$$\mathcal{U}(X\times S) \longrightarrow \dot{\prod_{k\geq 1}} \mathcal{U}_{\Sigma_k}(X^k\times S^k) \longrightarrow \dot{\prod_{k\geq 1}} \mathcal{U}_{\Sigma_k}(X\times S^k_{reg})$$

$$\longrightarrow \dot{\prod_{k\geq 1}} \mathcal{U}(X\times S^k_{reg}/\Sigma_k) \quad ?$$

$$S^k_{reg}\times S^l_{reg} \subset S^{k+l}$$

$$\Sigma_k \quad \Sigma_l$$

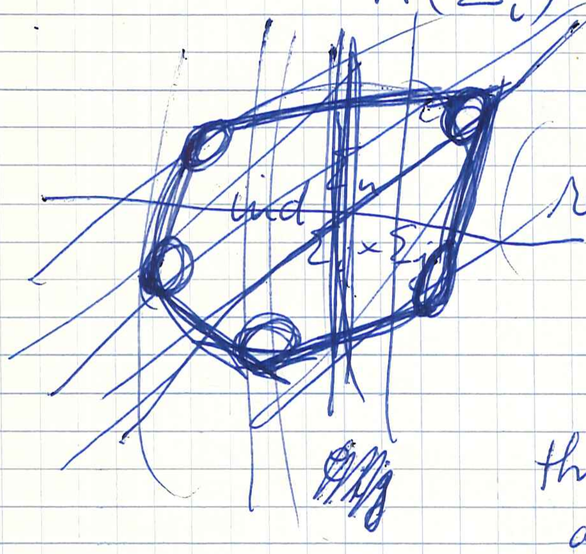
$$\begin{tikzcd}
 & S^{k+l}_{reg} \arrow[d] \\
\Sigma_{k+l}\times_{\Sigma_{k,l}}(S^k_{reg}\times S^l_{reg}) \arrow[r] \arrow[d, leftrightarrow] & B \arrow[d] \\
\Sigma_{k+l}\times_{\Sigma_k\times\Sigma_l}(S^k\times S^l) \arrow[r] & S^{k+l}
\end{tikzcd}$$

$$\dot{\prod_{n\geq 1}} R(\Sigma_n)\otimes \underline{K(X)}$$

$$\dot{\prod_{n\geq 1}} R(\Sigma_n)$$

$\underline{(\alpha_n)}$ $\underline{\mathrm{res}^{\Sigma_n}_{\Sigma_i\times\Sigma_j}\alpha_n = \alpha_i\otimes\alpha_j}$

$$R'(\Sigma_i)\otimes R'(\Sigma_j) \longrightarrow R'(\Sigma_n)$$



$$\left(\mathrm{res}^{\Sigma_n}_{\Sigma_i\times\Sigma_j}\alpha_n\right) = \alpha_n\,[\underbrace{\Sigma_n/\Sigma_i\times\Sigma_j}_{e_{ij}}]$$

think true that ~~at~~ at least after passing to filtration ~~we get~~

this means that α_n known for n not a power of a prime

$R(\Sigma_n)$ has bases $[\Sigma_n : \Sigma_{\omega_1}\times\cdots\times\Sigma_{\omega_r}]$ where $\omega_1,\dots,\omega_r$ is a partition of n.

$\mathbb{Z} + \prod_{n\geq 1} R(\Sigma_n)$ forms a ring with mult. given by induction

$$\mathbb{Z} + \prod_{n\geq 1} R(\Sigma_n) = \mathbb{Z}[e_1, e_2, \cdots]$$

$e_i = [\Sigma_i : \Sigma_1\times\cdots\times\Sigma_1]$. reg. repn of Σ_i

$$\prod_n R(\Sigma_n) \longrightarrow \prod_n \prod_{i+j=n} R(\Sigma_i') \otimes R(\Sigma_j)$$

should give us a diagonal on $\bigoplus_{n\geq 0} R(\Sigma_n')$

$$\Delta e_n = \sum_{i+j=n} \operatorname{res}^{\Sigma_n}_{\Sigma_i\times\Sigma_j} \operatorname{ind}^{\Sigma_n}_{(\Sigma_1)^n} 1$$

$$\begin{tikzcd} & 1 \arrow[d, "i"] \\ \Sigma_i\times\Sigma_j \arrow[r, "j"] & \Sigma_n \end{tikzcd}$$

$\boxed{g_{(\mu,\nu)} \quad (i,j)\text{-shuffle}}$

$K = \Sigma_i \times \Sigma_j$ reg. repn.

$$\frac{n!}{i!\,j!}\, e_i e_j$$

Thus it would seem that

$$\Delta e_n = \sum_{i+j=n} \binom{n}{i} e_i \otimes e_j$$

exactness special case of K theory and homology modulo p.

Is Δ a ring homomorphism ie.

$$\sum_{i+j=n} \operatorname{res}^{\Sigma_n}_{\Sigma_i\times\Sigma_j} \operatorname{ind}^{\Sigma_n}_{\Sigma_a\times\Sigma_b} x\otimes y$$

so I believe same argument works.

basic exact sequence

$$E_2^{pq} = H^p(X/G, x \mapsto Q^q(G_x))$$

so this corresponds to a genuine localization

Conclusion: for coh. theories in general one must consider the ~~coeff.~~ system

$$H \rightsquigarrow Q(G/H)$$

finite gps. Do two maps

$$G/H \rightrightarrows G/K$$

induce same map

$$Q(G/H) \leftleftarrows Q(G/K) \ ?$$

Question: Do inner autos of G act trivially on $\Omega_G(X)$?
(clear if G connected)

~~$\Omega_G(pt)$~~ $\quad R(G) \quad \cong V$

$$X \xrightarrow{\varphi} X^h$$

$$\varphi(gx) = hgh^{-1}\varphi(x)$$

$\Omega_G(pt)$

$$\begin{tikzcd} Q(G) & Q(G/H) \arrow[l] \arrow[d] \\ & Q(G/H) \arrow[ul] \end{tikzcd}$$

~~means that~~

$$G/H \xrightarrow{\theta} G/H$$

$$H \longmapsto gH$$

$$xH \longmapsto xgH$$

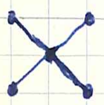
$$\begin{tikzcd} Q_G(G/H) \arrow[r, "\theta"] & Q_G(G/H) \\ Q_H(pt) \arrow[u] \arrow[r, "\theta_g"] & Q_H(pt) \arrow[u] \end{tikzcd}$$

$$HgH = gH \implies gHg^{-1} = H.$$

The ~~first~~ basic idea is to take the basic Steenrod operations

$$Q_n : U^*(X) \longrightarrow U(B\mathbb{Z}_n \times X)$$

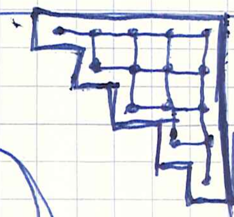
~~[illegible]~~ and take inverse limit of some kind to get a non-torsion operation.



systematic study of the operations

$$U(X) \longrightarrow U(B\Sigma_n \times X)$$

and ~~its~~ relation with Witt vectors



Key theme question: Analogy between typical reductions a la Cartier and the sylow reduction to the sylow subgroup.

Ideas:

$$Q_n : U(X) \longrightarrow U_{\Sigma_n}(X^n)$$

$$Q_n(x+y) = ?$$

$$(Z \cup W)^n \longrightarrow X^n$$

$n=i+j$ $\coprod_{i+j=n} \Sigma_n \times_{\Sigma_i \times \Sigma_j} (Z^i \times W^j)$ is orbit decomp.

$\Sigma_n \times_{(\Sigma_i \times \Sigma_j)} (Z^i \times W^j)$ is orbit where i come from Z
j —— W

This gives us the formula

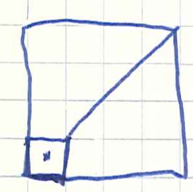
$$Q_n(x+y) = \sum_{i+j=n} \mathrm{ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} \{Q_i x \boxtimes Q_j y\} \in U_{\Sigma_n}(X^n)$$

Be more careful.

