Let $G$ be a finite group acting on a complex manifold $X$ preserving the structure. Then there is an index map

$$f! : K_G(X) \longrightarrow K_G(pt)$$

where $f : X \to pt$, defined by taking an equivariant bundle $E$ to $E \otimes (\mathcal{O})$ making a $G$-elliptic operator and taking index. It has the property that if $E$ is an equivariant holomorphic bundle, then

$$f! [E] = \sum_i (-1)^i [H^i(X,E)]$$

where $E$ denotes the sheaf of holomorphic $E$. Now we want a Riemann-Roch formula for the orbit variety $X/G$, stick to complex analytic case. That is, a formula for the map

$$Khol(X/G) \xrightarrow{\chi} \mathbb{Z}$$

$$E \in \mapsto \chi(X/G, E)$$

in terms of the characteristic classes of $E$. So we make some observations:

1) $H^*(X/G, E) = H^*(X, \pi^*E)^G$ where $\pi : X \to X/G$.

Indeed $\pi_* (\pi^*E)^G = E$ (local on $X/G$, hence can assume $E = \mathcal{O}$, whence it says that holomorphic functions on $X/G = \text{equivariant}$ holomorphic functions on $X$) and the two spectral sequences for equivariant cohomology degenerate because the coefficients are $\mathbb{Q}$ and
\pi is finite.

Hence the map we want is

\[
\begin{array}{ccc}
K_{\text{hol}}^*(X/G) & \xrightarrow{\chi} & \mathbb{Z} \\
\downarrow \pi^* & & \uparrow \text{inner product with trivial rep} \\
K_G^*(X) & \xrightarrow{f_1} & R(G)
\end{array}
\]

Thus we can extend \( \chi \) to \( K(X/G) \) satisfies Poincare duality.

2) \( H^*(X/G) = H^*(X)^G \) hence the formula we want is of the form

\[
\chi(X/G, E) = (\text{ch}(E) \cdot \text{Todd})[X/G]
\]

\[
= \text{ch}(E) \cap \{\text{Todd}\}[X/G]
\]

So thus the problem is to find the homology class

\[
\text{Todd} \cap [X/G] \subset H^*_*(X/G)
\]

(rational coeffs.)

Our problem thus becomes the following. We have the map \( \chi \)

\[
K(X/G) \xrightarrow{\pi^*} K_G^*(X) \xrightarrow{f_1} R(G) \xrightarrow{S_G} \mathbb{Z}
\]

and want to express it in the form

\[
\chi(E) = \text{ch}(E) \cap \alpha \quad \text{for some } \alpha \in H^*_\text{ev}(X/G).
\]
Example: Take $X$ to be the projective space $\mathbb{P}V$, $V$ a representation of $G$. Then

$$H^e(X/G) \xrightarrow{\sim} H^e_G(X) \xrightarrow{\sim} H^e(X) = \mathbb{Q}[H]/H^d \quad d = \dim V$$

$H = c_1(\mathcal{O}(1))$. Now note that a $G$-vector bundle $E$ on $X$ is of the form $\pi^*E$ iff the isotropy representations are trivial. Hence $\mathcal{O}(n)$ comes from $X/G$ where $n$ is the exponent of $G$. Now however one knows that $\text{ch}(\mathcal{O}(n) - 1) = e^{nH} - 1$ so the elements

$$\text{ch}(\mathcal{O}(n) - 1)^i \quad i = 0, \ldots, d - 1$$

form a $\mathbb{Q}$-basis for $H^e(X/G)$. It follows therefore that we ought to be able to compute $\chi$, because $\chi$ is completely determined by the formulas

$$e^{inH} \cap \chi = \dim \{ S_{\chi}(V)^G \}$$

for all $i \geq 0$.

Suppose take $G$ to be a cyclic group of order $p$ and $V$ a representation with the generator having eigenvalues. Computation should be manageable.
Let $X$ be a polyhedron. A function on $X$ will be called constructible if it is constant on each open simplex of some linear subdivision of $X$, and similarly for a sheaf. If we consider constructible sheaves of $A$-modules $A$ a field, say, then the Grothendieck ring $K$ of these sheaves may be identified with the ring of constructible integer-valued functions, the map assigning to a sheaf $F$, the function $x \mapsto \text{rank } F_x$. Let $R(X)$ be this ring.

There is a natural linear function $R(X) \to \mathbb{Z}$ which associates to $F$ the Euler characteristic $\chi(X, F)$. In terms of functions it assigns the value $(-1)^i$ to the characteristic function of an open $i$-simplex. More generally given a map $f : X \to Y$ of polyhedra (piecewise-linear on some subdivision of $X$) then we have $f^! : R(X) \to R(Y)$ defined as usual.

We think of $F \mapsto \chi(X, F)$ as an intrinsic "measure" on constructible functions. Call a function $f$ harmonic if for each point $x$

$$\sum (-1)^i \dim H^i_{f^!}(X, F) = f(x).$$

Suppose $f$ constant on a given subdivision and let $\sigma$ be the open simplex to which $x$ belongs. Then a neighborhood of $x$ is the product of $\sigma$ and the cone on the link of $\sigma$ and so $H^i_{f^!}(X, F)$ is the same as and $F$ constant on the $\sigma$ factor.
\[ H_i \left( \text{Cone } L(\sigma), F \right), \quad x = \dim \sigma. \]

Using the exact sequence:

\[ H^i \left( \text{Cone } L(\sigma), F \right) \rightarrow H^i \left( \text{Cone } L(\sigma), \tilde{F} \right) \rightarrow H^i \left( L(\sigma) \times \{0\}, \tilde{F} \right) \]

Together with the fact that \( \tilde{F} \) is constant along the generators of the cone, we see that

\[ \chi(L(\sigma), \tilde{F}) = \tilde{F}(\sigma) \]

Now use the exact sequence:

\[ H^i \left( \text{Cone } L(\sigma), F \right) \rightarrow H^i \left( \text{Cone } L(\sigma), F \right) \rightarrow H^i \left( L(\sigma) \times \{0\}, F \right) \]

Together with the fact that \( F \) is constant along the generators of the cone, one finds that

\[ f(x) = \chi(L(\sigma), F) + \sum (-1)^d \chi(H^d_{x[x]}(L(\sigma), F)) \]

\[ \dim F_x = \chi(L(\sigma), F) + (-1)^d \chi(H^d_{x[x]}(L(\sigma), F)) \]

For example, in an \( n \)-dimensional manifold with the function \( 1 \) and a vertex, we have

\[ 1 = \chi(S^{n-1}) + (-1)^n \]

which is okay, so we see that mod 2 harmonic is
equivalent with the link of every simplex having Euler characteristic zero. Note that the function $1$ on a manifold is harmonic iff the manifold has even dimension.

