Attempt to see what char classes of $GL_\infty(\mathbb{C})$ (discrete top) are ~~detected~~ by finite grps, e.g. abelian l-groups.

November 4, 1970:

I would like to understand better something that came up in conversation with John Mather. The problem is to compute $H^*(GL_\infty(\mathbb{R}), \mathbb{Z}/l\mathbb{Z})$ where $GL_\infty(\mathbb{R})$ has the discrete topology. Elements of this are characteristic classes which associate to a flat bundle $E \longrightarrow X$ a coh. class $u(E) \in H^*(X)$ in a natural way and which are stable: $u(E+\theta) = u(E)$.

Now the idea is to consider the restriction mapping from such classes to classes for flat bundles with finite structural group. In precise terms I want

$$\operatorname*{Hom}_{G \in \text{fin. gp.}}\left(\bar{R}_{\mathbb{R}}(G),\ H^*(G, \mathbb{Z}/l\mathbb{Z})\right)$$

and to compare this with

$$\operatorname*{Hom}_{G \in \text{gps.}}\left(\bar{R}_{\mathbb{R}}(G),\ H^*(G, \mathbb{Z}/l\mathbb{Z})\right)$$

which I know to be

$$\| $$

$$H^*(GL_\infty(\mathbb{R}), \mathbb{Z}/l\mathbb{Z})$$

by your deep(?) theorems.

Change $\mathbb{R}$ to $\mathbb{C}$. Now I ~~know that~~ can compute the first group, I think. Choose $p \neq l$ and let $k = \bar{\mathbb{F}}_p$, $\phi: k^* \hookrightarrow \mathbb{C}^*$. Then we get maps

$$R_k(G) \underset{\phi}{\overset{d}{\rightleftarrows}} R_{\mathbb{C}}(G)$$

(ϕ defined all groups G but d only for finite gps.)

making $R_k(G)$ a natural direct summand of $R_{\mathbb{C}}(G)$ in a

Lemma: Let $F : (\text{fin. gps.})^\circ \longrightarrow \text{sets}$. Then

$$\mathrm{Hom}_{\text{fin. gps.}}(F, H^*) \xrightarrow{\sim} \mathrm{Hom}_{\ell\text{-gps.}}(F, H^*).$$

Proof:

$$\begin{tikzcd}
F(G) \arrow[r] \arrow[d, dashed] & \varprojlim\limits_{P \to G} F(P) \arrow[d] \\
H^*(G) \arrow[r, "\sim"] & \varprojlim\limits_{P \to G} H^*(P)
\end{tikzcd}$$

where P runs over the category of ℓ-groups.

Influence of stability: see if you can use your good theorem. Now use structure provided by the Künneth formula, so I restrict attention to multiplicative ops. $\theta : \bar{R}_A \longrightarrow H^0(\ , S.)$

Unfortunately the restriction map to elementary abelian ℓ-subgps. ℓ is not injective because over a finite field $A = \mathbb{F}_q$ one has stable classes (the c_i'') which vanish on elementary ℓ-subgroups

Question: Can you compute $\mathrm{Hom}_{\text{abel. } \ell\text{-gps.}}(\bar{R}_A, H^*)$?

way compatible with products + λ-operations.
So what I'm getting at is the maps

$$BGL(k)^+ \longrightarrow BGL(\mathbb{C})^+ \longrightarrow B^{top}GL(\mathbb{C})$$

and I know this composition induces an isomorphism on cohomology for all $l \neq p$.

$$\begin{tikzcd}
\operatorname{Hom}_{G \text{ finite}}\left(\bar{R}_{\mathbb{C}}(G), H^*(G)\right) \arrow[r, shift left, "\phi^*"'] \arrow[d, hook] & \operatorname{Hom}_{G \text{ fin.}}\left(\bar{R}_k(G), H^*(G)\right) \arrow[l, shift left, "d^*"'] \arrow[d, hook] \\
\operatorname{Hom}_{G\ l\text{-gp}}\left(\qquad \right) \arrow[r, equal] & \operatorname{Hom}_{G\ l\text{-gp}}\left(\qquad \right)
\end{tikzcd}$$

This diagram shows that ϕ^* must be injective, hence $\bar{R}_{\mathbb{C}}(G)$ for G finite same as $\bar{R}_k(G)$ as far as cohomology mod l goes.

General principle: Let A be a ring. Then I can consider either natural transformations from $\bar{R}_A(G)$ to $H^*(G)$ where G is all gps, or all finite groups.

$$\begin{tikzcd}
\operatorname{Hom}_{\text{gps.}}(\bar{R}_A, H^*) \arrow[r] & \operatorname{Hom}_{\text{fin. gps.}}(\bar{R}_A, H^*) \arrow[d, hook, "\simeq \ \text{(see below)}"] \\
& \operatorname{Hom}_{l\text{-gps.}}(\bar{R}_A, H^*) \arrow[d] \\
& \operatorname{Hom}_{\text{abelian } l\text{-gps.}}(\bar{R}_A, H^*)
\end{tikzcd}$$

~~Suppose G finite and Λ is a Dedekind domain such that l is a unit. Then an~~

~~$B_\Lambda(G) \otimes_{\Lambda[l^{-1}]}$~~

$$\mathrm{Hom}_{\text{abel } l\text{-gps.}}(\bar{R}_\Lambda, H^*) = ?$$

Suppose l is a unit in Λ. and let $G \longrightarrow \mathrm{Aut}(E)$ be a representation of an abelian l-group where E is a finitely generated projective A-module. Suppose E indecomposable, and for simplicity that Λ is the ring of S-units in a number field K. Then $E \otimes_\Lambda K \cong K[\mu_{l^n}]$ where $\chi : G \longrightarrow \mu_{l^n}$ is surjective. So the invariants of E consist of this character, which is unique up to the Galois ~~[illegible]~~ group and the class of the invertible $\Lambda[\mu_{l^n}]$ ideal E in $E \otimes_\Lambda K = K[\mu_{l^n}]$. Thus it seems that

$$R_\Lambda(G) = \text{free abelian group on}$$

NO

$$\coprod_{n \geq 0} \left(\mathrm{Hom}_{surj}(G, \mu_{l^n}) \times \mathrm{Pic}\, \Lambda[\mu_{l^n}]\right) / \mathrm{Gal}$$

$$= \varinjlim \mathrm{Hom}(G, \mu_{l^n}) \times \mathrm{Pic}\, \Lambda[\mu_{l^n}]$$

where the limit is taken over the category of fields $K[\mu_{l^n}]$ and where Pic moves covariantly, i.e.

5

given $K[\mu_{\ell^m}] \longrightarrow K[\mu_{\ell^n}]$ we take the induced map

$$\mathrm{Pic}\,\Lambda[\mu_m] \longrightarrow \mathrm{Pic}\,\Lambda[\mu_n].$$

If you take $\Lambda = K$ so that $\mathrm{Pic} = 1$ then this is

$$\mathrm{Hom}(G, \mu_{\ell^\infty})/_{Gal}$$

and the homology of this functor is

$$H_*(\mu_{\ell^\infty})/_{Gal}$$

which if $\mu_\ell \subset \Lambda$ is a divided power algebra with one generator of degree 2. So we've made a mistake.

So try $\Lambda = K$. Then $R_\Lambda(G) \xrightarrow{\sim} R_{\bar\Lambda}(G)^{Gal}$ where $\bar\Lambda = \Lambda[\mu_{\ell^\infty}]$, and $R_{\bar\Lambda}(G) = \mathbb{Z}[\hat{G}]$. Our mistake is in **identifying** $\mathbb{Z}[\hat{G}/Gal]$ and $\mathbb{Z}[\hat{G}]^{Gal}$ in the obvious way which is not functorial in G.

Here's how **to** rectify things. Again write $R_\Lambda(G)$ as a quotient of

$$\mathbb{Z}\Big[\coprod_{n\geq 0} \big(\mathrm{Hom}(G,\mu_{\ell^n}) \times \mathrm{Pic}\,\Lambda[\mu_{\ell^n}]\big)_{Gal}\Big]$$

but now one must identify

$$\begin{array}{ccc} \mathbb{Z}[\mathrm{Hom}(G,\mu_{\ell^m}) \times \mathrm{Pic}\,\Lambda[\mu_{\ell^n}]] & \xrightarrow{\text{induced by } \mu_{\ell^m} \to \mu_{\ell^n}} & \mathbb{Z}[\mathrm{Hom}(G,\mu_{\ell^n}) \times \mathrm{Pic}\,\Lambda[\mu_{\ell^n}]] \\ \downarrow & & \\ \mathbb{Z}[\mathrm{Hom}(G,\mu_{\ell^m}) \times \mathrm{Pic}\,\Lambda[\mu_{\ell^m}]] & & \end{array}$$

