November 4, 1970:

I would like to understand better something that came up in conversation with John Mather. The problem is to compute \( H^*(\text{GL}_n(\mathbb{R}), \mathbb{Z}/\mathbb{Z}) \) where \( \text{GL}_n(\mathbb{R}) \) has the discrete topology. Elements of this are characteristic classes which associate to a flat bundle \( E \to X \) a coh. class \( u(E) \in H^*(X) \) in a natural way, and which are stable: \( u(E \oplus 0) = u(E) \).

Now the idea is to consider the restriction mapping from such classes to classes for flat bundles with finite structural groups. In precise terms I want

\[
\text{Hom}_{\text{G a fin. gp.}} \left( \overline{R}_R(G), H^*(G, \mathbb{Z}/\mathbb{Z}) \right)
\]

and to compare this with

\[
\text{Hom}_{\text{G a gpo.}} \left( \overline{R}_R(G), H^*(G, \mathbb{Z}/\mathbb{Z}) \right)
\]

which I know to be

\[
\overline{H}^*(\text{GL}_n(\mathbb{R}), \mathbb{Z}/\mathbb{Z})
\]

by your deep(?) theorems.

Now I change \( \mathbb{R} \) to \( \mathbb{C} \). Now I can compute the first group I think. Choose \( p \neq 1 \) and let \( k = \mathbb{F}_p \), \( \phi: k^* \to \mathbb{C}^* \). Then we get maps

\[
\mathbb{R}_k(G) \xrightarrow{\phi} \mathbb{R}_c(G)
\]

(\( \phi \) defined all groups \( G \) but only for finite gpos.)

making \( \mathbb{R}_k(G) \) a natural direct summand of \( \mathbb{R}_c(G) \) in a
Lemma: Let $F : (\text{fin.gps})^\circ \to \text{sets}$. Then

$$\text{Hom}_{\text{fin.gps}}(F, H^\ast) \sim \text{Hom}_{\text{l-gps}}(F, H^\ast).$$

Proof: 

\[
\begin{array}{ccc}
F(G) & \xrightarrow{\lim} & \lim_{P \to G} F(P) \\
\downarrow & & \downarrow \\
H^\ast(G) & \xleftarrow{\lim} & \lim_{P \to G} H^\ast(P)
\end{array}
\]

where $P$ runs over the category of l-groups.

Influence of stability: see if you can use your good theorem. Now use structure provided by the Künneth formula, so I restrict attention to multiplicative spes $\Theta : R^\vee_A \to H^0(S, \cdot)$. Unfortunately the restriction map to elementary abelian l-subgps. $H$ is not injective because over a finite field $A = F_p$ one has stable classes (the $c_i^u$) which vanish on elementary l-subgroups.

Question: Can you compute $\text{Hom}_{\text{abel. l-gps}}(R^\vee_A, H^\ast)$?
way compatible with products and $\Lambda$-operations.

So what I'm getting at is the maps

$$BGL(k)^+ \rightarrow BGL(C)^+ \rightarrow B^{\text{op}}GL(C)$$

and I know this composition induces an isomorphism on cohomology for all $l \neq p$.

$$\text{Hom}_{G \text{ finite}} \left( \frac{R_c(G)}{G^*}, H^*(G) \right) \cong \text{Hom}_{G \text{ fin.}} \left( \frac{R_k(G)}{G}, H^*(G) \right)$$

This diagram shows that $\phi^*$ must be surjective, hence $R_c(G)$ for G finite same as $R_k(G)$ as far as cohomology mod l goes.

General principle: Let $A$ be a ring. Then we can consider either natural transformations from $\frac{R_A(G)}{G^*}$ to $H^*(G)$ where $G$ is all gp's, or all finite groups.

$$\text{Hom}_{G \text{ gp}} \left( \frac{R_A}{G^*}, H^* \right) \cong \text{Hom}_{G \text{ fin. gp}} \left( \frac{R_A}{G}, H^* \right)$$

(see below)
Suppose $G$ finite and $A$ is a Dedekind domain such that $l$ is a unit in $A$ and let $G \to \text{Aut}(E)$ be a representation of an abelian $l$-group where $E$ is a finitely generated projective $A$-module. Suppose $E$ is indecomposable, and for simplicity that $A$ is the ring of $S$-units in a number field $K$. Then $E \otimes K \cong K[\mu_\infty]$ where $\chi : G \to \mu_\infty$ is surjective. To the invariants of $E$ consist of three characters, which is unique up to the Galois invariant groups and the class of the invertible $A[\mu_\infty]$ ideal $E$ in $E \otimes K = K[\mu_\infty]$. Thus it seems that

$$R_{A}(G) = \text{free abelian group on}$$

$$\bigoplus_{n \geq 1} \text{Hom}_{G}(G, \mu_\infty) \times \text{Pic } A[\mu_\infty] / \text{Gal}$$

$$= \lim_{\text{lim}} \text{Hom}_{G}(G, \mu_\infty) \times \text{Pic } A[\mu_\infty]$$

where the limit is taken over the category of fields $K[\mu_\infty]$ and where $\text{Pic }$ moves covariantly, i.e.
given \( K[\mu_{m}] \to K[\mu_{\infty}] \) we take the induced map 
\[ \text{Pic } A[\mu_{m}] \to \text{Pic } A[\mu_{\infty}] \].

If you take \( A = K \) so that \( \text{Pic } = 1 \) then this is 
\[ \text{Hom } (G, \mu_{\infty}) / \text{Gal} \]
and the homology of this functor is 
\[ H_{*} (\mu_{\infty}) / \text{Gal} \]
which if \( \mu_{\infty} \subset A \) is a divided power algebra with one generator of degree 2. So we've made a mistake.

To try \( A = K \). Then \( R_{A}\langle G \rangle \to R_{A} (G) / \text{Gal} \) where \( A = K[\mu_{m}] \) and \( B_{A}(G) = \mathbb{Z}[\hat{G}] / \text{Gal} \). Our mistake is in identifying \( \mathbb{Z}[\hat{G} / \text{Gal}] \) and \( \mathbb{Z}[\hat{G}] / \text{Gal} \) in the obvious way which is not functorial in \( G \).

Here's how to rectify things. Again write \( R_{A} (G) \) as a quotient of 
\[ \mathbb{Z} \left[ \bigoplus_{n>0} \left( \text{Hom} (G, \mu_{\infty}^{n}) \times \text{Pic } A[\mu_{\infty}^{n}] \right) \text{Gal} \right] \]
but now we must identify 
\[ \mathbb{Z} \left[ \text{Hom} (G, \mu_{\infty}^{m}) \times \text{Pic } A[\mu_{\infty}^{m}] \right] \to \mathbb{Z} \left[ \text{Hom} (G, \mu_{\infty}^{m}) \times \text{Pic } A[\mu_{\infty}^{m}] \right] \]
The vertical map sends $\chi : G \to \text{Aut}_\Lambda^\vee (E)$, $E$ an invertible $\Lambda[\mu_{p^n}]$-module into $E$ regarded as a sum of invertible $\Lambda[\mu_{p^n}]$-invertible modules. You are being stupid about the Picard group also. The point somehow is that

$$Z[\text{Hom}(G, \mu_{p^n})] \otimes K_0(\Lambda) \to R^\Lambda_\Lambda(G)$$

if $\mu_{p^n} \subset \Lambda$ and exponent of $G$ divides $p^n$. So now I want to write $R^\Lambda_\Lambda(G)$ as a quotient:

$$\bigoplus_{n \geq 0} Z[\text{Hom}(G, \mu_{p^n})] \otimes K_0(\Lambda[\mu_{p^n}]) \to R^\Lambda_\Lambda(G)$$

and it should be true that the equivalence relation is of form

$$\bigoplus \mathbb{Z} \left[ \text{Hom}(G, \mu_{p^n}) \right] \otimes K_0(\Lambda[\mu_{p^n}]) \xrightarrow{p_1} \bigoplus \mathbb{Z} \left[ \text{Hom}(G, \mu_{p^n}) \right] \otimes K_0(\Lambda[\mu_{p^n}])$$

where $p_1$ is induced by the given map $u : \mu_{p^n} \to \mu_{p^n}$, and where $p_2$ is induced by the norm

$$u_* : K_0(\Lambda[\mu_{p^n}]) \to K_0(\Lambda[\mu_{p^n}])$$


\[ V_n : K(\Lambda[\mu_{e^n}]) \rightarrow H^0(\mu_{e^n}, S)^\times \]

which are compatible with restriction on the right and norm on the left and Galois:

Now:

\[ s_0' = 1, \quad s_0'' = 0 \]

\[ H^0(\mu_{e^n}, S)^\times = (S, [\theta_S])^\times = \left\{ (\sum_{i \geq 0} s^i x^i + s'' x^{-1} y) \right\} \]