Let $\sigma$ be a simplex and let $U_\sigma$ be its star, i.e. points with positive coordinates at each vertex of $\sigma$. Then the

$$H^*(U_\sigma, F) = \begin{cases} \circ & * > 0 \\ F_\sigma & * = 0 \end{cases}$$
and \( H_c^*(U_\sigma, F) = H^*_{\text{barycenter of}}(X, F) \). By hypothesis then

\[
\chi(U_\sigma, F) = \chi_c(U_\sigma, F).
\]

Now given a subcomplex \( A \) of \( X \), consider the covering by \( U_\sigma \) where \( \sigma \) runs \( \bigwedge \) over vertices. Then

\[
\chi_A(\bigwedge X, F) = \chi_c(\text{star}(A), F) = \sum_{\sigma \in A} (-1)^{\dim \sigma} \chi_c(U_\sigma, F)
\]

and similarly

\[
\chi(A, F) = \chi(\text{star}(A), F) = \sum (-1)^{\dim \sigma} \chi(U_\sigma, F)
\]

(these are both consequences of Mayer-Vietoris)

Thus \( F \) harmonic \( \Rightarrow \)

\[
\chi(A, F) = \chi_A(X, F) \quad \text{A closed}
\]

\[
\chi(X, F) - \chi_c(X-A, F) \quad x(X, F) - x(X-A, F)
\]

hence

\[
\chi(U, F) = \chi_c(U, F) \quad U \text{ open}
\]

(immediate and simpler from Mayer-Vietoris relation.)
October 4, 1970

The relation \( X(U, F) = X_c(U, F) \) for a harmonic function is very reasonable if you think of the constant function 1 on an even dimensional manifold. Then one knows by duality that \( H^*_c(U) \) and \( H^*(U) \) have a non-singular pairing into \( H^{2n}_c(X) = \mathbb{R} \) and hence have same Euler Characteristic.

If \( f: X \to Y \), then \( f_! \) carries harmonic functions to harmonic functions, because

\[
X(V, f_! F) = X(f^{-1}V, F)
\]

\[
X_c(V, f_! F) = X_c(f^{-1}V, F)
\]

by Leray spectral sequence.

However harmonic functions are not closed under inverse image even when \( X, Y \) are manifolds. Indeed suppose in the plane \( = Y \) we take \( F \) to be the constant function on the graph of a function \( 0 \) for \( x < 0 \)

\[
\begin{array}{c}
X \\
\end{array}
\]

and \( X = \text{x-axis} \). Then pull-back is the function
which is clearly not harmonic as the link is wrong. Just as a check we compute the cohomology of the sheaf $$\mathbb{Z}_{(-\infty,0]}$$ at 0.

\[
\begin{align*}
H^*_{\{0\}}(\mathbb{Z}_{(-\infty,0]}) & \rightarrow H^*(\mathbb{Z}_{(-\infty,0]}) \rightarrow H^*(\mathbb{R}-0, \mathbb{Z}_{(-\infty,0]}) \\
H^*_{(-\infty,0], \mathbb{Z})} & \rightarrow H^*(-\infty,0], \mathbb{Z})
\end{align*}
\]

Thus the local cohomology at the point 0 is 0 so the $$x \neq 0$$ value at zero.

So what we want to prove is that the inverse image of a harmonic function $$F$$ by a map $$f: X \rightarrow Y$$ is again harmonic provided $$f$$ and $$F$$ are transversal in a sense to be made precise.

Suppose $$f$$ is a smooth map of manifolds, and $$F$$ is constant on the open simplices of a triangulation of $$Y$$. Suppose $$f$$ is transversal to the simplices of the triangulation. Then there is an induced stratification of $$X$$ and I claim that the inverse image of $$F$$ is harmonic on $$X$$. It's enough to worry about an embedding. Take $$x \in X$$ and suppose $$f(x) \in \sigma$$. 

\[\sigma\]
By transversality locally near \( x \), \( Y \) is \( L(\sigma) \times \sigma \) and \( X \) is \( L(\sigma) \times (X \cap \sigma) \) and \( F \) is constant in the \( \sigma \) direction. Thus the link condition remains the same and so \( F|_X \) is harmonic. The point to remember is that the normal structure of the strata around \( X \cap \sigma \) is exactly the same as the link of \( \sigma \).

Of course there is a dimension shift, so when working with integral valued harmonic functions it is necessary to require that the relative dimension of \( f \) be even.

Suppose now that \( F \) is a constructible function on \( X \) transversal to a submanifold \( Y \) of codimension \( 1 \) and that \( Y \) has an interior \( Z \), \( \partial Z = Y \). Then

\[
\int_Y F = 0
\]

because it is the difference of \( x(\mathbb{Z}, F) \) and \( x(\mathbb{Z}, F) \), which are the same by harmonicity. (To do anything cobordism - theoretically we have to work mod 2. In
effect you want to start with the function $1$ on an even dimensional manifold, restrict to odd dimensional submanifolds (so the cobordism can be used), and then take $X$ which of course gives zero.