The vertical map sends $\chi: G \to \mathrm{Aut}_\Lambda(E)$ (with χ factoring through μ_{ℓ^m}) E an invertible $\Lambda[\mu_{\ell^m}]$-module into E regarded as a sum of invertible $\Lambda[\mu_{\ell^m}]$ invertible modules. You are being stupid about the Picard group also. The point somehow is that

$$\mathbb{Z}[\mathrm{Hom}(G,\mu_{\ell^n})] \otimes K_0(\Lambda) \xrightarrow{\sim} R_\Lambda(G)$$

if $\mu_{\ell^n} \subset \Lambda$ and exponent of G divides ℓ^n. So now I want to write $R_\Lambda(G)$ as a quotient:

$$\bigoplus_{n\geq 0} \mathbb{Z}[\mathrm{Hom}(G,\mu_{\ell^n})] \otimes K_0(\Lambda[\mu_{\ell^n}]) \twoheadrightarrow R_\Lambda(G)$$

and it should be true that the equivalence relation is of form

$$\bigoplus_{u:\Lambda[\mu_{\ell^m}]\to\Lambda[\mu_{\ell^n}]} \mathbb{Z}[\mathrm{Hom}(G,\mu_{\ell^m})] \otimes K_0(\Lambda[\mu_{\ell^n}]) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \bigoplus_n \mathbb{Z}[\mathrm{Hom}(G,\mu_{\ell^n})] \otimes K_0(\Lambda[\mu_{\ell^n}])$$

where p_1 is induced by the given map $u: \mu_{\ell^m} \to \mu_{\ell^n}$, and where p_2 is induced by the norm

$$u_*: K_0(\Lambda[\mu_{\ell^n}]) \longrightarrow K_0(\Lambda[\mu_{\ell^m}])$$

~~Now for simplicity assume Λ such that $K_0(\Lambda[\mu_{\ell^n}])$~~ ~~$= \mathbb{Z}$, whence the norm maps are multiplication by degrees. The~~ ~~e.g. $\Lambda = \ldots$~~ So I think it should follow from the above formula for $R_\Lambda(G)$, assuming its true, that the multiplicative transformations we are interested in are simply families of homomorphisms

$$v_n : K_0(\Lambda[\mu_{\ell^n}]) \longrightarrow H^0(\mu_{\ell^n}, S_\cdot)^\times$$

which are compatible with restriction on the right and norms on the left and Galois.

Now $\qquad\qquad s_0' = 1, \; s_0'' = 0$

$$H^0(\mu_{\ell^n}, S_\cdot)^\times = (S_\cdot[y, x])^\times = \Big\{\Big(\sum_{i\geq 0} s_i' x^i + s_i'' x^{i-1} y\Big)\Big\}.$$

Suppose for simplicity that all the K-groups are $\mathbb{Z}$. and take g to be a generator the image of the Galois gp in $\mathbb{Z}_\ell^* = \mathrm{Aut}\,[\mu_{\ell^\infty}]$. I assume $g \equiv 1$ mod ℓ and put $a = v_\ell(g-1)$. Then the degree of $K[\mu_{\ell^n}]$ over K is least $r \ni \ell^n \mid g^r - 1$. But

$$n = v_\ell(g^r - 1) = v_\ell(r) + v_\ell(g-1) = v_\ell(r) + a$$

so degree $[K[\mu_{\ell^n}] ; K] = n - a$. So what happens is that

$$\begin{tikzcd}
K_0(\Lambda[\mu_{\ell^{a+2}}]) \arrow[r, phantom, "="] \arrow[d, "\ell"] & \mathbb{Z} \arrow[d, "\ell"] & \\
K_0(\Lambda[\mu_{\ell^{a+1}}]) \arrow[r, phantom, "="] \arrow[d, "\cdot\ell"] & \mathbb{Z} \arrow[r, "u_{a+1}"] \arrow[d, "\ell"] & H^0(\mu_{\ell^{a+1}}, S) \arrow[d] \\
K_0(\Lambda[\mu_{\ell^a}]) \arrow[r, phantom, "="] \arrow[d, equal] & \mathbb{Z} \arrow[r, "u_a"] \arrow[d, equal] & H^0(\mu_{\ell^a}, S) \arrow[d, "\mathrm{res}"] \\
K_0(\Lambda) & \mathbb{Z} \arrow[r, "u_{a-1}"] & H^0(\mu_{\ell^{a-1}}, S_\cdot)
\end{tikzcd}$$

compatibility conditions $\qquad u_a^\ell = \mathrm{res}\; u_{a+1}$

Unfortunately

$$\left(\sum s_i' x^i + s_i'' x^{i-1} y\right)^\ell = \sum (s_i')^\ell (x^i)^\ell$$

so it seems that there should be too many possibilities for the s_i'' and none for s_i'. ??

This is confusing but perhaps not incorrect. The point is that you don't really have very much control over abelian group of exponent $> \ell^q$. Precisely: suppose $\Lambda = \mathbb{F}_q$, $v_\ell(q-1) = a \geq 1$. I define a map $\Phi : R_\Lambda(G) \longrightarrow H^1(G)$ for all abelian ℓ-group. Given an irreducible repn: $\chi : G \twoheadrightarrow \mu_{\ell^b}$ well-determined up to Galois we associate the class $G \twoheadrightarrow \mu_{\ell^b} \xrightarrow{\ell^{b-1}} \mu_\ell$ if $b = a+1$ and zero otherwise. This defines Φ for each G; I claim its natural: Suppose have $G_1 \to G$ and $\chi : G \twoheadrightarrow \mu_{\ell^b}$. Cases: 1) $b = a+1$ and $G_1 \to G \to \mu_{\ell^b}$ onto; then OKAY. 2) $b > a+1$ and $G_1 \to G \to \mu_{\ell^b}$ has image $\mu_{\ell^{a+1}}$; in this case the representation of G_1 is ℓ^{b-a-1} times the $\mu_{\ell^{a+1}}$-repn of G_1, so both χ and $\Phi(\chi)$ pull-back to zero. 3) $G_1 \to G \to \mu_{\ell^b}$ not surjective with image not $\mu_{\ell^{a+1}}$; then both $\Phi(\chi)\,|\,G_2 = 0$ and $\Phi(\chi\,|\,G_2) = 0$.
So it seems that the above is OKAY, and that we should not consider natural transformations on abelian ℓ-groups.

November 5, 1970

Fix a field Λ, so that we have good control over $R_\Lambda(G)$ for G finite. Precisely, first suppose Λ of characteristic $p>0$. ~~when p~~ Let $\Lambda_0 \subset \Lambda$ be the largest subfield algebraic over $\mathbb{F}_p$. Then I claim that

$$R_{\Lambda_0}(G) \xrightarrow{\sim} R_\Lambda(G) \qquad G \text{ finite.}$$

Indeed I think we can show that

$$\begin{tikzcd}
R_{\Lambda_0}(G) \arrow[rr] \arrow[d, "\cong"] & & R_\Lambda(G) \arrow[d, "\cong"] \\
R_{\overline{\mathbb{F}}_p}(G)^{Gal(\overline{\mathbb{F}}_p, \Lambda_0)} \arrow[rr, "\cong"'] & & R_{\Lambda\overline{\mathbb{F}}_p}(G)^{Gal(\Lambda\overline{\mathbb{F}}_p, \Lambda)}
\end{tikzcd}$$

The point somehow is that the extensions $\Lambda_0 \to \overline{\mathbb{F}}_p$, $\Lambda_0 \to \Lambda$ and $\Lambda \to \Lambda\overline{\mathbb{F}}_p$ are separable, hence the only thing that can happen to an irreducible representation is that it splits into irreducibles corresponding to its endomorphism field splitting.

$$\Lambda_0 \longrightarrow \Lambda$$

E irred $\Lambda_0[G]$-module, $\quad \mathrm{End}(E) = \Lambda_1$

then Λ_1 is commutative, and by Galois nonsense $\Lambda_1 \cdot \Lambda$ is a field. Clear more or less.

November 13+14, 1970

Toward an understanding of Thompson's theorem.

G finite group. G_p a Sylow p-subgroup.

$J = J(G_p) =$ subgroup generated by abelian subgroups with maximal no. of generators.

$$\operatorname{rank}({}_pA) = \dim(G)$$

Thompson's hypothesis: $N_G(J)$ has normal p-complement

Idea: This implies that there is no fusion:

Suppose G has a normal p-complement

$$G = P \ltimes K$$

P = Sylow p-subgroup
K p'-Hall gp, normal.

$$\begin{tikzcd} P \arrow[r, hook] \arrow[dr, "\cong"'] & G \arrow[d] \\ & G^{\wedge} \end{tikzcd}$$

Then there is no fusion for subsets of P. Indeed if $S_1, S_2 \subset P$ and $g^{-1}S_1 g \subset S_2$, then write $g = kp$ and if $a \in S_1$, then

$$g^{-1}ag = p^{-1}k^{-1}akp \in P$$

$$\Rightarrow \underbrace{a^{-1}k^{-1}ak}_{\substack{\shortparallel \\ 1}} \in P \cap K \qquad (K \triangleleft)$$

$\Rightarrow$ k centralizes S_1 and $p^{-1}S_1 p \subset S_2$.

Consequently if we make the subsets of p-elements of G into a category, in which a morphism from S_1 to S_2 is a coset $C(S_1)g$, $g^{-1}S_1g < S_2$, then the categories for G + P are equivalent.

Conversely ~~Burnside~~ if there is no fusion of p-subgroups, then G has a normal p-complement.

Let's try p-subgroups: Idea is to show $H^*(G) \xrightarrow{\sim} H^*(P)$ and then apply Tate thm:

Start $g \in G$; we want to show that the two arrows

$$H^*(P) \rightrightarrows H^*(P \cap gPg^{-1})$$

~~coincide.~~

~~However we can consider the groups $Q = P \cap gPg^{-1}$ and P as objects of the category of p-subgroups of G, and then apply~~

~~However the two subgroups $P \cap gPg^{-1}$ and $g^{-1}Pg \cap P$ of P are conjugate in G, hence conjugate in P~~

coincide. But if $Q = P \cap gPg^{-1}$, then $Q < P$ and $g^{-1}Qg < P$, hence we have $g = kp$ where k centralizes

Q, so the map $g \mapsto g^{-1}qg$ from Q to P is the same as the map $q \mapsto p^{-1}qp$ from Q to P, whose effect on cohomology is the same as the inclusion $q \mapsto q$.