Suppose L line bundle over X.
Then you have

$$\begin{tikzcd}
{} \arrow[r] & U(DL, SL) \arrow[r] & U(DL) \arrow[r] & U(SL) \arrow[r, "\delta"] & {} \\
{} \arrow[r] & U(DL^k, SL^k) \arrow[r] \arrow[u] & U(DL^k) \arrow[r] \arrow[u] & U(SL^k) \arrow[r, "\delta"] \arrow[u] & {}
\end{tikzcd}$$

and hence a map of Gysin sequences



$$\begin{tikzcd}
{} \arrow[r] & U(X) \arrow[r, "e(L)"] & U(X) \arrow[r] & U(SL) \arrow[r, "\delta"] & {} \\
{} \arrow[r] & U(X) \arrow[r, "e(L^k)"] \arrow[u, "\frac{e(L^k)}{e(L)}"'] & U(X) \arrow[r] \arrow[u] & U(SL^k) \arrow[r, "\delta"] \arrow[u] & {}
\end{tikzcd}$$

This means that I ought to be able to take the inverse limit of the Gysin sequences

$$\xrightarrow{\delta} U(X) \xrightarrow{e(L^k)} U(X) \xrightarrow{\pi^*} U(SL^k) \xrightarrow{\delta}$$

The relation between the Steenrod operations

$$\cancel{U(X) \xrightarrow{\pi^*} U(SL^k)}$$

$$U_{S^1}(X) \longrightarrow U_{S^1}(S(X^n) \times X)$$
$$\cong$$
$$U_{\mu_n}(X)$$

$$U_{S^1}(\mathcal{S}(X^n) \times X) \longleftarrow U_{S^1}(\mathcal{S}(X^{\otimes n}) \times X)$$
$$\cong$$

Conclude that the map is the restriction

$$res: \quad \mathcal{U}_{\mu_{nq}}(X) \longrightarrow \mathcal{U}_{\mu_n}(X)$$

$$\boxed{\mu_n \hookrightarrow \mu_{nq}}$$

what happens for cohomology.

$$H^2(\mu_n, \mathbb{Z}) \simeq Hom(\mu_n, S^1) = \mathbb{Z}/n\mathbb{Z}$$

looks good because to rest. corresp surj.

what about higher cohomology ~~all zero?~~ X

$$\begin{tikzcd}
H^2(\mu_n, \mathbb{Z}) \arrow[r, phantom, "\cong"] & \mathbb{Z}/n\mathbb{Z} \\
H^2(\mu_m, \mathbb{Z}) \arrow[u, leftrightarrow] \arrow[r, phantom, "\cong"] & \mathbb{Z}/m\mathbb{Z} \arrow[u]
\end{tikzcd}$$

and this works after tensoring

Next can you do something with Steenrod operations.

$$\mathcal{U}_{\mu_n}(X) \xleftarrow{\text{rest}} \mathcal{U}_{\mu_{nq}}(X)$$

$$Q_{nq}$$

~~Difficulties~~

$\cancel{H_{\mathbb{Z}_2}(X^2)}$

so restrict to a given prime. Then one knows that

$$(\alpha_n) \qquad \operatorname{res}^{\Sigma_n}_{\Sigma_i\times\Sigma_j}(\alpha_n) = \alpha_i \otimes \alpha_j$$

$$\text{but} \qquad [\Sigma_n : \Sigma_i \times \Sigma_j] = \frac{n!}{i!\,j!} = \binom{n}{i}$$

if ~~$n \ne p^a$~~ $\qquad n =$

$$\underline{F(H)} \underset{\text{ind}}{\overset{\text{res}}{\rightleftarrows}} \underline{F(G)}$$

$$\text{ind} \circ \text{res} = [G:H].$$

$$\iota_*(\iota^* x) = \iota_* 1 \cdot x, \text{ basic fmla.}$$

Thus

$$\operatorname{ind}^{\Sigma_n}_{\Sigma_i\times\Sigma_j} \alpha_i \otimes \alpha_j = [\Sigma_n / \Sigma_i \times \Sigma_j] \cdot \alpha_n$$

unit in case $\binom{n}{i}$ prime to p.

But this will be the case for some $1 \le i \le n-1$ if n not a power of p.

$$n = p^a h \qquad (h,p) = 1$$

$$(1+X)^n \equiv (1+X^{p^a})^h \equiv 1 + hX^{p^a} + \cdots \qquad \text{mod } p$$

$$\binom{p^a h}{p^a} \equiv h \quad \text{mod } p.$$

This shows that

$F_4 \longrightarrow \tilde{F}_2$ onto ?

$\underset{3\quad\ 2\quad\ 1}{(\gamma, 0, \beta, 0)}$ $\quad (\beta, 0) \quad$ $\beta \in U_{\Sigma_2}(X^2)$ $\qquad \begin{array}{c} Z \times Z \\ \downarrow \\ X^2 \times X^2 \end{array}$

rest $\gamma = \beta \otimes \beta$? $\qquad \beta$

now we do have a $\beta \otimes \beta$ which is ~~is~~ $U_{(\Sigma_4)_2}(X^4)$

$\uparrow$ sylow 2 subgps

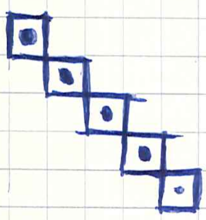
hence if we set $\gamma = \mathrm{ind}_{(\Sigma_4)_2 \to \Sigma_4} \beta \cdot \beta$

~~it is~~ then

$$\mathrm{res}^{(\Sigma_4)_2}_{\Sigma_2^{\times 2}} \, \mathrm{res}^{\Sigma_4}_{(\Sigma_4)_2} \gamma = \frac{4!}{8} \beta \otimes \beta = 3 \, \beta \otimes \beta$$

and so we can get $3(\beta \otimes \beta)$

Also



$$\mathrm{res}^{\Sigma_4}_{\Sigma_1 \times \Sigma_3} \gamma = \mathrm{res}^{\Sigma_4}_{\Sigma_1 \times \Sigma_3} \, \mathrm{ind}_{(\Sigma_4)_2 \to \Sigma_4}$$

= sum over double cosets

in this case $\exists$ single orbit of Σ_3 on $\Sigma_4 / (\Sigma_4)_2$

Σ_3 on Σ

thus $\mathrm{res}^{\Sigma_4}_{\Sigma_1 \times \Sigma_3} = \mathrm{ind}^{\beta \otimes \beta}_{\Sigma_1 \times \Sigma_1 \times \Sigma_1 \times \Sigma_1 \to \Sigma_1 \times \Sigma_3} = 0.$

would need $\frac{1}{3}\gamma$. Thus get onto provided can invert 3.

$U^{d*}_{\Sigma_d}(X^d)$

$\underline{X^d}$

conclusion is that

$$\tilde{Q} : U^*(X) \longrightarrow \dot{\prod_{n\geq 1}} U^{n*}_{\Sigma_n}(X^n)$$

is a well-defined ring homomorphism. | Gysin homom.

$$\dot{\prod_{n\geq 1}} U^{n*}_{\Sigma_n}(X^n) \xrightarrow[\text{ring hom.}]{\tilde{\Delta}} \dot{\prod_{n\geq 1}} U^{n*}_{\Sigma_n}(X) = F(X)$$

filtration of $F(X)$

$$F(X) \longrightarrow U(X).$$

various quotients of $F(X)$:

$$F_k(X) = \dot{\prod_{1\leq n\leq k}} U^{n*}_{\Sigma_n}(X)$$

gives an inverse system

$$\cdots \longrightarrow F_3(X) \longrightarrow F_2(X) \longrightarrow F_1(X) = U(X)$$

Question: Is this surjective?