Now work mod 2 and suppose $F$ is a harmonic function on a manifold $X$. Define now a map

$$\eta_*(X) \longrightarrow \mathbb{Z}_2$$

by

$$[Z \to X] \longmapsto \int f^*(F) \mod 2$$

As we've proved that transversal inverse images are harmonic and that integral vanishes on a boundary, it follows that this map is well-defined. On the other hand, if $M$ is a compact manifold, then this map associates to $[M \times \mathbb{Z} \to X]$ the number $\chi(M) \cdot \int f^*(F)$ so that it factors

$$\mathbb{Z}_2 \times \eta_*(X) \longrightarrow \mathbb{Z}_2$$

Now the significance of Deligne's question becomes apparent because he wanted to check that there is an isomorphism

$$\mathbb{Z}_2 \otimes \eta_*(X) \cong H_*(X)$$
Let $A$ be an elementary abelian subgroup of $G$ (compact Lie gp.), $N$ its normalizer, $p_A \subset H^*_G$ and $q_A \subset H^*_N$ the associated prime ideals. Claim that if $A$ is maximal among $[p]$-gps fixing points of $X$, then

\[
H^*_G(X)_{p_A} \sim \cdots
\]

First of all the localization theorem says

\[
H^*_G(X)_{p_A} \sim H^*_G(GX^A)
\]

because the third side of the triangle is a module over $H^*_G(X-GX^A) = 0$ since $A$ fixes no points of $X-GX^A$ hence the spectrum of this ring is $\emptyset$. Hence

\[
(A \triangleleft G \Rightarrow H^*_G(X)_{p_A} = H^*_G(X^A)_{p_A})
\]

This special case applied to $N$ shows that

\[
H^*_N(X^A)_{q_A} = H^*_N(X)_{q_A}
\]

hence will give an exact sequence when applied to the diagram

\[
X \leftarrow X \times F \leftarrow X \times F \times F.
\]

So this reduces us to the case where the isotropy groups...
are all $[p]$-groups, $X = GX^A$. It follows that

$$G \times ^N X^A \overset{\sim}{\rightarrow} X$$

and $N/A$ acts freely on $X^A$. (Any isotropy groups will contain a conjugate $x Ax^{-1}$ and as $N/A$ is a $[p]$-group and $A$ is maximal it will follow that it coincides with this conjugate.) Then

$$H^*_G(X) \overset{\sim}{\rightarrow} H^*_N(X^A)$$

If $B = xAx^{-1} \subset N, x \in G$, and $B \neq A$, then

$$H^*_N(X^A)_{\mathfrak{q}_B} = H^*_N(N \cdot (X^A)_B)_{\mathfrak{q}_B} = 0$$

because $AB > A$ is not a $[p]$-group. Hence the only prime in the support of $H^*_N(X^A)$ over $\mathfrak{p}_A$ is $\mathfrak{q}_A$. So

$$H^*_G(X)_{\mathfrak{p}_A} = H^*_N(X^A)_{\mathfrak{p}_A} = H^*_N(X^A)_{\mathfrak{q}_A}$$

which finishes the proof of: 

**Theorem:** If $A$ maximal $[p]$-group normalizer $N$, and $A$ determines $\mathfrak{p}_A$ in $H^*_G$ and $\mathfrak{q}_A$ in $H^*_N$, then

$$H^*_G(X)_{\mathfrak{p}_A} = H^*_N(X^A)_{\mathfrak{q}_A}.$$
The fundamental problem is to attach an Euler characteristic to $H^*_G(X)_{\mathcal{F}_A}$. The above thm. reduces this problem to a normal in $G$. The following example shows that the value of the Poincaré series at $t = -1$ is not additive.

$G = \mathbb{Z}/p^2\mathbb{Z}$, $\mathcal{F}_A = \mathbb{Z}/p\mathbb{Z}$, replace the map $G/\mathcal{F}_A \to \text{pt}$ by an inclusion $Y = G/\mathcal{F}_A \to \text{disk in a faithful repn. of } G/\mathcal{F}_A = X$. Then have

$$
\xymatrix{
\mathcal{S} \ar[r]^\delta & H^*_G(X, Y) \ar[r] & H^*_G(X) \ar[r] & H^*_G(Y) \ar[r] & \\
& H^*_G(Y) \ar[r]^-\text{res} & H^*_A
}
$$

$$
\xymatrix{
0 \ar[r] & k \ar[r]^\iota & k \ar[r]^-\cong & k \\
& k \ar[r]^-\cong & k \ar[r]^-\text{res} & k \\
& k \ar[r]^-\cong & k \ar[r]^-\iota & k \\
& k \ar[r]^-\iota & k \ar[r]^-0 & 0
}
$$

So

- $H^*_G(X) = \frac{1}{1-t}$
- $H^*_G(Y) = \frac{1}{1-t}$
- $H^*_G(X, Y) = \frac{t}{1-t}$

\[\frac{t}{1-t} = \frac{1}{2} \quad \text{at } t = -1\]

P.S. $\frac{1}{1-t} = \frac{1}{2}$ at $t = -1$

P.S. $\frac{t}{1-t} = -\frac{1}{2}$

and this isn't additive.
When $A < G$, and $X = X^A$, then we have Hochschild – Serre

$$E_2 = H^*(G/A, H^*_A \otimes H^*(X)) \Rightarrow H^*_G(X)$$

If $A$ maximal, then $H^*(G/A) \rightarrow H^*_G$ is zero in large degree, hence for $k$ large one expects $E_{2k}$ to be bounded horizontally. It perhaps is reasonable to conjecture that $H^*_G(X)$ might be a free module over a subring $\mathbf{P}$ of $H^*_G$ such that $H^*_A$ is also free over $\mathbf{P}$, possibly after localizing. If so one might then be able to define

$$\chi \left\{ H^*_G(X) : H^*_A \right\} = \frac{\chi \left\{ H^*_G(X) : \mathbf{P} \right\}}{\chi \left\{ H^*_A : \mathbf{P} \right\}}$$

It seems that one always has a spectral sequence

$$E^{p, q}_2 = R^p \lim_A \left\{ H^q_A \right\} \Rightarrow H^{p+q}_G$$

But there doesn't seem to be any reason for $E^{p, q}_2 = 0$ for $p \geq q$. This spectral sequence arises from a composite functor

$$\lim \left. M^A \right|_A = M^G.$$
October 13, 1970: On Thom's theorem realizing rational classes.

Let $x \in H^q(X)$ be a manifold. Thom's theorem asserts that $n \cdot x$ can be realized by an oriented submanifold of codimension $q$ for some $n$.

In terms of his realizability criterion, this means that $MSO_q \rightarrow K(Z, q)$ given by Thom class admits a section in the rational homotopy category. If $q$ is odd this is trivial as $S^q = K(Z, q)$, so one can realize $x$ by a framed submanifold. If $q = 2p$ we show $MU_p \rightarrow K(Z, 2p)$ admits a section.