I think it's also true that if there is no fusion of elements of P, then P has a normal p-complement. My reason for believing this is Atiyah's claim that G has a normal p-complement when one knows that ~~[illegible]~~ the map

$$\text{Conj. classes elts. of } P \longrightarrow \text{Conj. classes } p\text{-elts. of } G$$

is bijective, that is when $gxg^{-1} = y$ for $x, y \in P$ $\Longrightarrow pxp^{-1} = y$ some P.

(Does it follow from this that one can choose p so that gp^{-1} is a p'-element?)

First case: $P \triangleleft G$. Then I want to prove that if $g = pk = kp$ is the decomposition of g into its p-regular & p-singular parts, then $g \mapsto p = g_s$ is a homomorphism. However here if p is a p-elt + k is a p'-element, then they commute. Indeed $k^{-1}pk \in P$ and as ~~they~~ there is no fusion $k^{-1}pk = p_0^{-1}pp_0$. Thus $p_0k^{-1} \in \text{Cent } p$, so maybe can use induction?

Then one we define a map $G \longrightarrow P$ in general by sending g ??

Conjecture: No fusion for $[p]$-subgroups $\Longrightarrow$ normal p-complement

On normal p complements:

By ~~Frobenius's~~ Frobenius's theorem one has to understand p'-automorphisms of p-groups. So the ~~[illegible]~~ question is to find manageable criteria which guarantee that a p'-automorphism θ of a p-group P is trivial. This leads to

Problem: Determine those pairs (θ, P) where θ is a p'-automorphism of a p-group P ~~[illegible]~~ which are minimal, i.e. any θ-stable $P' < P$ is such that θ acts trivially on P'.

This forces $\Gamma_2^{(p)}P$ to be trivial and $gr_1 P = P/\Gamma_2^{(p)}P$ to be an irreducible representation of $\mathbb{Z}/m\mathbb{Z}$ ~~[illegible]~~ with generator θ of order m. Now $gr_2 P$ is a quotient of $\Lambda^2 gr_1 P$ (p odd) which by Schur's lemma has ~~[illegible]~~ at most one invariant. ~~[illegible]~~ Thus $\dim gr_2 P \leq 1$ and $gr_3 P$ being a quotient of

not alg closed

$$gr_1 P \otimes gr_2 P$$

will necessarily be zero. Thus P will either be abelian or an extra-special p-group without an element of order p^2 because the ~~squaring~~ pth operation is linear from $gr_1 P$ to $gr_2 P$ when P is odd.

When $p=2$ $gr_2 P$ is a quotient of $S_2(gr_1 P)$ which again will be one-dimensional by Schur's lemma and again $gr_3 P$ will be zero so again P will be

~~So go back to the original question and let Θ be a p'-auto of an p-group P. Now form a composition series stable under Θ~~

~~$0 \subset N_1 \subset N_2 \subset \cdots$~~

~~so N_1 is a minimal Θ-stable subgroup of P and N_2 is a minimal Θ-stable subgroup of P/N_1 etc. According to what we have just proved either N_1 or N_2 is non-trivial under Θ.~~ NO

So now given a p'-automorphism Θ of a p-group P let M be a minimal Θ-stable subgroup of P on which Θ acts non-trivially. If fact we should argue this way: Let M be a minimal member of the set of normal Θ-stable subgroups on which Θ acts non-trivial. Then subgp $[P,M]M^{(p)} \lneq M$ and is stable under Θ and normal in P, so has trivial Θ action. Any subgroup M' of M containing $[P,M]$ will be normal in P, hence conclude that Θ acts irreducibly on $M/[P,M]M^{(p)}$?

If M is a minimal Θ-stable subgroup, then for p odd at least Θ moves the $[p]$-subgroups of M around because every element is of order p. Now all I have to do is to see somehow that these motions won't be conjugate within P.

~~Let N_1 be a minimal~~

Let θ be a p'-auto of a p-group P and form a series

$$0 \subset N_1 \subset \cdots$$

where N_1 is a minimal normal subgroup of P stable under θ, N_2/N_1 ———————— P/N_1 ——— under θ, etc. Then $V_i = N_i/N_{i+1}$ is a representation of $(\theta) \tilde{\times} P$ over $\mathbb{Z}/p\mathbb{Z}$ which is irreducible. Claim P acts trivially on V. Indeed if I is the augmentation ideal of $\mathbb{Z}/p\mathbb{Z}[V]$, then $> I^2 V > \cdots$ is a $(\theta)\tilde{\times}P$ - filtration and $I^n V = 0$ for n large as V is a p-gp. Thus $IV = 0$. Thus θ acts irreducibly on V.

Now let M be a minimal member of the set of θ-stable normal subgroups of P on which θ acts non-trivially. Conclude ?

False — but contains an idea about extra-special p-groups — symmetry.

~~*Proposition*: Let θ be a p'-automorphism of a p-gp. P such that θ acts trivially on any $[p]$-subgroup A which it leaves stable. Then if p is odd, θ acts trivially on P.~~

Proof: Using induction on $|P|$ we can assume that θ acts trivially on any proper subgroup which it leaves stable. (Classical language: centralizes any proper subgroup it normalizes). In particular it acts trivially on the Frattini subgroup $\Gamma_2^{(p)} G$, so the fixed subgroup P^θ is normal and P/P^θ is an irreducible representation of θ over $\mathbb{Z}/p\mathbb{Z}$.

Counterexample: Assume $\mathbb{Z}/l\mathbb{Z}$ acts irreducibly on a $[p]$-group A, and $\Lambda^2 A$ has an invariant. This will happen when $[\mathbb{F}_p[\mu_l] : \mathbb{F}_p] = r$ is even. Indeed if ζ generates μ_l, then have eigenvalues

$$\zeta, \zeta^p, \ldots\ldots, \zeta^{p^{r-1}}$$

and as

$$p^{r/2} \overbrace{\equiv -1}^{\bmod l}, \text{ have } \zeta\,\zeta^{p^{r/2}} = 1.$$

Consequently if we form a central extension

$$1 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow G \longrightarrow A \longrightarrow 1$$

using an invariant element of $\Lambda^2 A^\vee \hookrightarrow H^2(A, \mathbb{Z}/p\mathbb{Z})$, then [p]-gp. we get a counterexample, because $\mathbb{Z}/p\mathbb{Z}$ is the only invariant

November 16, 1970

cohomology theories of symmetric product type

recall the Dold–Thom theorem:

$$\pi_k\Big(\bigcup_n SP^n(X)\Big) = H_k(X, \mathbb{Z}) \qquad X \text{ ptd. connected}$$

and your (and everybody elses) theorem

$$\pi_k\Big(\Big(\bigcup_n E\Sigma_n \times^{\Sigma_n} X^n\Big)^+\Big) = \pi^S_k(X)$$

The idea is to interpolate other cohomology theories in between these.

Idea: $SP^n(X) = X^n/\Sigma_n$

$$E\Sigma_n \overset{\Sigma_n}{\times} X^n = (E\Sigma_n \times X^n)/\Sigma_n$$

so what I want to do is to find a pleasant family of spaces Q_n with Σ_n action and maps

$$\mu_{m,n} : Q_m \times Q_n \longrightarrow Q_{m+n} \qquad \text{equivariant for}$$

$$\Sigma_m \times \Sigma_n \longrightarrow \Sigma_{m+n}$$

Requirements:

associativity ✓

commutativity

$$\begin{tikzcd} Q_m \times Q_n \arrow[r, "\mu_{mn}"] \arrow[d, "T"] & Q_{m+n} \arrow[d] \\ Q_n \times Q_m \arrow[r, "\mu_{nm}"] & Q_{m+n} \end{tikzcd} \qquad \begin{matrix} x \\ \downarrow \\ \sigma x \end{matrix}$$

where σ is the element of order 2 interchanging $1,\dots,m$ and $m+1,\dots,n$. Note that this diagram should be homotopy commutative. Note that if we take $Q_m = \Sigma_m$ with left translation action, then this isn't the case because

$$\begin{tikzcd} \Sigma_m \times \Sigma_n \arrow[r, hook] \arrow[d, "T"] & \Sigma_{m+n} \arrow[d, "\substack{x \\ \downarrow \\ \sigma x \sigma^{-1}}"] \\ \Sigma_n \times \Sigma_m \arrow[r, hook] & \Sigma_{m+n} \end{tikzcd}$$

and $\sigma x \sigma^{-1} \neq \sigma x$.

Examples: 1) Let Q_m be the set of conjugacy classes of elements in Σ_m i.e. partitions $\alpha = (\alpha_1 \geq \alpha_2 \geq \alpha_3 \cdots)$ with $\sum \alpha_i \leq m$. Then commutativity is clear.

2) Conjugacy classes of subgroups.

3) Conjugacy classes of subsets of m distinct elements in Σ_m; take this to be Q_m.

The above examples all have Q_m a trivial Σ_m-set, and hence contain the trivial example $Q_m = pt$ for all m, ~~indeed above~~ if $\exists\, x \in Q_1$.

(Actually as part of associativity one needs to have a unit element in Q_0.)

~~...~~ It is probably possible to classify the above examples as being built up out of the trivial examples $Q_m = pt$ for $m \equiv 0$ (d). But in practice there is a distinguished suspension element in Q_1 permitting one to put $Q_m \longrightarrow Q_{m+1}$. Thus the only examples ~~...~~ of this type are essentially trivial. In fact

$$Q_n \times^{\Sigma_n} X^n = Q_n \times (SP^n X)$$

so our theory is

$$\pi_* \left\{ Q_\infty \times SP^\infty(X) \right\}$$

the same as homology.

