\delta}(\mathbb{R}P^3) \xrightarrow{p_2^*} Sp^{\delta}(S^3 \times_{\mathbb{Z}_2} S^3) \xrightarrow{p_{1*}} Sp^{\delta-3}(\mathbb{R}P^3).$$

Using the isomorphism

$$\begin{tikzcd}
 & S^3 \times_{\mathbb{Z}_2} S^3 \arrow[dl, "p_2"'] \arrow[d, "p_1"] \arrow[dr, "\cong"] \arrow[rr, "\mu"] & & S^3 \arrow[dd] \\
 & & \mathbb{R}P^3 \times S^3 \arrow[ur, "pr_2"] \arrow[dl, "pr_1"] \arrow[dll, dashed, "\nu"'] & \\
\mathbb{R}P^3 & \mathbb{R}P^3 \arrow[rr] & & pt
\end{tikzcd}$$

where

$$\nu : \mathbb{R}P^3 \times S^3 \longrightarrow \mathbb{R}P^3$$
$$(\mathbb{Z}_2 x, y) \longmapsto \mathbb{Z}_2 x^{-1} y$$

we see Φ is also the map

$$Sp^{\delta}(\mathbb{R}P^3) \xrightarrow{\nu^*} Sp^{\delta}(\mathbb{R}P^3 \times S^3) \xrightarrow{pr_{2*}} Sp^{\delta-3}(\mathbb{R}P^3)$$

~~If I expect to use the method I worked out for U^* I must have the following fact~~

Conjectures: The map

$$Sp^*(pt) \longrightarrow KO^*(pt)[t_1, t_2, \dots] \qquad \deg t_i = -4i$$
$$x \longmapsto \mu(s_{\underline{t}}(x)) \qquad \mu = \text{Thom hom.}$$

I'd like to prove this using the method I worked out for U^*. Suppose q is the first dimension for which is fails to be injective, $q = 4k+l$ $0 \le l \le 3$, and let $x \in Sp^{-q}(pt)$ be in the kernel. Then $s_\alpha x = 0$ for $\alpha > 0$, so we get a relation

$$w^j\left(w^k Qx - x\right) = 0 \quad \text{in} \quad Sp^{-4k+4j}\left(B\mathbb{Z}_2 \times (S^l,*)\right) = Sp^{-q+4j}(B\mathbb{Z}_2)$$

for j large. So if I expect to do as for U^* it is necessary to bring j down to 1. So I need to know that

Conjecture: 1) Assume KO^* numbers determine x in dim $< q$. Then if $a \in \widetilde{Sp}^{-q+4j+3}(\mathbb{R}P^3)$ satisfies $\Phi(a) = 0$ and $j > 0$, we have $a = 0$.

2) Hypotheses as in 1), if $-x = \mathrm{res}_0\, \xi(a) = 2y + \langle \gamma, z \rangle$ had zero KO^*-numbers, ~~then~~ where $z \in \widetilde{Sp}^{-q+3}(\mathbb{R}P^2)$, ~~~~ then $\langle \gamma, z\rangle = 0$.

The problem is to calculate what Φ does to $KO^*(pt)[t_1, \cdot\,]$. The t's **don't enter** because the integration is over the framed sphere, or if you ~~~~ prefer, the map comes from the S-category.

Problem: What is the map

$$KO^*(\mathbb{R}P^3) \xrightarrow{\nu^*} KO^*(\mathbb{R}P^3 \times S^3)\ ?$$

$$\xrightarrow{2} KO^{q-2}(pt) \longrightarrow \widetilde{KO}^{q}(\mathbb{R}P^2) \longrightarrow KO^{q-1}(pt) \xrightarrow{2}$$

$q=2$	$\mathbb{Z}$	$\mathbb{Z}_2\omega$	0
1	$\mathbb{Z}_2\eta \hookrightarrow$	$\mathbb{Z}_2\eta\omega$	$\cancel{\mathbb{Z}}$
0	$\mathbb{Z}_2\eta^2 \hookrightarrow$	$\mathbb{Z}_4\gamma$	$\mathbb{Z}_2\eta$
-1	0	$\mathbb{Z}_2\eta\gamma$	$\mathbb{Z}_2\eta^2$
-2	$\mathbb{Z}\tau$	$\mathbb{Z}_2\tau\omega$	0
-3	0	0	$\cancel{\mathbb{Z}\tau}$
-4	0	0	0
-5	0	0	0
-6	$\mathbb{Z}\beta$	$\mathbb{Z}_2\beta\omega$	0