Suppose for simplicity that all the \(K\)-groups are \(\mathbb{Z}\), and take \(\theta\) to be a generator of the image of the Galois gp in \(Z^* = \text{Aut} [\mu_{e^n}]\). I assume \(\theta \equiv 1 \mod e\) and put \(a = \nu_e(\theta - 1)\). Then the degree of \(K[\mu_{e^n}]\) over \(K\) is least \(n \geq e^n | \theta - 1\). But

\[ n = \nu_e(\theta - 1) = \nu_e(\theta) + \nu_e(\theta - 1) = \nu_e(\theta) + a \]

so degree \([K[\mu_{e^n}] : K] = n - a\). So what happens is that

\[
\begin{array}{ccc}
K_0(\Lambda[\mu_{e^{a+2}}]) & \rightarrow & Z \\
\downarrow{l} & & \downarrow{l} \\
K_0(\Lambda[\mu_{e^{a+1}}]) & \rightarrow & H^0(\mu_{e^{a+1}}, S) \\
\downarrow{l} & & \downarrow{l} \\
K_0(\Lambda[\mu_{e^a}]) & \rightarrow & H^0(\mu_{e^a}, S) \\
\downarrow{l} & & \downarrow{l} \\
K_0(\Lambda) & \rightarrow & H^0(\mu_{e^0}, S) \\
\end{array}
\]

Compatibility conditions

\[ u_a = \text{res} u_{a+1} \]
Unfortunately

\[(\sum s_i x^i + s'' x^{i-1} y)^l = \sum (s'_i)^l(x^i)^l\]

so it seems that there should be too many possibilities for the \(s''\) and none for \(s'_i\). ??

This is confusing but perhaps not incorrect. The point is that you don't really have very much control over abelian groups of exponent \(> 1\). Precisely, suppose \(N = \mathbb{F}_p^d, \forall g \in \mathbb{F}_p, \langle g angle = a > 1\). I define a map \(\Phi: R^1(G) \to H^1(G)\) for all abelian \(l\)-groups. Given an irreducible repn \(\Phi: G \to \mu_b\) well-determined up to Galois we associate the class \(G \to \mu_a \to \mu_b\) if \(b = a+1\) and zero otherwise. This defines \(\Phi\) for each \(G\); I claim it is natural: Suppose have \(G \to G\).

\[\text{Cases: } 1) b = a+1\]

and \(G \to \mu_b\) onto; then OKAY, 2) \(b > a+1\) and \(G \to \mu_b\) has image \(\mu_{a+1}\); in this case the representation of \(G\) is \(l\) times the \(\mu\) repn of \(G\), so both \(X\) and \(\Phi(X)\) pull-back to zero. 3) \(G \to \mu_b\) not surjective with image not \(\mu_{a+1}\); then both \(\Phi(X)\) and \(\Phi(X|G_2) = 0\). So it seems that the above is OKAY, and that we should not consider natural transformations on abelian \(l\)-groups.
November 5, 1970

Fix a field $\Lambda$, so that we have good control over $R_\Lambda(G)$ for $G$ finite. Precisely, first suppose $\Lambda$ of characteristic $p > 0$. Let $\Lambda_0 \subset \Lambda$ be the subfield algebraic over $F_p$. Then I claim that

$$R_{\Lambda_0}(G) \cong R_\Lambda(G) \quad \text{G finite.}$$

Indeed, I think we can show that

$$\begin{align*}
R_{\Lambda_0}(G) & \cong \text{Gal}(F_p, \Lambda_0) \\
R_{\Lambda \bar{F}_p}(G) & \cong \text{Gal}(\Lambda \bar{F}_p, \Lambda)
\end{align*}$$

The point somehow is that the extensions $\Lambda_0 \rightarrow F_p$, $\Lambda_0 \rightarrow \Lambda$ and $\Lambda \rightarrow \Lambda \bar{F}_p$ are separable, hence the only thing that can happen to an irreducible representation is that it splits into irreducibles corresponding to its endomorphism field splitting.

$$\Lambda_0 \rightarrow \Lambda$$

Every $\Lambda_0[G]$-module, $\text{End}(E) = \Lambda_1$ then $\Lambda_1$ is commutative, and by Galois nonsense $\Lambda_1 \cdot \Lambda$ is a field. Clear more or less.
November 13-19, 1970

Toward an understanding of Thompson's theorem.

$G$ finite group, $G_p$ a Sylow $p$-subgroup,

$J = J(G_p) =$ subgroup generated by abelian subgroups with maximal no. of generators.

$\text{rank } (A) = \text{dim } (G)$

Thompson's hypothesis: $N_G(J)$ has normal $p$-complement.

Idea: This implies that there is no fusion.

Suppose $G$ has a normal $p$-complement.

$G = P \times K$

$P \triangleleft G$

$K$ is a $p'$-Hall subgroup, normal.

Then there is no fusion for subsets of $P$. Indeed if $S_1, S_2 \subset P$ and $gS_1g^{-1} \subset S_2$, then write $g = kp$

and if $a \in S_1$, then

$g^{-1}ag = p^{-1}k^{-1}akp \in P$

$\Rightarrow a^{-1}k^{-1}ak \in P \cap K$ \hspace{1cm} (K0)

$\Rightarrow k$ centralizes $S_1$ and $p^{-1}S_1p < S_2$. 
Consequently if we make the subsets of $p$-elements of $G$ into a category, in which a morphism from $S_1$ to $S_2$ for $G_+ p$, then the categories are equivalent.

Conversely, there is no fusion of $p$-subgroups, then $G$ has a normal $p$-complement.

Let's try $p$-subgroups: idea is to show $H^*(G) \to H^*(P)$ and then apply Tate's...外程起程

Start $g \in G$; we want to show that the two arrows $H^*(P) \to H^*(P \cdot g = P - 1)$ coincide.

However, we can consider $P = P \cdot g P^{-1}$ and the group $Q$ as objects of the category of $p$-subgroups of $G$.

And then apply... However, the two subgroups $P \cdot g P^{-1}$ and $g^{-1} P g P$ of $P$ are conjugate in $G$, hence conjugate in $P$.

coincide. But if $Q = P \cdot g P^{-1}$, then $Q \subseteq P$ and $g^{-1} Q g < P$, hence we have $g = kp$ where $k$ centralizes...
Q, so the map is the same as \( g \mapsto g^{-1}bg \) from \( Q \) to \( P \) from \( Q \) to \( P \), whose effect on cohomology is the same as the inclusion \( g \mapsto g \).

I think it's also true that if there is no fusion of elements of \( P \), then \( P \) has a normal \( p \)-complement. My reason for believing this is Atiyah's claim that \( G \) has a normal \( p \)-complement when one knows that the map

\[
\text{Conj. classes of } P \rightarrow \text{Conj. classes of } G
\]

is bijective, that is when \( gxg^{-1} = y \) for \( x, y \in P \) \( \Rightarrow pxp^{-1} = y \) some \( P \).

(Does it follow from this that one can choose \( p \) so that \( gp^{-1} \) is a \( p' \)-element?)

First case: \( P \triangleleft G \). Then I want to prove that if \( g = pk = kp \) is the decomposition of \( g \) into its \( p \)-regular + \( p \)-singular parts, then \( g \mapsto p^{-1}gp \) is a homomorphism. However here if \( p \) is a \( p \)-elt + \( k \) is a \( p' \)-element, then they commute. Indeed \( k'pk \in P \) and as there is no fusion \( k'pk = p'pp' \). Thus \( p'k' \in \text{Cent } p \), so maybe can use induction?

Then are we define a map \( G \rightarrow P \) in general by sending \( g \) ?

Conjecture: No fusion for \([p] \)-subgroups \( \rightarrow \) normal \( p \)-complement
On normal $p$ complements:

By Frobenius's theorem one has to understand $p'$-automorphisms of $p$-groups. So the question is to find manageable criteria which guarantee that a $p'$-automorphism $\Theta$ of a $p$-group $P$ is trivial. This leads to

**Problem:** Determine those pairs $(\Theta, P)$ where $\Theta$ is a $p'$-automorphism of a $p$-group $P$ which are minimal, i.e., any $\Theta$-stable $P' < P$ is such that $\Theta$ acts trivially on $P'$.

This forces $L_2(p^2)P$ to be trivial and $gr_1 P = P/L_2(p^2)P$ to be an irreducible representation of $\mathbb{Z}/m\mathbb{Z}$ with generator $\Theta$ of order $m$. Now $gr_2 P$ is a quotient of $L_2 gr_1 P$ (if odd) which by Schur's lemma has invariant at most one invariant. Thus dim $gr_2 P \leq 1$ and $gr_3 P$ being a quotient of $gr_1 P \otimes gr_2 P$ will necessarily be zero. Thus $P$ will either be abelian or an extra-special $p$-group without an element of order $p^2$ because the operation is linear from $gr_1 P$ to $gr_2 P$ when $P$ is odd.