Note that:

$$H^*(K(Z, p)) \rightarrow H^*(MU_p)$$

is a free map, consequently in the spectral sequence of $MU_p \rightarrow K(Z, 2p)$ the $E_2$-term is taken in the rational homotopy category. But

$$BU_p \rightarrow \prod_{i=1}^{p} K(Z, 2i)$$

is a rational equivalence and

$$BU_p \rightarrow MU_p \rightarrow \frac{K(Z, 2p)}{U}$$

so it's clear.
Actually we must be careful of Mumford's objection—all we get this way is a map $X \to \text{MU}_p \otimes \mathbb{Q}$. So what must be proved is that when dimension of $X$ is odd, $n$ can be found so dotted arrow exists in

$$
\begin{align*}
X & \longrightarrow K(n\mathbb{Z}, 2p) \\
\downarrow & \\
\text{BU}_p & \longrightarrow K(\mathbb{Z}, 2p).
\end{align*}
$$

More precisely, given $k$ we want to find $n$ so section

$$
\begin{align*}
K(n\mathbb{Z}, 2p) & \longrightarrow K(\mathbb{Z}, 2p) \\
\text{BU}_p & \longrightarrow \text{BU}_p
\end{align*}
$$

If $F$ is the fibre of $\text{BU}_p \to K(\mathbb{Z}, 2p)$, consider the Lustikov system of the map.

Better work with the map

$$
\text{BU}_p \to \prod_{i=1}^{p} K(\mathbb{Z}, 2i)
$$

and try for

$$
\prod_{i=1}^{p} K(n\mathbb{Z}, 2i) \longrightarrow \prod_{i=1}^{p} K(\mathbb{Z}, 2i)
$$

Now the homotopy groups of the fibre are finite, so
so what one needs to know is

\[
\lim_{n \to \infty} H^m(\prod_{i=1}^{p} K(n\mathbb{Z}, 2i), A) = 0 \quad m > 0
\]

**Lemma:** For any finite abelian group $A$

**Proof:** By dêvissage, one can assume $A = \mathbb{Z}/p\mathbb{Z}$, by Kunneth, one can worry about $\{K(n\mathbb{Z}, 2j)\}$, and then by the spectral sequence one can use induction on $j$. For $j = 1$

\[
H^*(K(n\mathbb{Z}, 1), \mathbb{Z}_p) \quad \text{OKAY.}
\]

In geometrical terms what we have just proved is that given $u \in H^{2p}(X, \mathbb{Z})$, then $E$ bundle $E$ such that $c_p(E) = n \cdot u$, $n$ universal depending on $\dim X$. 
October 13, 1970.

Need to understand the exponential map for $GL_n(\mathbb{C})$.

$$ A \mapsto e^A = \sum_{n>0} \frac{A^n}{n!} $$

$$ \exp: \text{gl}_n \rightarrow GL_n. $$

We begin by finding the singular values of $\exp$.

$$ \frac{d}{dt} e^{(A+\varepsilon B)t} = (A+\varepsilon B) e^{(A+\varepsilon B)t} $$

$$ = Ae^{(A+\varepsilon B)t} + \varepsilon Be^{At} \quad \varepsilon^2 = 0 $$

$$ \frac{d}{dt} \left[ e^{-At} e^{(A+\varepsilon B)t} \right] = \varepsilon e^{-At} Be^{At} $$

$$ = \varepsilon \sum_{n>0} (\text{ad } (-A))^n \cdot B \frac{t^n}{n!} $$

$$ \left[ \frac{d}{dt} (e^{-At} Be^{At}) = [-A, e^{-At} Be^{At}] \right] $$

so if $e^{-At} Be^{At} = \sum t^n \alpha_n$

$$ n \alpha_n = [-A, \alpha_{n-1}] \Rightarrow \alpha_n = \frac{1}{n!} (\text{ad } (-A))^n \alpha_0 $$

Then integrating

$$ e^{-At} e^{(A+\varepsilon B)t} = I + \varepsilon \sum_{n>0} (\text{ad } (-A))^n \cdot B \frac{t^n}{(n+1)!} $$

$$ \Rightarrow e^{A+\varepsilon B} = e^A + \varepsilon e^A \sum_{n>0} \frac{(\text{ad } -A)^n \cdot B}{(n+1)!} \quad \varepsilon^2 = 0 $$
\[ e^A de^A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} [A^{n+1} + A^{n+2} + \cdots] \quad B = dA. \]

\[ \text{tr} e^{-A} de^A = \text{tr} dA = d(\text{tr} A) \]

which agrees with formulæ

\[ e^{\text{tr} A} = \det(e^A) \]

\[ e^{\text{tr} A} \cdot \text{tr} dA = \text{tr}(e^{-A} de^A) \cdot \det A. \]

For what values of \( A \) is this transformation singular?

Suppose \( A \) diagonal eigenvalues \( \{\lambda_i\}_{i=1}^n \), then

\[ [A, e_{ij}] = (\lambda_i - \lambda_j) e_{ij} \]

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} (\det A)^n \cdot (e_{ij}) = \begin{cases} 1 - e^{\frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_j}} e_{ij} & \text{if } \lambda_i \neq \lambda_j \\ 0 & \text{if } \lambda_i = \lambda_j \end{cases} \]

with understanding that the coefficient is 1 if \( \lambda_i = \lambda_j \)

\[ de^{\left[ \lambda_1, \lambda_n \right]} + e e_{ij} = e^{\lambda_i - e^\lambda_j} e_{ij} \]

Therefore, the exponential map is singular at a diagonal matrix if two eigenvalues differ by \( 2n\pi \) in each \( n \in \mathbb{Z} \) and \( n \neq 0 \).
We want to start with a Bott cocycle and reconstruct the bundle it came from. Take a 2-cocycle. If we work with $SL_2$-bundles then the Bott 4-cocycle should determine the bundle up to torsion.

So we are given

$$h_{uvw} \in \Gamma(U \cap V, \Omega^2)$$

alternating,

$$k_{uv} \in \Gamma(U \cap V, \Omega^3)$$

so $\delta h = 0$, $dh = \delta k$, $dk = 0$.

