$\mathcal{K}$ fibred category of ~~[illegible]~~ virtual coverings
an object of $\mathcal{K}$ over X is a kind of cycles. It is a formal sum $\sum m_Y[Y]$ where $m_Y \in \mathbb{Z}$ and Y runs over the finite covering spaces of X. The concept of isomorphism class of such things is clear. And now an isomorphism ~~between two such~~ or cobordism should be what

Difference between this and algebraic geometry is that a cycle does not admit automorphisms so there is no isomorphism question involved with the equality of two cycles.

Cobordism of cycles: you ask for one on the product $X \times I$. This is same as equality

---

$\mathcal{K}$ fibred category in groupoids over $\mathcal{S}$
then if $f_0, f_1 : X \to Y$ are homotopic, then the homotopy gives rise to an isomorphism $f_0^* y \simeq f_1^* y$.
check fibre axiom.

Suppose $\theta : \mathcal{K} \longrightarrow E$ is a natural transformation i.e. for each $x \in \mathcal{K}(X)$ have $\theta_x : X \longrightarrow E$ and for each $x \xrightarrow{\lambda} y$ over $X \to Y$ have

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "\theta_x"] \arrow[dr, phantom, shift right=3, "\theta_\lambda \Downarrow"] & \\ Y \arrow[r, "\theta_y"'] & E \end{tikzcd}$$

satisfying evident compatibility conditions.

now you should also notice that there is a natural map

$$k(G \overset{G'}{\amalg} G) \times_{k(G)} k(G \overset{G'}{\amalg} G) \longrightarrow k(G \overset{G'}{\amalg} G)$$

Suppose you have an element of $k(G)$ i.e. a integral linear combination of ~~coverings~~ irreducible G-sets, write it as difference

$$S_0 - S_1$$

of positive disjoint G-sets. Now ~~if you~~ if this goes to zero in $k(G')$, then S_0 and S_1 are isomorphic. Choosing an isomorphism one gets an element of $k(G \overset{G'}{\amalg} G)$.

$$d(S_0, S_1, u) \in A$$

and $d(S_1, S_2, v)$

I want $\quad d(S_0, S_2, uv) = d(S_0, S_1, u) + d(S_1, S_2, v)$

so you need an algebraic K-theory to associate to the algebraic general linear groups over a scheme S, i.e. $B\underline{GL}_S$. What happens prime to the characteristic tends to be clear? Thus one tends to understand completely what happens prime to the characteristic, i.e. one gets that

$$\pi_{2q}(B\,\underline{GL}_S) = T^{\otimes q} \qquad T = \text{Tate motive.}$$

T is an inverse system of finite étale group schemes over S. hence this is the relative situation. Now you must understand what gives when you start taking sections of this animal and more generally "integrating it" over a map f_*

This style intuition is that alg. K generalizes cohomology. Kummer theory

$$0 \longrightarrow BGL(\mathbb{F}_q) \longrightarrow B\,\underline{GL}_S \xrightarrow{\Psi^q - id} B\,\underline{GL}_S \longrightarrow 0 \qquad ?$$

If S is a topological space, then $B(\underline{GL}_S) = S \times BU$ so the sections are space $\underline{Hom}(S, BU)$ whose homotopy groups are $K_*(S)$

Idea is that $\underline{GL}$ should be defined cat. of annele spaces and gives something over any annele topos. Even non-commutative ring. Something like a generalization of a sheaf.

functor from simplicial sets to simplicial sets which hopefully will ~~~~ give $\underline{\underline{k}}$ by truncation say.

Let us assume that we have a bisimplicial set $A_{\bullet\bullet}$ and we consider the functor from s. sets to s. sets represented by this:

$$X \longmapsto \underbrace{\mathrm{Hom}(X_{\bullet}, A_{\bullet\bullet})}_{} = Q(X).$$

Then I can ~~~~ define

$$k_i(X) = \pi_{i+1}(Q(X))$$

and I can still ask ~~~~ for ~~~~ the required universal ~~~~ property

---

so even if you can solve the category problem, it still remains to define ~~~~ $\underline{\underline{k}} \longrightarrow \underline{\underline{E\Psi^{\delta}}}$. Thus if X is a space and if we are given a word

