August 11, 1970

The immeuble for $\Sigma_n$: Recall that the immeuble for $\text{GL}_n(k)$ is the simplicial complex associated to the ordered set of proper subspaces of $k^n$. Thus, an $i$-simplex is an increasing family $0 < V_0 < \cdots < V_i < k^n$ of subspaces. This complex is of dimension $n-2$ and Serre, I believe, told me that it has the homotopy type of a bouquet of $(n-2)$-spheres.

The analogous simplicial complex for $\Sigma_n$ has for its $i$-simplices a chain $\emptyset < V_0 < \cdots < V_i < \{1, \ldots, n\}$ of proper subsets. Thus this simplicial complex is the barycentric subdivision of the simplicial complex of proper subsets of $\{1, \ldots, n\}$ and so it is $\Delta(n-1)^*$ which has homotopy type of $S^{n-2}$.
legal claims to prove that an invertible $H$-space
with traces $B_0$ extends to a $\Omega$-spectrum $\{B_n\}$, but
he doesn't show that the two categories are

equivalent. The first question is whether $\{B_n\} \rightarrow B_0$
is faithful, or equivalently given a map $\{B_n\} \rightarrow \{B_n'\}$
of connected $\Omega$-spectra such that $B_0 \rightarrow B_0'$ is the
gro-oro map, does it follow that $B_0 \rightarrow B_0'$ is zero for all

\[
\text{Related question: Let } X \text{ be a complex which begins in dimension } V \text{ (V-1)-connected). Then is } S^X = \{S^X\}
a retract of } \quad S^{S^X} \text{ for some } Y?\]
August 22, 1970: The Tits complex of $\text{GL}_n(k)$.

Let $V$ be an $n$-diml. vector space over $k$. The Tits complex $\Delta(V)$ is the simplicial complex of dim $n-2$ whose vertices are the proper subspaces of $V$ and whose $i$-simplices are chains $0 < W_0 < \cdots < W_i < V$ of $i+1$ proper subspaces. In other words, $\Delta(V)$ is the simplicial complex associated to the partially-ordered set of proper subspaces of $V$.

Claim: $\Delta(V)$ has the homotopy type of a bouquet of $(n-2)$-spheres. Assume this and let's compute the number for $k$ finite, card $(k) = q$, using Euler characteristics. We break up the simplices according to orbits under action of $\text{Aut}(V)$. One gets an orbit for each increasing sequence $0 < j_0 < j_1 < \cdots < j_r < n$ and the stabilizer of this is

\[
\begin{array}{ccc}
| j_0 | * & * & * \\
| j_1 - j_0 | * & * \\
| j_2 - j_1 | * \\
| j_r - j_{r-1} | *
\end{array}
\]

which has order $\prod (q^{a_i-1} - 1)^{\frac{n(n-1)}{2}}$ for $1 \leq a_i < b_i \leq a_i + 1$. 

\[
\alpha_0 = j_0 \\
\alpha_1 = j_1 - j_0 \\
\alpha_2 = j_2 - j_1 - 1 \\
\alpha_i
\]
So the Euler characteristic is something like
\[
\sum_{d=1}^{n-1} (-1)^{d+1} \sum_{d \geq 0} \frac{n}{b=1} g_{b-1}^{d+1} = \prod_{a=0}^{\ell-n} (g_{b-1})^{a_n}
\]

\[\sum a_n = n\]

E.g.
\[
\begin{align*}
n &= 2 & \chi &= g + 1 \\
n &= 3 & \chi &= -g^3 + 1 \\
n &= 4 & \chi &= g^6 + 1
\end{align*}
\]

Therefore we want to prove that \( T(e)(V) \) is of the homotopy type of a bouquet of \( 8 \) \( (n-2) \)-spheres.

We use induction on \( n \). Let us write \( V = L \oplus V' \). Let \( Z \subseteq T(e)(V) \) be the subcomplex with vertices

\[ Z : \{ W \mid W + L / L \text{ proper subspace of } V / L \} \]

I claim that \( Z \) is of the homotopy type of \( T(e)(V') \). Indeed there is an evident embedding of \( T(e)(V') \rightarrow Z \) and a retraction given by the projection of \( V \) on \( V' \). I claim this is a deformation retraction. Indeed given a vertex \( W \) of \( Z \) it can be joined to \( W + L \) which is joined
to \( f(W) \) where \( f(W) \in V' \) is \( f(W) = (W + L) \cap V' \).
Moreover a simplex \( \{ W_0, \ldots, W_i \} \) is homotopic to the simplex \( \{ W_0 + L, \ldots, W_i + L \} \) which is homotopic to \( \{ f(W_0), \ldots, f(W_i) \} \). More precisely define
\[
\begin{align*}
Z \times [0, 1] &\quad \xrightarrow{h} \quad Z' \times [0, 1] \\
\{ W, 0 \} &\quad \mapsto \quad W \\
\{ W, 1 \} &\quad \mapsto \quad W + L
\end{align*}
\]
and follow this homotopy by
\[
\begin{align*}
Z' \times [0, 1] &\quad \longrightarrow \quad Z \\
\{ W, 0 \} &\quad \longrightarrow \quad W \cap V' \\
\{ W, 1 \} &\quad \longrightarrow \quad W
\end{align*}
\]
where \( Z' = \{ W \in Z \mid W \supseteq L \} \).

It follows that the inclusion
\[
\text{Cone } \text{Tit}(V') \longrightarrow Z \times \{ V' \}
\]

is a homotopy equivalence, hence \( Z \times \{ V' \} \) is contractible.

Now to go from \( Z \times \{ V' \} \) we must add the vertices...
Let $H$ be a hyperplane $\neq V'$. Then for any $Z \cap V \subseteq \Gamma \subset Z$

$$\left(Z \times \{ \Gamma \} \right) \times \{ \{ H \} \} = \left(Z \times \{ \Gamma \} \right) U \cong \text{Tits}(H)$$

$\Gamma$ in adding a vertex $H$ we are attaching the cone on a wedge of $S^{n-3}$'s. So adding one $H$ at a time inductively one sees that $\Gamma$ is a wedge of $S^{n-2}$'s. ($\Gamma$ always $n-3$ connected).

So we get to $\mathcal{Z} - \{ L \}$, but adding $L$ is like putting the cone on $\text{Tits}(V\{1\}L)$ so again add more $S^{n-2}$'s. If total number of such spheres is $T_n$ then

$$T_n = \frac{1}{2} \left\{ \text{card } H + 1 \right\} T_{n-1}$$

$$= \left\{ \frac{n-1}{2} + 1 \right\} T_{n-1}$$

So

$$T_n = \frac{n(n-1)}{2}$$

as it should be.