\[
\begin{align*}
U(x \times S) & \rightarrow \prod_{k=1} U_{\Sigma_k} (x^k \times S^k) \\
& \rightarrow \prod_{k=1} U_{\Sigma_k} (x^k / S^k) \\
& \rightarrow \prod_{k=1} U(x^k / S^k)
\end{align*}
\]

\[
\begin{array}{c}
S^k \times S^l \\
\Sigma_k \Sigma_l \\
\Sigma_{k+l} \\ \\
\Sigma_{k+l} \times (S^k \times S^l) \\
\Sigma_{k+l} \Sigma_{k+l}
\end{array}
\]
\[ \bigoplus_{n \geq 1} R(\Sigma_n) \oplus K(X) \]

\[ \bigoplus_{n \geq 1} R(\Sigma_n) \]

\[ (d_n) \]

\[ \text{New } \Sigma_n d_n = \alpha_n \otimes d_j \]

\[ \Sigma_i \times \Sigma_j \]

\[ R'(\Sigma_i) \otimes R'(\Sigma_j) \rightarrow R'(\Sigma_n) \]

\[ \text{New } \Sigma_n \times \Sigma_n = \alpha_n \left[ \Sigma_n / \Sigma_i \times \Sigma_j \right] \]

\[ e_{i,j} \]

Think true that at least after passing to filtration we get this means that \( \alpha_n \) known for \( n \) not a power of a prime.

\[ R(\Sigma_n) \text{ has basis } [\Sigma_n : \Sigma_{\omega_1} \times \cdots \times \Sigma_{\omega_k}] \]

where \( \omega_1, \ldots, \omega_k \) is a partition of \( n \).

\[ \mathbb{Z} + \bigoplus_{n \geq 1} R(\Sigma_n) \]

forms a ring with unit, given by induction.

\[ \mathbb{Z} + \bigoplus_{n \geq 1} R(\Sigma_n) = \mathbb{Z}[e_1, e_2, \ldots, ] \]

\[ e_i = [\Sigma_i \times \Sigma_1 \times \cdots \times \Sigma_N] \text{ of degrees } \Sigma_i \]
\( \text{TR}(\Sigma_n) \rightarrow \text{TR}(\Sigma_i^r) \otimes R(\Sigma_j^y) \)

should give us a diagonal on \( \bigoplus_{n>0} R(\Sigma_n) \)

\[
\Delta e_n = \sum_{i+j=n} \text{res} \frac{\Sigma_i}{\Sigma_i^r} \otimes \frac{\Sigma_j}{\Sigma_j^y} \text{ind} \frac{\Sigma_n}{(\Sigma_i^r)^n} 1
\]

\[
\sum_i \sum_j \underbrace{1}_{i+j=n} \text{shuffle} (i, j)
\]

Thus it would seem that

\[
\Delta e_n = \sum_{i+j=n} (\binom{n}{i}) e_i \otimes e_j
\]

Is \( \Delta \) a ring homomorphism, i.e.

\[
\sum_{i+j=n} \text{res} \frac{\Sigma_i}{\Sigma_i^r} \otimes \frac{\Sigma_j}{\Sigma_j^y} \text{ind} \frac{\Sigma_n}{\Sigma_i^r \times \Sigma_j^y} \otimes y
\]

So I believe same argument works.
Basic exact sequence
\[ E_2^{pq} = H^p(X; G) \otimes Q^q(G) \]
so this corresponds to a genuine localization.

**Conclusion:** for coh. theories in general one must consider the coeff. system
\[ H \cong Q(G/H) \]
finite gfs. Do two maps
\[ G/H \to G/K \]
induce same map
\[ Q(G/H) \cong Q(G/K) \]?

**Question:** Do inner autos of \( G \) act trivially on \( \Omega G(X) \)? (clear if \( G \) connected)

\[ \Omega G(pt) \to R(G) \]
\[ \cong \mathcal{V} \]
\[ \times \phi \to \times h \]
\[ \phi(gx) = hg^{-1} \phi(x) \]

\[ Q(G) \leftarrow Q(G/H) \]
\[ Q(G/H) \]

\[ G/H \to G/H \]
\[ g H \to g H \]
\[ x H \to x g H \]

\[ Q_G(G/H) \]
The basic idea is to take the basic Steenrod operations

\[ Q_n : U^0(X) \rightarrow U(BZ_n \times X) \]

and take inverse limit of some kind to get a non-torsion operation.

Systematic study of the operations

\[ U(X) \rightarrow U(BZ_n \times X) \]

and the relation with Witt vectors.

Key theme question: Analogy between typical reduction à la Cartier and the Sylow reduction to the Sylow subgroup.

Ideas:

\[ Q_n : U(X) \rightarrow U_{Z_n}(X^n) \]

\[ Q_n(x+y) = ? \]

\[ (Z \cup W)^n \rightarrow X^n \]

\[ n = i + j \]

\[ \sum_{i+j=n} (Z_i \times W^j) \text{ is orbit decomp.} \]

\[ \sum_{i+j=n} (Z_i \times W^j) \text{ is orbit where } i \text{ come from } Z \text{ and } j \text{ from } W \]

This gives us the formula

\[ Q_n(x+y) = \sum_{i+j=n} \text{ind}_{Z_i \times W^j} \left( Q_i \times W \cup Q_j \right) \in U_{Z_n}(X^n) \]
Be more careful.

Suppose $L$ is a line bundle over $X$. Then you have:

\[
\begin{array}{c}
\rightarrow U(DL, SL) \rightarrow U(DL) \rightarrow U(SL) \rightarrow U(SL^k) \\
\uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \\
\rightarrow U(DL^k, SL^k) \rightarrow U(DL^k) \rightarrow U(SL^k) \rightarrow U(SL^k^k)
\end{array}
\]

and hence a map of Gysin sequences:

\[
\begin{array}{c}
\rightarrow U(X) \xrightarrow{e(L)} U(X) \rightarrow U(SL) \rightarrow U(SL^k) \\
\downarrow \hspace{1cm} \downarrow e(L^k) \hspace{1cm} e(L^k) \hspace{1cm} \downarrow \\
\rightarrow U(X) \xrightarrow{e(L^k)} U(X) \rightarrow U(SL^k) \rightarrow U(SL^k^k)
\end{array}
\]

This means that I ought to be able to take the inverse limit of the Gysin sequences.

\[
\begin{array}{c}
U(X) \xrightarrow{e(L^k)} U(X) \xrightarrow{\pi^*} U(SL^k) \rightarrow U(SL^k^k)
\end{array}
\]

The relation between the Steenrod operations:

\[
U_{S_1}(X) \rightarrow U_{S_1}(S(X^n) \times X) \rightarrow U_{S_1}(S(X^n) \times X)
\]

\[
U_{\nu_n}(X)
\]

\[
U_{S_1}(S(X^n) \times X) \leftarrow U_{S_1}(S(X^n) \times X)
\]
Conclude that the map is the restriction:

\[ \mu_n \rightarrow \mu_{n^0} \]

What happens for cohomology:

\[ H^2(\mu_{n^0}, \mathbb{Z}) \cong \text{Hom}(\mu_{n^0}, S^1) = \mathbb{Z}/n\mathbb{Z} \]

Looks good because to rest. covers surj.