Thus to a space X associate the Picard category $\underline{\underline{E\Psi^{\delta}}}(X)$ defined by rigidifying BU, etc. Then given an $\mathbb{F}_q$-vector bundle E over X choose a classifying map $f: X \longrightarrow BGL_n(\mathbb{F}_q)$ for E ie a pair

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d] & EGL_n(\mathbb{F}_q) \arrow[d] \\ X \arrow[r, "f"] & BGL_n(\mathbb{F}_q) \arrow[r, "\text{chosen}", "\text{already}"'] & \underline{E\Psi^{\delta}} \end{tikzcd}$$

and associate ~~~~ to an exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

of bundles

---

better start with $\quad BGL_n(\mathbb{F}_q) \xrightarrow{u_n} E\Psi^q$

and for n, m

$$\begin{tikzcd}
 & BGL_m(\mathbb{F}_q) \times BGL_n(\mathbb{F}_q) \arrow[r, "u_m \times u_n"] \arrow[d, Rightarrow] & E\Psi^q \times E\Psi^q \arrow[d, "\mu"] \\
B\Gamma_{mn}(\mathbb{F}_q) \arrow[ur, "(p_1, p_2)"] \arrow[r, "(in)"'] & BGL_{m+n}(\mathbb{F}_q) \arrow[r, "u_{m+n}"] & E\Psi^q
\end{tikzcd}$$

so choose such a homotopy. must check transitivity hence have to worry about the uniqueness of the homotopy

$$\begin{tikzcd}
 & U \arrow[d, "\Psi^q - id"] \\
 & U \arrow[d] \\
BG & E\Psi^q \arrow[d] \\
 & {}
\end{tikzcd}$$

but this should be OKAY again by Atiyah's theorem.

Thus by universal property of the functor $\underline{k}$ this part should be OKAY, hence if my universal property holds I get a map $B \longrightarrow E\Psi^q$ of spaces mult. char. classes

~~state~~ to simplify suppose A ~~k~~ is a field and suppose ~~k finite~~ $R = \bigoplus_{n\geq 0} R_n$ is a graded anti-commutative A-algebra. Then by a R-valued mult. char. class

$$\theta(E) = \sum \theta_n(E) \qquad \theta_n(E) \in H^n(X)\otimes R_n$$

$\exists$ ....

Somehow there should be ~~is~~ a universal one represented by $\mathbb{Z}[GQ]$? ~~Yes.~~

i.e. $\Gamma = \mathbb{Z}[GQ]$ is a simplicial ring

and in particular a DG ring ~~...~~ To each bundle $E \longrightarrow X$ we ~~get~~ want a coh. class

$$X \longrightarrow \Gamma$$

and to each exact sequence I want a homotopy class of maps!!!!!

~~...~~

~~...~~

If R is a simplicial ~~differential~~ graded ring

$$\cdots \longrightarrow R_2 \rightrightarrows R_1 \rightrightarrows R_0$$

not nec. commutative, then by a multiplicative charac. class with values in R I mean ultimately a hom.

$$\mathbb{Z}[GQ] \longrightarrow R$$

or

$$GQ \longrightarrow R^\times \qquad\qquad Q \longrightarrow \overline{W}R^\times$$

or

$$Q \xrightarrow{\tau} R^\times.$$

basic question which you would really like to solve is to produce something over $G(\mathbb{Q})$, ideally a perfect complex, which would possess Chern classes.

Thus suppose you form cokernel of

$$B\Gamma_{m,n} \rightrightarrows BGL_{m+n} \longrightarrow C$$

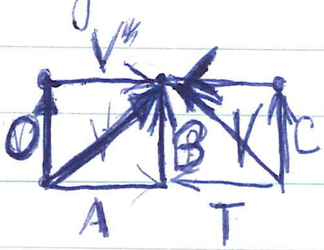
or better you have X with E' E'' over and an exact sequence $0 \to E' \to E \to E'' \to 0$, and you want to make $E = E' \oplus E''$. So you want to map $G_n \times G_m \longrightarrow \Gamma_{m,n}$ an equivalence.