$(\beta,\alpha) \qquad \alpha \qquad \text{rest}\,\beta = \alpha\otimes\alpha$

$$\begin{tikzcd}
0 \arrow[r] & \underbrace{\tilde{U}_{\mathbb{Z}_2}(X)}_{(\beta,0)} \arrow[r] & F_2(X) \arrow[r] & U(X) \arrow[r] & 0 \\
 & \arrow[r] & F_3(X) \arrow[u] \arrow[r] & U(X) \arrow[u] &
\end{tikzcd}$$

given $\quad \beta \in U_{\mathbb{Z}_2}(X^2) \ni \text{rest}\ \beta \overset{0\otimes 0}{=}$ in $U(X^2) = 0$

want $\quad \gamma \in U_{\Sigma_3}(X^3) \ni \text{rest}^{\Sigma_3}_{\Sigma_1\times\Sigma_2}\ \gamma = 0$

no problem. thus $F_3 \longrightarrow F_2$ onto

Program — Dyer-Lashof operations
~~Steenrod operations~~

basic periodicity, duality.

maybe you can try to define

$$U(X) \longrightarrow U(X \times BS^1)$$

by a limiting procedure.

$$\boxed{U_{\Sigma_p} \times U_{\Sigma_q}}$$

$$U(X) \longrightarrow U(X\times$$

$$U^{q-2}(BS^1\times X) \xrightarrow{uc(\theta(k))} U^q(BS^1\times X) \longrightarrow U^q(B\mu_k \times X) \xrightarrow{\delta} U^{q-1}(BS^1\times X)$$

let $k \longrightarrow \infty$ and note that

$$\begin{array}{ccccc} \mu_k & \longrightarrow & S^1 & \xrightarrow{k} & S^1 \\ \downarrow & & \downarrow 1 & & \downarrow m \\ \mu_{km} & \longrightarrow & S^1 & \xrightarrow{km} & S^1 \end{array}$$

commutes

$$U^*(BS^1\times X) \longrightarrow U^*(BS^1\times X) \longrightarrow$$

June 16, 1970:

exact sequences in alg. K-theory: Suppose A is a discrete val. ring with residue field k and quotient field L. Then I want a long exact sequence

$$\cdots \longrightarrow K_i(k) \longrightarrow K_i(A) \longrightarrow K_i(L) \xrightarrow{\partial} K_{i-1}(k) \longrightarrow \cdots$$

and this should result as the long exact homotopy sequence of a fibration.

first special case: If G is finite one has

$$R_k(G) \xrightarrow{i_*} R_A(G) \xrightarrow{j^*} R_L(G) \longrightarrow 0$$

defined as follows: j^* just $E \longmapsto E \otimes_A L$

i_*: given V over k we can find an E_0 over A (free as an A-module) $\ni$ $E_0 \longrightarrow V \longrightarrow 0$; this uses G-finite, I think, because you take $E_0 = A[G] \otimes_A P$ where $P \to V \to 0$ and P is a projective A-module. Then one has

$$0 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow V \longrightarrow 0$$

and define

$$i_*[V] = [E_0] - [E_1].$$

The fact that i_* is well defined is clear from basic properties of projective resolutions. This will all become much clearer maybe if one introduced the Grothendieck groups of complexes.

It is clear that $j^* i_* = 0$. To prove exactness one has a map p: Coker $i_* \longrightarrow R_L(G)$ and one produces a section s as follows. Given W over L one chooses an invariant lattice E and sets $s[W] = [E] + \mathrm{Im}\, i_*$. As in Serre's book two different choices E, E' have the property that $[E] - [E'] \in \mathrm{Im}\, i_*$, so s is well-defined and clearly a homomorphism. Also $ps = id$ and $sp = id$ are clear.

Now you want to understand this argument in general. You need to get rid of finiteness first of all. Now the only reasonable way of getting the desired fibration is to understand in advance what are the categories involved.

Thus to L we must associate the category of of f.g. L-modules. And we want to examine Euler characteristics. Thus given a simplicial set X we want to consider q-cochains associating to each q-simplex of X an L-module and to each $(q+1)$-simplex a long exact sequence such that certain compatibility conditions hold.

~~The problem is to show that all of these are distinct.~~
~~But suppose~~

what should the good proof be?

G alg. group over $S = \operatorname{Spec} A$

$$\begin{tikzcd} G_K \arrow[r] \arrow[d] & G \arrow[d] & G_k \arrow[l] \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} A & \operatorname{Spec} k \arrow[l] \end{tikzcd}$$

assume this is a smooth group like $G = \mathbb{G}_m$, more generally a torus T. Then I want to relate what goes on in char p with char. 0.

$k = \mathbb{F}_q$, T. Then have Frobenius endomorphism σ of T which is composition of q and an autom. θ of finite order

$$\begin{array}{c} T_{\mathbb{F}_q} \\ \downarrow \\ \operatorname{Spec} \mathbb{F}_q \end{array}$$

I originally wanted to understand cohomology of $T(\mathbb{F}_q)$ and ~~so~~ went to algebraic closure ~~and that~~ where you have exact sequence

$$0 \longrightarrow T(\mathbb{F}_q) \longrightarrow T \xrightarrow{\sigma - id} T \longrightarrow 0$$

What is a virtual bundle?

Let X be a nice space, then what is a virtual $\mathbb{R}$-vector bundle over X?

$$\begin{array}{c} k(S \times X) \\ \downarrow \varphi \\ H(S, A) \end{array}$$

φ natural defines a map from $K_a(X) \longrightarrow H^{8-a}(A)$

you want to work over the rationals in which case

$$K_*(X) \longrightarrow P H_*(B(X))$$

and so $K_*(X)$ appears as the universal ring hom

$$\boxed{ch_{univ} : k(S \times X) \longrightarrow H^*(S) \otimes K_*(X)}$$

Now the problem is to what extent such a thing is determined by line bundles.

$$ch : k_G(X) \longrightarrow H^*(G) \otimes K_*(X)$$

Suppose that E is a vector bundles over X, then

$$k_G(PE) = k_G(X)[T]/(\lambda_{-T}(E))$$

and so

$$K_*(PE) = K_*(X) \otimes_{K(X)} K(PE).$$

Question: Can you construct a Chern theory "over" X with coefficients in $K_*(X)$?

~~More precisely if $Y \xrightarrow{f} X$ is a morphism of ringed topoi and if E is a vector bundle over Y want~~

Thus I want to consider schemes of finite type over X and the universal Chern theory is functor

$$Y \longmapsto h(Y)$$

having Gysin homomorphism and satisfying the projective bundle theorem. Now if X is a field I know this universal animal is just the Chow ring, so I haven't taken into account the other K_i, i.e. the effect of Künneth and the group action.

$$k(S \times X) \longrightarrow H^*(S)$$

(additive) Thus if F is a field I want a universal homomorphism

$$kF(G) \longrightarrow H^*(G) \otimes K_*(F)$$

In particular $K_q(F) \otimes \mathbb{Q}$ should appear with a universal additive map

$$kF(G) \longrightarrow H^q(G) \otimes K_q(F).$$

better: $H^q(G, K_q(F)_{\mathbb{Q}})$

June 18, 1970:

In connection with your approach to K-theory and the recent success of the π_1-surgery ~~[illegible]~~ Novikov and Kirby's work, one would like to know just how much of a cohomology theory can be recovered from knowing it on spaces of the form BG. For example one knows that the suspension of T^n is a wedge of spheres, hence by just looking at the effect on $B\mathbb{Z}^n$ one recovers the homotopy groups.

Related problem: Let $\mathcal{C}$ be the category of groups up to inner autos, P a prime field, and denote by ~~[illegible]~~ $H.F$ the homology of a functor in $\mathcal{C}^\wedge$ as in K-theory $\mathbb{F}_q$ paper. Given a space Z, set $h_Z(G) = [BG, Z]$ (without fixing basepoint), then there's an obvious map

$$H^i(Z, M) \longrightarrow H^i(h_Z, M)$$

and the question is whether this is an isomorphism, i.e. is the map

$$H_i(h_Z) \longrightarrow H_i(Z)$$

an isomorphism.

Special case: $Z = K(P, n)$ Eilenberg-MacLane spaces. In this case write $h_Z = h^n$; then we want to know if

$$H_\bullet(h^n) \longrightarrow H_\bullet K(n) \qquad K(n) = K(P, n)$$

is an isomorphism of Hopf algebras. We will prove that it is surjective.