What about higher cohomology? All zero? X

\[ H^2(\mu_n, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \]

\[ H^0(\mu_n, \mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z} \]

And this works after tensoring.

Next can you do something with Steenrod operations.

\[ U_{\mu_n}(X) \overset{\text{rest}}{\rightarrow} U_{\mu_{n^0}}(X) \]

\[ \mathbb{Q}_{n^0} \]
This shows that

\[(p^m) = 1 \mod p.\]

\[\begin{align*}
(1 + x)^n & = \sum_{i=0}^{n} \binom{n}{i} x^i \\
& = 1 + \frac{\binom{n}{2} x^2}{2!} + \frac{\binom{n}{3} x^3}{3!} + \ldots
\end{align*}\]

Thus

\[\begin{align*}
\text{ind}_{p^3} 3 & = \frac{\binom{n}{3} x^3}{3!} \\
& = 1 + \frac{n}{2} + \frac{n}{3} + n
\end{align*}\]

\[\begin{align*}
\text{ind}_{p^2} 2 & = \frac{\binom{n}{2} x^2}{2!} \\
& = 1 + \frac{n}{2} + \frac{n}{3} + n
\end{align*}\]

\[\begin{align*}
\text{ind}_{p} 1 & = \frac{\binom{n}{1} x}{1!} \\
& = 1 + \frac{n}{2} + \frac{n}{3} + n
\end{align*}\]

Thus, we know that

\[\begin{align*}
\text{ind}_{p^3} 3 & = \frac{\binom{n}{3} x^3}{3!} \\
& = 1 + \frac{n}{2} + \frac{n}{3} + n
\end{align*}\]

\[\begin{align*}
\text{ind}_{p^2} 2 & = \frac{\binom{n}{2} x^2}{2!} \\
& = 1 + \frac{n}{2} + \frac{n}{3} + n
\end{align*}\]

\[\begin{align*}
\text{ind}_{p} 1 & = \frac{\binom{n}{1} x}{1!} \\
& = 1 + \frac{n}{2} + \frac{n}{3} + n
\end{align*}\]

... restrict to a given prime. Then we observe that

\[\begin{align*}
\sum_{i=0}^{n} \binom{n}{i} x^i & = 1 + \frac{n}{2} + \frac{n}{3} + n
\end{align*}\]
\[ F_4 \rightarrow F_2 \] onto?

\[
\begin{pmatrix}
(0,0,0,0) \\
(0,0)
\end{pmatrix}
\begin{pmatrix}
\beta \\
\beta
\end{pmatrix}
\text{ in } \mathbb{Z}_2^2(\mathbb{R}^2)
\]

Let \( \gamma = \beta \otimes \beta \)?

Now we do have a \( \beta \otimes \beta \) which is \( \text{ in } \mathbb{U}(\mathbb{Z}_2)(\mathbb{R}^4) \)

hence if we set \( \gamma = \text{ ind } (\Sigma_4) \otimes (\Sigma_1) \)

then

\[
\frac{(\Sigma_4) \times \Sigma_1}{\Sigma_2} \left\{ \frac{\text{res } \Sigma_4}{\text{res } \Sigma_1 \times \Sigma_3} \right\} \text{ in this case 3 single orbits of } \Sigma_3 \text{ on } \Sigma_4 / \Sigma_1
\]

\[
\sum_3 \text{ on } \Sigma_4
\]

\[
\text{thus res } \Sigma_4 = \text{ ind } (\Sigma_1) \otimes (\Sigma_1) \rightarrow \Sigma_1 \times \Sigma_1
\]

\[
\frac{\text{res } \Sigma_1 \times \Sigma_1 \times \Sigma_1}{\Sigma_1 \times \Sigma_1 \times \Sigma_1}
\]

would need \( \frac{1}{3} \beta \). Thus get onto provided can invert 3.

\[
\mathbb{U}(\Sigma_2)(\mathbb{R}^2)
\]

\[
\mathbb{R}^2
\]
Conclusion is that
\[ \hat{\mathcal{Q}} : U^*(X) \rightarrow \prod_{n \geq 1} U^R_n(X^n) \]
is a well-defined ring homomorphism.

\[ \prod_{n \geq 1} U^R_n(X^n) \xrightarrow{\text{ring hom.}} \prod_{n \geq 1} U^R_n(X) = F(X) \]
filtration of \( F(X) \)
\[ F(X) \rightarrow u(X). \]

Various quotients of \( F(X) \):
\[ F_k(X) = \prod_{k \leq n \leq k} U^R_n(X) \]
gives an inverse system
\[ \cdots \rightarrow F_3(X) \rightarrow F_2(X) \rightarrow F_1(X) = u(X) \]

Question: Is this surjective?
\[ (\beta, x) \quad \alpha \quad \text{rest } \beta = x \odot x \]

\[ 0 \rightarrow \tilde{U}_2(X) \rightarrow \tilde{F}_2(X) \rightarrow U(X) \rightarrow 0 \]
\[ \xrightarrow{(\beta, 0)} \]
\[ \rightarrow F_3(X) \rightarrow u(X) \]

given \[ \beta \in U_2^R(X^2) \]
\[ \text{rest } \beta = 0 \] in \( U_2^R(X^2) = 0 \)

want \[ \gamma \in U_3^R(X^3) \quad \text{rest } \frac{\gamma}{x^2} \] in \( U_3^R(X^3) = 0 \)

no problem, thus \( F_3 \rightarrow F_2 \) onto
June 16, 1970

exact sequences in alg. K-theory: suppose $A$ is a discrete val. ring with residue field $k$ and quotient field $L$. Then I want a long exact sequence

$$\cdots \rightarrow K_c(k) \rightarrow K_c(A) \rightarrow K_c(L) \rightarrow \cdots$$

and this should result in the long exact homotopy sequence of a filtration.

first special case: If $G$ is finite one has

$$R_k^c(G) \xrightarrow{i_*} R_A^c(G) \xrightarrow{j^*} R_L^c(G) \rightarrow 0$$

defined as follows: $j^*$ just $E \mapsto E \otimes_A L$.