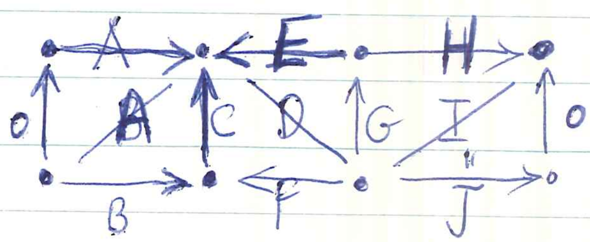
~~gad~~

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

$$\cancel{BR}\ G_m \times G_n \to \Gamma_{mn}.$$

form a 1-cocycle on $I \times X$





In this setup what I get is

$$A - E + H \qquad\qquad B - F + J$$

and a reason for ~~these~~ these being the same

~~Thus suppose X comp~~

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \\ Z \arrow[ur, dotted] & \end{tikzcd}$$

$\pi_1 W = 0$

The first obstruction to extending to Z ~~so in~~ are in $H^*(Z, X; \pi_*(W))$ so → This works as long as ~~$\pi_*(W)$ simple~~ **W simple.** ∎ ~~qed~~

~~$K(\pi_2 W$~~

$H^*(Z,X;\pi_*(W))$.

$$\begin{tikzcd} & K(\pi_2 W, 2) \arrow[d] & & \\ X \arrow[r] \arrow[d] & W_2 = \cancel{K(\pi_2 W, 2)} \arrow[d] & & * \arrow[d] \\ Z \arrow[r] \arrow[ur] \arrow[rrr, bend left] & W_1 = pt. \arrow[rr] & & K(\pi_2 W, 3) \end{tikzcd}$$

$$\overset{0}{\overset{\shortparallel}{H^3(Z/X, \pi_2 W)}} \longleftarrow H^2(X, \pi_2 W) \xleftarrow{\sim} H^2(Z, \pi_2 W) \longleftarrow \overset{0}{\overset{\shortparallel}{H^2(Z/X, \pi_2 W)}}$$

$$\begin{tikzcd} & K(\pi_3 W, 3) \arrow[d] & \\ X \arrow[r] \arrow[d] & W_3 \arrow[r] \arrow[d] & \cdot \\ Z \arrow[r] \arrow[ur, dotted] & W_2 \arrow[r] & K(\pi_3 W, 4) \end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & W \\
Z \arrow[ur] \arrow[d] & \\
Z/X & \vee S^2
\end{tikzcd}$$

so Z/X is contractible because it has no cohomology and is 1 connected since it has only 2+3 cells.

Suppose W is a 1-connected H-space $\pi_1(W)=0$

then one knows that

$$[Z,W] \longrightarrow [X,W]$$

~~Suppp~~ suppose W is a space, then

~~$[Z,W]$ [~~

$$[Z/X,\Omega W] = 0$$

$$\downarrow$$

$$[Z,\Omega W] \to [X,\Omega W] \longrightarrow [Z/X, W] = 0 \longrightarrow [Z,W] \longrightarrow [X,W]$$

hence $$[Z,\Omega W] = [X,\Omega W]$$

argument should be modifiable

killing fdl. gp. without changing H^*.

Suppose $H_1(X) = 0$ $\quad$ X connected

attach 2 cells to kill $\pi_1(X)$.

then get Y & $H_q(Y) = H_q(X)$

except $H_2(Y) = H_2(X) +$ free ab grp gen. by attached cells.

but $\pi_1(Y) = 0$ so these new elements are spherical so attaching 3 cells can obtain space Z with same homology.

~~If you want to~~

Is this construction functorial up to homotopy at least. Thus given

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \\ Y \arrow[ur, dotted] & \end{tikzcd}$$

with $\pi_1(W) = 0$

$$[\vee S^2, W] \longrightarrow [Y, W] \longrightarrow [X, W] \longrightarrow [\vee S^1, W]$$

so unique if $\pi_2(W) = 0$, similarly need $\pi_3(W) = 0$ to get a unique extension to Z.

