~~limit as $n \to \infty$~~

$$U^q(S^{2n-1} \times_{\mathbb{Z}_p} S^1)$$

$$U^q_c(S^{2n-1} \times_{\mathbb{Z}_p} \mathbb{C}^*) \longrightarrow U^q_c(S^{2n-1} \times_{\mathbb{Z}_p} \mathbb{C}) \longrightarrow U^q(S^{2n-1} \times_{\mathbb{Z}_p} \mathbb{C})$$

$$\begin{tikzcd}
U^{q+2}_c(L-X) \arrow[r] & U^{q+2}_c(L) \arrow[r] & U^{q+2}(L) \\
U^{q+1}(SL) \arrow[r] \arrow[u] & U^q(X) \arrow[r, "\omega"] \arrow[u] & U^{q+2}(X) \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
SL \arrow[r] \arrow[d] \arrow[dr] & L-X \arrow[d] \\
X \arrow[r] & L
\end{tikzcd}$$

properly homotopic

$$\begin{tikzcd}
U^q_c(S^{2n-1} \times_{\mathbb{Z}_p} \mathbb{C}^*) \\
U^q_c(S^{2n+1} \times_{\mathbb{Z}_p} \mathbb{C}^*) \arrow[u]
\end{tikzcd}$$

November 29, 1969:

Consider the symmetric group $\Sigma(n)$. We wish to determine the minimal prime in $H^*(B\Sigma(n))$ mod p cohomology ring. Such primes correspond to maximal elementary abelian p-subgroups of $\Sigma(n)$.

Theorem: Let π be a ~~[illegible]~~ p-adic partition of n, that is, π is ~~a partition [illegible]~~ decreasing sequence $p^{i_1} \geq p^{i_2} \geq \cdots$ whose sum is n. Let A_π be the subgroup of $\Sigma(n)$ ~~consisting of~~ which is the product of $(\mathbb{Z}_p)^{i_j}$ acting on the

$$\boxed{S^{2n-1} \longrightarrow S^n \longrightarrow X}$$

bordism definition of e-invariant

$$\begin{tikzcd}
S^{2n-1} \arrow[r] & S^n \\
M^{n-1} \arrow[u] \arrow[r] & pt \arrow[u]
\end{tikzcd}$$

$$M^{n-1} = \partial X^n$$

$$\underbrace{\text{Todd } X^n \in \mathbb{Q}/\mathbb{Z}}_{\text{e invariant}}$$

~~I want the H~~

do same in unoriented theory

$$M^{n-1} = \partial X^n$$

and you consider the intersection of M^{n-1} and X^n.

$$\cancel{M^{n-1} = \partial X^n}$$

take something like the whitney or maybe Wu genus.

$$w^{-g+r} Q(f_* 1) = \sum_{i=0}^{r} w^{r-i} \sum_{\langle\alpha\rangle = i} (a)^{\alpha}_* (s_\alpha f_* 1)$$

$L = U$ mod 2 torsion in degree $< g$.

$w^r(w^{-g} Q(f_* 1) - f_* 1) = \rho(\varphi(w))$ $\quad \varphi \in L[[X]]$.

r least. $r > 1$ $\quad \rho(\varphi(0)) = 0$ $\quad \cancel{\varphi(0) \in U}$ $\quad \rho : L \longrightarrow U$

~~by hypothesis~~ ρ injective by Lazard.

$\therefore \varphi(0) = 0$.

$$w^{r-1}(w^{-g} Q(f_* 1) - f_* 1) = \rho\left(\frac{\varphi(w)}{w}\right) + a$$

$a \in U(pt)$ lower degree contradiction

$\therefore r = 1$ $\quad \underline{f_* 1} = \rho(\underset{L}{\psi(0)}) + 2a$

$L/2L \longrightarrow U/2U$ onto

$U = C \oplus K$ $\quad K$ uniquely divisible

The basic formula is

$$\boxed{v^s(w^{-g} P(f_* 1) - f_* 1) = \sum_{\alpha > 0} \Phi^{(s)}_\alpha(v)\, s_\alpha(f_* 1)}$$

for s sufficiently large where $\Phi^{(s)}_\alpha(X) \in L[[X]]$ $\cancel{U(BZ_p)}$

case 1: Assume by induction that after localizing wrt (p) that $L \longrightarrow U$ is an isom. in dim $< g$ and that U has no p torsion in dim $< g$. True if $g = 0$ so suppose $g > 0$.

Let M be an oriented manifold and let G be a finite group acting on M. Then M/G is a rational homology manifold and has Pontryagin classes. Now

$$H^*(M/G) = H^*(M)^G = H^*_G(M).$$

Suppose M is a complex manifold on which G acts holomorphically. Then have a quotient manifold M/G and the analogue of the index is $\chi(M/G)$, arithmetic genus. Idea is that sheaf of holomorphic functions $\mathcal{O}_{M/G} \cong f_*(\mathcal{O}_M)^G$. Question: What are the Todd homology classes?

Same question for Stiefel-Whitney classes. According to Sullivan given a polyhedron such that any link has Euler char $\equiv 0$ (2) one can define Stiefel-Whitney homology classes.

$$\sum_{n\geq 0} t^n ch_n = \sum_{i=1}^{N} e^{tx_i}$$

$$ch_n = \cancel{\sum \frac{t^k}{k!} \sum_i x_i^k} \ \frac{1}{n!} \sum_{i=1}^{N} x_i^n$$

but by Newton formulas

$$(-1)^n n S_n = c_n \quad \text{modulo decomposable}$$

$$\log \frac{1}{\prod_{i=1}^{N}(1-tx_i)} = \sum_{i=1}^{N} \sum_k \frac{(tx_i)^k}{k}$$

$$\log \frac{1}{1 \cdot (-1)^{n-1} t^n c_n}$$

$$(-1)^{n-1} t^n c_n = \frac{1}{n} t^n S_n$$

$$\therefore \ S_n = (-1)^{n-1} n c_n \quad \text{mod decomp.}$$

$$\therefore \ ch_n = \frac{1}{n!}(-1)^{n-1} n c_n = \frac{(-1)^{n-1}}{(n-1)!} c_n \quad \text{mod decomp}$$

$$\therefore \ Ch = \sum_n \frac{(-1)^{n-1}}{(n-1)!} c_n \quad \text{mod decomp.}$$

$$\begin{tikzcd}
PH(U(n)) \arrow[d] & \tilde{H}(BU(n)) \arrow[l] \arrow[d] & ch - n \\
PH(T^n) & \tilde{H}(BT(n)) \arrow[l] & \Sigma'(e^{x_i}-1) \arrow[dll, bend left] \\
\Sigma'_i x_i & &
\end{tikzcd}$$

thus it seems that the basic element x in $K^{-1}(\Sigma' U(n))$ goes ~~...~~ under char into the $\varepsilon_1 \in H^1(U(n))$ and so we should have that

$$\psi^k x = k x$$

$$\text{ie} \quad \frac{\psi^k}{k}(x) = x. \qquad \times \quad ?$$

---

so I want to know what happens to $ch \in \tilde{H}^{ev}(BU)$ under the map $\tilde{H}^{ev}(BU) \longrightarrow H^{odd}(U)$ suspension

what can I say about ch modulo decomposables?