$i_*$: given $V$ over $k$ we can find an $E_0$ over $A$ (free as an $A$-module) $E_0 \rightarrow V \rightarrow 0$; this uses $G$ finite, I think, because you take $E_0 = A[G] \otimes_A P$ where $P \rightarrow \#V \rightarrow 0$ and $P$ is a projective $A$-module. Then one has

$$0 \rightarrow E_1 \rightarrow E_0 \rightarrow V \rightarrow 0$$

and define $i_*[V] = [E_0] - [E_1]$. The fact that $i_*$ is well defined is clear from basic properties of projective resolutions. This will all become much
clearer maybe if one introduced the Grothendieck groups of complexes.

It is clear that $g^* x = 0$. To prove exactness one has a map $p: \text{Coker } \delta_* \rightarrow \text{R}_L(G)$ and one produces a section as follows. Given $W$ over $L$ one chooses an invariant lattice $E$ and set $s_W = [E]$. As in Serre's book there are two different choices $E$, $E'$ have the property that $[E] - [E'] \in \text{Im } i_*$, so $s$ is well-defined and clearly a homomorphism. Also $ps = id$ and $sp = id$ are clear.

Now you want to understand this argument in general. You need to get rid of finiteness first of all. The only reasonable way of getting the desired fibration is to understand in advance what are the categories involved.

Thus to $L$ we must associate the category of f.g. $L$-modules. And we want to examine Euler characteristics. Thus given a simplicial set $X$ we want to consider $g$-cochains associating to each $g$-simplex of $X$ an $L$-module and to each $(g+1)$-simplex a long exact sequence such that certain compatibility conditions hold.
The problem is to show that all of these are distinct. What should the good proof be?

\[ \text{G alg group over } S = \text{Spec } A \]

\[ G_K \rightarrow G \leftarrow G_\ell \]
\[ \text{Spec } K \rightarrow \text{Spec } A \leftarrow \text{Spec } K \]

Assume this is a smooth group like \( G = \mathbb{G}_m \), more generally a torus \( T \). Then I want to relate what goes on in char 0 with char 0.

\[ k = \mathbb{F}_q, \quad T. \] Then have Frobenius endomorphism \( \sigma \) of \( T \) which is composition of \( \sigma_0 \) and an automorphism \( \Theta \) of finite order.

I originally wanted to understand cohomology of \( T(\overline{\mathbb{F}_q}) \) and so went to algebraic closure where you have exact sequence

\[ 0 \rightarrow T(\overline{\mathbb{F}_q}) \rightarrow T \rightarrow T \rightarrow 0 \]
What is a virtual bundle?

Let $X$ be a nice space, then what is a virtual $R$-vector bundle over $X$?

$k(S \times X) \overset{q}{\to} H^*(S, A)$ natural defines a map

from $K_\bullet(X) \to H^{8-q}(A)$

you want to work over the rationals in which case

$K_\bullet(X) \to PH_\bullet(B(X))$

and so $K_\bullet(X)$ appears as the universal ring hom

$\text{ch}_{\text{witt}} : k(S \times X) \to H^*(S) \otimes K_\bullet(X)$

Now the problem is to what extent such a thing is determined by line bundles.

$\text{ch} : k_G(X) \to H^*(G) \otimes K_\bullet(X)$

Suppose that $E$ is a vector bundle over $X$, then

$\otimes \quad k_G(PE) = k_G(X) [T] / A_T(E)$

and so $K_\bullet(PE) = K_\bullet(X) \otimes K(PE)$.\[\]
Question: Can you construct a Chern theory over $X$ with coefficients in $K_*(X)$?

More precisely, if $Y \to X$ is a morphism of ringed topos and if $E$ is a vector bundle over $Y$,

Thus I want to consider schemes of finite type over $X$ and the universal Chern theory is functor

$Y \to h(Y)$

having a ring homomorphism and satisfying the projective bundle theorem. Now if $X$ is a field I know this universal animal is just the Chow ring, so I haven't taken into account the other $K_i$, i.e. the effect of the group action.

$k(S \times X) \to H^*(S)$

Thus if $F$ is a field I want a universal ring homomorphism

$kF(G) \to H^*(G) \otimes K_*(F)$

In particular $K_0(G) \otimes \mathbb{Q}$ should appear with a universal additive map

$kF(G) \to H^0(G) \otimes K_0(G)$.

Better: $H^0(G, K_0(G) \otimes \mathbb{Q})$. 

|
June 18, 1970:

In connection with your approach to K-theory and the recent success of the π₁-surgery Norikiw and Kerby's work, we would like to know just how much of a cohomology theory can be recovered from knowing it on spaces of the form BΓ. For example one knows that the suspension of T^n is a wedge of spheres, hence by just looking at the effect on BZ^n one recovers the homology groups.

**Related problem.** Let C be the category of groups up to inner auto, P a prime field, and denote by H,F the homology of a functor in C^, as in K-theory paper. Given a space Z, set $h_Z(G) = [BG, Z]$ (without fixing basepoint), then there's an obvious map

$$H^i(Z, M) \rightarrow H^i(h_Z, M)$$

and the question is whether this is an isomorphism, i.e. is the map

$$H_2(h_Z) \rightarrow H_2(Z)$$

an isomorphism.

**Special case:** $Z = K(P, n)$ Eilenberg-MacLane spaces. In this case write $h_Z = h_Z^n$, then we want to know if

$$H_2(h_Z^n) \rightarrow H_2, K(n)$$

$K(n) = K(P, n)$
is an isomorphism of Hopf algebras. We will prove that it is surjective.

To prove surjectivity, it is enough to show

\[ \mathbb{L} \cdot (\mathbb{H}^n) \rightarrow \mathbb{L} \cdot \mathbb{H} \cdot \mathbb{K}(n) \]

is surjective, i.e., that any additive cohomology operation from \( \mathbb{H}^n \) is determined by its effect on spaces of the form \( B \mathbb{G} \). However, we know already that any additive

\[ R_\ast = \bigoplus_n \mathbb{L} \cdot \mathbb{H} \cdot \mathbb{K}(n) \]

is a ring in such a way that

\[ \Theta : \mathbb{H}^\ast(X) \rightarrow \mathbb{H}^\ast(X) \otimes R_\ast \]

is universal ring homomorphism, and that

\[ S[\mathbb{1}_0, \mathbb{1}_1, \ldots] \rightarrow R_\ast \]

where

\[ \Theta(x) = \sum_{i \geq 0} x^i \mathbb{1}_i \]

\( x \in \mathbb{H}^1 \)

(Here I take \( P = \mathbb{Z}/2 \), but analogous formulas hold for other \( P \).)