absolutely canonical

$\boxed{k \text{ field then } \mu^{\otimes i}}$

If k a field of char p, ~~then~~ and suppose one consider the maximal cyclotomic subfield k_0 say finite with q elements. Then

$$\mu^{\otimes i}_{q^i-1} \text{ is canonical.}$$

e.g. Galois group invariant

because

$$\begin{array}{ccc} k_0 & \longrightarrow & \bar{k}_0 \\ \downarrow & & \downarrow \\ k & \longrightarrow & \bar{k} \end{array}$$

$$Gal(\bar{k}/k) \twoheadrightarrow Gal(\bar{k}_0/k_0)$$

gen. by Frobenius. $x \mapsto x^q$

let $R = S(P) \otimes S(P)$

$\Delta : R \longrightarrow S(P)$

$$S(P) \otimes S(P) \overset{\Delta}{\underset{\Gamma}{\rightrightarrows}} S(P) \xrightarrow{\pi} S(P_0)$$

~~Ker~~ $\Delta(x \otimes 1) = \Delta(1 \otimes x) = x$

$\cancel{\Delta(x\otimes}$ $\Gamma(x \otimes 1) = x$

$\Gamma(1 \otimes x) = \sigma x.$

$I = \operatorname{Ker} \Delta$ gen. by $x \otimes 1 - 1 \otimes x$

~~So~~ now I have computed the Chern classes of ~~them~~ a character $G \longrightarrow \mathbb{F}_q(\mu_\ell)^*$.

basic classes $c'_{rj} \in H^{2rj}(BGL_n(\mathbb{F}_q), \mu_\ell^{\otimes rj})$ c''_{rj}

take values in ~~the~~ a certain ring

$$c_i(E) \in H^{2i}(Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q) \times G; \mu_\ell^{\otimes i})$$

$\alpha + \epsilon\beta$

this group is zero if $i \not\equiv 0$ $\uparrow \wr$ (α, β)

$$H^{2i}(G, \mu_\ell^{\otimes i}) \oplus H^{2i-1}(G, \mu_\ell^{\otimes i})$$

$$E_2^{pq} = H^p(G, H^q(Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q), \mu_\ell^{\otimes i})) \Longrightarrow H^{p+q}(Gal \times G, \mu_\ell^{\otimes i})$$

$$[\sigma(c_i), S^{2i-1}] = [c_i, S^{2i}]$$
$$= (i-1)!$$

$$\begin{tikzcd}
S^{2i-1} \arrow[r] \arrow[rrd, bend left] & U \arrow[d] \arrow[rd, "\sigma(c_i)"] & \\
& E \arrow[d] \arrow[r, "\Phi(c_i)"] & K(\mathbb{Z}_p, 2i-1) \\
& BU &
\end{tikzcd}$$

<u>question:</u> How bad is $(i-1)!$

in comparison with $q^i - 1$

<u>answer:</u> is very bad since as i goes to infinity

$$v_\ell((i-1)!) \sim (i-1)\left(\frac{1}{\ell} + \frac{1}{\ell^2} + \cdots\right)$$

$$\frac{i-1}{\ell}\left(\frac{1}{1-\frac{1}{\ell}}\right) = \frac{i-1}{\ell-1}$$

It is linear in i with slope $\dfrac{1}{\ell - 1}$

while $\quad v_\ell(q^i - 1) = i\, v_\ell(q-1)$

This is linear in i with slope $v_\ell(q-1)$. ~~Completely~~

Conclusion is that the ~~image~~ map

$$K_{2i-1}(\mathbb{F}_q) \longrightarrow \varprojlim_n H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), \mu_\ell^{\otimes i}) = \mu_{q^i-1}^{\otimes i}$$

$\cong$

$\mu_{q^i-1}^{\otimes i}$ is multiplication by $(i-1)!$

I have to produce an independent proof of the existence of the c_i' and c_i'' in part II

~~topoi~~ topoes

X polyhedron

Suppose you form inductive limit of various topois associated to finer & finer triangulations then do you get all sheaves. First take the usual limit and

$U \subset X$ open, then $\mathbb{Z}_U = \varinjlim_{V} \mathbb{Z}_V$

V open subset of an open subpolyhedron so it seems clear that one must obtain all sheaves.