$$\Delta\, ch = ch \otimes 1 + 1 \otimes ch \qquad \varepsilon(ch) = 0.$$

$$\Delta\, c_n = \sum c_i \otimes c_j$$

$$ch = \sum a_\alpha c^\alpha \qquad \text{want coeff of } c_n.$$

thus want to evaluate on the sphere S^{2n}

$$ch\, \beta^n = 1 \qquad \text{want coeff. of } c_n \text{ in } ch_n$$

need ~~to~~ to understand signs

$$\begin{tikzcd} X\times X' \arrow[r] \arrow[d, "id_X\times f'"'] \arrow[dr, "f\times f'"] & Y\times X' \arrow[d] \\ X\times Y' \arrow[r, "f\times id_{Y'}"'] & Y\times Y' \end{tikzcd}$$

$$(f\times f')_* = (f\times id)_* \circ (id\times f')_*$$

Definition: $\theta(f)\times\theta(f') \longrightarrow \theta(f\times f')$ is the composition

$$\begin{tikzcd} \theta(f)\times\theta(f') \arrow[r] \arrow[d] & \theta(f\times f') \arrow[d, phantom, "\cong"] \\ \theta(f\times id)\times\theta(id\times f') \arrow[r, "\circ"] & \theta((f\times id)\circ(id\times f')) \end{tikzcd}$$

$$\begin{tikzcd} X\times Y' \arrow[r] \arrow[d, "f\times id"] & X \arrow[d, "f"] \\ Y\times Y' \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} X\times X' \arrow[r] \arrow[d, "id\times f'"] & X' \arrow[d, "f'"] \\ X\times Y' \arrow[r] & Y' \end{tikzcd}$$

$$\begin{tikzcd} X\times X' \arrow[r, "\beta_2"] \arrow[d, "\alpha_2"] \arrow[dr] & Y\times X' \arrow[r] \arrow[d, "\alpha_1"] & X' \arrow[d, "\alpha"] \\ X\times Y' \arrow[r, "\beta_1"'] \arrow[d] & Y\times Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "\beta"'] & Y \arrow[r] & pt \end{tikzcd}$$

$$\nu_{gf} = \nu_f + f^*\nu_g \;!!$$

$$\frac{R(G)[T]}{T^n - \lambda^2(V)T^{n-1} + \cdots} \qquad \frac{\mathbb{C}[T]}{\prod_{i=1}^{n}(T-\alpha_i)} \qquad \alpha_i \; \cancel{g\text{th}} \text{ roots of unity}$$

$$R(G) \longrightarrow \mathbb{C}$$

$g \in G$

then

ψ^k acts for $k \equiv 1 \; (g)$

$$\psi^k T = T^k$$

$$\alpha_i^k = \alpha_i$$

$$\frac{\mathbb{C}[T]}{(T-\lambda)^n}$$

$$\psi^k(T) = T^k \qquad \lambda^k = \lambda.$$

$$T - \lambda \longmapsto T^k - \lambda^k = (T-\lambda)\left(T^{k-1} + T^{k-2}\lambda + \cdots + \lambda^{k-1}\right)$$

now set $(T-\lambda)^2 = 0$ and this is

$$k\lambda^{k-1} = k.$$

in other words when there is nilpotence one gets eigenvalues of k.

Tom Dieck claims that

$RP^n \times RP^n$ and $\underline{CP^n}$

are cobordant as $\mathbb{Z}_2$-manifold.

$\boxed{RP^n \times RP^n \text{ and } CP^n}$

so consider these as [illegible] in [illegible]

$$B \underset{A}{\otimes} A\Omega(X) \xrightarrow{\sim} B \otimes_k K(X)$$

$$B \otimes_A R \longrightarrow \underline{S}$$ S augmented B-algebra

$$B \otimes_A F_n R / F_{n+1} R \xrightarrow{\sim} F_n S / F_{n+1} S$$

$$B \otimes_A \underline{gr_1 R} \xrightarrow{\sim} \underline{F_1 S / F_2 S}$$ ~~free~~ finite type over B ~~proj. finitely gen over B~~

~~to calculate ho~~

as B/A is faithfully flat $gr_1 R$ must be proj. finite type

finite type: if $gr_1 R$ not f.t. has an inf. chain ~~$gr_1 R$~~ of submodules V_α with union $gr_1 R$

Same for $B \otimes_A V_\alpha$

proj. as above $gr_1 R$ is of fin pres. so

$$B \underset{A}{\otimes} \mathrm{Ext}^q_A(gr_1 R, M) = \mathrm{Ext}^q_B(gr_1 S, M) = 0.$$

It follows that $\Lambda gr_1 R \xrightarrow{\sim} R$. $\boxed{gr_1 R \mid \text{A-free?}}$

enough to compute action of Adams operations on $B \otimes_A gr_1 R$

so you get a map

$$Y^0 \wedge \Sigma^{12n} \longrightarrow MU$$

GOAL: ① Prove tom Dieck's localization theorem

② Generalize to localizing wrt a subgp H.

$$X \qquad \boxed{X^H}$$

X^G + G bundle ν_X

X^H + H-bundle

also can consider (X^H, ν_X) as an N space + bundle.

$$U_G(X) \longrightarrow U_N(X^H)$$

odd order $\Rightarrow$ $MSU(BG)$ $MU(BG)$

somehow similar

$MSU(BG)$
$MU(BG)$ somehow the same.

on the other hand one has that all Euler classes lie in $MSU(BG)$.

$$U_G(X) \longrightarrow U(X^0)$$

$$Y \xrightarrow{i} X \xleftarrow{j} (X-Y)$$

\begin{tikzcd}
Y \arrow[r] & E \\
Y^0 \arrow[u] \arrow[r] \arrow[ur, phantom, "\text{cart}"] & E^0 \arrow[u]
\end{tikzcd}

$\mathbb{P}(E)^0 \to X^0$

$X \longrightarrow E \longrightarrow Y$

$X \qquad E^0 \longrightarrow Y^0$ ✓

\begin{tikzcd}
V \arrow[r, hook] & W \\
V^0 \arrow[u] \arrow[r] \arrow[ur, phantom, "\text{cart.}"] & W^0 \arrow[u]
\end{tikzcd}

not trans cart.

start with X get X^0 and $\nu(X^0 \subset X)$ a G bundle over X^0.

$U(X^0)$ not exact. But it is both covariant + contravariant homotopy.

$h_t : X \longrightarrow Y$ G-maps

$X^0 \longrightarrow Y^0$

Next problem is to put in more information.