Next set

\[ R'_\ast = \bigoplus_n \mathbb{L} \cdot (\mathbb{H}^n) \]

and we see that the map \( R'_\ast \rightarrow R_\ast \) is onto as the \( \mathbb{1}_i \) can be defined in \( R'_\ast \).
Let $C_0$ be a full subcategory of $C$ consisting of elementary abelian 2 groups, and let $h_{0}^{n}$ denote restriction of $h^{n}(G) = H^{n}(G)$ to $C_0$. Then I claim that

$$H_{n} h_{0} \rightarrow H. K(n)$$

Indeed, we know it is surjective, and since $H^{*}(V) = S(V^{\vee})$ any ring operation

$$H^{*}(V) \rightarrow H^{*}(V) \otimes A,$$

is same as a additive operation

$$V^{\vee} \rightarrow H^{*}(V) \otimes A,$$

which is the same as a power series

$$1 \mapsto \sum_{i \geq 0} x^{i} a_{i}.$$ 

Thus

$$\bigoplus_{n} H^{*}(h_{0}^{n}) = \mathbb{Z}_{2}[t_{0}, \ldots]$$

so (**) is an isomorphism on indecomposable. Thus it will suffice to show that every element in the augmentation ideal of $H(h_{0}^{n})$ has square zero. But given an exp.

class

$$u: H^{*}(V) \rightarrow H^{*}(V) \otimes A,$$

write

$$u(x) = 1 + \sum \alpha_{i} x_{i} q_{i}(x)$$

$x_{i}$ basis for $H^{*}(V)$

then

$$1 = u(1) = u(x)^{2} = 1 + \sum \alpha_{i} x_{i}^{2} q_{i}(x)^{2}.$$
and as $X_i^2$ are independent $q_i(x)^2 = 0$. This finishes proof that (** is an isomorphism.

\[ \begin{align*}
\text{Conclusion: } & H_*(h^*) \xrightarrow{\cong} H_*(h^*) \xrightarrow{\cong} H_*(K(*)) \\
\end{align*} \]

and I do not see how to approach

i) showing $F = 0$ on $H_*(h^*)$ (know that $V = 0$

and $2 = FV$ so it would suffice to prove $V$ surjective.

Analogous to putting divided powers on $H_*(h^n)$, except in

topological case this seems to use $SP_1(X)$.

ii) showing that $2H_*(h^n) \xrightarrow{\cong} 2H_*(h^n)$. (Here

I have no idea at all of how to extend a cohomology

operation on groups to all spaces), or even the easier thing

of showing why an additive operation vanishing in dimension

1 and 2 vanishes identically.)
What I need to understand therefore is how to deduce the above results for $H, K(n)$ and $\prod 2H, K(n)$ without using Eilenberg-MacLane spaces. This brings me back to the old problem of doing this by using Steenrod operations.

I recall that one gets a map

$$\overline{\psi}_i : R_i = \bigoplus_{n \geq 0} 2H_n(K(n)) \longrightarrow \mathbb{Z}_2 [\omega_{2^{i-2} \cdot 2}, \ldots, \omega_{2^{i-1}}]$$

using the Steenrod operation

$$H(X) \rightarrow H(BZ_2^d, X) \xrightarrow{\text{GL}_d(F_2)} H(X) \left[ \omega_{2^{i-2} \cdot 2}, \ldots, \omega_{2^{i-1}} \right]$$

$$x \rightarrow \sum_{\nu=0}^{\frac{i}{2}} x^{2^\nu} \omega_{2^{i-2} \cdot 2\nu}$$

$$\overline{\psi}_i(\xi_y) = \omega_{2^{i-2} \cdot 2\nu} \quad 0 \leq \nu \leq i$$

$$= 0 \quad \nu > i$$

(Combine this with the operation of sending $u$ to $\xi^{\log_2(u)}$)
June 24-1970:

I want to understand the 2-category approach to homotopy theory.

Prime example: Let $\mathcal{H}$ be the 2-category of pointed spaces, where $\text{Hom}(X, Y)$ is the fundamental groupoid of the spaces of basepoint-preserving maps from $X$ to $Y$. Given a space $Z$, we consider the enriched object $\text{Hom}(?, Z)$ instead of $[?, Z]$.

Thus when we have maps $Z'_1 \rightarrow Z$ and $Z' \rightarrow Z$ we can form the homotopy fibre product

$$Z'_1 \times^h_Z Z_1 = Z'_1 \times_Z Z_1 \times^1_Z Z_1,$$

and we know there is a surjection

$$[?, Z'_1 \times^h_Z Z_1] \rightarrow [?, Z'_1] \times [?, Z_1].$$

I want the generalization of this.

Guess: The map of groupoids

$$\text{Hom}(X, Z'_1 \times^h_Z Z_1) \rightarrow \text{Hom}(X, Z'_1) \times \frac{\text{Hom}(X, Z_1)}{\text{Hom}(X, Z)}$$

is surjective on objects and morphisms (induces an isomorphism on $\pi_0$ and a surjection on all the $\pi_i$).
Proof. As \( \text{Hom}(X, \mathbb{Z}^2 \mathbb{Z}) = \pi_0(\mathbb{Z}^2 \mathbb{Z} \times X) \),
we can restrict to proving
\[
\pi(\mathbb{Z}^2 \mathbb{Z}) \to \pi\mathbb{Z}^2 \mathbb{Z} \times \pi\mathbb{Z}
\]
is surjective on morphisms. So suppose given points \( a', b' \in Z' \), \( a, b \in Z \),
\[
\begin{array}{ccc}
p(a') & \rightarrow & p(b') \\
\downarrow h & & \downarrow k \\
q(a) & \rightarrow & q(b)
\end{array}
\]
and maps \( a' \rightarrow b' \) and \( a \rightarrow b \), such that some compatibility conditions are satisfied, i.e. the square
\[
\begin{array}{ccc}
p(a') & \rightarrow & p(b') \\
\downarrow h & & \downarrow k \\
q(a) & \rightarrow & q(b)
\end{array}
\]
commutes. Now \( h, k, m' \) and \( m \) are homotopy classes of arcs, and commutation means on choosing representative arcs, we can make them into the boundary of a square. Thus
we get a map of \[ I \rightarrow Z' \times Z_1 \times Z \]
with components \( m', \sigma, m \) joining the points \((a, b, \sigma_1)\)
to \((b', k, b)\). This proves surjectivity.

So now the next problem is clear: we want
the exact sequence, which in the case of interest reads

\[
\begin{align*}
\pi_1(\text{Hom}(X, Z)) & \rightarrow \pi_1(\text{Hom}(X, Z) \times \text{Hom}(X, Z_1)) \rightarrow \pi_1(X, Z) \rightarrow \\
\pi_0(\text{Hom}(X, Z') \times \text{Hom}(X, Z_1)) & \rightarrow \pi_0(X, Z') \rightarrow \pi_0(X, Z_1) \rightarrow \pi_0(X, Z).
\end{align*}
\]
Next suppose \( \mathcal{F} \) is a fibred category over \( \mathcal{H} \), and set

\[
[F, Z] = \Pi_0 \text{Hom}_{\text{cat} / \mathcal{H}}(\mathcal{F}, h_Z)
\]

and see if you can prove half-exactness.