So if the covering consists of two elements $U, V$ then all we have is a single form $\omega \in \Omega^3(U \cap V)$ which we want to put in the form

$$\omega = tr(A^{-1}dA)^3$$

where $A : U \cap V \to SU_2 = S^3$. But $tr(A^{-1}dA)^3$ is a closed form on $S^3$, the invariant $\omega$ volume and so our problem is to construct a map $A : U \cap V \to S^3$ such that $\omega = A^*(\text{volume})$. Since any volume is locally $dx_1dx_2dx_3$, it's clear that this can't always be done, since there exist indecomposable closed 3 forms.

$$\dim \text{ Gr}_3(\mathbb{R}^n) = 3(n-3)$$

$$\dim \mathbb{P}(\Lambda^3(\mathbb{R}^n)) = \binom{n}{3} - 1$$
So unlike line bundles the form must be modified, and we see that the critical case is to understand the map

\[ [X, S^3] \longrightarrow H^3_{DR}(X), \]

which we know induces an isomorphism

\[ [X, S^3] \otimes \mathbb{C} \sim \longrightarrow H^3_{DR}(X) \]

(\(\otimes\) in the sense of Malcev, actually the non-abelian-ness is small since \(\pi_6(S^3) = \mathbb{Z}_{12}\).)

It seems reasonable to consider more generally the map

\[ [X, U(n)] \longrightarrow \prod_{i=1}^{\frac{n}{2}} H^{2i-1}_{DR}(X). \]

given by the map

\[ A \longrightarrow \text{tr} \ (A^{-1} dA)^{2i-1} \]
Let \( q \) be a power of \( p \) and \( l \) a prime \( \neq p \). Then I want to compute

\[
\lim_{\nu} H^*(\text{GL}_n(\mathbb{F}_q^\nu), \mathbb{Z}/l\mathbb{Z})
\]

Let \( r \mid l-1 \) be the order of \( q \) in \( (\mathbb{Z}/l\mathbb{Z})^* \). Then \( r \) is the same for \( q^\nu \) since \( l \) is prime to \( l-1 \). We know that

\[
H^*(\text{GL}_n(\mathbb{F}_q^\nu), \mathbb{Z}/l\mathbb{Z}) \rightarrow H^*(\{\mathbb{F}_{q^\nu}(\mu_l)^*\}^m, \mathbb{Z}/l\mathbb{Z})
\]

\[
m = \left[ \frac{m}{r} \right].
\]

Now:

\[
\mathbb{F}_q(\mu_l)^* \rightarrow \mathbb{F}_{q^\nu}(\mu_l)^* \rightarrow \mathbb{F}_{q^2}(\mu_l)^* \rightarrow \cdots
\]

cyclic order \( q \rightarrow q^2 \rightarrow q^4 \rightarrow \cdots \)

cyclic order \( q \rightarrow q^2 \rightarrow q^4 \rightarrow \cdots \)

cyclic order \( q \rightarrow q^2 \rightarrow q^4 \rightarrow \cdots \)

and

\[
\nu_{\mathbb{F}_q(\mu_l)}(q^\nu(\nu - 1)) = \nu_{\mathbb{F}_{q^\nu}(\mu_l)}(\nu) + \nu_{\mathbb{F}_{q^2}(\mu_l)}(\nu - 1)
\]

since

\[
\nu_{\mathbb{F}_{q^\nu}(\mu_l)}(q^\nu - 1) \geq 1
\]

and say \( l \neq 2 \). Okay once you take \( q^2 \).

So it seems then that

\[
\lim_{\nu} H^*(\{\mathbb{F}_{q^\nu}(\mu_l)^*\}^m, \mathbb{Z}/l\mathbb{Z}) = \mathbb{Z}/l\mathbb{Z}[x_1, \ldots, x_m]
\]

whence all the \( c_n(\mu_l) \) disappear in the limit.
October 14, 1970

Still want to understand

$$[X, S^3] \rightarrow H^3_{DR}(X).$$

The result is that if $\omega \in \Omega^3(X)$ is closed with integral periods, then for some $n$, $n\omega - d\eta = A^*V$, where $A : X \rightarrow S^3$ and $V$ is the volume element on $S^3$. Right invariant.

On $S^3$ there are three forms $\omega_1, \omega_2, \omega_3$ which satisfy the Maurer-Cartan formulas

$$d\omega_i = \sum_{jk} c_{ijk} \omega_j \wedge \omega_k,$$

where $c_{ijk}$ are the structural constants for the Lie algebra.

So the map $A : X \rightarrow S^3$ gives three one-forms on $X$ whose product is non-zero and which satisfy the Maurer-Cartan formulas.

Conversely, given $\lambda_i \in \Omega^3(X)$, $i=1,2,3$ satisfying M-C relations, we consider $X \times S^3$ and the ideal $I$ in the exterior algebra generated by $\mathfrak{pr}_1^*(\lambda_i) - \mathfrak{pr}_2^*(\lambda_i)$. This ideal will be stable under $d$, so defines a codimension 3 foliation of $X \times S^3$, which is taken over $X$ as the $\omega_i$ span the cotangent space of $S^3$ at each point. Note that foliation is right invariant under $S^3$ multiplication.

Consequently an integral leaf will be a covering space of $X$ mapping to $S^3$.

Conclusion: If $\pi_1(X) = 0$, then $\text{Map}(X, S^3)/S^3$ right multiplies is same as forms $\lambda_1, \lambda_2, \lambda_3 \in \Omega^3(X)$ satisfying M-C formulas.
Need to understand non-commutative integration a bit. Suppose $G$ is a nilpotent Lie group and connected. Then I claim that there are natural maps

$$G \times \cdots \times G \longrightarrow G$$

right equivariant which assigns to $\sum t_i g_i$ the appropriate center of gravity. Indeed, by induction using exact sequence

$$0 \longrightarrow \mathbb{R}^n \longrightarrow G \overset{T}{\longrightarrow} G' \longrightarrow 0$$

and we have

$$G \times \cdots \times G' \longrightarrow G'$$

which saying fixing $g_1, \ldots, g_n$ gives us $\sum t_i \Pi(\xi_i)$. Now have to check

$$\{0, \ldots, n\} \longrightarrow E \overset{F}{\longrightarrow} G \downarrow \Delta(\mathbb{R}^n) \longrightarrow G'$$

that if $f$ is an "affine" $\mathbb{R}^n$-bundle over $\Delta(\mathbb{R}^n)$ and if you give the liftings of the vertices then there is a canonical section. Need that transition functions are constant affine transformations.
Problem: To find out what is happening in the proof that $\chi: K(X) \otimes \mathbb{Q} \rightarrow H^\alpha(X, \mathbb{Q})$.