$X \hookleftarrow X^0$

$\boxed{\text{bundle}}\ \boxed{\text{cobordism:}}$

get both a manifold and a v.b. over that manifold.

honest positive v. bundle, not virtual

$$(Z, \nu_f) \xrightarrow{f} X$$

$$\begin{array}{ccc} X & \longmapsto & (X^0, \nu) \\ \downarrow & & \\ Y & & Y \end{array}$$

\begin{tikzcd}
X^0 \arrow[r] \arrow[d, "f^0"] & X \arrow[d, "f"] \\
Y^0 \arrow[r] & Y
\end{tikzcd}

Instead of $K(X)$ we use $K(pt) \otimes_L \mathcal{U}^*(X)$ which is ~~the~~ known to be $K(X)$ granted the periodicity theorem. ~~[illegible]~~ Assume that

$$K(pt) \otimes_L \mathcal{U}^*(X)$$

is a λ-ring (augmented over $H^0(X, \mathbb{Z})$) ~~and that~~ and introduce the ~~[illegible]~~ associated Chern ring/$\mathbb{Q}$. There is a good map

$$Ch\left(K(pt) \otimes_L \mathcal{U}^*(X)\right) \xrightarrow{\alpha} \mathcal{H}(X)$$

and the hope is that one can prove that ~~the [illegible] is [illegible] the map~~ there is a comm. diagram

$$\begin{tikzcd} & \mathcal{U}^*(X) \arrow[dl] \arrow[dr] & \\ Ch\left(K(pt) \otimes_L \mathcal{U}^*(X)\right) \arrow[rr, "\alpha"] & & \mathcal{H}(X) \end{tikzcd}$$

~~since both s~~ Then the map α must be surjective and so since Ch has only positive degree ~~negative~~ components the same holds for $\mathcal{H}(X)$.

Question: $K(X)$ is a λ-ring, hence $\exists\ \psi^k$ $k \in \mathbb{Z}$ does there exist a map

$$K(X) \longrightarrow \bigoplus_{n \geq 0} \mathbb{Z}\binom{T}{n} \otimes_{\mathbb{Z}} K(X) \qquad \text{ring hom.}$$

$$\ni \quad L \longmapsto \psi^T(L) = \text{[illegible]}\ L^T = (1 + L - 1)^T$$

$$= \sum_{n \geq 0} \binom{T}{n} (L-1)^n$$

yes assume $c_1(L)$ nilpotent.

~~Recall that~~

Thus I can form a ring of the form

$$\underbrace{Q^*_{\mathbb{Z}_2}(pt)} \times \underbrace{U^*(pt)}$$

$$(a,b) + (a',b') = (a+a', b+b'+\underline{aa'}) \qquad ?$$

$$(a,b)\cdot(a',b') = (aa', a^2b'+a'^2b-2bb')$$

then Q:

$$a * b = a + b + \bar{a}\bar{b}\xi \qquad \underline{no}$$

$$W_2(U(pt)) \longrightarrow U_{\mathbb{Z}_2}(pt)$$

$$(a,b) \longmapsto Qa + b\xi$$

The basic identity is therefore that $\xi . u = \xi . r(u)$

$$(a,b)+(a',b') \longmapsto Qa+b\xi + Qa'+b'\xi$$

$$\|$$

$$Q(a+a') - aa'\xi + (b+b')\xi$$

$$\|$$

$$(a+a', b+b'-aa') \longmapsto Q(a+a') + (b+b'-aa')\xi. \qquad \text{additive}$$

$$(a,b)\cdot(a',b') \longmapsto (Qa+\xi b)(Qa'+\xi b')$$

$$\|$$

$$Q(aa') + \underline{Qa\cdot\xi b'} + Qa'.\xi b + \xi^2 bb'.$$

$$(aa', a^2b' + a'^2b + 2bb')$$

---

question might be whether it is true that $Qa\cdot\xi = a^2\xi$

$$\xi^2 = 2\xi \longleftarrow \text{Clear}$$

May 5, 1970: Here are some notes about computational problems in cohomology of groups.

1.) mod p cohomology of $GL_n(\mathbb{F}_q)$. Sometime ask Mumford what he knows. ~~Milgram~~ Milgram said for $q=p$, he could compute the cohomology of the group of triangular matrices by using the permutation representation on the points of the vector space $\mathbb{F}_q^n$. ~~These~~ The point was that the wreath product in the symmetric group is easy to see.

Observe that for $GL_3(\mathbb{F}_p)$ the Sylow p-subgroup is the Heisenberg group: $x^p = y^p = z^p = 1 \quad (x,z) = (y,z) = 1$ $(x,y) = z$, and this might lead to a solution of the case of extra-special p-groups.

May 5, 1970 Attempt to do $H^*(GL_n(\mathbb{F}_q), \mathbb{Z}/2)$ in exceptional case by a similar method.

How to handle the exceptional case $l=2$, $q \equiv 3 \bmod 4$. The idea I had last night was to use the subgroup

$$N = \left[\Sigma_m \tilde{\times} (\pi \tilde{\times} \mathbb{F}_{q^2}^*)^m\right] \times (\mathbb{F}_q^*)^e \qquad \pi = \mathrm{gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$$

of $GL_n(\mathbb{F}_q)$, where $n = 2m+e$ $e=0,1$. Then the orders are

$$|GL_n(\mathbb{F}_q)| = q^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} (q^i - 1) = q^{\frac{n(n-1)}{2}} \prod_{1 \leq i \text{ odd} \leq n} (q^i-1) \prod_{\substack{i \text{ even} \\ i \leq n}} (q^i-1)$$

$$|N| = m! \left(2(q^2-1)\right)^m (q-1)^e$$

And these have same power of 2 since $v_2(q^{\text{odd}}-1) = 1$ and $v_2(q^{2i}-1) = v_2(i) + v_2(q^2-1)$. To make the argument work one needs to know whether $\pi \tilde{\times} C$ $C = \mathbb{F}_{q^2}^*$ has its mod 2 cohomology detected by elem. ab. 2 grps. Its Sylow 2 subgroup is

$$(\pi \tilde{\times} C)_2 = \langle y, x \rangle \qquad \begin{aligned} y^2 &= 1 \\ x^{2^{a+1}} &= 1 \\ yxy^{-1} &= x^{-1+2^a} \end{aligned}$$

Here y generates π and x is a primitive 2^{a+1} root of 1 where $a = v_2(q+1)$, $a+1 = v_2(q^2-1)$, and y acts on x by raising to the qth power and

$$q+1 \equiv 2^a \quad (2^{a+1})$$

since (both sides are $\not\equiv 0$ and) on multiplying by 2 they became zero.