When $p = 2$, $gr_2 P$ is a quotient of $S_2(gr_1 P)$ which again will be one-dimensional by Schur's lemma and again $gr_3 P$ will be zero so again $P$ will be
So go back to the original question and let $\Theta$ be a $P'$-auto of a $p$-group $P$. Now form a composition series stable under $\Theta$

$$0 < N_1 < N_2 < \cdots$$

so $N_1$ is a minimal $\Theta$-stable subgroup of $P$ and $N_2$ is a minimal $\Theta$-stable subgroup of $P/N_1$.

So now given a $P'$-automorphism $\Theta$ of a $p$-group $P$ let $M$ be a minimal $\Theta$-stable subgroup of $P$ on which $\Theta$ acts non-trivially. If fact we should argue this way: Let $M$ be a minimal member of the set of normal $\Theta$-stable subgroups in which $\Theta$ acts non-trivially. Then subgroup $[P, M]^{p^r} \leq M$ and is stable under $\Theta$ and normal in $P$, so has trivial $\Theta$ action. Any subgroup $M'$ of $M$ containing $[P, M]$ will be normal in $\Theta P$, hence conclude that $\Theta$ acts irreducibly on $M/[P, M]^{p^r}$.

If $M$ is a minimal $\Theta$-stable subgroup, then for $p$ odd at least $\Theta$ moves the $[p]$-subgroups of $M$ around because every element is of order $p$. Now all I have to do is to see somehow that these motions won't be conjugate within $P$. 
Let $0 \leq N_i C \ldots$

where $N_1$ is a minimal normal subgroup of $P$ stable under $\Theta$, $N_2/N_1 \subseteq \text{P/N}_1$ under $\Theta$, etc. Then $V_\Theta = N_i/N_{i+1}$ is a representation of $(\Theta \times P)$ over $\mathbb{Z}/p\mathbb{Z}$ which is irreducible.

Claim $P$ acts trivially on $V$. Indeed if $I$ is the augmentation ideal of $\mathbb{Z}/p\mathbb{Z}[V]$, then $> IV > \ldots$ is a $(\Theta \times P)$-filtration and $I^n V = 0$ for $n$ large as $V$ is a $p$-gp. Thus $IV = 0$. Thus $\Theta$ acts irreducibly on $V$.

Now let $M$ be a minimal member of

the set of $\Theta$-stable normal subgroups of $P$ on which $\Theta$ acts non-trivially. Conclude.
False but contains an idea about extra-special $p$-groups - symmetry.

**Proposition:** Let $\Theta$ be a $p'$-automorphism of a $p$-group $P$ such that $\Theta$ acts trivially on a $[p]$-subgroup $A$ which it leaves stable. Then if $p$ is odd, $\Theta$ acts trivially on $P$.

**Proof:** Using induction on $|P|$, we can assume that $\Theta$ acts trivially on any proper subgroup which it leaves stable. (Classical language: centralizes any proper subgroup it normalizes). In particular it acts trivially on the Frattini subgroup $P^{(p)}$, so the fixed subgroup $P^\Theta$ is normal and $P/P^\Theta$ is an irreducible representation of $\Theta$ over $\mathbb{Z}/p\mathbb{Z}$.

**Counterexample:** Assume $\mathbb{Z}/p\mathbb{Z}$ acts irreducibly on a $[p]$-group $A$, and $\Lambda^2 A$ has an invariant. This will happen when $\left[ [F_p[F_p,F_p] : F_p] = 2 \right.$ is even. Indeed if $I$ generates $F_p$, then have eigenvalues $j, j^p, \ldots, j^{p^{r-1}}$

and as $p^{1/2} \equiv -1 \pmod{p}$, have $j \cdot j^{p^{1/2}} = 1$.

Consequently if we form a central extension

$$1 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow G \rightarrow A \rightarrow 1$$

using an invariant element of $\Lambda^2 A^Y \rightarrow H^2(A,\mathbb{Z}/p\mathbb{Z})$, then we get a counterexample, because $\mathbb{Z}/p\mathbb{Z}$ is the only invariant $[p^{-1}]$. 
cohomology theories of symmetric product type

Recall the Dold-Thom theorem:

$$
\pi_k \left( \bigcup_{n} \Sigma^n X \right) = H_k(X, \mathbb{Z}) \quad X \text{ ptld. connected}
$$

and your (and everybody else) theorem

$$
\pi_k \left( \left( \bigcup_{n} \Sigma^n X \right)^o \right) = \pi_k \psi(X)
$$

The idea is to interpolate other cohomology theories in between these.

Idea: \( SP^n(X) = X^n/\Sigma^n \)

\( ES^n \times X^n = (E \Sigma_n \times X^n)/\Sigma_n \)

So what I want to do is to find a pleasant family of spaces \( Q_n \) with \( \Sigma_n \) action and maps

\[ \mu_{m,n}: Q_m \times Q_n \to Q_{m+n} \quad \text{equivariant for} \]

\[ \Sigma_m \times \Sigma_n \to \Sigma_{m+n} \]

Requirements:

- associativity
- commutativity

\[ Q_m \times Q_n \xrightarrow{\mu_{m,n}} Q_{m+n} \]

\[ Q_n \times Q_m \xrightarrow{\mu_{m,n}} Q_{m+n} \]
where \( T \) is the element of order 2 interchanging \( 1 \cdots m \) and \( m+1 \cdots n \). Note that this diagram should be homotopy commutative. Note that if we take \( Q_m = \Sigma_m \) with left translation action, then this isn't the case because

\[
\begin{array}{ccc}
\Sigma_m \times \Sigma_n & \xrightarrow{T} & \Sigma_{mn} \\
\downarrow & \downarrow & \downarrow \\
\Sigma_n \times \Sigma_m & \xrightarrow{\sigma \times \sigma^{-1}} & \Sigma_{m+n}
\end{array}
\]

and \( \sigma x \sigma^{-1} \neq \sigma x \).

**Examples:**
1) Let \( Q_m \) be the set of conjugacy classes of elements in \( \Sigma_m \) i.e. partitions \( \alpha = (x_1 \geq x_2 \geq x_3 \cdots) \) with \( \Sigma x_i \leq m \). Then commutativity is clear.

2) Conjugacy classes of subgroups.

3) Conjugacy classes of subsets of \( m \) elements in \( \Sigma_m \); take this to be \( Q_m \).

The above examples all have \( Q_m \) a trivial \( \Sigma_m \)-set, and hence contain the trivial example \( Q_m = \{ x \} \) for all \( m \), instead of \( \sigma x \neq x \) for \( x \neq 1_i \).

(Actually as part of associativity one needs to have...
It is probably possible to classify the above examples as being built up out of the trivial examples $Q_m = pt$ for $m = 0$. But in practice there is a distinguished suspension element in $Q_1$ permitting one to put $Q_m \rightarrow Q_{m+1}$. Thus the only examples of this type are essentially trivial. In fact

$$Q_n \times \Sigma^n X \cong Q_n \times \left( S^p X \right)$$

so our theory is

$$\pi_* \{ Q_\infty \times S^p \left( X \right) \}$$

the same as homology.

More sophisticated examples: suppose we take $Q_m$ to be the classifying space of some category of subgroups e.g. abelian subgroups. Thus

$$\begin{align*}
(\text{ab. subgps)} & \times (\text{ab. subgps)} \\
\text{of } \Sigma^m & \rightarrow (\text{ab. subgps)} \\
\text{of } \Sigma^m & \rightleftharpoons \text{conj} \text{ by } \sigma
\end{align*}$$

$$\begin{align*}
(\text{ab. subgps)} & \times (\text{ab. subgps)} \\
\text{of } \Sigma^n & \rightarrow (\text{ab. subgps)} \\
\text{of } \Sigma^{m+n} & \rightleftharpoons \text{conj} \text{ by } \sigma
\end{align*}$$
To keep from being confused consider the category with a single object, namely the torsor $\Sigma_m$ acting on itself from the right. Then $\mu_{mn}$ is the functor associating to $\Sigma_m$, $\Sigma_n$, $\Sigma_{m+n}$ and to a left operation on $\Sigma_m$ and $\Sigma_n$ the induced left operation on $\Sigma_{m+n}$.

$$
\begin{array}{ccc}
(S_m, pt) \times (S_n, pt) & \longrightarrow & (S_{m+n}, pt) \\
\downarrow & & \downarrow \\
(S_n, pt) \times (S_m, pt) & \longrightarrow & (S_{m+n}, pt)
\end{array}
$$

Thus in the case of abelian subgroups

$$
\begin{array}{ccc}
A \subset \Sigma_m, \ B \subset \Sigma_n & \longrightarrow & A \oplus B \subset \Sigma_{m+n} \\
\downarrow & & \downarrow \text{conj. by } \sigma \\
B \subset \Sigma_m, \ A \subset \Sigma_n & \longrightarrow & B \oplus A \subset \Sigma_{m+n}
\end{array}
$$

so you want to see that conjugation by $\sigma$ is isomorphic to identity. Precisely the functor

$$
A \subset G \longmapsto \sigma^{-1} A \sigma \subset G
$$

from subgps to subgps is isomorphic to the identity the isomorphism being

$$
A \longmapsto \sigma^{-1} A \sigma.
$$
Thus to $\Sigma_m$ I associate the category of homogeneous spaces $\Sigma_m/A$ with specified isotropy groups.