More sophisticated examples: suppose we take Q_m to be the classifying space of some category of subgroups e.g. abelian subgroups. Thus

$$\begin{tikzcd}
\begin{pmatrix}\text{ab. subgp} \\ \text{of } \Sigma_m\end{pmatrix} \times \begin{pmatrix}\text{ab. subgps.} \\ \text{of } \Sigma_n\end{pmatrix} \arrow[r] \arrow[d, "T"] & \begin{pmatrix}\text{ab. subgps} \\ \text{of } \Sigma_{m+n}\end{pmatrix} \arrow[d, "\text{conjugation by } \sigma"] \\
\begin{pmatrix}\text{ab. subgps.} \\ \text{of } \Sigma_n\end{pmatrix} \times \begin{pmatrix}\text{ab. subgroup} \\ \text{of } \Sigma_m\end{pmatrix} \arrow[r] & \begin{pmatrix}\text{ab. subgrps} \\ \text{of } \Sigma_{m+n}\end{pmatrix} \quad ?
\end{tikzcd}$$

~~[illegible]~~

To keep from being confused consider the category with a single object, namely the torsor Σ_m acting on itself from the right. Then μ_{mn} is the functor associating to Σ_m, Σ_n Σ_{m+n} and to a left operation on Σ_m and Σ_n the induced left operation on Σ_{m+n}.

$$\begin{tikzcd}
(\Sigma_m, pt) \times (\Sigma_n, pt) \arrow[r] \arrow[d] & (\Sigma_{m+n}, pt) \arrow[d, "g \mapsto \sigma^{-1} g \sigma"] \\
(\Sigma_n, pt) \times (\Sigma_m, pt) \arrow[r] & (\Sigma_{m+n}, pt)
\end{tikzcd}$$

this functor is isomorphic to the identity.

Thus in the case of abelian subgroups ~~groups~~

$$\begin{tikzcd}
A \subset \Sigma_m,\ B \subset \Sigma_n \arrow[r, mapsto] \arrow[d, mapsto] & A \oplus B \subset \Sigma_{m+n} \arrow[d, "\text{conj. by } \sigma"] \\
B \subset \Sigma_m,\ A \subset \Sigma_n \arrow[r, mapsto] & B \oplus A \subset \Sigma_{m+n}
\end{tikzcd}$$

so you want to see that conjugation by σ is isomorphic to identity: Precisely the functor

$$A \subset G \longmapsto \sigma^{-1} A \sigma \subset G$$

~~[illegible]~~ from subgps to subgrps is isomorphic to the identity the isomorphism being

$$A \xrightarrow{x \mapsto \sigma^{-1} x \sigma} \sigma^{-1} A \sigma .$$

Thus to Σ_m I associate the category of homogeneous spaces Σ_m/A ~~[illegible]~~ with specified isotropy groups.

Example 1: If you ~~[illegible]~~ take the trivial homogeneous space, you get Σ_m/Σ_m and the ~~[illegible]~~ functor is then

$$X \longmapsto SP^\infty(X)$$

2) If you take the principal homogeneous space you get

$$E\Sigma_n \times^{\Sigma_n} (X^n).$$

So now you would like to understand some intermediate categories if possible

November 26, 1970: K of a local field (cont.)

Some basic remarks about cristalline Chern classes:

Let $Z \longrightarrow X$ be a nilpotent extension with divided powers and E a vector bundle on Z. Then I would like formulas for Chern classes

$$(*) \qquad c_i(E) \in H^{2i}(I^i)$$

where I is the ideal defining Z, and the cohomology is taken over Z or X, there's no difference. For line bundles these are obtained from

$$\begin{tikzcd}
1 \arrow[r] & 1+I \arrow[r] \arrow[d, "\log"] & \mathcal{O}_X^* \arrow[r] & \mathcal{O}_Z^* \arrow[r] & 1 \\
& I & & &
\end{tikzcd}$$

and coboundary $H^1(\mathcal{O}_Z^*) \xrightarrow{\delta} H^2(1+I)$. The point is that $(*)$ are the restriction to the "open set" X of the cristalline topos of Z of the absolute cristalline Chern classes. Roughly they should be definable by taking the projective bundle $\mathbb{P}E$ over Z and computing its cristalline cohomology relative to X, which roughly should be the same as an extension of $\mathbb{P}E$ over X which needn't exist except cohomologically, then using standard "coefficient-of-relation" formulas.

I would love to be able to define $(*)$ using some non-commutative logarithm for matrices:

$$1 \longrightarrow GL_n(\mathcal{O}_X, I) \longrightarrow GL_n(\mathcal{O}_X) \longrightarrow GL_n(\mathcal{O}_Z) \longrightarrow 1.$$

But this seems difficult. Look at the easier problem of defining the images

$$(**) \qquad c_i(E) \in H^{2i}(I^i/I^{i+1}).$$

There are two (probably equivalent) way of getting such classes. First suppose Z smooth over k and then we have exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^1_{Z/k} \longrightarrow 0$$

and Atiyah-classes

$$c_i^A(E) \in H^i(\Omega^i_{Z/k}).$$

The classes $(**)$ should be obtained from the Atiyah classes using

$$0 \longrightarrow I^i/I^{i+1} \longrightarrow \cdots\cdots\cdots \longrightarrow \Omega^i_X \longrightarrow \Omega^i_Z \longrightarrow 0$$

Secondly there should be a canonical element in

$$\boxed{K \in H^2(I/I^2 \otimes \operatorname{End} E)}$$

such that

$$c_i(E) = \varphi_i(K)$$

$\varphi_i : \operatorname{End}(E) \longrightarrow \mathcal{O}_Z$ i-th symmetric fn. of eigenvalues, i.e. $\operatorname{tr}(\Lambda^i A)$.

Now consider a ring situation

$$1 \to I/I^2 \otimes \mathfrak{gl}_n(A/I) \longrightarrow GL_n(A/I^2) \longrightarrow GL_n(A/I) \longrightarrow 1$$

which gives us a canonical element in

$$H^2\left(GL_n(A/I) ,\ I/I^2 \otimes \mathfrak{gl}_n(A/I)\right),$$

and hence leads to Chern classes

$$c_k \in H^{2k}\left(GL_n(A/I),\ I^k/I^{k+1}\right).$$

If I has divided powers these should generalize to classes

$$c_k \in H^{2k}\left(GL_n(A/I),\ I^k\right).$$

Next we try to compute cohomology for $\mathbb{Z}_p = A$. We know there should be cristalline classes

$$c_k^{(n)} \in H^{2k}\left(GL(\mathbb{Z}/p^n),\ p^{nk}\mathbb{Z}_p\right)$$

I conjecture this class lifts back to a canonical

$$\tilde{c}_k^{(n)\prime\prime} \in H^{2k-1}\left(GL(\mathbb{Z}/p^n),\ p^k\mathbb{Z}_p/p^{nk}\mathbb{Z}_p\right)$$

satisfying $\beta\, \tilde{c}_k^{(n)\prime\prime} = c_k^{(n)}$, β being the relevant Bockstein. Now the existence of some $c_k^{(n)\prime\prime}$ is clear as follows: One must show $c_k^{(n)}$ is killed by $p^{nk}\mathbb{Z}_p \hookrightarrow p^k\mathbb{Z}_p$. But

The class K should be the ~~coboundary~~ image of the class in $H^1(\Omega_Z \otimes \mathrm{End}(E))$ represented by Atiyah ext.

$$0 \to \Omega^1_Z \otimes \mathrm{End}(E) \longrightarrow \mathrm{Hom}(E, J_1 E) \longrightarrow \mathrm{End}(E) \longrightarrow 0$$

by the coboundary operator of the exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^1_{Z/k} \longrightarrow 0 .$$

Given

$$1 \to I/I^2 \otimes \mathfrak{gl}_n(\mathcal{O}_Z) \longrightarrow GL_n(\mathcal{O}_X/I^2) \longrightarrow GL_n(\mathcal{O}_Z) \longrightarrow 1$$

$$[E] \in H^1(GL_n(\mathcal{O}_Z))$$

~~then [E] defines a torsor over Z for $GL_n(\mathcal{O}_Z)$.~~

Somehow K is the cup product of these 1-dimensional classes:

If we are given a G-torsor $P \to X$ and an extension

$$0 \to M \to G' \to G \to 0$$

where M is a G-module, does this lead to an element of $H^2(X, P \times^G M)$. Topologically the extension gives an element of $H^2(BG, M)$ and the torsor gives a map $X \to BG$ so you pull the class back. This is clear in principales.

the formation of classes should be compatible with the morphism $A/I^n \longrightarrow A/I^m$. Thus $c_k^{(n)}$ and $c_k^{(m)}$ are

$$\begin{array}{ccc} c_k^{(n)} \in H^{2k}(GL(A/I^n), I^{kn}) & \longrightarrow & H^{2k}(GL(A/I^n), I^{km}) \\ & & \uparrow \\ & & c_k^{(m)} \in H^{2k}(GL(A/I^m), I^{km}) \end{array}$$

compatible. But then for $m = 1$ one knows there are no stable p-torsion classes. Now taking the direct limit under inflation gives element in

$$\varprojlim_N \varinjlim_n H^{2k-1}(GL(\mathbb{Z}/p^n\mathbb{Z}), p^k\mathbb{Z}_p/p^N\mathbb{Z}_p)$$

$$\|$$

$$c_k^{(\infty)''} \in H^{2k-1}(GL(\mathbb{Z}_p), p^k\mathbb{Z}_p).$$

November 27, 1970.