Let G denote this group $\langle x, y\rangle$ and lets first determine its conjugacy classes of elementary abelian 2-subgroups. The elements of order 2 are

$$x^{2^a} \qquad x^{2i}y \qquad i = 0, 1, .., 2^a - 1$$

$$(x^{2i}y)(x^{2j}y) = x^{2i-2j}$$

hence $x^{2i}y$ and $x^{2j}y$ commute $\Leftrightarrow$ $2i \equiv 2j \mod 2^a$. Therefore the maximal elementary abelian 2-subgroups are

$$\langle x^{2^a}, x^{2i}y\rangle \qquad i = 0, 1, .., 2^{a-1} - 1$$

We know that in the dihedral group $\langle x^2, y\rangle$ these subgroup fall into 2-conjugacy classes depending on whether i is even or odd. In the whole group

$$x^j \langle x^{2^a}, x^{2i}y\rangle x^{-j} = \langle x^{2^a}, x^{2i+2j(1-2^{a-1})}y\rangle$$

since $\quad x^j y x^{-j} = x^j x^{(-1+2^a)(-j)}y = x^{2j-2^a j}y = x^{2j(1-2^{a-1})}y$

As $a \geq 2$ (since $q+1 \equiv 0$ (4)), $1-2^{a-1}$ is a unit mod 2^a, hence $2j(1-2^{a-1})$ is any even number, and we conclude that all of these elementary 2-subgroups are conjugate.

Conclusion: Only one max. elementary abelian 2-subgp. of G up to conjugacy and its of rank 2.

This has some implications when we try to compute the

G is an extension of one cyclic group by another and I think Wall has published computation in his article on extensions. (see Layard for reference).

cohomology of G by means of the Hochschild-Serre spectral sequence associated to the extension

$$0 \longrightarrow \langle x^2 \rangle \longrightarrow \langle x, y \rangle \longrightarrow \langle \bar{x}, \bar{y} \rangle \longrightarrow 0$$

With the notation of the Adams conjecture paper we know that $d_2 v = t_1^2 + t_1 t_2$ since mod x^4 the 2^a drops out. If $d_3 u = 0$, then we would get same cohomology as dihedral gp which isn't true since there would have to be two minimal prime ideals. Hence $d_3 u \neq 0$. If $d_3 u$ were a non-zero-divisor in $S[t_1, t_2]/(t_1^2 + t_1 t_2)$ then

$$E_4 = E_4^{*0} \otimes S[u^2]$$

~~And~~ ~~E_4^{*0} would have Betti numbers 1, 2, 2, 1, 0 ...~~ ~~since it has rank 2.3 by Bezout. Hence $d_5(u^2)$ ...~~ ~~dimensional ... so the dimension would~~ would be of dimension 1 contradicting rank 2. Thus $d_3 u$ is either divisible by t_1 or by $t_1 + t_2$ and one can ~~check easy enough~~ ~~over which elementary abelian~~ see that it is t_1 since $\langle x^2, y \rangle$ acts on the plane.

Perhaps ~~possible this~~ one can go on to show that u^4 is an infinite cycles and compute the cohomology. $\circledast$

It doesn't do me any good because the element $t_1 \in \mathrm{Hom}(G, \mathbb{Z}/2)$ with $t_1(x) = 1$ $t_1(y) = 0$ dies on all the elementary abelian 2-subgps.

<u>Conclusion:</u> Mod 2 cohomology of the Sylow 2-subgp of $GL_2(\mathbb{F}_q)$ $q \equiv 3 \mod 4$ is not detected by elementary abelian 2-subgroups.

May 7, 1970. On $H^*(GL_n(\mathbb{F}_q), \mathbb{Z}/p)$ as $\mathbb{F}_q \nearrow \overline{\mathbb{F}}_p$.

Suppose $n=2$, then the Sylow p-subgroup of $GL_2(\mathbb{F}_q)$ is the group $U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ $* \in \mathbb{F}_q$. Each conjugate corresponds to a line in $V = \mathbb{F}_q^2$, i.e. to the line L one associates U_L = those linear transf inducing identity on L and on V/L. The intersection of two distinct conjugates must induces identity on two independent lines and so is the identity. Hence ~~one~~ has (coeff in $\mathbb{Z}/p$)

$$H^*(GL_2(\mathbb{F}_q)) \xrightarrow{\sim} H^*(U)^B$$

where B = Borel subgroup $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ is the normalizer of U. (Actually this formula holds as the Sylow subgp. is abelian) Thus we wish to compute the invariants in

(p odd) $$H^*(\mathbb{F}_q, \mathbb{Z}/p) = \Lambda[\mathrm{Hom}(\mathbb{F}_q, \mathbb{Z}/p)] \otimes S[\mathrm{Hom}(\mathbb{F}_q, \mathbb{Z}_p)]$$

under the action of $\mathbb{F}_q^*$. I am going to show that if $q = p^r$, then first non-zero invariant in $\tilde{H}^*(\mathbb{F}_q)$ occurs in dimension $r(2p-3)$.

Extending the bases to $\overline{\mathbb{F}}_q$ commutes with taking invariants and

$$\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{F}_q, \overline{\mathbb{F}}_q) \cong \bigoplus_{a=0}^{r-1} \overline{\mathbb{F}}_q \sigma^a$$

where σ is Frobenius. If $\lambda \in \mathbb{F}_q^*$ then

$$\lambda^*(\sigma^a) = \sigma^a \circ \text{mult by } \lambda = \lambda^{p^a} \cdot \sigma$$

as transf. of $\mathbb{F}_q$. Thus if L denotes the standard representation of $\mathbb{F}_q^*$ on $\mathbb{F}_q$, we have

$$\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{F}_q, \overline{\mathbb{F}}_q) = \bigoplus_{a=0}^{r-1} L^{\otimes p^a}$$

ignore

and the Poincaré series of $H^*(\mathbb{F}_q, \mathbb{F}_q)$ as a representation of $\mathbb{F}_q^*$ is

$\otimes$ = product in repn. ring of $\mathbb{F}_q^*$ over $\mathbb{F}_q$.

$$\bigotimes_{a=0}^{r-1} \left(1 + tL^{\otimes p^a} + t^2 L^{\otimes p^a} + t^3 L^{\otimes 2p^a} \cdots\right)$$

A typical term of this is of the form

$$\bigotimes_{a=0}^{r-1} t^{2n_a - \varepsilon_a} L^{\otimes n_a p^a} = t^{\sum(2n_a - \varepsilon_a)} L^{\otimes \sum n_a p^a}$$

where $\varepsilon_a = 0, 1$ and $\varepsilon_a = 0$ if $n_a = 0$. This will give an invariant iff $\sum n_a p^a \equiv 0 \mod p^r - 1$. Now I wish to look for the first such invariant of positive degree. Suppose a_0 such that $0 \le a_0 < r$ and $n_{a_0} > p-1$. Can assume $\varepsilon_{a_0} = 1$. If $a_0 < r-1$, then changing

$$n_{a_0} \longmapsto n_{a_0} - p \qquad \varepsilon_{a_0} \longmapsto 0$$

ignore

$$n_{a_0+1} \longmapsto n_{a_0+1} + 1$$

$$n_{a_0} p^{a_0} + n_{a_0+1} p^{a_0+1} = (n_{a_0} - p) p^{a_0} + (n_{a_0+1} + 1) p^{a_0+1}$$

but $\sum 2n_a - \varepsilon_a$ changes by $-2p + 2 + 1 < 0$.