Suppose we are given functors

\[
A': \mathcal{F} \to h_{Z'}
\]

\[
A_i: \mathcal{F} \to h_{Z_i}
\]

and an isomorphism of \( pA' \simeq qA_i \). This means that for each \( (\xi, X) \in \mathcal{F}(X) \), I have a class of homotopies

\[
\Theta(\xi): pA'(\xi) \to qA_i(\xi)
\]

such that for any maps \( (\tilde{f}, f): (\xi, X) \to (\eta, Y) \), we have

\[
\begin{align*}
PA'(\xi) & \xrightarrow{\Theta(\xi)} QA_i(\xi) \\
pA'(\xi) & \quad \downarrow pA'(f) \\
 & \quad \downarrow QA_i(f) \\
pA'(\eta) & \xrightarrow{\Theta(\eta)} QA_i(\eta)
\end{align*}
\]

commuting. Now the problem is to raise these classes of homotopies to actual homotopies so as to obtain
\[
B(\eta) \in \text{Hom}(X, \mathbb{Z}^2 / \mathbb{Z}^2)
\]
and
\[
B(f) : B(\iota) \to B(\eta)
\]
such that \( B(fg) = B(f) B(g) \).

Choose the \( \Theta(\iota) \) for every \( \iota \) in some way, and also choose actual homotopies \( A(f) \) and \( \tilde{A}(f) \).

Because the square (\( \star \)) commutes we can find a map from \( I^2 \) with given boundary; in other words we choose \( B(f) : B(\iota) \to B(\eta) \) so as to cover our choices \( \tilde{A}(f), A(f), \tilde{\Theta}(\iota) \).

So the only problem is why is \( B(fg) \sim B(f) B(g) \).

But the left triangle with sides \( pA(f), p\tilde{A}(g) \), and \( p\tilde{A}(fg) \) fills in as one knows that \( A(f) \tilde{A}(g) = A(fg) \), so the whole prism fills in giving the desired homotopy.

Thus it seems to work.
$k \leftarrow A \rightarrow L$

four abelian categories

\[ \text{Modf}(k) \rightarrow \text{Modf}(A) \rightarrow \text{Modf}(A) \rightarrow \text{Modf}(L) \]

Now the point to show is that the last three give rise to a long exact sequence and the first two give an isomorphism.

The fact that

\[ K_0(G; \text{Modf}(k)) = K_0(G; \text{Modf}(A)) \]

is clear, hence no problems with the beginning. Long exact sequence: For this you want fibre of map

\[ \begin{array}{cccc}
\mathcal{M}(L) & \xrightarrow{F} & B_A & \xrightarrow{B_L} & B_k^2 \xrightarrow{\text{k}[T]}
\end{array} \]

Start with the relations involving $X, Y, Z$.

must start with representations over $L$

\[ \begin{array}{c}
\lambda^*_X = \\
\lambda^*_Y = \\
\lambda^*_Z = \\
A^n \rightarrow \mathbb{R}^n
\end{array} \]

\[ \begin{array}{c}
\lambda^*_X = s_X \cdot \iota_1 \\
\lambda^*_Y = s_Y \\
\lambda^*_Z = s_Z
\end{array} \]

\[ \begin{array}{c}
G \\
\text{GL}_n(A) \xrightarrow{S} \text{GL}_n(k) \\
G
\end{array} \]

\[ \begin{array}{c}
A^n \\
\mathbb{R}^n
\end{array} \]

\[ \begin{array}{c}
K_2(k) \rightarrow K_2(A) \rightarrow K_2(L) \rightarrow K_{n-1}(k) \rightarrow
\end{array} \]
The great hope is to be able to do higher $K$-theory using a 2-category approach. Grothendieck derives the homotopy type of a space from the underlying category of open sets. 

**General scheme:** To each topos $X$ should be associated a category of sheaf spectra $S(X)$ which behaves like the derived category. Intuitively what's involved is *all kinds of mucky glueing data.* Thus an object $E'$ of $S(X)$ has homotopy sheaves which for sake of uniform notation we denote $H^1(X, E')$. However the Postnikov invariants can be quite a bit richer, $\pi_{\infty}$ as one sees immediately from the considerations of two stage Postnikov systems. Now let us assume that $E(X)$ has been defined. Then given a map $f: X \to Y$, one expects a functor $\mathcal{F}_{f*}: S(X) \to S(Y)$ and a Leray spectral sequence

$$E_2^{pq} = H^p(Y, R^q\mathcal{F}_*E') \quad H^{p+q}(X, E')$$

I am confused. Thus I would expect to have a spectral sequence

$$H^{p+q}(E') = H^p(X, H^q(E'))$$

So it's okay with $R^q\mathcal{F}_*E' = H^q(\mathcal{F}_*E')$. Again the point is to think of $E'$ as a generalized complex of sheaves, but with non-abelian Postnikov invariants. $X$

Now it is necessary to understand what are the omega spectra. This is interesting these are things $E'$ with $H^q(E') = 0$ for $q$ positive. As a check you note that

$$H^n(X, E') = \oplus H^{n-q}(X, H^q(E')) \quad \text{over } \mathbb{Q}$$

so once $n$ exceeds the dimension of $X$ you get $0$. Thus $H^n(E') = \pi_{-q}E'$. Thus the good complexes are above. The next point was to have an analogue of $\mathbb{E}_m$ which to every ringed space $X$ assigns an object of $S(X)$. I want to call this animal $\mathbb{G}_L$. It's like $\mathbb{E}_m$ which as an object of $S(X)$ is determined by a single group in dimension $0$. This seems pretty strange
general nonsense about alg. K-theory and how to go about formulating it.

Idea first of all is that to each scheme $X$ there should be an object of a suitable derived category $\mathcal{G}_X$ whose cohomology groups are the $K$-groups:

$$K^*_i(X) = H^*_i(\mathcal{G}_X)$$

$\mathcal{G}_X$ somehow is built up out of $\mathfrak{f}$-ad. on $\mathfrak{f}$ loc. free $\mathcal{O}_X$-modules of finite rank. $\mathcal{G}_X$ is a spectrum over $X$. It is not clear to me whether it should be corrected or not!