Example 1: If you take the trivial homogeneous space, you get $\Sigma_m/\Sigma_m$ and the functor is then

$$X \mapsto \Sigma^\infty(X)$$

2) If you take the principal homogeneous space you get

$$E\Sigma_n \times \Sigma^n(X^n).$$

So now you would like to understand some intermediate categories if possible.
November 26, 1970: K of a local field (cont.)

Some basic remarks about crystalline Chern classes:

Let $\mathbf{Z} \rightarrow \mathbf{X}$ be a nilpotent extension with divided powers and $E$ a vector bundle on $\mathbf{Z}$. Then I would like formulas for Chern classes

$(*) \quad c_i(E) \in H^{2i}(I^i)$

where $I$ is the ideal defining $\mathbf{Z}$, and cohomology is taken over $\mathbf{Z}$ or $\mathbf{X}$, there's no difference. For line bundles these are obtained from

$$
1 \rightarrow 1 + I \rightarrow \mathcal{O}_X^* \rightarrow \mathcal{O}_Z^* \rightarrow 1 \quad \text{log}
$$

and coboundary $H^1(\mathcal{O}_Z^*) \xrightarrow{\delta} H^2(1 + I)$. The point is that $(*)$ are the restriction to the open set $\mathbf{X}$ of the crystalline topos of $\mathbf{Z}$ of the absolute crystalline Chern classes. Roughly they should be definable by taking the projective bundle $\mathcal{P}E$ over $\mathbf{Z}$ and computing its crystalline cohomology relative to $\mathbf{X}$, which roughly should be the same as an extension of $\mathcal{P}E$ over $\mathbf{X}$ which needn't exist except cohomologically, then using standard "coefficient of relation" formulas.

I would love to be able to define $(*)$ using some non-commutative logarithm for matrices.
1 \rightarrow GL_n(O_X, I) \rightarrow GL_n(O_X) \rightarrow GL_n(O_Z) \rightarrow 1.

But this seems difficult. Look at the easier problem of defining the images

\((**)\) \quad c_i(E) \in H^{2i}(I^i/I^{i+1}).

There are two (probably equivalent) ways of getting such classes. First suppose \(Z\) smooth over \(k\) and then we have exact sequence

\[0 \rightarrow I/I^2 \rightarrow \Omega_{X/k}^1 \rightarrow \Omega_{Z/k}^1 \rightarrow 0\]

and Atiyah classes

\[c_i^A(E) \in H^i(\Omega_{Z/k}^i).\]

The classes \((**)\) should be obtained from the Atiyah classes using

\[0 \rightarrow I^i/I^{i+1} \rightarrow \cdots \rightarrow \Omega_X^i \rightarrow \Omega_Z^i \rightarrow 0\]

secondly there should be a canonical element in

\[K \in H^2(I/I^2 \otimes \text{End} E)\]

such that

\[c_i(E) = \varphi_i(K)\]

\(\varphi_i : \text{End}(E) \rightarrow O_Z^i\) is the symmetric fn. of eigenvalues, \(\text{i.e. } \text{tr}(I^i)\).
Now consider a ring situation
\[ 1 \to I/I^2 \otimes \text{gl}_n(A/I) \to \text{GL}_n(A/I^2) \to \text{GL}_n(A/I) \to 1 \]
which gives us a canonical element in
\[ H^2(\text{GL}_n(A/I), I/I^2 \otimes \text{gl}_n(A/I)) , \]
and hence leads to Chern classes
\[ c_k \in H^{2k}(\text{GL}_n(A/I), I^k/I^{k+1}) . \]
If \( I \) has divided powers these should generalize to classes
\[ c_k \in H^{2k}(\text{GL}_n(A/I), I^k) . \]

Next we try to compute cohomology for \( \mathbb{Z}_p = A \). We know there should be crystalline classes
\[ c_k^{(m)} \in H^{2k}(\text{GL}(\mathbb{Z}_p^n), \mathbb{p}^{nk}\mathbb{Z}_p) \]
I conjecture this class lifts back to a canonical
\[ \beta c_k^{(m)} \in H^{2k-1}(\text{GL}(\mathbb{Z}/p^n), \mathbb{p}^k\mathbb{Z}_p/\mathbb{p}^{nk}\mathbb{Z}_p) \]
satisfying \( \beta c_k^{(m)} = c_k^{(m)} \), \( \beta \) being the relevant Bockstein.
Now the existence of some \( c_k^{(m)} \) is clear as follows: One must show \( c_k^{(m)} \) is killed by \( \mathbb{p}^{nk}\mathbb{Z}_p \to \mathbb{p}^k\mathbb{Z}_p \). But
The class $K$ should be the image of the class in $H^1(\Omega^1 \otimes \text{End}(E))$ represented by Atiyah ext.

$$0 \rightarrow \Omega^1 \otimes \text{End}(E) \rightarrow \text{Hom}(E, J_1 E) \rightarrow \text{End}(E) \rightarrow 0$$

by the coboundary operator of the exact sequence

$$0 \rightarrow I/I^2 \rightarrow \Omega^1_{X/k} \rightarrow \Omega^1_{Z/k} \rightarrow 0.$$ 

Given

$$1 \rightarrow I/I^2 \otimes \text{gl}_n(\mathcal{O}_Z) \rightarrow \text{GL}_n(\mathcal{O}_X/I^2) \rightarrow \text{GL}_n(\mathcal{O}_Z) \rightarrow 1$$

$[E] \in H^1(\text{GL}_n(\mathcal{O}_Z))$.

Somehow $K$ is the cup product of these 1-dimensional classes.

If we are given a $G$-torsor $P \rightarrow X$ and an $E_\mathbb{Z}$-extension

$$0 \rightarrow M \rightarrow G' \rightarrow G \rightarrow 0$$

where $M$ is a $G$-module, does this lead to an element of $H^2(X, \mathbb{Z} \times_\mathbb{Z} P \times_\mathbb{Z} M)$? Topologically, the extension gives an element of $H^2(BG, M)$ and the torsor gives a map $X \rightarrow BG$ so you pull the class back. This is clear in principle.
The formation of classes should be compatible with the morphism \[ A/I^n \to A/I^m. \] Thus \( c_k^{(n)} \) and \( c_k^{(m)} \) are compatible. But then for \( m = 1 \) one knows there are no stable \( p \)-torsion classes. Now taking the direct limit under inflation gives element in

\[
\lim_{\to N} \lim_{\to n} H^{2k-1}(\text{GL}(\mathbb{Z}/p^n\mathbb{Z}), \mathbb{P}^k\mathbb{Z}_p'/p^n\mathbb{Z}_p)
\]\
\subseteq H^{2k-1}(\text{GL}(\mathbb{Z}_p), \mathbb{P}^k\mathbb{Z}_p).
\]
November 27, 1970.

Problem: To define classes

\[ c_k' \in H^{2k}(GL_\mathbb{Q}(\mathbb{Z}_p), \mathbb{Z}/p\mathbb{Z}) \]
\[ c_k'' \in H^{2k-1}(GL(\mathbb{Z}_p), \mathbb{Z}/p\mathbb{Z}) \]

satisfying the analogues of the formulas you have proved for finite fields. Thus

\[ c(E) = \sum (c_k'(E) + c_k''(E) \varepsilon^k) \varepsilon^k \varepsilon \neq 0 \]

should satisfy a product formula and for a one dimensional representation \( G \rightarrow \mathbb{Z}_p^* \)

I conjecture there exist basic classes

\[ c_k'' \in H^{2k-1}(GL(\mathbb{Z}_p), \mathbb{Z}/p\mathbb{Z}) \]

integral classes which are primitive and which come from \( GL(\mathbb{Q}_p) \). Moreover \( c_k'' \) reduced in \( \mathbb{Z}/p\mathbb{Z} \)
comes from \( GL(\mathbb{Z}/p^n\mathbb{Z}) \). It is not yet clear to me whether these conjectures are reasonable, except I can check them for \( k = 1 \). Then

\[ c'_1 : GL(\mathbb{Z}_p) \rightarrow \mathbb{Z}_p \]

is the homomorphism \( A \rightarrow \log (\det A) \) where one removes off the \( \mathbb{Z}/p^{-1}\mathbb{Z}^*/ \) before taking the logarithm, i.e.

\[ \frac{1}{p-1} \log (\det A)^{p-1}. \]
November 25, 1970.