~~[illegible]~~ If $a_0 = r-1$, then changing

$$n_{r-1} \longmapsto n_{r-1} - p \qquad \varepsilon_{r-1} \longmapsto 0$$

$$n_0 \longmapsto n_0 + 1$$

so

$$n_{r-1} p^{r-1} + n_0 \equiv (n_{r-1} - p) p^r + (n_0 + 1)$$

stays same, but $\sum 2n_a - \varepsilon_a$ changes by $-2p + 2 + 1 < 0$. Therefore the minimal situation occurs with $0 \le n_a < p$. However $\sum n_a p^a = p^r - 1$ plus uniqueness of ~~[illegible]~~ p-adic expansion implies that $n_a = p-1$ and hence all $\varepsilon_a = 1$ and so the first term is of degree

$$\sum_{a=0}^{r-1} 2(p-1) - 1 = (2p-3)r$$

which proves our assertion

It seems unlikely that $H^*(U)^B$ is a polynomial ring when $q = p^r$ and $r \ge 2$. Thus if $p = 2$ we want the invariants in just the symmetric algebra i.e. ~~[illegible]~~ and Poincaré series is

$$\sum_{n_0, \ldots, n_{r-1} \ge 0} t^{\sum n_a} L^{\otimes \sum n_a 2^a}$$

So if $r = 2$ the Poincare series of the invariants is

$$\sum_{\substack{n_0 + 2n_1 \equiv 0\ (3) \\ n_0, n_1 \ge 0}} t^{n_0 + n_1} = \sum_{\substack{m_0 \ge 0 \\ m_1 \ge 0}} t^{3m_0 + 3m_1} \left\{ 1 + t^2 + t^4 \right\}$$

$$= \frac{1+t^2+t^4}{(1-t^3)^2}$$

which is not the Poincare series of a polynomial ring with 2 generators. This calculation tends to suggest great difficulty exists in finding a closed formula for $H^*(GL_n(\mathbb{F}_q))$ even for $n=2$.

<u>Corollary</u> of this calculation

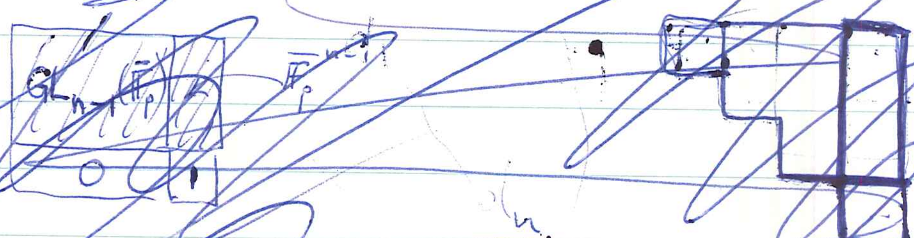
$$\tilde{H}^a(GL_2(\mathbb{F}_{p^r}), \mathbb{Z}/p) = 0 \qquad \text{if} \quad a < r(2p-3).$$

$$\tilde{H}^*(GL_2(\bar{\mathbb{F}}_p), \mathbb{Z}/p) = 0$$

<u>Question:</u>

$$\tilde{H}^*(GL_n(\bar{\mathbb{F}}_p), \mathbb{Z}/p) = 0 \quad ?$$

~~Proof: By induction on n. Let~~ $\cancel{\Gamma = GL_{n-1}(\bar{\mathbb{F}}_p) \ltimes \bar{\mathbb{F}}_p^{n-1} \subset GL_n(\bar{\mathbb{F}}_p)}$ ~~as the matrices~~



~~The~~ $\cancel{\Gamma}$ ~~detects mod p cohomology of~~ $\cancel{GL_n(\bar{\mathbb{F}}_p)}$ ~~as it contains a Sylow p subgroup of~~ $\cancel{GL_n(\mathbb{F}_{p^r})}$ ~~for all r. By~~

? ~~Proof~~: We (try to) show that the **Borel subgroup** B of upper triangular matrices has no mod p cohomology and we use induction on n. So B acts on a vector sp-

V of dimension n and preserves a flag. Let B' be the subgroup inducing the identity on the bottom line of the flag so that $B = \overline{\mathbb{F}}_p^* \tilde{\times} B'$. The cohomology of B injects into that of B' and is invariant under the action of $\overline{\mathbb{F}}_p^*$. But $B' \cong B_{n-1} \tilde{\times} \overline{\mathbb{F}}_p^{n-1}$ and B_{n-1} has no cohomology mod p, hence none in that of $\overline{\mathbb{F}}_p^{n-1}$ as one sees by using a composition series. So finally one reduces to show that there is no cohomology of $\overline{\mathbb{F}}_p^{n-1}$ invariant under the multiplication by $\overline{\mathbb{F}}_p^*$.

To be more precise use homology, whence the homology mod L of $V = \overline{\mathbb{F}}_p^{n-1}$ as an abelian group is

$$\Gamma(V) \otimes \Lambda(V)$$

Γ and Λ being taken over $\mathbb{Z}/p$, $\overline{\mathbb{F}}_p$. I want to show there are no co-invariants for the multiplication action of $\overline{\mathbb{F}}_p^*$. Can extend the base

$$V \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p = V \otimes_{\overline{\mathbb{F}}_p} (\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$$

and

$$\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \xrightarrow{\sim} \mathrm{Homcont}(\hat{\mathbb{Z}}, \overline{\mathbb{F}}_p)$$

$$\lambda \otimes \mu \longrightarrow (\sigma \longmapsto \lambda^\sigma \mu)$$

and $\overline{\mathbb{F}}_p$ acts on a function by

$$(\lambda \cdot f)(\sigma) = \lambda^\sigma f(\sigma).$$

~~What~~ So I go back to calculation in the case $\overset{n-1}{=}1$ and I'm now looking for a minimal $\sum 2n_{ai} - \varepsilon_{ai}$ with

$$\sum n_{ai}\, p^a \equiv 0 \quad (p^r - 1)$$

where $a = 0, \dots, r-1$ and $i = 1, \dots, n-1$. But if the first sum is less than $2N$, then each $n_{ai} \leq N$ and so

$$\sum n_{ai}\, p^a \leq N(n-1)\frac{p^r - 1}{p-1}$$

which shows that there should be trouble with $n = p$ or $p+1$.

Part II. last section

May 11, 1970

Heuristic calculation part of paper:

~~The preceding sections have some interesting heuristic consequences for algebraic K theory.~~

The cohomological calculations of the preceding sections suggest some natural conjectures concerning algebraic K theory and what the groups $K_i(\mathbb{F}_q)$, suitably defined for all $i \geq 0$, ought to be.