When one has a fibration

$$\eta \quad \Rightarrow \quad \mathcal{G}_X \quad \Rightarrow \quad \mathcal{G}_X$$

where $E^{\Phi \theta}$ is a sheaf over $X$ with

$$\pi_1(E^{\Phi \theta}) = \mu_{q^{-1}}$$

sheaves for the étale topology, so you must have $q^{-1}$ prime to residual characteristics, hence $q$ must be nilpotent on $X$. 
are there any coincidences

\[ K_a(X) \xrightarrow{c_i^\#} H^{2i-a}(X, \mu_a \otimes i) \]

no coincidences

\[ K_a(X) \xrightarrow{c_i^\#} H^{i-a}(X, \mathbb{Q}/\mathbb{Z}) \]

for K-theory I want \( c_i^\# = (i-1)! \cdot ch_i \) ?

\[ K_a(Y) \xrightarrow{i_*} K_a(X) \xrightarrow{j^*} K_a(U) \]

\[ \downarrow ch_i \quad \downarrow ch_i \]

\[ \rightarrow H^{2i-a}(X, \mu_a) \rightarrow H^{2i-a}(U, \mu_a) \]

\( ch_i (i_! 1) \quad \text{Todd class + Thom class} \)

\( ch (i_! 1) = c_i (\text{Todd } \nu_i) \)
Characteristic classes of $E$-vector bundles, where $E$ is a finite field. Let $A$ be an abelian group and consider the natural transformations from $X$ to $H^q(J,A)$. Something I did today suggests that such a natural transformation $G$ must also be considered a group-$k$ for relative representations. Thus if $G'$ is a subgroup of $G$ we have defined a group $k(G,G')$ of relative representations; it is a quotient of the Grothendieck group of the representation of the group $G+G$.

We want to see what we need to prove the exactness axiom for maps into cohomology. Thus suppose that $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ is an exact sequence of abelian groups and suppose I am given a natural transformation $\alpha$ from virtual bundles to cohomology with coefficients in $A$ of dimension $q$. This means that to each virtual representation of a group $G$ I am given a cohomology class $\alpha(x)$ in $H^q(G,A)$ in such a way as to be compatible with pull-backs. I suppose that $\alpha(x)$ goes to zero under the map induced by $A \rightarrow A''$. The problem arises from the fact that

What is a stable vector bundle over a topological space $X$? First idea is that of a sheaf of some sort, e.g., to every open subset of $X$ one should be able to associate the sections of the virtual bundle and this sheaf possibly with some extra structure should characterize the virtual bundle and hence this could be taken as a definition.

Example: A vector bundle of dimension $n$ can be defined as a locally free sheaf of modules for the structural sheaf. One should observe that whereas topologically one thinks of vector bundles of dimension $n$ in terms of the classifying space $BU_n$, i.e., the definition of the fibred category of such bundles so that the fibre over $X$ is equivalent to the fundamental groupoid of $X$ on $BU_n$. However this latter fibred groupoid is not a stack (champ en Francais) whereas the actual category of vector bundles is. Question: Does there exist a stack of virtual vector bundles over the category of topological spaces $\text{XX}$? Fix a classifying space within a category of spaces $\text{XX}$ e.g., paracompact ones. Then one obtains a stack, the fibred category $\text{XX}$ given by the functor represented by the classifying space.

It seems that virtual bundles do not glue.
exact sequence to generalize

\[ K_0(Y) \rightarrow K_0(X) \rightarrow K_0(U) \rightarrow 0 \]

In case of a Dedekind ring

In order to get I would want to get a filtration

\[ GL(k) \rightarrow G(A) \rightarrow G(L) \]

In terms of characteristic classes what we have is various reps. of a group G. In other words we have a rep. of G

So the only method I have at the moment is to form the fibre and try to discover its properties

So the above is typical of

\[ A[t] \leftarrow A \rightarrow A[t-1] \]

Situation.

Thus the problem might be the description of the representations. In different forms it might be as follows: (anyway)

Only chance seems to work in immeasurable. Idea is that this way can relate K-theory of A, K, k.

The method seems to be very mystifying.

Thus what is at stake is the action of GL_n(k) on the immeasurable (contractible after B-T) and having the indicated stabilization.
get straight chain classes for finite representations with arbitrary coefficients. The problem is that

$$\text{Hom}_n(D, H^i(G, M)) = \lim_n H^i(G \mathcal{L}_n, M)$$

whereas what we are after is $H^i(G \mathcal{L}_\infty, M)$. So the point is instead of a natural transf.

from $D$ to cohomology we want to consider isomorphism classes of natural transformations from stable representations to cohomology classes.

Thus we want to study a function which associates to each rep. $E$ of $G$ a cohomology $Z(E)$ of $G$, in such a way that the following holds.

Thus we have a 2-fibred category over the 2-category of groups, and we have a functor from

$$\mathcal{G} \rightarrow \mathcal{S}$$

and over spaces we have a fibred category represented by $K(M, i)$, so we pull this back, and then we want to consider isomorphism classes of cartesian sections.

It would seem that the 2-category approach might work with groups; unclear where the 2+3 cells come from.
Intuition about the stability theorems: It is a problem of showing that a map $S^i \rightarrow BGL^+_{\mathbb{W}}$ can be deformed into $BGL^+_{\mathbb{W}}$. My idea is that such a map can be realized as a kind of bundle over $S^2$ with a certain ramification of subvarieties of codimensions 2 and 3.

The method of proof would put then consist of carrying out the general position arguments to the ramification in a low end of the infinite general linear group. The first thing to understand is this: Given a manifold $Y$, attach a 2-cell by means of an embedded $S^1$ and determine general position for maps into $Y \times S^2$.

So at the moment we see that we get a submanifold of codimension 2 and some kind of ramification behavior along that submanifold. It seems necessary to take the cyclic permutation and write it as a product of commutators, which can be done already in $E_3$.

However after you attach the 2-cell, the new 2-dimensional homology class doesn't become spherical under you get into $E_5$. So what you need to see is the map $S^2 \rightarrow Y \times S^2$. Thus I must take the 2-sphere and find an interesting 5-fold covering with ramification points somehow related to the way you express the cycle $(123)$ as a product of commutators in the alternating group on 5-latters. For example let the icosahedral group act on the 2-sphere $S^2$ and form the quotient. If a point is fixed then it lies on the middle of a simplex on the icosahedron.

So the idea is that the 2-sphere has over it a branched covering with a finite number of branch points $S_1$ and now when you attach a 3-disk you should join these points to the center. So what one gets is a
Thus it appears that an element of $\mathfrak{M}_n^2(\mathfrak{gl}_n^+)$ can be realized by the following data: a subvariety of $S^2$ consisting of two strata one of which is a framed submanifold of codimension 2 and the other a framed submanifold of codimension 3, where the normal structure is that of a cone on a finite set of points on $\mathbb{S}^2$. On the complement I must give a representation of the fundamental group in the general linear group with monodromy transformation around the 2 manifold and relations around the 3 manifold prescribed. Now to prove the stability theorem what I want to do is to deform the representation of the fundamental group into matrices of a given size. What does one know about the homology of the complement of this subvariety? Alexander says that the homology of the complement is related by duality to that of the subvariety which it would seem is almost arbitrary. Thus it seems unlikely that the representation of the fundamental group could be deformed without changing the subvariety.