Let \( \lfloor F: \mathbb{Q}_p \rfloor < \infty \). I want to determine the p-primary part of \( K_i(F) \). Now I am going to assume that it is possible to talk about continuous cohomology of \( \text{GL}_n(F) \) and to define continuous \( K \) groups \( K_i(F) \). For example

\[
K_1(F) = F^* 
\]

is a locally compact group. In addition I will assume that Grothendieck construction of Chern classes produces continuous cohomology classes \( \mathcal{C}_i \in H^2i(\text{GL}_n(F) \times \text{Gal}(F/F), \mu_{m}^{\otimes i}) \)

and I want to compute the groups

\[
\lim_{\frac{0}{\Delta}} H^i(\text{Gal}(F/F), \mu_{m}^{\otimes i}),
\]

using local class field theory, or better Tate duality.

**Tate duality:** Let \( M \) be a finite Galois module. Then \( H^i(M) = H^i(\text{Gal}(F/F), M) \) is finite and cap product

\[
H^i(M) \times H^{2d-i}(\text{Hom}(M, F^*)) \rightarrow H^{2}(F^*) \xrightarrow{\text{can}} \mathbb{Q}/\mathbb{Z}
\]

is a perfect duality. (In particular \( H^i(M) = 0 \) \( i > 2 \))

One must always think of this along with
Tate – Riemann - Roch:
\[
\frac{h^0(M) h^2(M)}{h^1(M)} = \text{normalized absolute value of } \text{card}(M) \text{ in } F
\]
\[
= \frac{1}{\text{card} \{ \mathfrak{p} | \text{card}(M) \mathfrak{p} \}}
\]

It seems to me that this duality theorem implies the reciprocity law. Indeed, in dimension 1 it says that for a trivial finite Gal-module
\[
H^1(M) = \text{Hom}(\text{cent}(\text{Gal}_{ab}, M))
\]

is isomorphic to
\[
\text{Hom}(H^1(\otimes \text{Hom}(M, F^*)), \mathbb{Q}/\mathbb{Z})
\]

Now write \( M \)
\[
0 \rightarrow P_i \rightarrow P_0 \rightarrow M \rightarrow 0
\]
where \( P_i \) are free f.t. abelian groups and we have by Hilbert th 90
\[
H^1(F^*) = 0
\]
\[
0 \leftarrow H^1(\text{Hom}(M, F^*)) \leftarrow \text{Hom}(P_i, F^*) \leftarrow \text{Hom}(P_0, F^*)
\]
\[
0 \leftarrow H^1(M) \leftarrow \text{Hom}(F^*, P_i \otimes \mathbb{Q}/\mathbb{Z}) \leftarrow \text{Hom}(F^*, P_0 \otimes \mathbb{Q}/\mathbb{Z})
\]
\[ \text{Homcont } (\text{Gal}_{ab}(F), M) \cong \text{Hom} (F^*, M) \]

Thus \( (F^*)^\wedge \cong \text{Gal}_{ab} \), which seems to be both the reciprocity law and the existence theorem.

Recall that inverse limits are exact for inverse systems of finite groups, hence I can extend cohomology continuously to profinite Galois modules.

Let \( \chi : \text{Gal}(F/F) \to \mathbb{Z}/p^\infty \) be the Tate character on the \( p \)-th power roots of unity. Instead of \( \mu^\infty \) we write \( \mathbb{Z}/p^\infty \). Then we set

\[ H^i(\mathbb{Z}/p^j) = \varprojlim_n H^i(\mathbb{Z}/p^{jn}) \]

and as remarked already the exact sequence

\[ 0 \to \mathbb{Z}/p^j \xrightarrow{P^m} \mathbb{Z}/p^j \to \mathbb{Z}/p^m \to 0 \]

gives rise to a long exact sequence in cohomology:

\[ 0 \to H^0(\mathbb{Z}/p^j) \xrightarrow{P^m} H^0(\mathbb{Z}/p^j) \to H^0(\mathbb{Z}/p^m) \]

\[ \to H^1(\mathbb{Z}/p^j) \xrightarrow{P^m} H^1(\mathbb{Z}/p^j) \to H^1(\mathbb{Z}/p^m) \]

\[ \to H^2(\mathbb{Z}/p^j) \xrightarrow{P^m} H^2(\mathbb{Z}/p^j) \to H^2(\mathbb{Z}/p^m) \to 0 \]
"General" case: \( \mu_p \subset K \), \( \text{Im } \chi = 1 + p^d \mathbb{Z}_p \) and we assume \( d \geq 2 \) if \( p = 2 \). Take \( m=1 \)

\[
H^0(\mathbb{Z}/p) = \mathbb{Z}/p \cong H^2(\mathbb{Z}/p) \quad \text{because } \mu_p \cong \mathbb{Z}/p
\]

\[
H^1(\mathbb{Z}/p) = H^1(\mu_p) \cong F^*/(F^*)^p
\]

Now

\[
F^* = \mathbb{Z} \times \mathfrak{o}^* \cong \mathbb{Z} \times \mu(F) \times \mathbb{Z}_p^{[K:\mathbb{Q}_p]}
\]

so

\[
F^*/(F^*)^p \text{ has rank } 2 + [K:\mathbb{Q}_p]
\]

Conclude \( H^2(\mathbb{Z}/p(1)) \), being pro-\( p \)-abelian with \( \otimes \mathbb{Z}/p \) of rank 1, is cyclic. In fact

\[
H^2(\mathbb{Z}/p(1)) = \varprojlim H^2(\mathbb{Z}/p^n(1))
\]

is dual to \( \varprojlim H^0(\mathbb{Z}/p^n(1-j)) = H^0(\mathbb{Q}_p/\mathbb{Z}_p(1-j)) \)

Elements of Galois are acting on \( \mathbb{Q}_p/\mathbb{Z}_p(1-j) \) by multiplying by \( \chi(\sigma)^{-j} \), hence acting by \( (1 + p^d \mathbb{Z}_p)^{-j} \).

Thus invariants cyclic

\[
H^2(\mathbb{Z}_p(1))^v \cong H^0((\mathbb{Q}_p/\mathbb{Z}_p)(1-j)) \cong \mathbb{Z}_p \left[ 1/p^d + \mathbb{Z}_p^{(g-1)} \right]/\mathbb{Z}_p
\]

so

\[
H^2(\mathbb{Z}_p(1)) \cong \mathbb{Z}_p \left. \right|_p \left[ 1/p^d + \mathbb{Z}_p^{(g-1)} \right]/\mathbb{Z}_p
\]
Similarly

\[ H^0(\mathbb{Z}/p^m \hat{\otimes} \mathbb{Z}_p(j)) \cong \begin{cases} \mathbb{Z}/p^m & m \leq d+v_p(j) \\ \mathbb{Z}/p^{d+v_p(j)} & m > d+v_p(j) \end{cases} \]

do from the exact sequence we see that

\[ H^1(\mathbb{Z}_p(j)) \cong \begin{cases} \mathbb{Z}/p^{d+v_p(j)} \oplus \mathbb{Z}_p & j \neq 0, 1 \\ \mathbb{Z}_p^{1+[K:Q_p]} & j = 0 \\ \mathbb{Z}_p \oplus \mathbb{Z}_p^{1+[K:Q_p]} & j = 1 \end{cases} \]

Indeed if \( j \neq 1 \), then \( H^2(\mathbb{Z}_p(j)) \) has a subgroup of order \( p \) so \( H^1(\mathbb{Z}_p(j)) \) has rank \( \{H^1(\mathbb{Z}_p)\} - 1 = 1 + [K:Q_p] \) generators. Otherwise \( 2 + [K:Q_p] \) generators. Its torsion subgroup is non-trivial for \( j \neq 0 \) and has order \( p^{d+v_p(j)} \).

I conjecture that the etale character induces an isomorphism of the pro-\( p \) completion of \( K_2(F) \) with etale cohomology. Thus

\[ \mathbb{Z}_p^2 + \mathbb{Z}/p^{d+v_p(j)} \cong K_2(F)^\wedge \overset{c_1^\#}{\longrightarrow} H^1(\mathbb{Z}_p(1)) \cong \mathbb{Z}_p^{d+v_p(j)} \]

OKAY

\[ K_2(F)^\wedge \overset{c_2^\#}{\longrightarrow} H^2(\mathbb{Z}_p(2)) \cong \mu_p^d \]

OKAY

\[ K_3(F)^\wedge \overset{(2!)^{-1}c_3^\#}{\longrightarrow} H^3(\mathbb{Z}_p(3)) \cong \mathbb{Z}/p^{d+v_p(3-j)} \oplus [K:Q_p] \mathbb{Z}_p \]
Now let's show these maps are onto modulo torsion. The point is that we have

$$\sum_n x(K^*)^n \cong GL_n(K)$$

as the normalizer of the maximal elementary abelian $p$-subgroup $(\mu_p)^n$. Now the $K$-theory associated to the first group is stable cohomotopy of

$$BK^* = B\mathbb{Z} \times B\mathbb{Z}/p^d \times B\mathbb{Z}_p^{[k:q]}$$

and the rational groups are non-trivial only in dimension 1. Nuts, but you are being stupid to expect the torus to generate the homotopy of $BU$, i.e. $BN \to BU$ pretty lousy on homotopy. However we can make sensible conjectures maybe in this direction.