Let R be a ring which need not be commutative and let X be a space. By an R-vector bundle over X I mean a locally-trivial fibre bundle E whose fibers are projective finitely-generated R-modules. If X is connected with basepoint x and locally-simply connected so that $\pi_1(X,x)$ exists, then ~~[illegible]~~ then the category of R-vector bundles over X is equivalent to the category of projective finitely generated R-modules endowed with an action of $\pi_1(X,x)$, i.e. representations of $\pi_1(X,x)$ over R. Let $kR(X)$ be the Grothendieck group of the category of R-vector bundles over X. Then $kR(X)$ is a contravariant functor on the homotopy category (of CW complexes say). Consider the following statement.

Statement: There is a universal morphism of functors on the homotopy category

$$kR \longrightarrow [\ ,B_u]$$

of kR to a representable functor

since kR admits a decomposition

$$kR(X) = [X, K_0(R)] \times \widetilde{kR}(X)$$

it follows that B_u admits a product decomposition

$$B_u \simeq K_0(R) \times B.$$

The groups $K_i(R)$ are defined by

$$K_i(R) = \pi_i(B_u) \qquad i \geq 0.$$

Suppose R is commutative. Then given i one has the ith arithmetic Chern class

$$c_i : \widetilde{kR}(X) \longrightarrow H^{2i}(X, \operatorname{Spec} R; \mu_n^{\otimes i})$$

---

where the cohomology involves the etale cohomology of Spec R. Let $I^\cdot$ be an injective resolution of $\mu_n^{\otimes i}$ in the category of sheaves for the etale topology on Spec R, and set

$$C^\cdot(R, \mu_n^{\otimes i}) = I^\cdot(\operatorname{Spec} R)$$

Then one knows that

$$H^q(X, \operatorname{Spec} R; \mu_n^{\otimes i}) = H^q(X, C^\cdot) \qquad \text{(hypercohomology)}$$
$$= [X, K(q, C^\cdot)]$$

where $K(q, C^\cdot)$ denotes the simplicial abelian group associated to the chain complex $S^q C^\cdot$, $C^\cdot$ shifted up q units.

By the universal property we get a map

$$B_u \longrightarrow K(2i, C^\cdot(\mathrm{Spec}\, R, \mu_n^{\otimes i}))$$

and taking homotopy groups, we get maps

$$c_i^\# : K_a(R) \longrightarrow H^{2i-a}(\mathrm{Spec}\, R, \mu_n^{\otimes i}).$$

(Problem: work out ring properties; one would expect lots of $(i-1)!$ factors.) The above homomorphisms ~~could~~ might generalize the maps found by **Bass-Tate** in low dimensions.

Now suppose $R = \mathbb{F}_q$. ~~[illegible]~~ As the Brauer map

$$\widetilde{k\mathbb{F}_q}(X) \longrightarrow [X, E\Psi^q]$$

is a map to a representable functor, there is by the universal property a definite map

$$B \longrightarrow E\Psi^q$$

~~[illegible]~~ By the universal property an element of $H^q(B, A)$, A an abelian group, is the same thing as a natural transformation from $\widetilde{k\mathbb{F}_q}$ to $H^q(\ , A)$ and we have seen this is the same as a cohomology class of $E\Psi^q$ if [illegible] $A = \mathbb{Z}/l$. Thus the map induces an isomorphism on H^* with coefficients in $\mathbb{Z}/l$ for all primes l. This makes it reasonable to conjecture that

$$(*) \qquad \pi_i(B) \xrightarrow{\sim} \pi_i(E\Psi^q).$$

Now the latter can be computed using the homotopy long exact sequence of the fibration

$$E\Psi^q \longrightarrow BU \xrightarrow{\Psi^q - id} BU$$

and one finds

$$\pi_{2i}(B) = 0 \qquad i \geq 0$$

$$0 \longrightarrow \underset{\mathbb{Z}}{\underset{\|}{\pi_{2i}(BU)}} \xrightarrow{q^i - 1} \underset{\mathbb{Z}}{\underset{\|}{\pi_{2i}(BU)}} \longrightarrow \pi_{2i-1}(E\Psi^q) \longrightarrow 0$$

so that $\pi_{2i-1}(B)$ is cyclic of order $q^i - 1$.

Now the isomorphism $*$ depends on the lifting $\phi : \bar{\mathbb{F}}_q^* \longrightarrow \mathbb{C}^*$, and it would be nice to have a functorial formula for $K_i(\mathbb{F}_q)$ independent of ϕ. [illegible] ϕ is a generator of the dual

$$\mathrm{Hom}(\bar{\mathbb{F}}_q^*, \mathbb{C}^*) = \varprojlim_{(n,p)=1} \check{\mu}_n$$

which is non-canonically isomorphic to

$$\varprojlim_{(n,p)=1} \mathbb{Z}/n\mathbb{Z} \;\cong\; \prod_{l\neq p} \mathbb{Z}_l \, .$$

We wish to define an action of the group of units of this ring on $E\Psi^q$. Given an integer k because Ψ^k commutes with Ψ^q it induces a well-defined map $\Psi^k : E\Psi^q \longrightarrow E\Psi^q$, which is the identity of course, if k is a power of q. On the homotopy groups

$$\Psi^k = \text{mult. by } k^i \text{ on } \pi_{2i-1}(E\Psi^q)$$

hence if k and k' are congruent modulo some large integer prime to p they induce the same transformation of a given Postnikov part of $E\Psi^q$. Thus we get an action of (mult. monoid of) the above profinite ring on $E\Psi^q$. Clearly then if $\gamma \in$ that ring then

$$\begin{tikzcd} & E\Psi^q \arrow[d, "\Psi^\gamma"] \\ k\mathbb{F}_q \arrow[ur, "\phi"] \arrow[r, "\phi^\gamma"'] & E\Psi^q \end{tikzcd}$$

is commutative, because one can use a density argument.

Let $\zeta_{\phi,q}$ be the unique element of $\mathbb{F}_q^*$ such that

$$\phi(\zeta_{\phi,q}) = \exp 2\pi i/q-1$$

Then we have a commutative diagram

$$\begin{tikzcd}
 & \mathbb{Z}/q^i-1 \arrow[r, "\zeta_{\phi,q}^{\otimes i}"] \arrow[d, "k^i"] & \mu_{q^i-1}^{\otimes i} \arrow[d, equal] \\
\pi_{2i-1}(B) \arrow[ur, "\phi"] \arrow[r, "\phi^k"'] & \mathbb{Z}/q^i-1 \arrow[r, "\zeta_{\phi^k,q}^{\otimes i}"] & \mu_{q^i-1}^{\otimes i}
\end{tikzcd}$$

for k a unit in $\mathbb{Z}/q-1$

$$\phi^k\left(\zeta_{\phi,q}^{k^{-1}}\right) = \phi(\zeta_{\phi,q}) = \exp 2\pi i/q-1$$

One concludes that there is a canonical isomorphism

$$\pi_{2i-1}(B) = \boxed{K_{2i-1}(\mathbb{F}_q) \xrightarrow{\sim} \mu_{q^i-1}^{\otimes i}}$$

independent of the choice of ϕ.