Problem: To define classes

$$c_k' \in H^{2k}(GL(\mathbb{Z}_p), \mathbb{Z}/p\mathbb{Z})$$
$$c_k'' \in H^{2k-1}(GL(\mathbb{Z}_p), \mathbb{Z}/p\mathbb{Z})$$

satisfying the analogues of the formulas you have proved for finite fields. Thus

$$c(E) = \sum (c_k'(E) + c_k''(E)\varepsilon) t^k \qquad \varepsilon^2 = 0$$

should satisfy a product formula and for a one dimensional representation $G \longrightarrow \mathbb{Z}_p^*$????

I conjecture there exist basic classes

$$c_k'' \in H^{2k-1}(GL(\mathbb{Z}_p), p^k\mathbb{Z}_p)$$

integral classes which are primitive and which come from $GL(\mathbb{Q}_p)$. Moreover c_k'' reduced in $p^k\mathbb{Z}/p^{kn}\mathbb{Z}$ comes from $GL(\mathbb{Z}/p^n\mathbb{Z})$. It is not yet clear to me whether these conjectures are reasonable, except I can check them for $k=1$. Then

$$c_1'' : GL(\mathbb{Z}_p) \longrightarrow p\mathbb{Z}_p$$

is the homomorphism $A \longmapsto \log(\det A)$ where one removes off the $\mu_{p-1} \subset \mathbb{Z}_p^*$ before taking the logarithm, i.e.

$$\frac{1}{p-1} \log (\det A)^{p-1}.$$

<u>November 25, 1970</u>. K of a local field.

Let $[F:\mathbb{Q}_p] < \infty$. I want to determine the p-primary part of $K_i(F)$. Now I am going to assume that it is possible to talk about continuous cohomology of $GL_n(F)$ and to define continuous K groups $K_i(F)$. For example

$$K_1(F) = F^*$$

is a locally compact group. In addition I will assume that Grothendieck construction of Chern classes produces continuous cohomology classes:

$$c_i \in H^{2i}(GL_n(F) \times Gal(\bar{F}/F), \mu_m^{\otimes i})$$

and I want to compute the groups

$$\varprojlim_n H^i(Gal(\bar{F}/F), \mu_{p^n}^{\otimes j}),$$

using local class field theory, or better Tate duality.

~~Remarks~~

<u>Tate duality</u>: Let M be a finite Galois module. Then $H^i(M) = H^i(Gal(\bar{F}/F), M)$ is finite and cup product

$$H^i(M) \times H^{2-i}(Hom(M, \bar{F}^*)) \longrightarrow H^2(\bar{F}^*) \overset{\text{can}}{\underset{\text{isom}}{=}} \mathbb{Q}/\mathbb{Z}$$

is a perfect duality. (In particular $H^i(M)=0 \quad i>2$)

One must always think of this along with

<u>Tate - Riemann - Roch:</u>

$$\frac{h^0(M)\, h^2(M)}{h^1(M)} = \text{normalized absolute value of card } M \text{ in } F$$

$$= 1/\operatorname{card}\{\mathcal{O}/\operatorname{card}(M)\mathcal{O}\}$$

It seems to me that this duality theorem implies the reciprocity law. Indeed in dimension 1 it says that for a trivial finite Gal-module

$$H^1(M) = \operatorname{Homcont}(\mathrm{Gal}_{ab}, M)$$

is isomorphic to

$$\operatorname{Hom}\left(H^1(\operatorname{Hom}(M, \bar{F}^*)), \mathbb{Q}/\mathbb{Z}\right)$$

Now write M

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where P_i are free f.t. abelian gps and we have by Hilbert th 90 $H^1(\bar{F}^*) = 0$

$$0 \longleftarrow H^1 \operatorname{Hom}(M, \bar{F}^*) \longleftarrow \operatorname{Hom}(P_1, \bar{F}^*) \longleftarrow \operatorname{Hom}(P_0, \bar{F}^*)$$

so taking duals

$$0 \longrightarrow H^1(M) \longrightarrow \operatorname{Hom}(F^*, P_1 \otimes \mathbb{Q}/\mathbb{Z}) \longrightarrow \operatorname{Hom}(F^*, P_0 \otimes \mathbb{Q}/\mathbb{Z})$$

i.e.

$$\mathrm{Homcont}(\mathrm{Gal}_{ab}, M) \cong \mathrm{Hom}(F^*, M)$$

Thus $(F^*)^\wedge \cong \mathrm{Gal}_{ab}$, which seems to be both the reciprocity law and the existence theorem.

Recall that inverse ~~systems are exact~~ limits are exact for inverse systems of finite groups, hence I can extend cohomology continuous to profinite Galois modules.

Let $\chi : \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathbb{Z}_p^*$ be the Tate character on the pth power roots of unity. Instead of $\mu_{p^n}^{\otimes j}$ we write $\mathbb{Z}/p^n(j)$. Then we set

$$H^i(\mathbb{Z}_p(j)) = \varprojlim_n H^i(\mathbb{Z}/p^n(j))$$

and as remarked already ~~[illegible]~~ the exact sequence

$$0 \longrightarrow \mathbb{Z}_p(j) \xrightarrow{p^m} \mathbb{Z}_p(j) \longrightarrow \mathbb{Z}/p^m(j) \longrightarrow 0$$

gives rise to a long exact sequence in cohomology:

$$\begin{aligned}
0 \longrightarrow H^0(\mathbb{Z}_p(j)) &\xrightarrow{p^m} H^0(\mathbb{Z}_p(j)) \longrightarrow H^0(\mathbb{Z}/p^m(j)) \\
\longrightarrow H^1(\mathbb{Z}_p(j)) &\xrightarrow{p^m} H^1(\mathbb{Z}_p(j)) \longrightarrow H^1(\mathbb{Z}/p^m(j)) \\
\longrightarrow H^2(\mathbb{Z}_p(j)) &\xrightarrow{p^m} H^2(\mathbb{Z}_p(j)) \longrightarrow H^2(\mathbb{Z}/p^m(j)) \longrightarrow 0
\end{aligned}$$

"General" case: $\mu_p \subset K$, $\mathrm{Im}\,\chi = 1 + p^d \mathbb{Z}_p$
and we assume $d \geq 2$ if $p = 2$. Take $m = 1$

$$H^0(\mathbb{Z}/p) = \mathbb{Z}/p \cong H^2(\mathbb{Z}/p) \qquad \text{because } \mu_p \cong \mathbb{Z}/p$$

$$H^1(\mathbb{Z}/p) = H^1(\mu_p) \cong F^*/(F^*)^p$$

Now

$$F^* = \mathbb{Z} \times \mathcal{O}^* \cong \mathbb{Z} \times \mu(F) \times \mathbb{Z}_p^{[K:\mathbb{Q}_p]}$$

so

$$F^*/(F^*)^p \quad \text{has rank} \quad 2 + [K:\mathbb{Q}_p]$$

Conclude $H^2(\mathbb{Z}_p(j))$, being a pro-p-abelian with $\otimes \mathbb{Z}/p$ of rank 1, is cyclic. In fact

$$H^2(\mathbb{Z}_p(j)) \; \cancel{} = \varprojlim H^2(\mathbb{Z}/p^n(j))$$

is dual to $\varinjlim H^0(\mathbb{Z}/p^n(1-j)) = H^0((\mathbb{Q}_p/\mathbb{Z}_p)(1-j))$

Elements σ of Galois are acting on $\mathbb{Q}_p/\mathbb{Z}_p\,(1-j)$ by multiplying by $\chi(\sigma)^{1-j}$, hence acting by $(1+p^d\mathbb{Z}_p)^{1-j}$ $= 1 + p^{d+v_p(j-1)}\mathbb{Z}_p$. Thus invariants cyclic

$$H^2(\mathbb{Z}_p(j))^\vee \cong H^0((\mathbb{Q}_p/\mathbb{Z}_p)(1-j))$$

$$\cong \mathbb{Z}_p[1/p^{d+v_p(j-1)}]/\mathbb{Z}_p$$

so

$$\boxed{H^2(\mathbb{Z}_p(j)) \cong \mathbb{Z}_p / p^{d+v_p(j-1)}\mathbb{Z}_p}$$

Similarly
$$\boxed{H^0(\mathbb{Z}/p^m(j)) \cong \begin{cases} \mathbb{Z}/p^m & m \leq d+v_p(j) \\ \mathbb{Z}/p^{d+v_p(j)} & m \geq \text{"} \end{cases}}$$

so from the exact sequence we see that

$$\boxed{H^1(\mathbb{Z}_p(j)) \cong \begin{cases} \mathbb{Z}/p^{d+v_p(j)} \oplus (\mathbb{Z}_p)^{[K:\mathbb{Q}_p]} & j \neq 0,1 \\ \mathbb{Z}_p^{1+[K:\mathbb{Q}_p]} & j=0 \\ \mathbb{Z}/p^d + \mathbb{Z}_p^{1+[K:\mathbb{Q}_p]} & j=1 \end{cases}}$$

Indeed if $j \neq 1$, then $H^2(\mathbb{Z}_p(j))$ has a subgp. of order p so $H^1(\mathbb{Z}_p(j))$ has rank $\{H^1(\mathbb{Z}/p)\} - 1 = 1 + [K:\mathbb{Q}_p]$ generators, otherwise $2 + [K:\mathbb{Q}_p]$ generators. Its torsion subgroup is non-trivial for $j \neq 0$ and has order $p^{d+v_p(j)}$.