Suppose $l$ is a prime $\neq p$ with $\mu_l \subset K$, i.e. $l \mid q - 1$ where $q = \text{card (res. field)}$. Then we conjecture that the $l$-primary components are finite and are given by

$$K_{2j}^*(K) \to K_{2j-1}^*(k) = (\mathbb{Z}/l^j \alpha_i)^{(l)} = \mathbb{Z}/l^{j+\alpha_i}$$

$$K_{2j-1}^*(K) \to K_{2j-1}^*(k) = \mathbb{Z}/l^{d_x + \alpha_j}$$

where $d_x = \epsilon(q - 1)$ or $\text{Im } x = (1 + l^{d_x})^x$. Consistent pattern.
"Special" cases include $p=2$, $\mu_4 \not\in F$ which we shall ignore and concentrate instead on $p$ odd and $\mu_p \not\in F$. The image of $X: \text{Gal} \rightarrow \mathbb{Z}_p^*$ is again cyclic; denote a generator of the image by $g$, and let $r$ be the least integer $> 0 \Rightarrow g^r \equiv 1 \pmod{p}$; $r \mid p-1$. We are assuming that $r > 2$.

\[ h^0(\mathbb{Z}/p\mathbb{Z}(j)) = 1 \iff j = 0 \pmod{r} \]
\[ h^2(\mathbb{Z}/p\mathbb{Z}(j)) = h^0(\mathbb{Z}/p(1-j)) = 1 \iff j = 1 \pmod{r} \]

and
\[ h^1(\mathbb{Z}/p\mathbb{Z}(j)) = [K: \mathbb{Q}_p] + h^0(\mathbb{Z}/p(j)) + h^2(\mathbb{Z}/p(j)) \]

where $h^i$ denotes the length of $H^i$.

\[ H^2(\mathbb{Z}_p(j)) = \lim_{n \to \infty} H^2(\mathbb{Z}/p^n(j)) \]

dual to \[ \lim_{n \to \infty} H^0(\mathbb{Z}/p^n(1-j)) \]

\[ = H^0(\mathbb{Q}_p/\mathbb{Z}_p(1-j)) \]

\[ \cong Z_\rho \cdot \frac{1}{q^{d-1}} \bigg/ Z_\rho \]

So

\[ H^2(\mathbb{Z}_p(j)) \cong \mathbb{Z}_\rho / (q^{d-1} - 1) \]
We have then

\[
H^1(\mathbb{Z}_p(j)) = \begin{cases} 
\mathbb{Z}_p / \theta^j - 1 + \mathbb{Z}_p & j \leq 0 \\
\mathbb{Z}_p [K : \mathbb{Q}_p] & j = 1 \\
\mathbb{Z}_p (K : \mathbb{Q}_p) & j > 1 \\
\mathbb{Z}_p [K : \mathbb{Q}_p] & j \geq 0, 1
\end{cases}
\]

so we have the following uniform formulas:

**Theorem:** Assume image of \( \chi : \text{Gal} \rightarrow \mathbb{Z}_p^* \) is cyclic generated by \( \theta \). Then for \( j > 1 \) we have

\[
\begin{align*}
H^1(\mathbb{Z}_p(j)) &\cong \mathbb{Z}_p / (\theta^j - 1) + \mathbb{Z}_p [K : \mathbb{Q}_p] \\
H^2(\mathbb{Z}_p(j)) &\cong \mathbb{Z}_p / (\theta^{j-1} - 1)
\end{align*}
\]

\[
\begin{align*}
H^1(\mathbb{Z}_p(1)) &\cong \mathbb{Z}_p / (\theta - 1) + \mathbb{Z}_p [K : \mathbb{Q}_p] + 1 \\
H^2(\mathbb{Z}_p(1)) &\cong \mathbb{Z}_p
\end{align*}
\]
so we conjecture in general that

\[ j > 1: \quad K_{2j}(F)_\rho \xrightarrow{(j+1)! c_{2j+1}^\#} H^2(Z_{p}(j+1)) = \mathbb{Z}/(q^{j+1}) \]

\[ j > 1: \quad K_{2j-1}(F)_\rho \xrightarrow{(j-1)! c_{2j}^\#} H^1(Z_{p}(j)) = \mathbb{Z}/(q^{j-1}) + \mathbb{Z}_{p}[K: Q_{p}] \]

are isomorphisms for \( j > 1 \). (I expect this anomaly in dimension 1 to disappear by taking the building without \( \det \) piece, and I hope that it is unnecessary to take the \( p \)-completion, i.e. that the groups \( K_i(F) \) except for the \( K_2(F) \) are already profinite.)

\[ \text{Idea:} \quad K_i(\mathcal{O}) \otimes \mathbb{Q} \rightarrow K_i(F) \otimes \mathbb{Q} \]

and \( K_i(\mathcal{O}) \otimes \mathbb{Q} \) can be computed via Lazard. In fact Lazard shows that the cohomology of \( GL_n(\mathcal{O}) \) tensored with \( \mathbb{Q}_p \) over \( \mathbb{Z}_p \) is the same as the cohomology of the Lie algebra which one knows via Koszul is an exterior algebra with generators of degrees \( 1, 3, \ldots, 2n-1 \). Therefore we see quite clearly that this can be made into a proof that

\[ K_{2j-1}^{\text{top}}(\mathcal{O}) \otimes \mathbb{Q} = F \]

\[ K_{2j}^{\text{top}}(\mathcal{O}) \otimes \mathbb{Q} = 0 \]
It is reasonable to expect the Hodge type classes to give these rational K-groups.
November 29, 1970.

Gran conjecture. Let \( l \) be a prime number which is invertible over \( A \). By the Kunneth formula

\[
H^* (\text{Spec} A, G; \mu^\otimes_i) = H^* (G, H^* (\text{Spec} A; \mu^\otimes_i))
\]

one can associate to each linear function \( \Lambda : H^* (\text{Spec} A; \mu^\otimes_i) \rightarrow \mathbb{Z} \) of cohomology class \( \Lambda (c_i (E)) \in H^{2i-1} (G, \mathbb{Z} / l \mathbb{Z}) \), if \( E \) is a representation of \( G \) over \( A \). The conjecture asserts that \( H^* (\text{GL}(A), \mathbb{Z} / l \mathbb{Z}) \) is generated by these Chern class components.

Thus if \( A \) is a strictly local ring the conjecture asserts that

\[
H^* (\text{GL}(A), \mathbb{Z} / l \mathbb{Z}) = \mathbb{Z} / l \mathbb{Z} [c_1, \ldots]
\]

In particular for \( A \) an algebraically closed field.

F field to fix the ideas. Then we get a "Rogul" group scheme affine over \( \mathbb{Z} / l \mathbb{Z} \)

\[
H^* (\text{Spec} S) = \text{rep. classes of rep. over } F \text{ coeff. in } S
\]

\[
= \text{Hom}_{\text{rep. g.p. functors}} ( R_F (?), H^* (? , S))
\]

\[
= \text{Hom}_{\text{Z/2Z-algs, graded anti-comm.}} ( H^* (\text{GL}(F)), S)
\]
Then to a field extension $u: F_1 \to F_2$ is associated extension of scalars:

$$
\begin{array}{ccc}
R_{F_1}(G) & \xrightarrow{u^*} & R_{F_2}(G) \\
\downarrow{u_*} & & \\
R_{F_2}(G) & \xleftarrow{u_*} & R_{F_1}(G)
\end{array}
$$

restriction of scalars:

If $u$ finite such that

$$u_*(u^* x) = [F_2:F_1] x$$

$$u^*(u_* y) = \sum_{\sigma \in Gal(F_2/F_1)} \sigma^* y \quad \text{if } F_2/F_1 \text{ is Galois.}$$

Then $u^*$ induces a homomorphism of group schemes which we denote

$$G_{F_2} \xrightarrow{u_*} G_{F_1}$$

and similarly $u_*$ for $u$ finite induces a homomorphism

$$G_{F_1} \xrightarrow{u_*} G_{F_2}$$

such that the same formulas hold.

Thus these group schemes behave covariantly with respect to $\mathcal{O}_{\text{Spec}(F)}$. 
Now the Galois cohomology functor assigns to $F$ the graded anti-commutative \( \mathbb{Z}/l \mathbb{Z} \)-algebra

\[ \bigoplus_{i \geq 0} H^{2i-k} (\text{Gal}(F_0/F), \mu_l^{\otimes i}) = C^*_k(F) \]

whose "affine spectrum" will be denoted $C_F$. The total Chern class $c = \sum_{i \geq 0} c_i$ is an exponential class hence gives a map

\[ C_F \longrightarrow \mathcal{S}_F \]

which is functorial in $\text{Spec } F$. There probably is little hope in connecting up this map with the restriction-of-sheaves without coming to grips with Chern classes of induced reps.
\[ F \] again a finite extension of \( \mathbb{Q}_p \). Assume \( \mu_l \subset F \) and \( l \neq p \) to fix the ideas.