To prove surjectivity, it is enough to show

$$2H_\cdot(h^n_{\mathbb{M}}) \longrightarrow 2H_\cdot K(n)$$

is surjective, i.e. that any additive cohomology operation from H^n is determined by its effect on spaces of the form BG. ~~However one knows already that any additive operation is~~ Recall that

$$R_\cdot = \bigoplus_n 2H_\cdot K(n)$$

is a ring in such a way that

$$\theta : H^\cdot(X) \longrightarrow H^\cdot(X) \hat{\otimes} R.$$

is universal ring homomorphism, and that

$$S[\xi_0, \xi_1, \ldots\ldots] \xrightarrow{\sim} R_\cdot \qquad \text{where}$$

$$\theta(x) = \sum_{i\geq 0}' x^{2^i}\xi_i \qquad x \in H^1$$

(here I take $P = \mathbb{Z}/2$, but analogous formulas hold for other P.)

~~Next let Suppose~~ Next set

$$R'_\cdot = \bigoplus_n 2H_\cdot(h^n_{\mathbb{M}})$$

and we see that the map $R'_\cdot \longrightarrow R_\cdot$ is onto as the ξ_i can be defined in $R'_\cdot$.

Let C_0 be full subcategory of C consisting of elementary abelian 2 groups, and let h_0^n denote restriction of $h^n(G) = H^n(G)$ to C_0. Then I claim that

$$(**) \qquad H_\cdot h_0^n \xrightarrow{\sim} H_\cdot K(n)$$

Indeed [redacted] we know it is [redacted] surjective, [redacted] and since $H^\cdot(V) = S(V^\wedge)$ any ~~ring~~ operation

$$H^\cdot(V) \longrightarrow H^\cdot(V) \otimes A.$$

is same as a additive operation

$$V^\wedge \longrightarrow H^\cdot(V) \otimes A.$$

which is the same as a power series

$$\lambda \longmapsto \sum_{i \geq 0} \lambda^{2^i} a_i .$$

Thus

$$\bigoplus_n 2\, H_\cdot(h_0^n) = \mathbb{Z}_2[\xi_0, \ldots]$$

so $(**)$ is an isomorphism on indecomposable. [redacted] Thus it will suffice to show that every element in the augmentation ideal of $H_\cdot(h_0^n)$ has square zero. But given an exp. class

$$u : H^n(V) \longrightarrow H^\cdot(V) \otimes A.$$

write

$$u(x) = 1 + \sum x_i\, a_i(x) \qquad x_i \text{ basis for } H^+(V)$$

then

$$1 = u(2x) = u(x)^2 = 1 + \sum x_i^2\, a_i(x)^2$$

and as the x_i^2 are independent $a_i(x)^2 = 0$. This finishes proof that $(**)$ is an isomorphism.

<u>Conclusion:</u>

$$\begin{tikzcd} H_\cdot(h_0^*) \arrow[r, hook] \arrow[rr, bend left, "\cong"] & H_\cdot(h^*) \arrow[r, two heads] & H_\cdot(K(*)) \end{tikzcd}$$

and I do not see how to approach

i) showing $F = 0$ on $H_\cdot(h^*)$ (I know that $2 = 0$ and $2 = FV$ so it would suffices to prove V surjective. Analogous to putting divided powers on $H_\cdot(h^n)$, except in topological case this seems to use $SP_n(X)$.)

ii) showing that $2H_\cdot(h_0^n) \twoheadrightarrow 2H_\cdot(h^n)$. (Here I have no idea at all of how to extend a cohomology operation on groups to all spaces, or even the easier thing of showing why an additive operation vanishing in dimension 1 and 2 vanishes identically.)

What I need to understand therefore is how to deduce the above results for $H_\bullet K(n)$ and $\bigoplus_{n\geq 0} 2H_\bullet(K(n))$ without using Eilenberg - Maclane spaces. This brings me back to the old problem of ~~[illegible]~~ doing this by using Steenrod operations.

I recall that one gets a map

$$\Psi_i : R_\bullet = \bigoplus_{n\geq 0} 2H_\bullet(K(n)) \longrightarrow \mathbb{Z}_2[\omega_{2^i-2^{i-1}}, \dots, \omega_{2^i-1}]$$

using the Steenrod operation

$$H(X) \longrightarrow H(B\mathbb{Z}_2^i \times X)^{GL_i(\mathbb{F}_2)} = H(X)[[\omega_{2^i-2^{i-1}}, \dots, \omega_{2^i-1}]]$$

$$x \longmapsto \sum_{\nu=0}^{i} x^{2^\nu} \omega_{2^i-2^\nu}$$

$$\Psi_i(\xi_\nu) = \omega_{2^i-2^\nu} \qquad 0 \leq \nu \leq i$$

$$= 0 \qquad \nu > i$$

(Combine this with the operation of sending u to $t^{\deg(u)}u$)

June 24, 1970:

I want to understand the 2-category approach to homotopy theory.

Prime example: Let $\mathcal{H}$ be the 2-category of pointed spaces, where $\underline{\mathrm{Hom}}_{\mathcal{H}}(X,Y)$ is the fundamental groupoid of the space of basepoint-preserving maps from X to Y. Given a space Z we consider the enriched object $\underline{\mathrm{Hom}}(?,Z)$ instead of $[?,Z]$. Thus when we have maps $Z_1 \longrightarrow Z$ and $Z' \longrightarrow Z$ we can form the homotopy fibred product

$$Z' \overset{2}{\times}_Z Z_1 = Z' \times_Z Z^I \times_Z Z_1$$

and we know there is a surjection

$$[?, Z' \overset{2}{\times}_Z Z_1] \twoheadrightarrow [?, Z'] \times_{[?,Z]} [?, Z_1].$$

I want the generalization of this:

Guess: The map of groupoids

$$\underline{\mathrm{Hom}}(X, Z' \overset{2}{\times}_Z Z_1) \longrightarrow \underline{\mathrm{Hom}}(X,Z') \overset{2}{\times}_{\underline{\mathrm{Hom}}(X,Z)} \underline{\mathrm{Hom}}(X,Z_1)$$

surjective on objects and morphisms. (so induces an isomorphism on π_0 and a surjection on all the π_1.)

Proof: As $\underline{Hom}(X, Z' \overset{2}{\times}_Z Z_1) = \pi\left((Z'^X) \overset{2}{\times}_{(Z^X)} (Z_1^X)\right)$

we can restrict to proving

$$\pi(Z' \overset{2}{\times}_Z Z_1) \longrightarrow \pi Z' \times_{\pi Z} \pi Z_1$$

is surjective on morphisms. So suppose given points $a', b' \in Z'$, $a_1, b_1 \in Z_1$

$$\begin{array}{cc} p(a') & p(b') \\ \downarrow h & \downarrow k \\ q(a_1) & q(b_1) \end{array}$$

and maps $a' \xrightarrow{m'} b'$ $a_1 \xrightarrow{m_1} b_1$ such that some compatibility conditions are satisfied, i.e. the square

$$\begin{tikzcd} p(a') \arrow[r, "p(m')"] \arrow[d, "h"'] & p(b') \arrow[d, "k"] \\ q(a_1) \arrow[r, "q(m_1)"'] & q(b_1) \end{tikzcd}$$

commutes. Now h, k, m' and m_1 are homotopy classes of arcs and commutation means on choosing representative arcs, we can make them into the boundary of a square! Thus we get a map of $I \longrightarrow Z' \times_Z Z^I \times_{Z_1} Z_1$ with components m', σ, m_1 joining the points (a', h, a_1) to (b', k, b_1). This proves surjectivity.

So now the next problem is clear: we want the exact sequence which in the case of interest reads

$$\pi_1\left(\underline{\mathrm{Hom}}(X,Z') \overset{2}{\times}_{\underline{\mathrm{Hom}}(X,Z)} \underline{\mathrm{Hom}}(X,Z_1)\right) \hookrightarrow \pi_1(X,Z') \times \pi_1(X,Z_1) \longrightarrow \pi_1(X,Z)$$

$$\longrightarrow \pi_0(X, Z' \overset{2}{\times}_Z Z_1) \longrightarrow \pi_0(X,Z') \times \pi_0(X,Z_1) \longrightarrow \pi_0(X,Z).$$

Next suppose $\mathcal{F}$ is a fibred category over $\mathcal{H}$, and set

$$[\mathcal{F}, Z] = \pi_0 \, \underline{\mathrm{Hom}}_{Cat/\mathcal{H}}(\mathcal{F}, \underline{\underline{h}}_Z)$$

and see if you can prove half-exactness.