Existence of c_i' c_i'' topologically

$$\begin{tikzcd} & E\Phi^g \arrow[d] \\ BG \arrow[ur] & BU \arrow[d] \\ & BU \end{tikzcd}$$

so you produce elements c_i' $c_i'' \in$ $H^*(E\Phi^g)$ by you method

so you **must know** how these behave under sums

$$\begin{tikzcd} E\Phi^g \times E\Phi^g \arrow[r] \arrow[d] & E\Phi^g \arrow[d] \\ BU \times BU \arrow[r] \arrow[d, shift right] \arrow[d, shift left, "(\Phi^g \times \Phi^g)"] & BU \arrow[d, shift right, "\Phi^g"'] \arrow[d, shift left, "id"] \\ BU \times BU \arrow[r] & BU \end{tikzcd}$$

~~And the problem I ran run into is uniqueness of the lifting.~~

$$\begin{tikzcd}
k \arrow[dr] & & K(\pi_n X, n) \arrow[d] \\
& B \arrow[r, shift left] \arrow[r, shift right] & X_{(n)} \arrow[d] \\
& & X_{(n)}
\end{tikzcd}$$

~~Let us first suppose that the difference of these two maps from B to $K(\pi_n X, n)$ is an H-space map, i.e.~~

and the problem is commutativity of

$$\begin{tikzcd}
& K(\pi_n X, n) \arrow[d] \\
k \arrow[r] \arrow[d] & X_{(n)} \arrow[d] \\
B \arrow[ur] \arrow[r] & X_{(n-1)}
\end{tikzcd}$$

point is that we don't yet know that

$$\mathrm{Hom}(k, K(\pi_n X, n)) \longrightarrow \mathrm{Hom}(k, X_{(n)}) \longrightarrow \mathrm{Hom}(k, X_{(n-1)})$$

is exact.

Thus suppose for each $E \longrightarrow Y$ you give a homotopy class.

Maybe you have to formulate the desired condition 2-universally, e.g.

The idea is that a cartesian functor

$$\begin{tikzcd} \underline{k} \arrow[rr] \arrow[dr, "\text{univ one}"'] & & \underline{\underline{E\Phi^\delta}} \\ & \underline{B} \arrow[ur] & \end{tikzcd}$$

$k \longrightarrow [\,,E\Phi^\delta]$

what you have already is a functor, but you need a **one** of fibred cats.

The other thing you want to be able to handle is cohomology, and for this you must know that

$$\pi_0 \operatorname{Homcart}(\underline{k}, \underline{K}(A,\delta)) = \operatorname{Hom}(k, H^\delta(\,,A)).$$

Modulo these two unknowns the theory should work.

Back to simplicial setup. Thus we study

$Q(X)_1$ = set of R-v. bundles over X.

but instead we take $\qquad B_0 = B\{0\}$

$= \operatorname{Hom}(X, B_1) \qquad B_1 = B\{Cat(V)\}$

$Q(X)_2$ = set of exact sequences

instead take

$= \operatorname{Hom}(X, B_2) \qquad B_2 = B\{Cat(\text{exact sequences})\}.$

and similarly **for** all $Q(X)_n$. Thus get a scratch:

terrifically confusing.

I have been looking at category of R-vector bundles over X, which is not a functor of X in the strict sense. make it so, at least the objects. Choose representatives for the proj. f.t. R-modules; form a set ∇. then consider

$$\coprod_{P \in \nabla} B\,\mathrm{Aut}(P)$$

and form category of exact sequences

~~Classifying space of the category of exact sequences.~~

analogy is this: think of k as $H_0(L.)$ where $L.$ is a chain complex in an abelian category. Think of $\underline{\underline{k}}$ as $L.$ itself and k_1 as $H_1(L.)$. Then I want to think of B as a complex $B.$ and what I think I should gain by working with 2-cats, etc. is embodied by the difference between

$$\mathrm{Hom}_D(L., B.) \quad \text{and} \quad \mathrm{Hom}(H_0(L.), H_0(B.)).$$

keep up the analogy:

Go on to what you need to finish off your theorem. Thus you have constructed a natural transformation $k \longrightarrow [\,,E\Phi^8]$ but you need more, namely a functor

$$\underline{\underline{k}} \longrightarrow \underline{\underline{E\Phi^8}}$$

Suppose I have two cartesian functors

$$\theta_i : \underline{\underline{k}} \longrightarrow \underline{\underline{X}},$$

such that $f\theta_1$ is isomorphic to $f\theta_2$. What θ_1 does is to for each space S associate a functor

$$\underline{k}(S) \longrightarrow \mathrm{Hom}(S, X)$$

plus functorality for all maps in $\underline{k}$. Thus if I am given a map $u : S' \longrightarrow S$ and a map of virtual bundles $v : x' \longrightarrow x$ over u, then I want to have a homotopy $\theta_i(v)$ making