Now I want to understand better my conjecture that the mod \( l \) cohomology of \( \text{GL}_n(F) \) is detected on \( (F^*)^n \). Let's begin by computing the subring of \( H^*(\mu_l)^n \) (mod \( l \) coeff.) generated by the Chern class components. Start with canonical isomorphism

\[ F^* / (F^*)^l \cong H^1(\mu_l) \]

Choose a uniformizing \( \pi \) and a generator \( v \) for \( \mu_{ld} \subset F \), where \( \mu_{ld} \) is the \( l \)-primary part of \( \mu_l(F) \), equivalently

\[ \text{Im} \, \chi = 1 + l^d \mathbb{Z}_l \subset \mathbb{Z}_l^* \]

where \( \chi : \text{Gal}(F/\mathbb{Q}) \rightarrow \mathbb{Z}_l^* \) is the Tate character. Then the images of \( \pi \) and \( v \) in \( F^*/(F^*)^l \) generate it.

Until we find a better notation, let \( c_1(\pi), c_1(v) \) denote the elements of \( H^1(\mathbb{Z}/l\mathbb{Z}) \) defined by the coboundary

\[ C \rightarrow F^* \xrightarrow{S} H^1(\mu_l) \rightarrow 0 \]

is \( H^1(\mathbb{Z}/l\mathbb{Z}) \)

together with isomorphism furnished by \( S \). Thus

\[ c_1(\pi) \cdot 1 = S(\pi) \quad c_1(v) \cdot 1 = S(v) \]
Moreover \( c_1(\pi), c_1(\mathcal{X}) \in H^2(\mathbb{Z}/\ell\mathbb{Z}) \cong \mathbb{Z}/\ell\mathbb{Z} \) is a generator of this group. Now there is a canonical map

\[ H^2(\mu_\ell) = \mathbb{Z}/\ell\mathbb{Z} \]

given by the invariant, we have to check that \( \mathcal{X} \) and \( \pi \) are independent. The point is that we have two elements

\[ \delta \pi, \delta \mathcal{X} \in H^2(\mu_\ell^3) \]

where \( \text{can} \) is the canonical class. Therefore having chosen \( \mathcal{X} \in \mu_\ell \) one wants to choose \( \pi, \mathcal{X} \) so that

\[ \delta \pi \cdot \delta \mathcal{X} = 1 \cdot (\text{can}) \]

(a bit confused)

Now we have the basis

\[ 1 \in H^0(\mathbb{Z}/\ell\mathbb{Z}) \]

\[ c_1(\pi), c_1(\mathcal{X}) \in H^1(\mu_\ell^3) \]

\[ c_1(\pi), c_1(\mathcal{X}) \in H^2(\mu_\ell^3) \]

and to save writing we put \( c_1(\pi) = \alpha, c_1(\mathcal{X}) = \beta \). As cup product is skew-symmetric we have

\[ \alpha^2 = \beta^2 = 0 \]

\[ \alpha \beta + \beta \alpha = 0 \]

Perhaps it is reasonable to make the connection...
with the Hilbert symbol. Thus if \( \mu_m = \mu(\mathbb{F}) \) one has
\[
(a, b) \in \mu_m
\]
defined by
\[
(a, b) \cap = \delta a \cup \delta b \in H^2(\mathbb{F}, \mu_m \otimes \mathbb{Q}_2)
\]
or perhaps using
\[
\forall : H^2(\mathbb{F}, \mu_m \otimes \mathbb{Q}_2) \to \mu_m \otimes (\mathbb{Q}_2)^{-1}
\]

to denote the canonical map, we have
\[
(a, b) = \forall \delta a \circ \delta b
\]

For the tame symbol one composed with the surjection
\[
\mu_m \longrightarrow (\mathfrak{p} \text{-resfd})^*
\]
and it's known then that
\[
(a, b) = (-1)^{v(a) v(b)} \frac{a}{b^{v(a)}} \quad \text{reduced mod } \pi.
\]

So now that we understand the Galois cohomology, we can investigate the Chern classes.
Review of Kummer theory in the appropriate way.

Suppose \( \mu_m \subset F \), \( (m, \text{char } F) = 1 \). Then there is a canonical element

\[
c_1 \in H^2(\mathbb{F}^* \times \text{Gal}(\overline{F}/F), \mu_m)
\]

which is the coboundary of the homomorphism

\[
\mathbb{F}^* \times \text{Gal}(\overline{F}/F) \rightarrow \mathbb{F}^* \subset \mathbb{F}^*
\]

for the exact sequence

\[
0 \rightarrow \mu_m \rightarrow \mathbb{F}^* \rightarrow [m] \mathbb{F}^* \rightarrow 0.
\]

I want to generalize my calculation for a finite field to the general case.

**Proposition:** Let \( \eta \in H^2(\mathbb{F}^*, \mu_m) \) be the canonical element represented by the extension

\[
0 \rightarrow \mu_m \rightarrow (\mathbb{F}^*)^{\mu_m} \xrightarrow{[m]} \mathbb{F}^* \rightarrow 0
\]

(\( (\mathbb{F}^*)^{\mu_m} = \{\eta \in \mathbb{F}^* \mid \eta^m \in F \}\)). Equivalently, \( \eta \) is the geometric first Chern class of the canonical representation of \( \mathbb{F}^* \). Let \( K \) be the Kunneth isomorphism

\[
K : H^1(\mathbb{F}^*, H^1(\text{Gal}, \mu_m)) \rightarrow H^2(\mathbb{F}^* \times \text{Gal}, \mu_m)
\]

which in good cases, would be the composite...
Let \( c_{an} \in H^1(F^*, H^1(\text{Gal}, \mu_m)) \) be the canonical element furnished by the Hilbert map

\[
F^* \longrightarrow F^*(F^*)^m \longrightarrow H^1(\text{Gal}, \mu_m)
\]

Then the arithmetic Chern class is the sum

\[
c_1 = u + K(c_{an}).
\]

Proof is essentially obvious. As in your finite fields paper, the class \( c_1 \) is represented by the cocycle

\[
(\delta h')(g_1, x_1, g_2, x_2) = s(g_2)^{x_1 - 1} \frac{(\delta s)(g_1, g_2)}{u}
\]

where \( s(g_2)^e = g_2 \in F^* \). Thus

\[
s(g_2)^{x_1 - 1} = (v g_1)^{x_1 - 1}
\]

is the Hilbert pairing of \( \text{Gal} \) and \( F^* \) with values in \( \mu_m \).

Now we want to use this in our calculations when \( F \) is a local field, and \( m = e \) is \( \neq p \). Then
We choose \( \mathfrak{g} \) as a generator of \( H_1(\text{Gal}(\mu), \mathbb{Z}) \) and \( \mathfrak{h} \) the dual basis.

Actually we choose \( \mathfrak{g} \) and \( \mathfrak{h} \) dual bases of \( H_1(\text{Gal}(\mu), \mathbb{Z}) \).

We choose a generator \( F^{*}\mathfrak{g} \in H_2(\text{Gal}(\mu), \mathbb{Z}) \) and it is clear that \( F^{*}\mathfrak{g} \) is the dual basis of \( \mathfrak{g} \).

Since \( x, \mathfrak{g} \in H_2(\text{Gal}(\mu), \mathbb{Z}) \) and \( x, \mathfrak{h} \in H_1(\text{Gal}(\mu), \mathbb{Z}) \), we have the following diagram:

\[
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\cdot x \cdot \mathfrak{g}} & \mathbb{Z} \\
\uparrow & & \downarrow \\
\mathbb{Z} & \xrightarrow{\cdot x \cdot \mathfrak{h}} & \mathbb{Z}
\end{array}
\]

where the vertical arrows are the dual bases.

And the element which is the image of the geometric Chern class is \( \sigma \in H^2(\text{Gal}(\mu), \mathbb{Z}) \xrightarrow{\cdot (\mathfrak{h} \cdot x)} H_2(\text{Gal}(\mu), \mathbb{Z}) \).

As we are assuming \( \mu \subset F \) and have a generator \( \mathfrak{h} \) we are looking at the element.