For example start with formula

$$\left[ X, \mathbb{C}P^\alpha \right] \rightarrow H^2(X, \mathbb{Z})$$

But

$$\left[ X, \mathbb{S}P^\alpha(\mathbb{C}P^3) \right]$$

What makes this result true? For example suppose we have a complex-analytic manifold $X$. Then an analytic map $X \rightarrow \mathbb{S}P^\alpha(\mathbb{C}P^3) = \mathbb{C}P^n$ is a line bundle together with $n+1$ generating sections. Thus the proof of this formula requires something about $C^\infty$ functions.

A better understanding is achieved by use of sheaf theory. Thus one looks at

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_X \xrightarrow{\exp 2\pi i} \mathcal{O}_X^* \rightarrow 0$$

and gets a long exact sequence

$$H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}_X)$$

where the ends vanish by partitions of unity. This proves the isomorphism for $X \cong C^\infty$ and of course it works to do the analytic case also.
Symmetric product

The basic idea:  \( H^*(X, \mathbb{C}(Y)) = [X, \mathbb{S}P^\infty(Y)] \) ignoring basepoint

Barry's formulation: An element of

\[
[X, \mathbb{S}P^\infty(Y)] \otimes \mathbb{Q} = \prod_{i} \text{Hom}(H_i(X), H_i(Y))
\]

map of degree zero from \( X \) to \( Y \).

Can you algebraically define a map from

\[
H^*(Y) \longrightarrow H^*(X)
\]

for each map \( X \rightarrow \mathbb{S}P^\infty(Y) \).

From the rational point of view this is easy because

\[
H^*(\mathbb{S}P^\infty(Y)) = S[H^*(Y)] \quad \text{as Hopf algebras}
\]

\( \mathbb{S} \) being the geometric Pontrjagin ring.

\[
H^*(X) \leftarrow \text{ring hom} \quad S[H^*(Y)] \quad \leftarrow \quad H^*(Y)
\]

So the correspondence is fairly clear.

Bott has produced a formula for the Chern classes of a vector bundle in terms of the transition functions for the bundle, which I want to understand.

The idea: start with $E \to X$ a complex bundle over a manifold or scheme. Then form the bundle $Y$ over $X$ whose sections are connections, i.e. the bundle of splittings of

$$0 \to E \otimes T^* \to J_1(E) \to E \to 0$$

Let $f: Y \to X$ be the canonical map. Then $f^*(E)$ has a canonical connection and so global De Rham classes representing the Chern classes. Now if we are given local trivializations

$$s_u: U \times \mathbb{C}^n \to E$$

for some covering $U$, then over each $U$ we have a canonical section $D_U$ of $f^*(E)$, hence can pull back these classes. For example $\text{tr} \{K^n_U\}$ pulls back to give an element in $\mathcal{C}^o(U, \Omega^{2n}_X)$ (which is zero because $D_U$ is flat). On $U \cap V$, then we have a family $t_0 D_U + t_1 D_V$ of connections, hence a formula

$$\text{tr} \{K^n_U\} - \text{tr} \{K^n_V\} = d \int_{t_0}^{t_1} \{\star \omega\} = d h_{U,V}^{\mathcal{A}^4}$$
and the \( h^{(n)}_{uvw} \) define an element of \( \mathcal{C}^1(V, \Omega^{2n-1}_X) \). On \( U \cap V \cap W \) we get the family \( t_0D_u + t_1D_v + t_2D_w \) of connections which should produce an element \( h^{(n)}_{uvw} \in \Gamma(U \cap V \cap W, \Omega^{2n-2}_X) \) such that

\[
dh^{(n)}_{uvw} = h^{(n)}_{vw} - h^{(n)}_{uw} + h^{(n)}_{uv}.
\]

In general one gets by this process a Čech cochain \( h^{(n)}_i = \{ h^{(n)}_{u_0 \cdots u_i} \} \in \mathcal{C}^i(V, \Omega^{2n-i}_X) \), satisfying

\[
dh^{(n)}_i = \delta h^{(n)}_{i-1}.
\]

It's more or less clear that (still needs proof to be sure)

\[
h^{(n)}_{u_0 \cdots u_i} = \int_{t_i + t_{i+1} \leq 1} \{ tr(K^n_t) \}
\]

where

\[
K_t = d\Theta_t + \Theta_t \Theta_t
\]

\[
\Theta_t = \sum_{j=1}^{\# t} t_j g^{-1}_{u_ju_0} du_ju_0
\]

(Here I recall that \( s_u = s_v g_{uv} \) and that the connection \( D_v \) is given relative to
the connection $D_u$ by the form $\theta^D_u$ determined by

$$D_v s_u = s_u \theta^D_v$$

$$D_v(s_v g_{vu}) = s_v d g_{vu} = s_u g^{-1} u d g_{vu}$$

i.e.

$$\theta^D_u = g^{-1} u d g_{vu}$$

so that family joining the $D_u$ is relative to $D_0$ given by the connection form

$$\theta^D_t = \sum_{j=1}^i t_j g^{-1} u_j d g_{u_j} u_0.$$
Computations for $n=1,2$ ignoring signs.

\[ h^{(1)}_{uv} = \text{tr} \left\{ A^{-1}dA \right\} \quad A = g_{uv} \]

\[ \begin{cases} 
  h^{(2)}_{uv} = \frac{1}{3} \text{tr} \left( A^{-1}dA \right)^3 \\
  h^{(2)}_{uvw} = \text{tr} \left\{ A^{-1}dA \cdot B^{-1}dB \right\} \quad A = g_{uv}, B = g_{uw} 
\end{cases} \]

\[ \text{d} h^{(2)}_{uv} = -\text{tr} \left( A^{-1}dA \right)^4 = 0 \]

\[ \text{d} h^{(2)}_{uvw} = h^{(2)}_{vw} - h^{(2)}_{uw} + h^{(2)}_{uv} \]

\[ (A^2 2BA') = A \left[ B' A - A' B \right] A' \]