Suppose we are given functors

$$A' : \mathcal{F} \longrightarrow \underline{\underline{h}}_{Z'}$$
$$A_1 : \mathcal{F} \longrightarrow \underline{\underline{h}}_{Z_1}$$

and an isomorphism θ of $pA' \simeq qA_1$. This means that for each (ξ, X) $\xi \in \mathcal{F}(X)$ I have a class of ~~homotopies~~ homotopies

$$\theta(\xi) : pA'(\xi) \longrightarrow qA_1(\xi)$$

such that for any maps $(\tilde{f}, f) : (\xi, X) \longrightarrow (\eta, Y)$ we have

$$(*) \qquad \begin{tikzcd} pA'(\xi) \arrow[r, "\theta(\xi)"] \arrow[d, "pA'(\tilde{f})"] & qA_1(\xi) \arrow[d, "qA_1(\tilde{f})"] \\ pA'(\eta) \arrow[r, "\theta(\eta)"] & qA_1(\eta) \end{tikzcd}$$

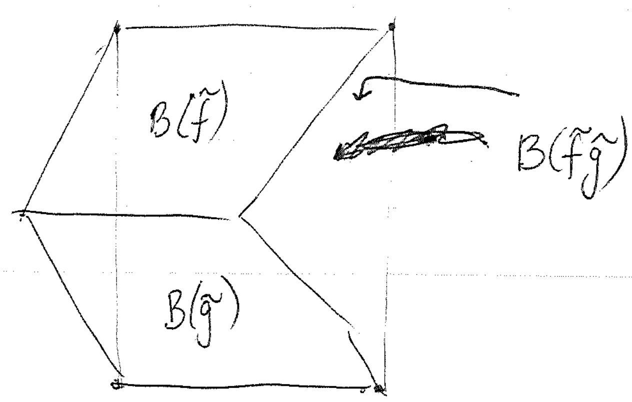
commuting. Now the problem is to raise these classes of homotopies $\theta(\xi)$ to actual homotopies $\tilde{\theta}(\xi)$ so as to obtain

$B(\xi) = \left(A'(\xi),\ \widetilde{\theta(\xi)},\ A_1(\xi)\right)$ $\quad B(\xi) \in Hom(X, Z'^2 \times_Z Z_1)$ and

$$B(\tilde{f}) : B(\xi) \longrightarrow B(\eta)$$

such that $B(\tilde{f}\tilde{g}) = B(\tilde{f})\, B(\tilde{g})$.

Choose $\widetilde{\theta(\xi)}$ for every ξ in some way, so get and also choose actual homotopies $\tilde{A}'(\tilde{f})$ and $\tilde{A}_1(\tilde{f})$. Because the square $(*)$ commutes we can find a map from I^2 with given boundary; in other words we choose $\vec{B}(\tilde{f}) : \vec{B}(\xi) \longrightarrow \widetilde{B(\eta)}$ so as to cover our **choices** of $\tilde{A}'(\tilde{f})$, $\tilde{A}_1(\tilde{f})$, $\widetilde{\theta(\xi)}$. So the only problem is why is $B(\tilde{f}\tilde{g}) \simeq B(\tilde{f})\, B(\tilde{g})$.



But the left triangle with sides $p\tilde{A}'(\tilde{f})$, $p\tilde{A}'(\tilde{g})$, and $pA'(\tilde{f}\tilde{g})$ fills in as one knows that $A'(\tilde{f})\, A'(\tilde{g}) = A'(\tilde{f}\tilde{g})$, so whole prism fills in giving the desired homotopy.

Thus it seems to work

$k \longleftarrow A \longrightarrow L$

~~[illegible]~~ four abelian categories

$$\mathrm{Modf}(k) \hookrightarrow \mathrm{Modfl}(A) \hookrightarrow \mathrm{Modf}(A) \xrightarrow{\otimes_A L} \mathrm{Modf}(L)$$

~~[illegible]~~ now the point to show is that ~~[illegible]~~ the last three give rise to ~~[illegible]~~ a long exact sequence and the first two give an isomorphism.

The fact that

$$K_0(G; \mathrm{Modf}(k)) = K_0(G; \mathrm{Modfl}(A))$$

is clear, hence no problems with the beginning. Long exact sequence: For this you want fibre of map

$$\underline{GL(L)} \to F \longrightarrow B_A \longrightarrow B_L \longrightarrow B^2_k$$

$k[[t]]$

Start with the relations involving X, Y, Z.

must start with ~~[illegible]~~ (the category of) representations over L

G

k^n

$$GL_n(A) \overset{s}{\underset{\cancel{A}}{\rightleftarrows}} GL_n(k)$$

$$i_* x = \underline{\cancel{\$}^* x \circ i_* 1} \circ$$

$\shortparallel$

$i_*(i^* s^* x)$

$$A^n \longrightarrow k^n$$

$$K_i(k) \xrightarrow{i_* = 0} K_i(A) \longrightarrow K_i(L) \longrightarrow K_{i-1}(k) \longrightarrow$$

The great hope is to be able to do higher K-theory using a 2-category approach. Grothendieck derives the homotopy type of a space from the underlying category of open sets.

General scheme: To each topos X should be associated a category of sheaf spectra $S(X)$ which behaves like the derived category. Intuitively what's involved is all kinds of mucky glueing data. Thus an object $E^\bullet$ of $S(X)$ has homotopy sheaves which for sake of uniform notation we denote $\underline{\underline{H}}^i(E^\bullet)$. However the Postnikov invariants can be quite a bit righer, as one sees immediately from the considerations of two stage Postnikov systems. Now let us assume that $S(X)$ has been defined. Then given a map $f: X \longrightarrow Y$, one expects a functor $\underline{R}f_*: S(X) \quad S(Y)$ and a Leray spectral sequence

$$E_2^{pq} = H^p(Y, R^q f_*(E^\bullet)) \qquad H^{p+q}(X, E^\bullet)$$

I am confused. Thus I would expect to have a spectral sequence

$$[X,E^\bullet]^{p+q} \Longleftarrow H^p(X, \underline{\underline{H}}^q(E^\bullet))$$

So its okay with $R^q f_* E^\bullet = \underline{\underline{H}}^q(f_*(E^\bullet))$. Again the point is to think of $E^\bullet$ as a generalized complex of sheaves, but with non-abelian Postnikov invariants.

Now it is necessary to understand what are the omega spectra. Thus these are things $E^\bullet$ with $H^q(E^\bullet) = 0$ for q positive. As a check you note that

$$H^n(X,E^\bullet) = \oplus\, H^{n-i}(X, H^i(E^\bullet)) \qquad \text{over } \mathbb{Q}$$

so once n exceeds the dimension of X you get 0. Thus $H^q(E^\bullet) = \pi_{-q} E^\bullet$.

Thus the good complexes are bounded above.

The next point was to have an analogue of $\mathbb{G}_m$ which to every ringed space X assigns an object of $S(X)$. I want to call this animal $\mathbb{GL}$. It's like $\mathbb{G}_m$ which as an object of $S(X)$ is determined by a single group in dimension $\neq 0$. This seems pretty strange

~~Cohomology~~ $\cancel{H_i(GL(\mathbb{F}_q))}$

general nonsense about alg. K-theory and how to go about formulating it.

Idea first of all is that to each scheme X there should be ~~a complex~~ an object of a suitable derived category GL_X over X whose cohomology groups are the K-groups.

$$K_i(X) = H^{i}(GL_X)$$

GL_X somehow is built up out of repn. on loc. free $\mathcal{O}_X$-modules of finite rank. GL_X is a spectrum over X. It is not clear to me whether it should be connected or not!