No hope.

Formula for $H_*(R_\mathbb{A})$: Let $A^\infty$ denote the free $A$-module with basis indexed by the positive $\mathfrak{m}$ indices $N$ and $GL_\infty(A)$ the group of automorphisms of $A^\infty$ whose matrices are equal to the identity except for a finite number of entries. Let $GL_{n,m}(A)$ be the group of $A$-linear automorphisms of the free $A$-module $A^n \otimes A^m$. Using the isomorphism of $A^n \otimes A^m$ with $A^{n+m}$, one obtains a homomorphisms $GL_{n,m}$, $GL_{n+m}$, and $GL_{n,m+1}$ permitting one to define the limit group

$$GL_{\infty,\infty}(A) = \lim \ GL_{n,m}(A)$$

as those automorphisms of the direct sum $A^\infty \otimes A^\infty$ which are matrices almost everywhere the identity, and which carry the first factor into itself.

Then choosing an isomorphism of the direct sum $A^n \otimes A^m$ with $A^{n+m}$ one obtains two homomorphisms of $GL_{n,m}$ into $GL_{n+m}$. Let $GL_{n,m}$ be the group of $A$-module $A^n \otimes A^m$ which carry the subspace...
the

Goal: To simultaneously generalize/generalized cohomology theories of algebraic topology and the derived category in algebraic geometry. Key ideas involve associating to any topos $X$ a triangulated category $\mathcal{T}(X)$ whose objects should be thought of as generalized cohomology theories over $X$. Perhaps $\mathcal{T}(X)$ should be the homotopy category associated to the category of simplicial spectra of groups in the topos $X$. Given a morphism $f : X \to Y$ one expects to have a functor $f_* : \mathcal{T}(X) \to \mathcal{T}(Y)$ and a functor $f^\# : \mathcal{T}(Y) \to \mathcal{T}(X)$ which should be some kind of derived functor extension of the similarly denoted such functors on sheaves of groups. The setting up of this theory should encompass the derived category $\mathcal{S}$ of sheaves over a topos and the theory of generalized cohomology theories. More precisely we should have a cartesian square of theories

\[
\begin{array}{ccc}
\text{derived category of abelian sheaves over } X & \text{?} \\
\text{chain complexes of abelian groups} & \text{gen. coh. theories or Boardman spectra} \\
\end{array}
\]

The principal reason for introducing the above God-awful machinery is to habile the example of algebraic $K$-theory, i.e. stable groups. Thus over a point everything comes from the family of symmetric groups. (Key problem: any idea of how to describe the category of spectra in terms of the symmetric groups:?)

Goal 2: Algebraic $K$-theory. To any ringed topos one has the object $G_m$ of the derived category; similarly one should have an object $\mathcal{G}_m$ in $\mathcal{T}(X)$, where which when applied to a point give the $K$-groups of the ringed topos $X$. One expects to be able to define $\mathcal{K}$ trace and Gysin homomorphisms for relative proper schemes of finite Tor dimension. Essentially this theory should be (strictly) $\mathcal{K}$-gen. strictly degree zero phenomena? I don't know whether to expect a type of periodicity Gysin homomorphism (Perhaps this is the really key result of the good theory a non-trivial result relating $\mathcal{K}$-groups of different dimensions.
I see at the moment that the definition of $T(X)$ is an extension of the derived category theory and no more in the case of characteristic zero where generalized cohomology theories and chain complex theories ought to coincide. (Question: Let $G$ be a group and consider the homotopy category of $G$-sets. Is the usual degeneracy over $Q$ valid?)

The fundamental problem remains how to work with objects in the derived category which do not glue? Here Deligne's approach seems to be the key.

Produce a generalization of $\mathbb{G}_m$ which associates to a ringed space $X$ a sheaf spectrum $\mathbb{GL}_X$ whose cohomology

$$H^i(X, \mathbb{GL}_X) = K_{-i}(X)$$

are the $K$-groups. Notice that the determinant should give rise to a map

$$\varprojlim_n \Omega^n \mathbb{GL}_X \longrightarrow B \mathbb{G}_m X$$

hence to maps

$$K_{-i}(X) \longrightarrow H^{i+1}(X, \mathbb{G}_m X)$$

(not too clear).

Given $g \mapsto g^{i-1}$ unit on $X$ for all $i$, e.g. $g$ nilpotent or $X$ of characteristic zero, then have a fibration

$$E_X \longrightarrow GL_X \xrightarrow{g^{i-1}} GL_X$$

where $E_X$ is a sheaf spectrum over $X$ such that

$$\pi_{2i}(E_X) = 0$$

$$\pi_{2i-1}(E_X) = \mathbb{M} g^{i-1}$$

hence we expect maps
Some ideas: Let $X$ be a projective variety over $k$, let $F$ be a coh. sheaf on $X$, and let $G$ act linearly on $F$. Then we can write $F$ as the quotient of a $G$-vector bundle

$$
\mathcal{O}(n) \otimes \Gamma(X, F(n)) \longrightarrow F
$$

Therefore in computing $K$-groups one notes that they depend only on the algebraic hull of $G$.

For example if $X$ projective over $\mathbb{F}_p$, then the higher $K$-groups come from finite group representations of finite groups. Thus:

Example: Suppose $X$ is a projective variety over $\mathbb{F}_p$, and we define $K(X) \otimes \mathbb{Q}$ in terms of a universal additive class

$$
R(G; X) \longrightarrow H^*(G, K(X) \otimes \mathbb{Q})
$$

Then as the group of automorphisms of a bundle $E$ is finite it follows that rational classes are trivial $\Rightarrow K(X) \otimes \mathbb{Q} = 0$. 
\[
K_a(X) \xrightarrow{\partial^1} K_a(X) \xrightarrow{\tilde{\alpha}} H^1(X, E_X) \\
\xrightarrow{c_{\tilde{\alpha}}^{-1}} \\
H^{2\tilde{\alpha} - 1}(X, \mu_{\tilde{\alpha}^{-1}}).
\]

Now suppose we have 
\[
y \subset X \subset U
\]
with \(y\) and \(X\) non-singular. Then there should be a long exact sequence of \(K\)-groups. This might come about as follows: quite generally one should have a triangle

\[
0 \rightarrow i^! GL_X \xrightarrow{j^*} GL_X \xrightarrow{i^* GL_X} 0
\]

and one should have isomorphisms

\[
j^* GL_X = GL_U \quad \text{(trivial as } j^* GL_{m,x} = GL)\]

\[
i^* GL_X = GL_Y \quad \text{periodicity theorem.}
\]
June 27, 1990

I recall that I have shown that if $X$ is a projective variety over $\mathbb{F}_p$, then $K_*(X) \otimes \mathbb{Q} = 0$. I think this implies that the character

$$
\chi : K_*(X) \otimes \mathbb{Q} \rightarrow \left[ \lim_{\leftarrow} H^{2*-d}(X, \mu_{2^d}) \right] \otimes \mathbb{Q}
$$

is quite far from being injective, i.e., it is not quasi-isomorphism. Thus I want to show that

$$
\lim_{\leftarrow} H^{2*-1}(X, \mu_{2^d}) \otimes \mathbb{Q}
$$

is not in general non-zero. No.