\[ \begin{align*}
  \text{d} h^{(2)}_{uvw} &= \text{tr} \left( (A^{-1}dA)^2 (B^{-1}dB) \right) - \text{tr} \left( (A^{-1}dA)(B^{-1}dB)^2 \right) \\
  \text{d} h^{(2)}_{uvw} &= \frac{1}{3} \left[ \text{tr} \left( (BA^{-1})'d(BA^{-1}) \right)^2 \right] - \frac{1}{3} \text{tr} (B^{-1}dB)^3 + \frac{1}{3} \text{tr} (A^{-1}dA)^3 \\
  \text{d} h^{(2)}_{uvw} &= \frac{1}{3} \text{tr} \left( B^{-1}dB - A^{-1}dA \right)^3 \\
  \text{d} h^{(2)}_{uvw} &= \text{tr} \left( (A^{-1}dA)^2 (B^{-1}dB) \right) - \text{tr} \left( (A^{-1}dA)(B^{-1}dB)^2 \right) \\
  \text{d} h^{(2)}_{uvw} &= A = g_{u_1 u_6}, \quad B = g_{u_2 u_6}, \quad C = g_{u_3 u_6} 
\end{align*} \]
\[ A(B^{-1}dA - A^{-1}dA) \left( C^{-1}dC - A^{-1}dA \right) A^{-1} \]

\[ \delta h^{(2)}_2 = \text{tr} \left[ (BA^{-1})^{-1} \cdot (BA^{-1}) \cdot (CA^{-1})^{-1} \cdot (CA^{-1}) \right] \]

\[ - \text{tr} \left[ B^{-1}dB \cdot C^{-1}dC \right] \]

\[ + \text{tr} \left[ A^{-1}dA \cdot C^{-1}dC \right] \]

\[ - \text{tr} \left[ A^{-1}dA \cdot B^{-1}dB \right] = 0 \]

The above shows that normalized group cocycles might be very ugly.

However in unnormalized term we have associated to the two simplex \((A_0, A_1, A_2)\) of PG the element

\[ \text{tr} \left[ (A_1A_0^{-1})^{-1} \cdot (A_1A_0^{-1}) \cdot (A_2A_0^{-1})^{-1} \cdot (A_2A_0^{-1}) \right] \]

\[ = \text{tr} \left[ (A_1^{-1}dA_1 - A_0^{-1}dA_0)(A_2^{-1}dA_2 - A_0^{-1}dA_0) \right] \]

\[ = \text{tr} \left( A_1^{-1}dA_1 \cdot A_2^{-1}dA_2 \right) - \text{tr} \left( A_0^{-1}dA_0 \cdot A_2^{-1}dA_2 \right) \]

\[ + \text{tr} \left( A_0^{-1}dA_0 \cdot A_2^{-1}dA_2 \right) \]

The first formula makes visible the right invariance, and the last the fact it is a cocycle.
Quite generally in an exterior algebra we have the identity
\[
(\mathbf{z}_1 - \mathbf{z}_0) \wedge \cdots \wedge (\mathbf{z}_n - \mathbf{z}_0) = \sum_{i=0}^{n} (-1)^i \mathbf{z}_0 \wedge \cdots \wedge \hat{\mathbf{z}}_i \wedge \cdots \wedge \mathbf{z}_n
\]

(induction on \(n\):
\[
\sum_{i=0}^{n} (-1)^i \mathbf{z}_0 \wedge \cdots \wedge \hat{\mathbf{z}}_i \wedge \cdots \wedge \mathbf{z}_n \wedge (\mathbf{z}_{n+1} - \mathbf{z}_0)
\]
\[
\sum_{i=0}^{n} (-1)^i \mathbf{z}_0 \cdots \hat{\mathbf{z}}_i \cdots \mathbf{z}_{n+1} + \sum_{i=0}^{n} (-1)^i \mathbf{z}_0 \wedge \cdots \wedge \mathbf{z}_n
\]

Hence denoting by
\[
\varphi_n(\mathbf{z}_1, \ldots, \mathbf{z}_n) = \sum_{\sigma \in \Sigma_n} (-1)^{\pi_1(\sigma)} \mathbf{z}_{\sigma(1)} \cdots \mathbf{z}_{\sigma(n)}
\]
we have
\[
\varphi_n(\mathbf{A}_1^{-1} \mathbf{d}(\mathbf{A}_1), \ldots, \mathbf{A}_n^{-1} \mathbf{d}(\mathbf{A}_n))
\]
\[
= \varphi_n(\mathbf{A}_1^{-1} \mathbf{d} \mathbf{A}_1, \ldots, \mathbf{A}_n^{-1} \mathbf{d} \mathbf{A}_n)
\]
\[
= \sum_{i=0}^{n} (-1)^i \varphi_n(\mathbf{A}_0^{-1} \mathbf{d} \mathbf{A}_0, \ldots, \mathbf{A}_i^{-1} \mathbf{d} \mathbf{A}_i, \ldots, \mathbf{A}_n^{-1} \mathbf{d} \mathbf{A}_n)
\]

(Note that \(\varphi\) is a finitely exterior algebra because it vanishes if two \(z_i\) are equal). The first formula shows that \(\varphi_n\) is invariant under right multiplication and the last one shows it is a cocycle.
Question: Does there exist a relative K-group?

The conjecture to make is that for any R over K, the maps are isomorphisms, i.e.,

$$\chi_i : H^i (\Omega^i, \mathbb{K}) \to \mathbb{K}$$

for all $$i \geq 0$$.

Now the usual formula in that ring R they usually release

$$\chi_i \in H^i (\mathbb{K}, \mathbb{R} \to \mathbb{R}^{*})$$

And hence maps

$$\mathbb{K} \to H^i (\mathbb{K}, \mathbb{R} \to \mathbb{R}^{*})$$

Now this gives the Hodge components (up to scalars) of

$$\text{tr} (\Delta A^{-1} A \Delta A^{-1}) = \text{tr} (\Delta A^{-1} - A^{-1} \Delta A) \Delta A^{-1}$$

The next most important is the $i^{th}$ component.
to realize algebraically the idea of the topology on $\text{GL}_n(R)$ forcing one to use a different kind of classifying space. So instead of thinking of $K_*(R)$ as related to $K(S \times R) \to H^*(S)$ where $S$ is a variable topos we want to allow $S$ to be an arbitrary $k$-scheme.
Recall the Dold-Thom theorem: Let $X$ be a connected space with basepoint. Then
\[ \pi_i \; SP^\infty(X) \cong \tilde{H}_i(X; \mathbb{Z}) \]
They prove this by showing that given a cofibration
\[ Y \to X \to X/Y \]
(but both $Y$, $X$ are pointed and connected) then
\[ SP^\infty(X) \to SP^\infty(X/Y) \]
is a quasi-fibration with fiber $SP^\infty(Y)$, hence one gets a long exact sequence
\[ \to \pi_i SP^\infty(Y) \to \pi_i SP^\infty(X) \to \pi_i SP^\infty(X/Y) \to \cdots \]