~~Suppose ... prime to the residual characteristics of X ...~~. One has a fibration

$$(E\Psi^q)_X \longrightarrow GL_X \xrightarrow{\Psi^q - 1} GL_X$$

where $E\Psi^q$ is a sheaf spectrum over X with

$$\pi_{2i-1}(E\Psi^q) = \mu_{q^i-1}^{\otimes i}$$

sheaves for the étale topology, so you must have q^i-1 prime to residual characteristics, hence q must be nilpotent on X.

are there any coincidences

$$K_a(X) \xrightarrow{c_i^\#} H^{2i-a}(X, \mu_\ell^{\otimes i})$$

no coincidences.

$$K_a(X) \xrightarrow{c_i^\#} H^{i-a}(X, \Omega^{i}_{X/\mathbb{Z}})$$

---

for K-theory I want

$c_i^\# = (i-1)!\, ch_i$?

$$\begin{array}{ccccc} K_a(Y) & \xrightarrow{i_*} & K_a(X) & \xrightarrow{j^*} & K_a(U) \\ & & \downarrow ch_i & & \downarrow ch_i \\ & \longrightarrow & H^{2i-a}(X, \mu^{\otimes i}) & \longrightarrow & H^{2i-a}(U, \mu^{\otimes i}) \end{array}$$

$ch_i(i_* 1)$ Todd class + Thom class

$$ch(i_* 1) = i_*(\text{Todd } \nu_i)$$

Characteristic classes of R-vector bundles, where R is a finite field. Let A be ~~XXXXXXXXXX~~ abelian group and consider the natural transformations from K k to $H^q(\;, A)$~~xxxxxxx~~. Something I did today suggests that such a natural transformation θ must also be considered ~~x~~ for relative represnetations. Thus if G' is a subgroup of G we have defined a group k(G,G') of relative representations; it is a quotient of the Grothendieck group of the ~~xxxxxx~~ representation of the group $G \underset{G'}{+} G$.

We want to ~~check~~ see what we need to prove the exactness axiom for maps into cohomology. Thus suppose that $0 \quad A' \quad A \quad A'' \quad 0$ is an exact sequence of abelian groups and suppose I am given a natural transformation α from virtual bundles to cohomology with ~~xxxx~~ coefficients in A of dimension q. This means that to each virtual ~~xxxxxxxxxxxxxx~~ representation x of a group G I am given a cohomology class ~~XXXX~~ α(x) in $H^q(G,A)$ in such a way as to be compatible with pull-backs. I suppose that α(x) goes to zero under the map induced by A A'' . The problem arises ~~fxxx~~ from the fact that

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What is a stable vector bundle over a topological space X? First idea is that of a sheaf of some sort, e.g. to every open subset of X one should be able to associate the sections of the virtual bundle and this sheaf possibly with some extra structure should characterize the virtual bundle and hence this could be taken as a definition. Example: A vector bundle of dimension n can be defined as a locally free sheaf of modules for the structural sheaf. One should observe that whereas topologically one thinks of vector bundles of dimension n in terms of the classifying space BU_n, i.e. one rigs the definition of the ~~xxxxx~~ fibred category of such bundles so that the fibre over X is equivalent to the fundamental groupoid of Hom(X,BU_m). However this latter fibred groupoid is not a stack (champ en Francais) whereas the actual category of vector bundles is. Question: Does there exist a stack of virtual vector bundles over the category of topological spaces? ~~XXXXXXXXXXXXXXX~~ Fix a classifying space within a category of spaces ~~xxxx~~ e.g. paracompact ones. Then one obtains a stack, the discrete fibred category ~~xx~~ given by the functor represented by the classifying space.

It seems that virtual bundles do not glue.

exact sequence to generalize

$$K_0(Y) \longrightarrow K_0(X) \longrightarrow K_0(U) \longrightarrow 0$$

in case of a Dedekind ring.
in order to get I would want to get a fibration!!!!!!

$$GL(k) \longrightarrow G(A) \longrightarrow G(L)$$

In terms of characteristic classes what we have is various repns. of a group G. In other words we have a repn. of G

So the only method I have at the moment is to form the fibre and try to discover its properties.
So the above is typical of $A/t \longleftarrow A \longrightarrow A[t^{-1}]$ situation.

Thus the problem might be the discription of the representations. In different forms it might be as follows: anyhow.

only chance seems to work in immeuble. Ideas is that this way can relate K-theory of $\hat{A}, K, k$.
The method seems to be very mystifying.

Thus what is at stake is the action of $GL_n(K)$ on the immeuble (contractible after B-T) and having the indicated stabilizers get straight char. classes for fine representations with arbitrary coefficients. The problem is that

$$\mathrm{Hom}_{\mathcal{C}}(I_\infty, H^i(\ ,M)) = \varprojlim_{n} H^i(GL_n, M)$$

whereas what we are after is $H^i(GL_\infty, M)$.

So the point is instead of a natural transf. from I_∞ to cohomology we want to consider isomorphism classes of natural transformations from stable representations to cohomology classes.

Thus we want to study a function which associates to each repn. E of G a ~~cohomology class~~ a cocycle $Z(E)$ of G, in such a way that the following holds.

Thus we have a 2-fibred category $\mathcal{G}$ over the 2-category of groups, and we have a functor from

$$\mathcal{G} \longrightarrow \text{Spaces}$$

and over Spaces we have a fibred category represented by $K(M,i)$; so we pull this back, and then we want to consider isomorphism classes of cartesian sections.

It would seem that the 2-category approach might work with groups. unclear where the 2 & 3 cells come from.

Intuition about the stabilty theorems: It is a poblem of showing that a map S^i BGL^+ can be deformed into BGL_n^+. My idea is that such a map can be realized as a kind of bundle over S^i with a certain ramification along subvarieties of codimensions 2 and 3. The method of proof would then consist of carrying out the general position arguments to put the ramification in a low end of the infinite general linear group. The first thing to understand is this: Given a manifold Y, attach a 2-cell by means of an embedded S^1 and determine general position for maps into $Y \cup e_2$

So at the moment we see that we get a framed submanifold of codimension 2 and some king of ramification behavior along that submanifold. It seems necessary to take the cyclic permutation and write it as a product of commutators, which can be done already in Σ_3. However after you attach the 2-cell, the new 2-dimensional homology class doesn't become sperical under you get into Σ_5. So what you need to see is the map S^2 $Y \cup e_2$. Thus I must take the 2-sphere and find an interesting 5-fold branched covering, with ramification points somehow related to the way you express the cycle (123) as a product of commutators in the alternating group on 5-letters. For example let the icosahedral group I act on the 2-sphere S^2 and form the quotient. If a point is fixed then it lies on the middle of a simplex on the icosahedron.

So the idea is that the 2-sphere has over it a branched covering with a finite number of brach points z_i and now when you attach a 3-disk you should join these points to the center. So what one gets is a

Bass Stability

Thus it appears that an element of $\pi_i(BGL_n^+)$ Z can be realized by the following data: a subvariety of S^i consisting of two strata one of which is a framed submanifold of codimension 2 and the other a framed submanifold of codimension 3, where the normal structure is that of a cone on a finite set of points on the 2-sphere. On the complement I must give a a rperesentation of thefundamental group in the general linear group with monodeomy transformation around the 2 manifold and relatio ns around the 3 manifold prescribed. Now to prove the stablilty theorem what I want to do is to deform the representation of the fundamental group into matrices of a given size. What does one know about the homology of the complement of this subvariety Alexander says that the homology of the complement is related by duality tb that of the subvariety which it would seen is alsmot arbitrary. Thus it seems unlikely that the representation of the fundamental group could be deformed without changing the subvariet No hope.

Formula for $H_*(\overline{R}_A)$: Let A^∞ denote thefree A-module with basis indexed by the positive integers N and $GL_\infty(A)$ the group of auomorphisms of A^∞ whose matrices are equal to the identity expept for a finite number of entries. Let $GL_{n,m}(A)$ be the group of A-linear automorphisms of the free A-module $A^n \oplus A^m$ which carry the first factor into itself. Using the standard isomorphism of $A^n \oplus A$ with A^{N+1} one obtaines a homomorphisms $GL_{n,m}$ $GL_{n+1,m}$ and $GL_{n,m+1}$ permitting one to define the limit group

$$GL_{\infty,\infty}(A) = \lim GL_{b,m}(A)$$

as those automorphisms of the direct sum $A^\infty \oplus A^\infty$ which as matirices are almost everywhere the identity, and which carry the first factor into itself.