$$\begin{tikzcd}
S' \arrow[rr, "u"] \arrow[dr, "\theta_i(x')"'] & \arrow[d, phantom, "\overset{\theta_i(v)}{\Rightarrow}"] & S \arrow[dl, "\theta_i(x)"] \\
& X &
\end{tikzcd}$$

commute, plus transitivity, etc. Now suppose ξ is an isomorphism from $f\theta_1$ to $f\theta_2$, so for each pair (S, σ) I have a homotopy $\xi(S, \sigma)$

$$\begin{tikzcd}
S \arrow[r, shift left, "f\theta_1(S,\sigma)"] \arrow[r, shift right, "f\theta_2(S,\sigma)"'] & Y
\end{tikzcd}.$$

Then let $\beta(S,\sigma) = \big(\theta_1(S,\sigma), \xi(S,\sigma), \theta_2(S,\sigma)\big) : S \longrightarrow X \times_Y Y^I \times_Y X$

~~side what I probably need is to express the~~

I form a fibred category over the homotopy category, associating to each X the Picard category $\underline{\underline{k}}(X)$ generated by R-vector bundles over X. An object of $\underline{\underline{k}}(X)$ might be defined to be an element of the free group generated by the set of R-vector bundles over X. On the other hand given a space B I can form the fibered category $\underline{\underline{B}}$ over the homotopy category where $\underline{\underline{B}}(X) = \mathrm{Hom}(X, B)$ and maps are homotopy classes of homotopies. Note that a map $B_1 \longrightarrow B_2$ induces a functor $\underline{\underline{B}}_1 \longrightarrow \underline{\underline{B}}_2$ over $\mathcal{H}o$ and that homotopic maps give rise to isomorphic functors.

The basic question is now whether there exists a morphism of fibered categories $\underline{\underline{k}} \longrightarrow \underline{\underline{B}}$ which is universal, i.e. given $\underline{\underline{k}} \longrightarrow \underline{\underline{B}}'$ there is a map $f : B \longrightarrow B'$ and an isomorphism θ rendering

$$\begin{tikzcd} \underline{\underline{k}} \arrow[r] \arrow[dr] & \underline{\underline{B}} \arrow[d, "f"] \arrow[d, phantom, shift right=3ex, "\overset{\theta}{\Rightarrow}"] \\ & \underline{\underline{B}}' \end{tikzcd}$$

and the pair (θ, f) is unique in some sense.

The semi-simplicial approach is wrong! Only a universal approach is categorically acceptable.

So before giving up the simplicial stuff observe that if $Q(X)$ is the simplicial set constructed out of R-vector bundles over X, then there is a map

$$Q(X) \longrightarrow \underline{\mathrm{Hom}}(X, Q)$$

or equivalently a map

$$X \times Q(X) \longrightarrow Q.$$

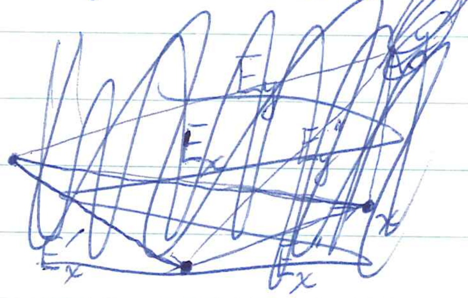
Indeed a map $Y \longrightarrow X \times Q(X)$ is the same thing as a pair consisting of a map $u: Y \longrightarrow X$ and a 1-cocycle f on Y with values in R-vector bundles over X. So if y is a 1-simplex in Y, then associate to y the R-module $f'(y) = f(y)_{u(y)}$ i.e. the fiber of $f(y)$ over the 1-simplex $u(y)$. Then f' is a 1-cocycle on Y with values in R-modules, so we have a map $Y \longrightarrow Q$.