I "conjecture" that the etale character "induces" an isomorphism of the pro-p-completion of $K_i(F)$ with etale cohomology. Thus

$$\mathbb{Z}_p + \mathbb{Z}/p^d + \mathbb{Z}_p^{[K:\mathbb{Q}_p]} = K_1(F)^\wedge \xrightarrow{c_1^\#} H^1(\mathbb{Z}_p(1)) \cong \mathbb{Z}/p^d + \mathbb{Z}_p^{1+[K:\mathbb{Q}_p]}$$

OKAY

$$K_2(F)^\wedge \xrightarrow{c_2^\#} H^2(\mathbb{Z}_p(2)) \cong \mu_{p^d}$$ OKAY

$$K_3(F)^\wedge \xrightarrow{(2!)^{-1} c_3^\#} H^1(\mathbb{Z}_p(3)) \cong \mathbb{Z}/p^{d+v_p(3-1)} + \mathbb{Z}_p^{[K:\mathbb{Q}_p]}$$

$$\boxed{\begin{array}{l} K_{2j}(F)^{\wedge} \xrightarrow{(j!)^{-1} c_{j+1}^{\#}} H^2(\mathbb{Z}_p(j+1)) \cong \mathbb{Z}/p^{d+v_p(j)} \\ K_{2j-1}(F)^{\wedge} \xrightarrow[(j\geq 2)]{((j-1)!)^{-1} c_j^{\#}} H^1(\mathbb{Z}_p(j)) \cong \mathbb{Z}/p^{d+v_p(j)} + \mathbb{Z}_p^{[K:\mathbb{Q}_p]} \end{array}}$$

Now let's show these maps are onto mod torsion.(?) The point is that we have

$$\Sigma_n \tilde{\times} (K^*)^n \hookrightarrow GL_n(K)$$

as the normalizer of the maximal elementary abelian **p-subgroup** $(\mu_p)^n$. Now the K-theory associated to the first group is stable cohomotopy of

$$BK^* = B\mathbb{Z} \times B\mathbb{Z}/p^d \times B\mathbb{Z}_p^{[K:\mathbb{Q}_p]}$$

and the rational groups ~~[illegible]~~ are non-trivial only in dimension 1. Nuts, but you are being stupid to expect the torus to generate the homotopy of BU ie. $BN \longrightarrow BU$ pretty lousy on homotopy. However we can make sensisble conjectures maybe in this direction.

Suppose l is a prime ~~[illegible]~~ $\neq p$ with $\mu_l \subset K$ i.e. $l \mid q-1$ where $q =$ card (res. field). Then we conjecture that the l-primary components are finite and are given by

$$K_{2j}(K)_{(l)} \xrightarrow{\sim} K_{2j-1}(k)_{(l)} = (\mathbb{Z}/q^j-1)_{(l)} = \mathbb{Z}/l^{d_l+v_l(j)}$$

$$K_{2j-1}(K)_{(l)} \xleftarrow{\sim} K_{2j-1}(k)_{(l)} = \mathbb{Z}/l^{d_l+v_l(j)} \qquad (j\geq 2)$$

where $d_l = v_l(q-1)$ or $\operatorname{Im} \chi = (1+l^{d_l}\mathbb{Z}_l)^{\times}$. <u>Consistent pattern</u>

"Special" cases include $p=2$, $\mu_4 \not\subset F$ which we shall ignore and concentrate instead on p odd and $\mu_p \not\subset F$. The image of $\chi : \mathrm{Gal} \longrightarrow \mathbb{Z}_p^*$ is again cyclic; denote a generator of the image by q and let r be the least integer >0 $\ni$ $q^r \equiv 1 \ (p)$; $r \mid p-1$. We are assuming that $r \geq 2$.

$$\tilde{h}^0(\mathbb{Z}/p\mathbb{Z}(j)) = 1 \iff j \equiv 0 \quad (r)$$
$$\tilde{h}^2(\mathbb{Z}/p\mathbb{Z}(j)) = \tilde{h}^0(\mathbb{Z}/p(1-j)) = 1 \iff j \equiv 1 \ (r)$$

and

$$\tilde{h}^1(\mathbb{Z}/p\mathbb{Z}(j)) = [K:\mathbb{Q}_p] + \tilde{h}^0(\mathbb{Z}/p(j)) + \tilde{h}^2(\mathbb{Z}/p(j))$$

where $\tilde{h}^i$ denotes the length of H^i.

$$H^2(\mathbb{Z}_p(j)) = \varprojlim_n H^2(\mathbb{Z}/p^n(j))$$

$$\text{dual to } \varinjlim_n H^0(\mathbb{Z}/p^n(1-j))$$
$$= H^0(\mathbb{Q}_p/\mathbb{Z}_p(1-j))$$
$$\cong \mathbb{Z}_p \cdot \frac{1}{q^{j-1}-1} \Big/ \mathbb{Z}_p$$

so

$$\boxed{H^2(\mathbb{Z}_p(j)) \cong \mathbb{Z}_p/(q^{j-1}-1)}$$

$j > 0$ always.

We have then

$$H^1(\mathbb{Z}_p(j)) \cong \begin{cases} \mathbb{Z}_p/q^j-1 + \mathbb{Z}_p^{[K:\mathbb{Q}_p]} & j \equiv 0 \ (r) \\ \mathbb{Z}_p^{[K:\mathbb{Q}_p]} & \begin{cases} j \equiv 1 \ (r) \\ j > 1 \end{cases} \\ \mathbb{Z}_p^{[K:\mathbb{Q}_p]} & j \not\equiv 0, 1 \ (r) \\ \mathbb{Z}_p^{1+[K:\mathbb{Q}_p]} & j = 1 \end{cases}$$

so we have the following uniform formulas:

<u>Theorem</u>: Assume image of $\chi : Gal \longrightarrow \mathbb{Z}_p^*$ is cyclic generated by q. Then for $j > 1$ we have

$$j > 1 \qquad \boxed{\begin{aligned} H^1(\mathbb{Z}_p(j)) &\cong \mathbb{Z}_p/(q^j-1) + \mathbb{Z}_p^{[K:\mathbb{Q}_p]} \\ H^2(\mathbb{Z}_p(j)) &\cong \mathbb{Z}_p/(q^{j-1}-1) \end{aligned}}$$

$$H^1(\mathbb{Z}_p(1)) \cong \mathbb{Z}_p/(q-1) + \mathbb{Z}_p^{[K:\mathbb{Q}_p]+1}$$

$$H^2(\mathbb{Z}_p(1)) = \mathbb{Z}_p$$

So we conjecture in general that

$$j \geq 1: \quad K_{2j}(F)_{(p)} \xrightarrow{(j!)^{-1} c_{j+1}^{\#}} H^2(\mathbb{Z}_p(j+1)) \cong \mathbb{Z}_p/(q^j-1)$$

$$j > 1: \quad K_{2j-1}(F)_{(p)} \xrightarrow{((j-1)!)^{-1} c_j^{\#}} H^1(\mathbb{Z}_p(j)) \cong \mathbb{Z}_p/(q^j-1) + \mathbb{Z}_p^{[K:\mathbb{Q}_p]}$$

are isomorphisms for $j > 1$. (I expect this anomaly in dimension 1 to disappear by taking the building without $|\det|$ piece, and I hope that it is unnecessary to take the p-completion, i.e. that the groups $K_i(F)$ except for the $K_1(F)$ are already profinite.)

<u>Idea</u>: $K_i(\mathcal{O}) \otimes \mathbb{Q} \xrightarrow{\sim} K_i(F) \otimes \mathbb{Q}$

and $K_i(\mathcal{O}) \otimes \mathbb{Q}$ can be computed via Lazard. In fact Lazard shows that the cohomology of $GL_n(\mathcal{O})$ tensored with $\mathbb{Q}_p$ over $\mathbb{Z}_p$ is the same as the cohomology of the Lie algebra which one knows via Koszul is an exterior algebra with generators of degrees $1, 3, \ldots, 2n-1$. Therefore we see quite clearly that ~~this~~ can be made into a proof that

$$K_{2j-1}^{top}(\mathcal{O}) \otimes \mathbb{Q} \cong F$$

$$K_{2j}^{top}(\mathcal{O}) \otimes \mathbb{Q} = 0$$

It is reasonable to expect the Hodge type classes to give these rational K-groups.

<u>November 29, 1970:</u>

Grand conjecture. Let ℓ be a prime number which is invertible over A. By the Kunneth formula

$$H^*(\operatorname{Spec} A, G; \mu_\ell^{\otimes i}) = H^*(G, H^*(\operatorname{Spec} A; \mu_\ell^{\otimes i}))$$

one can associate to each linear function $\lambda : H^\nu(\operatorname{Spec} A; \mu_\ell^{\otimes i}) \to \mathbb{Z}/\ell\mathbb{Z}$ a cohomology class $\lambda(c_i(E)) \in H^{2i-\nu}(G, \mathbb{Z}/\ell)$, if E is a representation of G over A. The conjecture asserts that $H^*(GL(A), \mathbb{Z}/\ell\mathbb{Z})$ is generated by these Chern class components.

Thus if A is a strictly local ring the conjecture asserts that

$$H^*(GL(A), \mathbb{Z}/\ell\mathbb{Z}) = \mathbb{Z}/\ell\mathbb{Z}[c_1, \ldots\ldots]$$

In particular for A an <u>algebraically</u> closed field.