The identity map the two spaces being naturally dual.
\[ c_1 = u + \hat{\alpha} \cdot \alpha + \hat{\beta} \cdot \beta \in H^2(F) \oplus H^1(F^*) \otimes H^1(\text{gal}) \]

Now recall

\[ H^*(F^*) = \mathbb{Z}/l [\hat{\alpha}, \hat{\beta}, u] \]

\[ H^*(\mathbb{F}^*)^n = \mathbb{Z}/l [\hat{\alpha}_i, \hat{\beta}_i, u_i] \]

and the total Chern class of the standard representation is

\[ \prod_{i=1}^n \left( 1 + u_i + \hat{\alpha}_i \cdot \alpha + \hat{\beta}_i \cdot \beta \right) \]

so what you need to recognize the subring of \( H^*(\mathbb{F}^*)^n \) generated by the coefficients of the various elements 1, \( \alpha \), \( \beta \), \( \alpha \beta \).
November 30, 1970. My education in group theory:

**Sylow theorem:**

\[ N = \text{Norm}(Z) \text{ in } G. \]  

Form the categories of \( p \text{-groups} \) in the usual way (morphisms are homomorphisms).

Then if \( G \) is \( p \text{-normal} \):

\[ Z < gPg^{-1} \implies Z = gZg^{-1}, \]

we have

\[ \text{Cat}(N) \longrightarrow \text{Cat}(G) \]

is an equivalence of categories.

**Proof.** One has to show that if \( g^{-1}Mg < P \) and \( M < P \), then \( \exists \ Y \in \text{Cent}(M) \) \( \implies Yg \in \text{Norm}(Z) = N \).

But

\[ M < gPg^{-1} \implies gZg^{-1} \subset \text{Cent}(M) \]

\[ M < P \implies Z \subset \text{Cent}(M) \]

and both being \( p \text{-groups} \) \( \exists Q \subset \text{Cent}(M) \) Sylow \( p \text{-subgroups} \) and \( Q \supset Z \), \( Z < Q \), \( gZg^{-1} \subset Y^{-1}QY \) so

\[ Z, YgZ(Yg)^{-1} \subset Q \subset \text{Sylow grp of } G. \]

Hyp.

\[ Z = YgZ(Yg)^{-1} \]

so we are done.

---

**Remark 1:** One must work in \( \text{Cent}(M) \) in order to keep category unchanged. Possibly in \( M \cdot \text{Cent}(M) \), but there is no diff. because \( M \subset N \).

It is enough to have \( Z \) central in \( P \) and "weakly closed" i.e. \( g^{-1}Zg < P \implies g^{-1}Zg = Z \).
Corollary: If $N$ is the normalizer of a central weakly closed subgroup $Z$ of $P$, then

$$H^*(G) \rightarrow H^*(N)$$

($p$-torsion cohomology). Hence $G$ has a normal $p$-complement iff $N$ does (by Tate or Frobenius).

So now if $G$ is a group such that $P$ and $G$ have same categories of elementary abelian $p$-subgroups, then take $Z$ to be the elements of order $p$ in the center of $P$. Then $Z$ is central and weakly closed for there is no fusion of elementary abelian $p$-subgroups. So by Gr"{u}nn we can replace $G$ by $N$ if we wish to show $p$-nilpotence. Hence can assume $Z \triangleleft G$, whence $Z$ is central in $G$.

Now can divide out by largest normal $p$-subgroup, whence can assume $\text{Center}(G) = \text{Center}(P)$. In more detail, the $p'$ part of $\text{center}(G)$ is $\text{Center}(P)$, hence $\text{Center}(G) \leq \text{Center}(P)$, and any $p'$-element of $\text{Norm}\{\text{Center}(P)\}$ acts trivially on $Z$, the "bottom" of $\text{Center}(P)$.

Hence $\text{Norm}\{\text{Center}(P)\} = (\text{Center}(P))^P$.

So $Z = \text{Z}(G) < \text{Z}(G) < \text{Z}(P)$. 

Proof that a maximal normal elem. ab. subgroup is maximal elem. abelian:

Let $\Theta$ be an automorphism of an abelian $p$-group $A$ such that (i) $\Theta^p = 1$ (ii) $(\Theta - 1)$ kills $\Omega_1 A$. Then if $p$ is odd $(\Theta - 1) A \leq \Omega_1 A$.

Proof: $1 = (1 + (\Theta - 1))^p$ \\

So $p(\Theta - 1) + \binom{p}{2}(\Theta - 1)^2 + \cdots + (\Theta - 1)^p = 0$.

Let $d$ be largest such that $(\Theta - 1) \Omega_d A \subseteq \Omega_1 A$ assuming that $(\Theta - 1) A \not\subseteq \Omega_1 A$. Then $(\Theta - 1) \Omega_{d+1} A \not\subseteq \Omega_1 A$

so $\exists z \in \Omega_{d+1} A$ with $p(\Theta - 1)z \neq 0$, $p^2(\Theta - 1)z = 0$.

Now $(\Theta - 1) \Omega_1 A = 0$ $\Rightarrow$ $(\Theta - 1) \Omega_{d+1} A \subseteq \Omega_d A$ \\
$\Rightarrow$ $(\Theta - 1)^2 \Omega_{d+1} A \subseteq \Omega_1 A$ \\
$\Rightarrow \begin{cases} p(\Theta - 1)^2 \Omega_{d+1} A = 0 \\
(\Theta - 1)^3 \Omega_{d+1} A = 0 \end{cases}$ as $p \geq 3$

Hence $p(\Theta - 1)z = 0$ a contradiction.

In brief: 

$p(\Theta - 1) \Omega_d A = 0$ $\Rightarrow$ $(\Theta - 1) \Omega_d A \subseteq \Omega_1 A$ \\
$\Rightarrow$ $(\Theta - 1)^2 \Omega_d A = 0$ \\
$\Rightarrow$ $(\Theta - 1)^2 \Omega_{d+1} A \subseteq \Omega_1 A$ \\
$\Rightarrow$ $(\Theta - 1)^3 \Omega_{d+1} A = 0$
Hence have \((\Theta - 1)^2\) and \(p(\Theta - 1)^2\) kill \(\Omega_{d+1}A\), so by identity
\[ C = \Theta^p - 1 = (1 + (\Theta - 1))^p - 1 \equiv p(\Theta - 1) \mod (p(\Theta - 1)^2) \]
I have
\[ p(\Theta - 1)^{d+1} = 0. \]
Thus by induction conclude that \(p(\Theta - 1)A = 0\).

Next step: A maximal normal abelian subgroup of the p-group \(P\), \(p\) odd. Claim that the elements of order 1 or \(p\) in \(\text{Cent}_p(A)\) form a subgroup. Suppose not and let \(x, y \in \text{Cent}_p(A)\) be such that \(x^p = y^p = 1\) but \((xy)^p \neq 1\) and such that \(<x, y>\) has least possible order. Then as \(<x, y>\) is not cyclic \(<y^{-1}x^{-1}yx, x^{-1}yx> < <x, y> \) so \(y^{-1}x^{-1}yx\) has order \(p\).

But \(x, y\) stabilize \(A \supset \Omega_1A \supset 1\) hence the commutator \(y^{-1}x^{-1}yx\) centralizes \(A\), so as \(A\) is maximal \(y^{-1}x^{-1}yx \in \Omega_1A\). Hence \(<x, y>\) is of class 2 so as \(p\) is odd every element of \(<x, y>\) is of order \(p\), a contradiction. This proves claim. We have proved:

Thus \(\Omega_1\text{Cent}_p(\Omega_1A)\) is of exponent \(p\) and if \(\Omega_1A\) is not a maximal elementary abelian group properly enlarge it to a maximal elementary abelian subgroup.

Prof. If \(A\) is a maximal normal abelian subgroup of \(P\), \(p\) odd, then \(\Omega_1\text{Cent}_p(\Omega_1A)\) is of exponent \(p\). Consequently if
$\Omega_1 A$ is a max. normal elem. ab. subgp. of $P$; then $\Omega_1 A$ is a max. elem. ab. subgp. of $P$.

Corollary. Any max. norm. elem. ab. subgp. of a $p$-gp, $p$ odd, is a max. elem. ab. subgroup.

(These are the analogues of the Klein group in $(\mathbb{Z}/p)^n$)

If $B$ is a max. norm. ab. subgroup, choose $A$ max. norm. ab. $A > B$. Then $B = \Omega_1 A$ so

$$\Omega_1 \text{Cent}_P (B) = B$$

by the preceding proposition. So $B$ is max elem. ab.

On the contrary the normalizer of the Klein group $(\mathbb{Z}/p)^n < \text{Syl}_p (\Sigma_p^n)$ has order

$$p^{n(n-1)/2} \prod_{i=1}^{n} (p^{i-1})$$

so isn't normal in $P$. The group $(\mathbb{Z}/p)^{p^{n-1}} c (\Sigma_p^n) c \Sigma_p^n$ has normalizers $\Sigma_{p^{n-1}}, S \mathbb{Z}/p$ whose order has

$$\nu_p \left( | \Sigma_{p^{n-1}}, S \mathbb{Z}/p | \right) = p^{n-1} + \nu_p (p^{n-1}) = p^{n-1} + \ldots + 1$$

$$= \nu_p (p^n)$$

and so it is normal in some Sylow group.