Ideas: to $Q(X)$ I associate the Picard category $\underline{\underline{k}}(X)$, say defined as the Postnikov part of $G(Q(X))$. Then what I want to produce is a map of Picard categories from

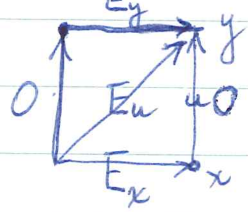
$$\underline{\underline{k}}(X) \longrightarrow \underline{\Pi}(X, E\Phi^{\theta})$$

when I have a finite field. ~~But from the theoretical~~

~~bundle over X. If $0 \to E' \to E \to E'' \to 0$ is an exact sequence then you form a simplicial set $E^2 X$~~



Perhaps it is slightly better to use the suspension $\Sigma X = \Delta(1) \times X / \dot{\Delta}(1) \times X$ and to associate the 1-cocycle which over the product of $\Delta(1)$ and a 1-simplex u looks like



The reason is that if $0 \to E' \to E \to E'' \to 0$ is an exact sequence of R-bundles over X, then we get a map

$$\Delta(2) \times X / \dot{\Delta}(2) \times X \longrightarrow Q$$

whose faces are the maps associated to E', E, and E''. This shows that we get a map

$$k(X) \longrightarrow \pi_1 \, \underline{\mathrm{Hom}}(X, Q)$$

for any "space" X (note that the Grothendieck group as a group defined by generators + relations is necessarily abelian since $E' \oplus E''$ fits into two exact sequences.)

Situation:

For each $n \geq 0$ I have a simplicial set (n-reduced) $Q(n)$ and probably maps $\Sigma Q(n) \longrightarrow Q(n+1)$ which come from the fact an exact sequence

$$0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_{n+1} \longrightarrow 0$$

can be extended one more step by adding 0 at the end (or ~~[illegible]~~ beginning depending on how $\Sigma Q(n)$ is defined).

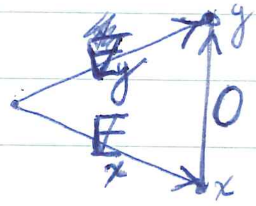
Problem 1: Is induced map $\pi_i\, Q(n) \longrightarrow \pi_{i+1}\, Q(n+1)$ an isomorphism?

If so then the cup product maps

$$Q_R(p) \times Q_S(q) \longrightarrow Q_{R \otimes S}(p+q)$$

will give the appropriate product structures on K-groups. (How about λ-operations? Seem OKAY in char 0!)

Suppose E is a (R-) vector bundle over X. Then over each vertex you get a vertex so over ΣX you get for each 1 simplex a vector space



this is a typical 2-simplex in the cone on X.

This shows that there is a map

$$\Sigma X \longrightarrow Q = Q(1)$$

~~[illegible]~~ associated to an R-vector bundle over X.

part of paper on algebraic K-theory

~~[illegible]~~ theorems:

I. $BGL^{+} = \Omega B\left(\coprod_n BG_n\right)_0$

II. $K_i'A \simeq K_iA$.

---

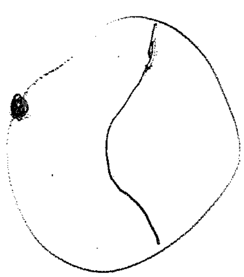
stability: manifold X compact with basepoint then get a framed codim 2 variety Z + must be done in a certain order.

$$\begin{tikzcd}
0 \arrow[r] & E' \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & E'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A\otimes_{\mathbb{Z}} E' \arrow[r] \arrow[d] & ? \arrow[r] \arrow[d] & E'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A\otimes_{\mathbb{Z}} E' \arrow[r] & A\otimes_{\mathbb{Z}} E \arrow[r] & A\otimes_{\mathbb{Z}} E'' \arrow[r] & 0
\end{tikzcd}$$

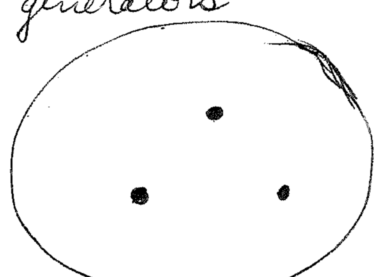
don't

and a representation of $\pi_1(X-Z)$ except that certain ramification behavior is to take place.

~~Each of the little ramification pieces is clear + refined.~~



so over the 2-sphere I get
so the fundamental group is free group on generators



cobordism is a bit amusing for then singularities come in.