$\tau^2 = 4\beta$ $\qquad$ $\eta^2\omega = 2\gamma$ $\qquad$ (since $\eta^2\omega \neq 0$)

$\eta^2\gamma = \ ?$

$$\widetilde{KO}^{q}(\mathbb{R}P^3) \xrightarrow{\Phi} KO^{q-3}(\mathbb{R}P^2)$$

(assume $f_*\Phi = 0$)

$q=2$	$\iota_*\mathbb{Z}_2\eta + \mathbb{Z}_2\omega$	$\mathbb{Z}_2\eta + \mathbb{Z}_2$
1	$\iota_*\mathbb{Z}_2\eta^2 + \mathbb{Z}_2\eta\omega$	$\mathbb{Z}_2\eta^2 + \mathbb{Z}_2$
0	$0 + \mathbb{Z}_4\gamma$	$0 + 0$
-1	$\iota_*\mathbb{Z}\tau + \mathbb{Z}_2\eta\gamma$	$\mathbb{Z}\tau + 0$
	$0 + \mathbb{Z}_2\tau\omega$	$0 + 0$
	$0 + 0$	$0 + \mathbb{Z}_2$
	$0 + 0$	$0 + \mathbb{Z}_2$
	$\iota_*\mathbb{Z}\beta + 0$	$\mathbb{Z}\beta + \mathbb{Z}_4$

This shows that the map Φ can't be injective on the KO level.

<u>Notes on KO:</u> Set $k(X) = KO(X) \oplus KSp(X)$.
So we study complex bundles endowed with a conjugate linear automorphism $\sigma \ni \sigma^4 = 1$. Observe that $k(X)$ is a λ-ring. Thom isomorphism for KO says that if L is an S^3 bundle over X, then the relative element λ_L of $KSp(L, L-X)$ defined by the complex $1 \to \pi^* L$ is a Thom class. ie.

$$k(X) \xrightarrow[\cdot \lambda_L]{\sim} k(L, L-X)$$

Hence there is a Thom homomorphism

$$Sp(X) \longrightarrow k(X) \qquad \left(Sp(X) = Sp^{4*}(X) \right)$$
$$e(L) \longmapsto L - 1$$

this is τ

$$k(pt) = \mathbb{Z}[\tau] / \tau^2 = 4$$

<u>Question:</u> Take $X = \mathbb{H}P^\infty \times \mathbb{H}P^\infty$ and L_i the two canonical line bundles. We can form $L_1 \cdot L_2$ in $k(X)$. Does $\sum t_j (L_1 \cdot L_2 - 1)^{j+1}$ lie in the image of

$$Sp(X) \longrightarrow k(X)[t] \; ?$$

Why your technique fails to determine the image of $Sp^*(pt)$ in $N^*(pt)$: The idea would have been to use the formula, $x \in Sp^{-4k}(pt)$

$$(*) \qquad w^n(w^k Qx - x) = \sum_{0<\ell(\alpha)\leq n}' w^{n-\ell(\alpha)} a^\alpha s_\alpha(x) \qquad n \gg k$$

or rather its image in $N^*(BZ_2)$, and to argue that the image is at most the set of $y \in N^{4*}(pt)$ satisfying this formula. Then the hope would be that this would be the image. However this set is the subring of 4th powers in $N^*(pt)$. This one can see more or less in principle; on one hand taking the 4th power of the squaring equation for an element y of $N^*(pt)$ shows that y^4 satisfies $(*)$ and on the other hand the image of $Sp^*(pt)$ is contained in the ring of 4th powers.

If $x \in Sp^{-4}(pt)$, then $s_1 x \equiv 0 \ (2)$.

$$w(wQx - x) = a_1(s_1 x) + \xi(b) \qquad b \in Sp^3(\mathbb{R}P^3)$$

Now restrict to a point of BZ_2. Since

$$e(\eta \otimes L) = w + a_1 e(L) + \cdots$$

on restricting to a point $a_1 \mapsto 1$. Also $\xi(b) \mapsto 2f_*(b)$ so

$$s_1 x + 2f_*(b) = 0$$

$$\Longrightarrow s_1 x \equiv 0 \ (2).$$