F field to fix the ideas. Then we get a "Koszul" group scheme affine over $\mathbb{Z}/\ell\mathbb{Z}$

$$\begin{aligned} \mathcal{G}_F(S) &= \text{exp. classes of rep. over } F \text{ coeff. in } S \\ &= \operatorname{Hom}_{\substack{\text{ab. gp functors} \\ \text{on groups}}}(R_F(?), H^0(?, S)) \\ &= \operatorname{Hom}_{\substack{\mathbb{Z}/\ell\mathbb{Z}\text{-algs} \\ \text{graded anti-comm.}}}(H_*(GL(F)), S) \end{aligned}$$

Then to a field extension $u : F_1 \longrightarrow F_2$ is associated

extension of scalars: $$R_{F_1}(G) \xrightarrow{\;u^*\;} R_{F_2}(G)$$

restriction of scalars: if u finite $$\xleftarrow{\;u_*\;}$$

such that

$$u_*(u^* x) = [F_2 : F_1]\, x$$
$$u^*(u_* y) = \sum_{\sigma \in Gal(F_2/F_1)} \sigma^* y \qquad \text{if } F_2/F_1 \text{ is Galois.}$$

Then u^* induces a homomorphism of group schemes which we denote

$$\mathcal{G}_{F_2} \xrightarrow{\;u_*\;} \mathcal{G}_{F_1}$$

and similarly u_* for u finite induces a homo.

$$\mathcal{G}_{F_1} \xrightarrow{\;u^*\;} \mathcal{G}_{F_2}$$

such that the same formulas hold.

Thus the group scheme $\mathcal{G}_F$ behaves covariantly with respect to Spec(F).

Now the Galois cohomology functor assigns to ~~F~~ F the graded anti-commutative $\mathbb{Z}/\ell\mathbb{Z}$-algebra

$$\bigoplus_{i\geq 0} H^{2i-*}\left(\mathrm{Gal}(F_s/F),\ \mu_\ell^{\otimes i}\right) = C_*(F)$$

whose "affine spectrum" will be denoted $\mathcal{C}_F$. The total Chern class $c = \sum_{i\geq 0} c_i$ is an exponential class hence gives a ~~[illegible]~~ map

$$\mathcal{C}_F \longrightarrow \mathcal{G}_F$$

which is functorial in Spec F. There probably is little hope in connecting up this map with the restriction-of-scalars without coming to grips with Chern classes of induced reps.

F again a finite extension of $\mathbb{Q}_p$. Assume $\mu_\ell \subset F$ and $\ell \neq p$ to fix the ideas. Now I want to understand better my conjecture that the mod ℓ cohomology of $GL_n(F)$ is detected on $(F^*)^n$. Let's begin by computing the subring of $H^*((F^*)^n)$ (mod ℓ coeff.) generated by Chern class components. Start with canonical isomorphism

$$F^*/(F^*)^\ell \xrightarrow{\sim} H^1(\mu_\ell)$$

Choose a uniformizant π and a generator γ for $\mu_{\ell^d} \subset F$, where μ_{ℓ^d} is the ℓ-primary part of $\mu(F)$, equivalently

$$\operatorname{Im} \chi = 1 + \ell^d \mathbb{Z}_\ell \subset \mathbb{Z}_\ell^*$$

where $\chi : \operatorname{Gal}(\bar{F}/F) \longrightarrow \mathbb{Z}_\ell^*$ is the Tate character. Then the images of π and γ in $F^*/(F^*)^\ell$ generate it.

Until we find a better notation, let $c_1(\pi)$, $c_1(\gamma)$ denote the elements of $H^1(\mathbb{Z}/\ell\mathbb{Z})$ defined by the coboundary

$$\xrightarrow{\ \ell\ } F^* \xrightarrow{\ \delta\ } H^1(\mu_\ell) \longrightarrow 0$$
$$\wr\!\wr$$
$$H^1(\mathbb{Z}/\ell\mathbb{Z})$$

ζ generates μ_ℓ, e.g. $\zeta = \gamma^{(\ell^{d-1})}$

together with isomorphism furnished by ζ. Thus

$$c_1(\pi) \cdot \zeta = \delta(\pi) \qquad c_1(\gamma) \cdot \zeta = \delta(\gamma).$$

Moreover $c_1(\pi)\cdot c_1(\gamma) \in H^2(\mathbb{Z}/\ell\mathbb{Z}) \cong \mathbb{Z}/\ell\mathbb{Z}$ is a generator of this group. Now there is a canonical isom.

$$H^2(\mu_\ell) = \mathbb{Z}/\ell\mathbb{Z}$$

given by the invariant, ~~so the next problem is to find~~ so have to check that γ and π are independent. The point is that we have two elements

$$\delta\pi\cdot\delta\gamma \in H^2(\mu_\ell^{\otimes 2})$$

$$\zeta\cdot(\mathrm{can}) \in$$

where can is the canonical class. Therefore having chosen $\zeta \in \mu_\ell$ one wants to choose π, γ so that

$$\boxed{\delta\pi\cdot\delta\gamma = \zeta\cdot(\mathrm{can})}$$

(a bit confused)

Now we have the basis

$$\begin{aligned} 1 &\in H^0(\mathbb{Z}/\ell) \\ c_1(\pi),\ c_1(\gamma) &\in H^1(\ ``\ ) \\ c_1(\pi)\,c_1(\gamma) &\in H^2(\ ``\ ) \end{aligned}$$

and to save writing we put $c_1(\pi)=\alpha$, $c_1(\gamma)=\beta$. As cup product is skew-symmetric we have

$$\alpha^2 = \beta^2 = 0.$$

$$\alpha\beta + \beta\alpha = 0.$$

Perhaps it is reasonable to make the connection with the Hilbert symbol. Thus if $\mu_m = \mu(F)$ one has

$$(a,b) \in \mu_m$$

defined by

$$(a,b)\,\mathrm{can} = \delta a \cup \delta b \in H^2(F, \mu_m^{\otimes 2})$$

or perhaps using

$$\int : H^2(F, \mu_m^{\otimes j}) \longrightarrow \mu_m^{\otimes (j-1)}$$

to denote the canonical map, we have

$$\boxed{(a,b) = \int \delta a \cdot \delta b}\,.$$

For the tame symbol one composes with the surjection

$$\mu_m \twoheadrightarrow (\text{res. fd})^*$$

and it's known then that

$$(a,b) = (-1)^{v(a)v(b)} \frac{a^{v(b)}}{b^{v(a)}} \quad \text{reduced mod } \pi.$$

So now that we understand the Galois cohomology, we can investigate the Chern classes.

Review of Kummer theory in the appropriate way:

Suppose $\mu_m \subset F$, $(m, \operatorname{char} F) = 1$. Then there is a canonical element

$$c_1 \in H^2(F^* \times \mathrm{Gal}(\bar{F}/F), \mu_m)$$

which is the coboundary of the homomorphism

$$F^* \times \mathrm{Gal}(\bar{F}/F) \longrightarrow F^* \hookrightarrow \bar{F}^*$$

for the exact sequence

$$0 \longrightarrow \mu_m \longrightarrow \bar{F}^* \xrightarrow{\ m\ } \bar{F}^* \longrightarrow 0.$$

I want to generalize my calculation for a finite field to the general case.

<u>Proposition:</u> Let $u \in H^2(F^*, \mu_m)$ be the canonical element represented by the extension

$$0 \longrightarrow \mu_m \longrightarrow (F^*)^{1/m} \xrightarrow{\ m\ } F^* \longrightarrow 0$$

(where $(F^*)^{1/m} = \{ z \in \bar{F}^* \mid z^m \in F \}$). Equivalently u is the geometric first Chern class of the canonical representation of F^*. Let K be the Kunneth isomorphism

$$K: H^1(F^*, H^1(\mathrm{Gal}, \mu_m)) \xrightarrow{\ \sim\ } H^2(F^* \times \mathrm{Gal}, \mu_m)$$

which in good cases would be the composite

$$\begin{tikzcd}
H^1(F^*, H^1(\mathrm{Gal}, \mu_m)) \arrow[ddr, bend right, "K"'] & H^1(F^*, \mathbb{Z}/m) \otimes H^1(\mathrm{Gal}, \mu_m) \arrow[l, "\sim"'] \arrow[dd, "\cup"] \\
& \\
& H^2(F^* \times \mathrm{Gal}, \mu_m).
\end{tikzcd}$$

Let $can \in H^1(F^*, H^1(\mathrm{Gal}, \mu_m))$ be the canonical element furnished by the Hilbert map

$$F^* \longrightarrow F^*/(F^*)^m \xrightarrow{\delta} H^1(\mathrm{Gal}, \mu_m)$$

Then the arithmetic Chern class is the sum

$$c_1 = u + K(can).$$

Proof is essentially obvious. As in your finite fields paper, the class c_1 is represented by the cocycle

$$(\delta h')((g_1, \alpha_1), (g_2, \alpha_2)) = s(g_2)^{\alpha_1 - 1} \underbrace{(\delta s)(g_1, g_2)}_{\overset{\shortparallel}{u}}$$

where $s(g_2)^{\ell} = g_2 \in F^*$. Thus

$$s(g_2)^{\alpha_1 - 1} = (\sqrt[m]{g_1})^{\alpha_1 - 1}$$

is the **Hilbert** pairing of Gal and F^* with values in μ_m.

Now we want to use this in our calculations when F is a local field, and $m = \ell$ is $\neq p$. Then

$$c_1 \in H^2(F^* \times Gal, \mu_\ell)$$

is the sum of the geometric Chern class

$$u \in H^2(F^*, \mu_\ell)$$

and the element which is the image under cup product

$$H^1(F^*, \mathbb{Z}/\ell\mathbb{Z}) \otimes H^1(Gal, \mu_\ell) \longrightarrow H^2(F^* \times Gal, \mu_\ell)$$

of the identity map, the two spaces being naturally dual.