Recall

$$
\begin{array}{ccc}
X & \leftarrow & \overline{X} \\
\downarrow & & \downarrow \\
\mathbb{F}_p & \leftarrow & \overline{\mathbb{F}_p}
\end{array}
$$

$$
E^{p_0} = H^p(\text{Gal}(\mathbb{F}_p/\mathbb{F}_q), H^q(\overline{X}, \mu_{2^d}^{\otimes i})) \rightarrow H^q(X, \mu_{2^d}^{\otimes i}).
$$

As the groups in this spec. seq. are finite, we can pass to limit over $v$ and then tensor with $\mathbb{Q}$ obtaining

$$
0 \rightarrow \left[ H^{d-1}(X)(i) \right]_F \rightarrow \left[ \lim_{\leftarrow} H^d(X, \mu_{2^d}^{\otimes i}) \right] \otimes \mathbb{Q} \rightarrow [H^d(X)(i)]^F \rightarrow 0
$$

where as usual $H^d(X)(i) = \left( \lim_{\leftarrow} H^d(X, \mu_{2^d}^{\otimes i}) \right) \otimes \mathbb{Q}$.
and \( F \) is Frobenius. Now the Weil conjectures imply that \( H^d(X)(i)^F \neq 0 \) only if \( j = 2i \), and similarly for \( H^d(X)(i)^F \) which has the same dimension as the invariants. Hence we see that

\[
\left[ \lim_{\nu} H^d(X, \mu_{2\nu}^{\infty}) \right] \otimes \mathbb{Q} = \begin{cases} 
\mathbb{Q} & j \neq 2i, 2i+1 \\
H^{2i}(X)(i)^F & j = 2i \\
H^{2i}(X)(i) & j = 2i+1 
\end{cases}
\]

Thus there is no contradiction as we thought.

**Conclusion:** Weil-Tate conjectures imply that

\[
\text{ch}: K^*(X) \otimes \mathbb{Q} \rightarrow \bigoplus_{i} H^{2i-\alpha}(X, \mu_{2\nu}^{\infty}) \otimes \mathbb{Q}
\]

is surjective for \( \alpha > 0 \) and an isomorphism for \( \alpha > 1 \).

(In fact maybe an isomorphism for \( \alpha = 0 \) if it should be true that any cycle homologically equivalent to zero is torsion element of \( K(X) \), which is reasonable if one believes that such cycles are points of \( \mathbb{Q} \)-varieties.)
June 26, 1970:

Here is an example which lends support to the conjecture that $H^*_c(GL(F_2)) = 0$. I suppose $q = 2^r$ and let $a \in GL_n(F_2)$ be an element of order 2. By Jordan canonical form $a$ is determined up to conjugacy by the dimension $d$ of the image of $(a-1)$. First suppose $d = 1$. Then

$$a = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

hence $a$ belongs to the elementary 2 group of rank $n-1$

$$\begin{bmatrix} 1 & * & * & * & * & * \\ & 1 & & & & \end{bmatrix}$$

whose normalizer produces any linear transformation on $A$.

But

$$H^*_c(GL(A)) = \mathbb{Z}_2[\omega_{2^{-1}r_1}, \ldots, \omega_{2^{-1}r_n}] \quad \text{if} \quad r = \text{rank } A$$

so as $n \to \infty$ it follows that the map

$$H^*_c(GL) \longrightarrow H^*_c(Z_2)$$

is zero.
For general $d$ we note that $\alpha$ is conjugate to $\beta$ contained in the subgroup

\[
\begin{array}{c}
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\end{array}
\]

whose normalizer contains $GL_\ell(\mathbb{F}_q)$ and one might hope no invariants remain as $r \to \infty$. 
Better thing is to take elementary subgroup $A$. Let $\begin{array}{c|c|c}
1 & a & b \\
\hline
a & b & b \\
\hline
\end{array}$

whose normalizer contains all linear transformations of $A$. 
June 28, 1970:

More evidence for $\tilde{H}_x(\text{GL}(\mathbb{F}_q), \mathbb{Z}_p) = 0$. I claim that the map

$$H_+ (\Sigma_{\infty}) \longrightarrow H_+ (\text{GL}(\mathbb{F}_q))$$

is zero, or equivalently if $W$ is a characteristic class for $\overline{\mathcal{R}}$, then $W(E) = \mathcal{O} W(0)$ if $E$ is a permutation representation. The point is to introduce the kernel

$$JG \longrightarrow \overline{\mathcal{R}}(G) \longrightarrow \overline{H}^0 (G, H_{\text{GL}})$$

consisting of the five virtual representations which are killed by all unstable characteristic class. Claim $JG$ is closed under multiplication. In effect the product on $\overline{\mathcal{R}}$ induces a map

$$H_{\text{GL}} \otimes H_{\text{GL}} \longrightarrow H_{\text{GL}}$$

showing that there is a formula of the form

$$W(\xi \cdot \eta) = \sum \omega_i^{\prime}(\xi) \omega_i^{\prime}(\eta)$$

(all $\omega_i^{\prime}, \omega_i^{\prime\prime}$, of positivity)

if $\omega \in H^n(\text{GL})$. Thus if $\omega(\xi) = 0$ for all $\xi$ we have same for $\omega(\xi \cdot \eta) = 0$, hence $JG$ is an ideal in $\overline{\mathcal{R}}(G)$.

Now use the fact that the coh. of $\Sigma_m$ is detected by elementary abelian groups. The permutation representation is an exact sum of regular representations.
and a regular representation is a tensor product of smaller ones. Hence we are reduced to checking that

\[ w(\text{reg } \mathbb{Z}_2) = 0, \quad \text{and} \quad w(E \otimes F) = 0 \quad \text{if} \quad w(E) = w(F) = 0. \]

But

\[
(E \otimes F) - d(E) \cdot d(F) = (E - d(E))(F - d(F)) + d(E)(F - d(F)) + d(F)(E - d(F))
\]

and all these go to zero. Finally for reg \( \mathbb{Z}_2 \) we did this before.