Then choosing an isomorphism of the direct sum $A^n \oplus A^m$ with A^{n+m} one obtai ns two homomorphisms of $Gl_{n,m}$ into Gl_{n+m}

Let Gln.m be the group of autos of the A modlle An+Am wihch carry the subspace

Goal: To simultaneously generalize the generalized cohomology theories of algebraic topology and the derived category in algebraic geometry. Key ideas involve associating to any topos X a triangulated category T(X) whose objects should be thought of as generalized cohomology theories over X. Perhaps T(X) should be the homotopy category associated to the category of simplicial spectra of groups in the topos X. Given a morphism of topi f:X Y one expects to have a functor f_*:T(X) T(Y) and a functor f^*: T(Y) T(X) which should be some kind of derived functor extension of the similarly denoted such functors on sheaves of groups. The setting up of this theory should encompass the derived category of sheaves over a topos and the theory of generalized cohomology theories. More precisely we should have a cartesian square of theories

derived category of abelian sheaves ofer X | ?

(restriction of above over a point) chain complexes of abelian groups | gen.coh. theories or Boardman spectra

The principal reason for introducing the above God-awful machinery is to handle the example of algebraic K-theory, i.e. stable groups. Thus over a point everything comes from the family of symmetric groups. (Key problem: any idea of how to describe the category of spectra in terms of the symmetric groups:?)

Goal 2: Algebraic K-theory. To any ringed topos one has the object G_m of the derived category; similarly one should have an object $\underline{BGL}_X$ in T(X), which when applied to a point give the K-groups of the ringed topos X. One expects to be able to define trace and Gysin homomorphisms for relative proper schemes of finite Tor dimension. Essentially this theory should be a strictly degree zero phenomenon; I don't know whether to expect a type of periodicity Gysin homomorphism (Perhaps this is the really key result of the good theory a non-trivial result relating K-groups of different dimesnions.

I see at the moment ~~xxxxxx~~ that the ~~xxxxxxxx~~ definition of $T(X)$ is an extension of the derived category theory and no more in the case of characteristic zero where generalized cohomology theories and chain complex theories ought to coincide. (Question: Let G be a group and consider the homotopy category of G-sets. Is the usual degeneracy over $\mathbb{Q}$ valid:?)

The fundamental problem remains how to work with objects in the derived category which do not glue? Here Deligne's approach seems to be the key.

June 26, 1970. Grand scheme for alg. K-theory.

Produce a generalization of $\mathbb{G}_m$ which associates to a ringed space X a sheaf spectrum $\mathbb{GL}_X$ whose cohomology

$$H^i(X, \mathbb{GL}_X) = K_{-i}(X)$$

are the K-groups. Notice that the determinant should give rise to a map

$$\varinjlim_n \Omega^n \mathbb{GL}_X^{(n)} \longrightarrow B\,\mathbb{G}_{m,X}$$

hence to maps

$$K_i(X) \longrightarrow H^{1-i}(X, \mathbb{G}_{m,X})$$

(not too clear).

Given $q \ni q^i - 1$ unit on X for all i, e.g. q nilpotent or X of characteristic zero, then have a fibration

$$E_X \longrightarrow \mathbb{GL}_X \xrightarrow{\Psi^q - 1} \mathbb{GL}_X$$

where E_X is a sheaf spectrum over X such that

$$\pi_{2i}(E_X) = 0$$

$$\pi_{2i-1}(E_X) = \mu_{q^i-1}^{\otimes i},$$

hence we expect maps

Some ideas: Let X be a projective variety over k, let $\mathcal{F}$ be a coh. sheaf on X, and let G act linearly on $\mathcal{F}$. Then we can write $\mathcal{F}$ as the quotient of a G-vector bundle

$$\mathcal{O}(-n) \otimes \Gamma(X, \mathcal{F}(n)) \longrightarrow \mathcal{F}$$

Therefore in computing K-groups one notes that they depend only on the algebraic hull of G.

For example if X projective over $\mathbb{F}_q$, then the higher K-groups come from ~~[illegible]~~ representations of finite groups. Thus:

~~Example: Suppose we consider the category~~

Example: Suppose X is a projective variety over $\mathbb{F}_q$, and we define $K_\cdot(X) \otimes \mathbb{Q}$ in terms of a universal additive class

$$R(G; X) \longrightarrow H^0(G, K_\cdot(X) \otimes \mathbb{Q})$$

Then as the group of automorphisms of a bundle E is **finite** it **follows** that rational classes are trivial $\Longrightarrow$ $K_+(X) \otimes \mathbb{Q} = 0$.

$$
\begin{tikzcd}
K_a(X) \arrow[r, "\Phi^q - 1"] & K_a(X) \arrow[r] & H^{-a+1}(X, E_X) \arrow[d, "c_i^{\#}"] \\
& & H^{2i-a}(X, \mu_{q^i-1}^{\otimes i}).
\end{tikzcd}
$$

~~[illegible]~~

Now suppose we have
$$Y \overset{i}{\subset} X \overset{j}{\supset} U$$
with Y and X non-singular. Then there should be a long exact sequence of K-groups. This might come about as follows: Quite generally one should have a triangle
$$0 \longrightarrow i_* i^! GL_X \longrightarrow GL_X \longrightarrow j_* j^* GL_X \longrightarrow 0$$

and one should have isomorphisms
$$j^* GL_X = GL_U \qquad (\text{trivial as } j^* \mathbb{G}_{m,X} = \mathbb{G}_{m,U})$$
$$i^! GL_X = GL_Y \qquad \text{periodicity theorem.}$$

June 27, 1990

I recall that I have shown that if X is a projective variety over $\mathbb{F}_q$, then $K_4(X)\otimes\mathbb{Q}=0$. ~~[illegible]~~
I think this implies that the character

NO

$$ch: K_a(X)^{\otimes\mathbb{Q}_\ell} \longrightarrow \left[\bigoplus_i \varprojlim_\nu H^{2i-a}(X, \mu_{\ell^\nu}^{\otimes i})\right]\otimes\mathbb{Q}$$

is quite far from being surjective. ~~an isomorphism~~ Thus I want to show that

$$\varprojlim_\nu H^{2i-1}(X, \mu_{\ell^\nu}^{\otimes i}) \otimes \mathbb{Q}$$

is in general non-zero. NO.

Recall

$$\begin{array}{ccc} X & \longleftarrow & \bar{X} \\ \downarrow & & \downarrow \\ \mathbb{F}_q & \hookleftarrow & \bar{\mathbb{F}}_q \end{array}$$

$$E_2^{pq} = H^p\left(Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q), H^q(\bar{X}, \mu_{\ell^\nu}^{\otimes i})\right) \Longrightarrow H^j(X, \mu_{\ell^\nu}^{\otimes i}).$$

As the groups in this spec. seq. are finite we can pass to limit over ν and then tensor with $\mathbb{Q}$ obtaining

$$0\to\left[H^{j-1}(\bar{X})(i)\right]_F \longrightarrow \left(\varprojlim_\nu H^j(X, \mu_{\ell^\nu}^{\otimes i})\right)\otimes\mathbb{Q} \longrightarrow \left[H^j(\bar{X})(i)\right]^F \to 0$$

where as usual

$$H^j(\bar{X})(i) = \left(\varprojlim_\nu H^j(\bar{X}, \mu_{\ell^\nu}^{\otimes i})\right)\otimes\mathbb{Q}$$

and F is Frobenius. Now the Weil conjectures imply that $H^j(\bar{X})(i)^F$ $\neq 0$ only if $j = 2i$, and similarly for $H^j(\bar{X})(i)_F$ which has the same dimension as the invariants. Hence we see that

$$\Big[\varprojlim_{\nu} H^j(X, \mu_{\ell^\nu}^{\otimes i})\Big] \otimes \mathbb{Q} = \begin{cases} 0 & j \neq 2i, 2i+1 \\ H^{2i}(\bar{X})(i)^F & j = 2i \\ H^{2i}(\bar{X})(i)_F & j = 2i+1 \end{cases}$$

Thus there is no contradiction as we thought.

<u>Conclusion:</u> Weil-Tate conjectures imply that

$$ch : K_a(X) \otimes \mathbb{Q}_\ell \longrightarrow \bigoplus_i \Big[\varprojlim_{\nu} H^{2i-a}(X, \mu_{\ell^\nu}^{\otimes i})\Big] \otimes \mathbb{Q}$$

is surjective for $a \geq 0$ and an isomorphism for $a \geq 1$

(In fact maybe an isomorphism for $a = 0$ if it should be true that any cycle homologically equivalent to zero is a torsion element of $K(X)$, which is reasonable if one believes that such cycles are rational points of comm. group varieties.)