As we are assuming $\mu_\ell \subset F$ and have chosen $\zeta \in \mu_\ell$ a generator, this means we are looking at the element

$$\hat{\alpha} \cdot \alpha + \hat{\beta} \cdot \beta \quad \in H^1(F^*, \mathbb{Z}/\ell) \otimes H^1(Gal, \mathbb{Z}/\ell)$$

where $\hat{\alpha}, \hat{\beta} : F^* \longrightarrow \mathbb{Z}/\ell$ are the dual basis to $\pi, \gamma \in F^*/(F^*)^\ell$.

<u>Conclusion:</u> We choose a generator ζ of μ_ℓ and a basis α, β of $H^1(Gal, \mathbb{Z}/\ell)$ such that

$$\alpha \cdot \beta \cdot \zeta \in H^2(Gal, \mu_\ell)$$

is canonical generator. Actually we choose first a generator γ for $\mu(F)$ and a uniformizant π so that γ, π give a basis of $F^*/(F^*)^\ell$ and then we choose α and β so that γ, π and α, β are dual bases. (ugly). Then the point is that

$$c_1 = u + \hat{\alpha}\cdot\alpha + \hat{\beta}\cdot\beta \in H^{2}(F^*) + H^1(F^*)\otimes H^*(\mathrm{gal})$$

Now recall

$$H^*(F^*) = \mathbb{Z}/\ell[\hat{\alpha},\hat{\beta},u] \quad \text{so}$$

$$H^*((F^*)^n) = \mathbb{Z}/\ell[\hat{\alpha}_i, \hat{\beta}_i, u_i]$$

and total Chern class of standard ~~representation~~ is

$$\prod_{i=1}^{n}\left(1 + u_i + \hat{\alpha}_i\alpha + \hat{\beta}_i\beta\right)$$

so what you need now is to recognize the subring of $H^*((F^*)^n)$ generated by the coefficients of the various elements $1, \alpha, \beta, \alpha\beta$

<u>November 30, 1970:</u> My education in group theory:

<u>Grün theorem:</u> P Sylow p-subgp. of G, $Z = Z(P)$ $N = \text{Norm } Z$ in G. Form the categories of p-subgroups in the usual way (morphisms are homomorphisms.) Then if G is p-normal: $Z \subset gPg^{-1} \Rightarrow Z = gZg^{-1}$, we have

$$Cat(N) \longrightarrow Cat(G)$$

is an equivalence of categories.

Proof. One has to show that if $g^{-1}Mg \subset P$ and $M \subset P$, then $\exists\ \gamma \in Cent(M) \ni \gamma g \in Norm(Z) = N$. But

$$M \subset gPg^{-1} \implies gZg^{-1} \subset Cent(M)$$
$$M \subset P \implies Z \subset Cent(M)$$

and both being p-gps $\exists\ Q \subset Cent(M)$ Sylow p-gp and $\gamma \ni Z \subset Q$, $gZg^{-1} \subset \gamma^{-1}Q\gamma$ so $Z, \gamma g Z(\gamma g)^{-1} \subset Q \subset$ Sylow grp of G. Hyp. $\Rightarrow$

$$Z = \gamma g Z(\gamma g)^{-1}$$

so we are done.

Remark 1: One must work in $Cent(M)$ in order to keep category unchanged. Possibly in $M \cdot Cent(M)$, but there is no diff. because $M \subset N$.

It is enough to have Z central in P and "weakly closed" i.e. $g^{-1}Zg \subset P \Rightarrow g^{-1}Zg = Z$

Corollary: If N is the normalizer of a central weakly closed subgroup Z of P, then

$$H^*(G) \xrightarrow{\sim} H^*(N)$$

(p-torsion cohomology). Hence G has a normal p-complement iff N does (by Tate or Frobenius).

So now if G is a group such that P and G have same categories of elementary abelian p-subgroups, then take Z to be the elements of order $|p$ in the center of P. Then Z is central and weakly closed for there is no fusion of elementary abelian p-subgroups. So by Grün we can replace G by N if we wish to show p-nilpotence. Hence can assume $Z \triangleleft G$, whence Z is central in G.

Now can divide out by largest normal p'-subgroup, ~~whence can assume Center(G) = Center(P). In more detail the p' part center(G) is 1, hence Center(G) $\subset$ Center(P). Any p'-element of Norm{Center(P)} acts trivially on Z the "bottom" of Center(P). Hence Norm{Center(P)} = (Center P)·P~~ ? ?

so $$Z = {}_pZ(P) \subset Z(G) \subset Z(P).$$

Proof that a **maximal** normal elem. ab. subgroup is maximal elem. abelian:

Let θ be an automorphism of an abelian p-group A such that (i) $\theta^p = 1$ (ii) $(\theta - 1)$ kills $\Omega_1 A$. Then if p is odd $(\theta-1)A \subset \Omega_1 A$.

<u>Proof:</u> $$1 = \big(1 + (\theta - 1)\big)^p$$

so $$p(\theta-1) + \binom{p}{2}(\theta-1)^2 + \cdots\cdots + (\theta-1)^p = 0.$$

Let d be largest such that $(\theta-1)\,\Omega_d A \subset \Omega_1 A$ assuming that $(\theta-1)A \not\subset \Omega_1 A$. Then

$$(\theta-1)\,\Omega_{d+1} A \not\supset \Omega_1 A$$

so $\exists\ z \in \Omega_{d+1} A$ with $p(\theta-1)z \neq 0$, $p^2(\theta-1)z = 0$.

Now $$(\theta-1)\,\Omega_1 A = 0 \implies (\theta-1)\,\Omega_{d+1}A \subset \Omega_d A$$
$$\implies (\theta-1)^2\,\Omega_{d+1}A \subset \Omega_1 A$$
$$\implies \begin{cases} p(\theta-1)^2\,\Omega_{d+1}A = 0 \\ (\theta-1)^p\,\Omega_{d+1}A = 0 \quad \text{as } p \geq 3 \end{cases}$$

Hence $p(\theta-1)z = 0$ a contradiction.

<u>In brief:</u>

$$p(\theta-1)\,\Omega_d A = 0 \implies (\theta-1)\,\Omega_d A \subset \Omega_1 A$$
$$\implies (\theta-1)^2\,\Omega_d A = 0$$
$$\implies (\theta-1)^2\,\Omega_{d+1} A \subset \Omega_1 A$$
$$\implies (\theta-1)^3\,\Omega_{d+1} A = 0$$

Hence have $(\Theta-1)^3$ and $p(\Theta-1)^2$ kill $\Omega_{d+1}A$, so by identity

$$0 = \Theta^p - 1 = (1+(\Theta-1))^p - 1 \equiv p(\Theta-1) \mod \left(p(\Theta-1)^2, (\Theta-1)^3\right)$$

I have

$$p(\Theta-1)\Omega_{d+1}A = 0.$$

Thus by induction conclude that $p(\Theta-1)A = 0$.

Next step: A maximal normal abelian subgroup of the p group P, p odd. Claim that the elements of order 1 or p in $\mathrm{Cent}_p(\Omega_1 A)$ form a subgroup. Suppose not and let $x, y \in \mathrm{Cent}_p(A)$ be such that $x^p = y^p = 1$ but $(xy)^p \neq 1$ and such that $\langle x, y\rangle$ has least possible order. Then as $\langle x, y\rangle$ is not cyclic $\langle y, x^{-1}yx\rangle < \langle x, y\rangle$ so $y^{-1}x^{-1}yx$ has order p.

But x, y stabilize $A \supset \Omega_1 A \supset 1$ hence the commutator $y^{-1}x^{-1}yx$ centralizes A, so as A is maximal $y^{-1}x^{-1}yx \in \Omega_1 A$. Hence $\langle x, y\rangle$ is of class 2 so as p is odd every element of $\langle x, y\rangle$ is of order p, a contradiction. This proves claim. We have proved:

~~Thus $\Omega_1 \mathrm{Cent}_p(\Omega_1 A)$ is of exponent p and so if $\Omega_1 A$ is not a maximal elementary p-abelian one can properly enlarge it to a normal elementary abelian p subgroup.~~

Prop: If A is a max. normal ab. subgroup of P, p odd, then $\Omega_1 \mathrm{Cent}_p(\Omega_1 A)$ is of exponent p. Consequently if

$\Omega_1 A$ is a max. normal elem. ab. subgp. of P, then $\Omega_1 A$ is a max. elem. ab. subgp. of P.

Cor. Any max. norm. elem. ab. subgp. of a p-gp, p odd, is a max. elem. ab. subgroup.

(These are the analogues of the Klein group in $(\mathbb{Z}/p)^n \subset Syl_p(\Sigma_{p^n})$.)

NO, NO*

If B is a max. norm. elem. ab. subgroup choose A max. norm ab $\supset B$. Then $B = \Omega_1 A$ so

$$\Omega_1 \, Cent_p(B) = B$$

by the preceding proposition. So B is max elem. ab.

On the contrary the normalizer of the Klein group $(\mathbb{Z}/p)^n \subset Syl_p(\Sigma_{p^n})$ has order

$$p^{\frac{n(n+1)}{2}} \prod_{i=1}^{n} (p^i - 1)$$

so isn't normal in P. The group $(\mathbb{Z}/p)^{p^{n-1}} \subset (\Sigma_p)^{p^{n-1}} \subset \Sigma_{p^n}$ has normalizer $\Sigma_{p^{n-1}} \int \mathbb{Z}/p$ whose order has

$$v_p\left(\left|\Sigma_{p^{n-1}} \int \mathbb{Z}/p\right|\right) = p^{n-1} + v_p(p^{n-1}!) = p^{n-1} + \cdots + 1$$

$$= v_p(p^n!)$$

and so it is normal in some Sylow group.