Thus the functor $F_*(X) = \pi_*(SP^\infty(X))$ for pointed connected spaces is a generalized homology theory, and the only thing left is to identify $SP^\infty(S^1)$. But $S^1$ being a topological abelian group one knows there are maps
\[ S^1 \to SP^\infty(S^1) \to S^1 \]
which one would like to know are homotopy equivalence. Doesn't seem to be entirely trivial, however $SP^n(C^*)$ can be identified with monic polynomials $Z^n + q_1Z^{n-1} + \cdots + q_n$ where an is a unit. This gives a fibration $SP^n(C^*) \to C^n$. 

October 24, 1970: On symmetric products.
whose fiber is a vector bundle of dimension n-1. So now everything is clear.

Next I want to see that

\[
[Y; \mathcal{SP}^\infty X]_0 = \text{Hom}_{D(\mathbb{A}^n)}(\tilde{\mathcal{C}}(Y), \tilde{\mathcal{C}}(X))
\]

enough to define the map really and that works this way

\[
[Y; \mathcal{SP}^\infty X]_0 \rightarrow \text{Hom}_{D(\mathbb{A}^n)}(\tilde{\mathcal{C}}(Y), \tilde{\mathcal{C}}(SP^\infty X))
\]

so we need a map

\[
\tilde{\mathcal{C}}(SP^\infty X) \rightarrow \tilde{\mathcal{C}}(X).
\]

But semi-simplicially this is obvious, namely you have dimension-wise a map

\[
SP^\infty(X) \rightarrow \tilde{\mathbb{Z}}X \quad (\tilde{\mathbb{Z}}X = \mathbb{Z}/\mathbb{Z}^*),
\]

which extends to

\[
\tilde{\mathbb{Z}} SP^\infty(X) \rightarrow \tilde{\mathbb{Z}}X
\]

in a canonical way.

To see if this can be understood geometrically. Thus if X is a space I want to define a map

\[
\tilde{H}^\mathbb{R}_*(SP^\infty X) \rightarrow \tilde{H}_*(X)
\]
This must be something like the transfer in the Borel book. I recall this.

Suppose $G$ finite acts on $X$ Hausdorff. Then for $F$ on $X/G$ we have

$$(f_x f^* F)_y = \prod_{x \in f^{-1}(y)} F_y$$

and we want to define a map

$$f_x f^* F \longrightarrow F.$$ 

The obvious thing to try is the sum map

$$\prod_{x \in f^{-1}(y)} F_y \longrightarrow F_y$$

$$(a_x) \longrightarrow \sum a_x$$

Unfortunately, if $F = \mathbb{Z}$, then we have that the composite map

$$\mathbb{Z} \longrightarrow f_x f^* \mathbb{Z} \longrightarrow \mathbb{Z}$$

is multiplication by $\text{card } f^{-1}(y)$ on fibers over $y$ which won't be locally constant. Hence we need a multiplicity function $x \mapsto m(x)$ which gives the multiplicity of $x \mapsto$ in the fiber $f^{-1}(y)$.

Thus I need to have a continuous map

$$X/G \longrightarrow SP^n(X)$$
\[ y \mapsto f^{-1}(y) \] counted with multiplicity which assigns to \( y \in X/G \) the divisor
\[ \sum_{x \in f^{-1}(y)} m(x) \]

But if \( n = |G| \) then the obvious multiplicity function is
\[ m(x) = \card G_x \]

but the most efficient multiplicity function it appears is when
\[ n = \text{l.c.m.} \left\{ \card f^{-1}(y) : y \in Y \right\} \]

and then
\[ m(x) = \frac{\card G_x}{\text{g.c.d.} \left\{ \card G_x \right\}} \]

Thus when \( X \) has one orbit type \( G/H \) we can take all \( m(x) = 1 \).

So now given a multiplicity fn. \( m(x) \) we define the trace
\[ f \mapsto F \xrightarrow{\text{tr}} F \]

by
\[ Ty \in X \mapsto F_y \quad \text{for} \quad \{ f \in \mathcal{F} : y \in f^{-1}(y) \} \]

\[ (a_x) \mapsto \sum m(x) a_x . \]

To see this is well-defined we have to show
that it maps continuous sections of \( f^* F \) to continuous sections of \( F \). Work near \( y_0 \) and suppose we have elements \( a_x \in F_{y_0} \) for \( x \in f^{-1}(y_0) \). Then over some subnb. \( U \) of \( y_0 \) we get sections \( s_x \in \Gamma(U, F) \Rightarrow s_x(y_0) = a_x \) for \( x \in f^{-1}(y_0) \).

I want to show that

\[
\sum_{x \in f^{-1}(y)} m(x') s_x(y')
\]

is continuous for \( y' \) near \( y_0 \). However if \( y' \) is really close to \( y_0 \), then each \( x \) is closed to only one \( x' \) and then

\[
\sum_{x \text{ closed to } x'} m(x) = m(x)
\]

so this is all clear.

---

**Conclusion:** Whenever a finite group \( G \) acts on a space \( X \) there is a natural trace map on cohomology

\[
f_* : H^i(X; \Lambda) \longrightarrow H^i(X/G; \Lambda)
\]

satisfying \( f_* f^* = |G| \) \( f^* f_* = \sum_{g \in G} g^* \). The universal situation is to define

\[
H^i(X; \Lambda) \longrightarrow H^i(\# S^p(X); \Lambda)
\]

which is the equivariant sum \( u \mapsto \sum p_x^* u \).
Definition: Recall that
\[ F \sim (f^*f^*)^G \]
and summing over the group defines a map
\[ f^*F \sim (f^*f^*)^G F. \]
Thus one gets the required map
\[ H^i(X; f^*F) \leftarrow H^i(X/G, f^*f^*F) \]
\[ \xrightarrow{\sigma} \]
\[ H^i(X/G, F) \]