Problems: Suppose $K(X)$ is a representable contravariant functor from finite complexes to $\mathcal{Ab}$ endowed with traces for finite coverings. Does K extend to a generalized cohomology theory? (Segal needs $K(X)$ defined for all CW complexes X, I think.)

Suppose $K(X)$ has a natural ring structure such that the projection formula holds. Then does the cohomology theory $K^*(X)$ admit products?

In the case of the connected theory $k^*(X,A)$ does Ψ^{ℓ} extend to a stable cohomology operation when denominators are introduced? And how about the filtration properties signalled by Milgram as a possible method of attack on the stability problem? Any relation between the two kinds of stability?

If $\mathcal{A}$ is a small abelian category and we form $\Omega|C.|$ is its homology the same as what one gets from characteristic classes?

Connection of algebraic K-theory and the theory of motives: Fix some groundfield k and consider algebraic schemes over k. Assume that k is small, i.e. of finite type whence one expects that absolute etale cohomology is a good functor in the sense that it detects lots of stuff. To each scheme X, let $K_\bullet(X)$ be the rational K-groups. Then $K_\bullet(X)$ should satisfy the rpojective bundle theorem, and this suggests stronly that it has a Gysin homomorphism; assume so. Wait. Let's review the proj. bundle thm. Thus if E is a vector bundle over X, one knows that $R(G,\mathbb{P}E)$ is a free module over $R(G,X)$ with standard generators, and this isomorphism is compatible with changes in G. Thus the universal map

$$\underline{o}(1) \qquad R(G,\mathbb{P}E) \qquad H^0(G,K_\bullet(\mathbb{P}E))$$

Thus $\underline{o}(1)$ defines an element $K_0(\mathbb{P}E)$ and $K_\bullet(\mathbb{P}E)$ is a free module over $K_\bullet(X)$ with usual basis and relation. So what we're getting is the various difference in weights already in K_0. Thus if there is a Gysin morphism it is of degree 0 for the grading but changes the weights in the appropriate way.

Conjecture which should be answerable: Show that the Adams operations on $K_\bullet$ admit eigenspaces of the standard sort.

What's missing is a typr of periodicity result which would connect up K-groups of different dimensions.

Grand conjecture would run like this: Take the groups $K_\bullet$ and rearrange the grading so as to be by weights Vaguely the K-groups should measure the deficiency between motives and cohomology. Nonsense.

~~The reason that $BGL_\infty(A)^+$ is an H-space should follow exactly the way it does for BU once you work with the spaces $BGL_n(A)^+$ and the fact that these pointed spaces have the homotopies that you need. Stability theorem should be absolutely formal due to the fact that the cells you a mixing with are of dimensions 2 adn 3. Thus an honest bundle has the property that~~

$$X^n \qquad BGL_n^+ = BGL_n \cup e_2 \cup e_3$$

~~Suppose that X^n is an n-manifold. Then a map to BGL_N^+ should be first of all a~~

Program: Characterisitic classes with field coefficients. Formula for the rational K groups.

Universal property of $K_*(A)_{\mathbb{Q}}$: There is a morphism of functors from $\underline{C}$ to $\underline{Ab}$

$$R_A(G) \longrightarrow H^0(G, K_*(A)_{\mathbb{Q}})$$

which is a universal additive transformation from R_A to cohomology with coefficients in a vector space over $\mathbb{Q}$.

Corollary: If A is commutative, then there is a unique ring structure on $K_*(A)_{\mathbb{Q}}$ such that the above is a ring homomotrphism.

What about the ψ^k ? They must act on $K_*(A)_{\mathbb{Q}}$; is it reasonable to expect that they give a weight decomposition of this space, i.e. does it split up as a direct sum of eigenspaces for the various characters of $\mathbb{G}_m$. For example there is the homo

$$ch : K_*(A)_{\mathbb{Q}} \longrightarrow \bigoplus_i H^{2i-*}(\mathrm{Spec}\ A, T^{\otimes i})_{\mathbb{Q}}$$

and one knows how the ψ^k act on the ch_i . Thus if the homo. above is something like an isomorphism, then

$$c_i \quad K_a(X)_{\mathbb{Q}} \longrightarrow H^{2i-a}(\mathrm{Spec}\ A, T^{\otimes i})$$

should be the projection on the part of weight i .

~~Connection with the theory of motives is becoming pretty clear now.~~