Our next problem is to see how the groups $K_i(\mathbb{F}_q)$ vary. So consider an extension

$$\simeq \left(\Psi^{q^{d-1}}x + \cdots + \Psi^{q}x\right) + \Psi^{q^d}x$$

$$\overset{id+\theta}{\simeq} \Psi^{q^{d-1}}x + \cdots + x$$

so β fits into a morphism of fibration sequences

$$\begin{tikzcd}
E\Psi^{q^d} \arrow[r] \arrow[d, "\beta"] & BU \arrow[r, "\Psi^{q^d}-1"] \arrow[d, "\Psi^{q^{d-1}}+\cdots+1"] & BU \arrow[d, "1"] \\
E\Psi^{q} \arrow[r] & BU \arrow[r, "\Psi^{q}-1"] & BU
\end{tikzcd}$$

Using these morphisms of fibrations one sees that

$$\begin{tikzcd}
K_{2i-1}(\mathbb{F}_q) \arrow[r, "\phi", "\sim"'] \arrow[d, "f^*"] & \mathbb{Z}/q^i-1 \arrow[r, "j^{i}_{\phi,q}", "\sim"'] \arrow[d, "\alpha = 1+q^i+\cdots+q^{(d-1)i}"] & \mu^{\otimes i}_{q^i-1} \arrow[d] \\
K_{2i-1}(\mathbb{F}_{q^d}) \arrow[r, "\phi", "\sim"'] & \mathbb{Z}/q^{di}-1 \arrow[r, "j^{i}_{\phi,q^d}", "\sim"'] & \mu^{\otimes i}_{q^{di}-1}
\end{tikzcd}$$

see p. 9

commutes since

$$\phi\left(j_{\phi,q^d}\right) = \exp 2\pi i/q^d-1$$

$$\Longrightarrow \left(j_{\phi,q^d}\right)^{q^{d-1}+\cdots+1} = j_{\phi,q}.$$

Also

$$\begin{tikzcd}
K_{2i-1}(\mathbb{F}_{q^d}) \arrow[r, "\phi"] \arrow[d, "f_*"] & \mathbb{Z}/q^{di}-1 \arrow[r, "j^{\otimes i}_{\phi,q^d}"] \arrow[d, "\cdot 1"] & \mu^{\otimes i}_{q^{di}-1} \arrow[d, "\mathrm{Norm}^{\otimes i}"] \\
K_{2i-1}(\mathbb{F}_q) \arrow[r, "\phi"] & \mathbb{Z}/q^{i}-1 \arrow[r, "j^{\otimes i}_{\phi,q}"] & \mu^{\otimes i}_{q^{i}-1}
\end{tikzcd}$$

OKAY

commutes. So we conclude that

$$\begin{tikzcd}
K_{2i-1}(\mathbb{F}_q) \arrow[r, "\sim"] \arrow[d, shift right, "f^*"'] & \mu_{q^i-1}^{\otimes i} \arrow[d, shift right, "\alpha"'] \\
K_{2i-1}(\mathbb{F}_{q^d}) \arrow[r, "\sim"] \arrow[u, shift right, "f_*"'] & \mu_{q^{di}-1}^{\otimes i} \arrow[u, shift right, "\mathrm{Norm}^{\otimes i}"']
\end{tikzcd}$$

where α is the unique map such that

$$\alpha\left[(\mathrm{Norm}\,\xi)^{\otimes i}\right] = \left(q^{(d-1)i} + \cdots + q^i + 1\right)\xi^{\otimes i}$$

for $\xi =$ a primitive ~~[illegible]~~ $(q^{di}-1)^{\text{th}}$ root of 1. Perhaps a better way of putting it is to say that

$$\boxed{f^* f_* z = \sum_{\sigma \in \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)} z^\sigma}$$

or

$$f^*\left(f_*\,\xi^{\otimes i}\right) = \left(\sum_{a=0}^{d-1} q^{ai}\right)\xi^{\otimes i} = \frac{q^{di}-1}{q^i-1}\cdot\xi^{\otimes i}$$

Note also that

$$\boxed{f_* f^* x = d\cdot x}$$

Since f^* is injective on passage to the limit we obtain non-canonical isomorphisms

$$K_{2i-1}(\overline{\mathbb{F}}_q) \cong \bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell \qquad K_{2i}(\overline{\mathbb{F}}_q) = 0 \quad i>0$$

The last thing we want to compute is the integral arithmetic Chern class map

$$c_i : K_a(\mathbb{F}_q) \longrightarrow \varprojlim_{(n,p)=1} H^{2i-a}(\overset{\pi}{\cancel{Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)}}, \mu_n^{\otimes i})$$

where $\pi = Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.

Now one computes easily that

$$\varprojlim_n H^i(\overset{\pi}{\cancel{Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q}}, \mu_n^{\otimes i}) = \begin{cases} 0 & i \neq 1 \\ \mu_{q^i-1}^{\otimes i} & i = 1 \end{cases}$$

(Recall how this goes: $\pi = \hat{\mathbb{Z}}$ with Frobenius σ for generator and φ acts on μ_n by multiplying by q. Now

$$H^1(\pi, M) \simeq M/(\sigma-1)M$$

a functorial isomorphism. Hence

$$H^1(\pi, \mu_n^{\otimes i}) \cong \mu_n^{\otimes i}/(q^i-1)\mu_n^{\otimes i} \longrightarrow \mu_{q^i-1}^{\otimes i}$$

provided q^i-1 divides n. This is compatible as $n \to \infty$.

What we want to know therefore is that

$$\begin{array}{ccc} k\mathbb{F}_q(X) & \xrightarrow{\phi} & E\Phi^q(X) \\ {\scriptstyle \text{arith}}\downarrow{\scriptstyle c_i} & & \downarrow \text{— map induced by } \Phi c_i \text{ in } H^*(\ , \mathbb{Z}/q^i-1) \\ H^{2i-1}(X; \mu_{q^i-1}^{\otimes i}) & \cong & H^{2i-1}(X, \mathbb{Z}/q^i-1) \end{array}$$

$\uparrow$ isom furnished by ϕ

commutes. If we suppose this is true, then we can compute the map

$$c_i^{\#} : K_{2i-1}(\mathbb{F}_q) \longrightarrow \mu_{q^i-1}^{\otimes i}$$

because it is isomorphic to the map

(*) $$\pi_{2i-1}(E\Psi^q) \longrightarrow \mathbb{Z}/q^i-1$$

which is obtained by evaluating $\Phi(c_i)$ on a spherical homology class. Actually it is better to see this by writing the diagram

$$\begin{tikzcd}
k\mathbb{F}_q(X) \arrow[r, "\phi"] \arrow[d] & E\Psi^q(X) \arrow[d] \\
{[X, B_{\mathbb{F}_q}]} \arrow[ur, "\phi", "\cong"'] \arrow[d] & H^{2i-1}(X, \mathbb{Z}/q^i-1) \\
H^{2i-1}(X, \mu_{q^i-1}^{\otimes i}) \arrow[ur, "\cong", "\phi"'] &
\end{tikzcd}$$

and this gives a square by putting $X = S^{2i-1}$.