June 26, 1970:

Here is an example which lends support to the conjecture that $H_+(GL(\mathbb{F}_q)) = 0$. I suppose $q = 2^\nu$ and let $a \in GL_n(\mathbb{F}_q)$ be an element of order 2. By Jordan canonical form a is determined up to conjugacy by the dimension d of the image of $(a-1)$. First suppose $d = 1$. Then

$$a = \begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix}$$ rest are zeros.

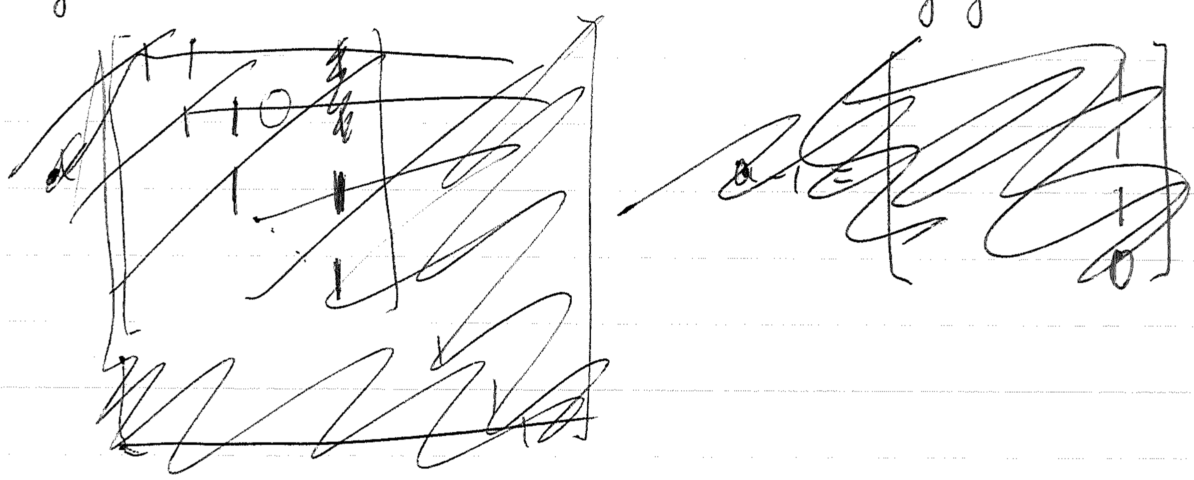
hence a belongs to the elementary 2 group A of rank $n-1$

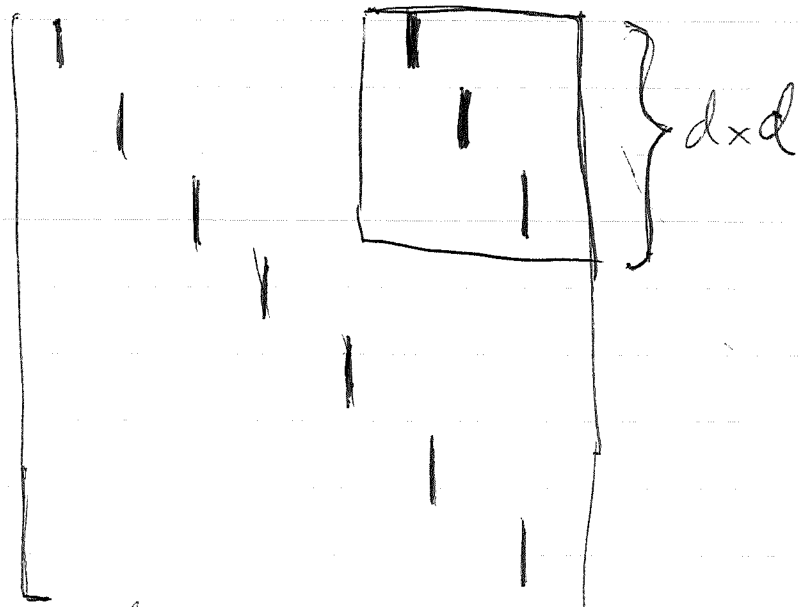
$$\begin{bmatrix} 1 & * & * & * & * & * & * \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & & & & \end{bmatrix}$$

whose normalizer produces any linear transformation on A.
But
$$H^*(A)^{GL(A)} = \mathbb{Z}_2[w_{2^r-2^{r-1}}, \ldots, w_{2^r-1}] \qquad \text{if } r = \text{rank } A,$$

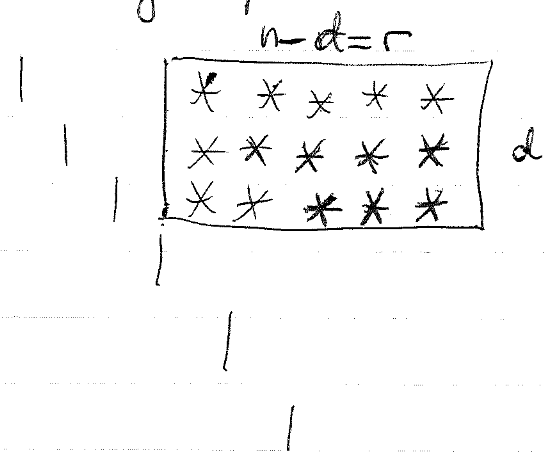
so as $n \longrightarrow \infty$ it follows that the map
$$H^+(GL) \longrightarrow H^+(\mathbb{Z}_2)$$
is zero.

For general d we note that a is conjugate to





This is contained in the elementary abelian subgroup



whose normalizer contains $GL_r(\mathbb{F}_q)$ and one might hope no invariants remain as $r \to \infty$.

Better thing is to take elementary abelian 2-subgroup A.

$$1 \qquad \boxed{\begin{matrix} a & & \\ & a & \\ & & a \end{matrix}}\boxed{\begin{matrix} b & & \\ & b & \\ & & b \end{matrix}} \cdot \text{---} \boxed{\begin{matrix} c & & \\ & c & \\ & & c \end{matrix}}$$

Whose normalizer contains all linear transformations of A.

June 28, 1970:

More evidence for $\tilde{H}_*(GL(\mathbb{F}_q), \mathbb{Z}/p) = 0$. I claim that the map

$$H_+(\Sigma_\infty) \longrightarrow H_+(GL(\mathbb{F}_q)) \qquad \text{coeff mod } p$$

is zero, or equivalently if w is a characteristic class for $\bar{R}'$, then $w(E) = w(0)$ if E is a permutation representation. The point is to introduce the kernel

$$JG \longrightarrow \bar{R}'(G) \xrightarrow{w_{\bar{R}'}} H^0(G, H.GL)$$

consisting of the fine virtual representations which are killed by all stable characteristic class. Claim JG is closed under multiplication. In effect the product on $\bar{R}'$ induces a map

$$H.GL \otimes H.GL \longrightarrow H.GL$$

showing that there is a formula of the form

$$w(\xi \cdot \eta) = \sum w_i'(\xi)\, w_i''(\eta) \qquad \left(\begin{array}{c}\text{all } w_i', w_i'' \\ \text{of positive deg.}\end{array}\right)$$

if $w \in H^n(GL)$. Thus if $w(\xi) = 0$ for all ξ we have same for $w(\xi \cdot \eta) = 0$, hence JG is an ideal in $\bar{R}'(G)$.

Now use the fact that the coh. of Σ_n is detected by elementary abelian groups. The permutation representation is an external direct sum of regular representations,

and a regular representation is a tensor product of smaller ones. Hence we are reduced to checking that $w(\text{reg}\, \mathbb{Z}_2) = 0$, and $w(E \otimes F) = 0$ if $w(E) = w(F) = 0$.
But

$$(E \otimes F) - d(E) \cdot d(F) = (E - d(E))(F - d(F)) + d(E)(F - d(F))$$
$$+ d(F)(E - d(F))$$

and all these go to zero. Finally for reg $\mathbb{Z}_2$ we did this before