$$\begin{tikzcd}
\pi_{2i-1}(B_{\mathbb{F}_q}) \arrow[r, "\simeq", "\phi"'] \arrow[d, "c_i^{\#}"] & \pi_{2i-1}(E\Psi^q) \arrow[d, "(\Phi c_i)^{\#}"] \\
\mu_{q^i-1}^{\otimes i} \arrow[r, "\simeq", "\phi"'] & \mathbb{Z}/q^i-1
\end{tikzcd}$$

To compute the map $(*)$ is easy because the generator of $\pi_{2i-1}(E\Phi^\delta)$ comes from the generator of $\pi_{2i}(BU) \simeq \pi_{2i-1}(U)$.

$$
\begin{tikzcd}
U \arrow[d, "j"] \\
E\Phi^\delta \arrow[d, "i"] \\
BU
\end{tikzcd}
$$

One knows $j^*\Phi(c_i)$ = suspension of $c_i \in H^{2i}(BU, \mathbb{Z}/q^i-1)$.
This is naturality of Φ with respect to the morphism of squares

$$
\begin{tikzcd}
U \arrow[r] \arrow[d] & BU^I \arrow[d] \\
pt \arrow[r] & BU \times BU
\end{tikzcd}
\longrightarrow
\begin{tikzcd}
E\Phi^\delta \arrow[r] \arrow[d] & BU^I \arrow[d, "\Delta"] \\
BU \arrow[r, "\Gamma"] & BU \times BU
\end{tikzcd}
$$

So ~~[illegible] we conclude that~~

$$\left\langle j_*(\text{gen. of } \pi_{2i-1}(U)), \Phi(c_i) \right\rangle$$

$$= \left\langle \text{gen. of } \pi_{2i-1}(U), \text{susp. } c_i \right\rangle$$

$$= \left\langle \text{gen. of } \pi_{2i}(BU), c_i \right\rangle = (-1)^{i-1}(i-1)!$$

Conclusion (conjectural): Both groups $K_{2i-1}(\mathbb{F}_q)$ and $\varprojlim_{(n,p)=1} H^1(\pi, \mu_n^{\otimes i})$ are canonically isomorphic to $\mu_{q^i-1}^{\otimes i}$, and the map

$$c_i^\# : K_{2i-1}(\mathbb{F}_q) \longrightarrow \varprojlim_{(n,p)=1} H^1(\pi, \mu_n^{\otimes i})$$

Crazy idea: barycentric coordinates

$$\sum_{i=0}^{n} t_i = 1$$

Think of t_i as the energy of the i-th vertex. ~~Is there any point to introducing~~

Better: Think of the $\{t_i\}$ as a probability measure on the vertices.

Is there any point in introducing a wave function, i.e. a complex valued function ψ on the vertices so that $|\psi(i)|^2 = t_i$?

May 14, 1970

To understand fibre-bundle theory as it occurs in homotopy theory with the goal of finding the correct formulation of representing a fibre-bundle theory.

I want to start with the simplest example: principal G-bundles where G is a discrete group. For each space X let $\mathcal{K}(X)$ be the category of principal G-bundles over X. Then $\mathcal{K}$ is a fibred category over the category $\mathcal{S}$ of topological spaces.

Next I want to consider $\mathcal{S}$ as a 2-category where a 2-morphism is a homotopy class of homotopies. Then the pull-back should ~~define~~ give rise to a functor

$$\begin{array}{ccccc} \mathcal{K}(Y) & \times & \underline{\mathrm{Hom}}(X,Y) & \longrightarrow & \mathcal{K}(X) \\ \xi & & f & \longmapsto & f^*(\xi) \end{array}$$

in some way to be made precise. Now although $f^*(\xi)$ is not determined it is determined up to canonical isomorphism in $\mathcal{K}(Y)$, and similarly if $h : f \Rightarrow g$ is a homotopy, then there is a definite isomorphism $f^*(\xi) \to g^*(\xi)$ associated to it. I recall that the good way to think of h is as coming from inverting a homotopy equivalence, i.e.

$$\begin{tikzcd} X \times I \arrow[r, "h"] \arrow[d, "pr_1"] & Y \\ X \arrow[u, shift left=3, "i_0"] \arrow[u, shift left=1.5, "i_1"'] & \end{tikzcd}$$

So let's try only to use that f^* is an equivalence of categories when f is a heq, or more precisely that if f is a homotopy equivalence then there are equivalences of categories

$$\mathcal{K}(X) \xleftarrow{\text{source}} \overset{\text{in } \mathcal{K}}{\text{Arrows}}/f \xrightarrow{\text{target}} \mathcal{K}(Y)$$

(Note that for a general fibred category one has an equivalence of categories

$$\{\text{Cartarrows in } \mathcal{K}/f\} \xrightarrow{\text{target}} \mathcal{K}(Y)$$

which on choosing an inverse gives an f^*. For the $\mathcal{K}$ under consideration all arrows are cartesian.)

$$\begin{tikzcd} & \mathrm{Cart}/i_0 \arrow[dl, "\text{targ.}"'] \arrow[dr, "\text{source}"] & \\ \mathcal{K}(X \times I) & & \mathcal{K}(X) \\ & \mathrm{Cart}/i_1 \arrow[ul, "\text{targ}"] \arrow[ur, "\text{source}"'] & \end{tikzcd}$$

(seems to be a mess.)

Next let's see what it means to represent this bundle theory by BG. First of all there is a universal bundle ξ over BG and so for any X we can consider the category $\mathcal{K}(X)/\xi$

$$\mathcal{K}(X) \xleftarrow{\text{source}} \mathcal{K}(X)/\xi$$

(To get any bearing, suppose $\mathcal{K}$ is a discrete fibred category hence $\mathrm{Ob}\,\mathcal{K}(X) = \{\xi \in F(X)\}$, only identity morphisms. Then for ξ over B to represent F means that $\xi \in \mathrm{Ob}\,\mathcal{K}$ is the final object of $\mathcal{K}$ and hence

$$\mathcal{K}(X) \longleftarrow \mathcal{K}(X)/\xi$$

is an equivalence of categories.)

Back to principal G-bundles. An object of $\mathcal{K}(X)/\xi$ is a square

$$\begin{array}{ccc} \eta & \longrightarrow & \xi \\ & & \\ X & \longrightarrow & B \end{array}$$

and a morphism is a map of squares inducing the identity on X, B, ξ. Now given two maps $\eta \to \xi$ and $\eta' \to \xi$ over the same $f: X \to B$, one knows that there can be only one map from η to η' and vice versa. Thus the category $\mathcal{K}(X)/\xi$ is discrete.

So what's really going on is that whereas in the discrete case one got $\mathcal{K}(X)$ by letting the discrete category $\mathrm{Hom}(X,B)$ act on ξ, here we get the full groupoid $\mathcal{K}(X)$ by letting the groupoid $\underline{\mathrm{Hom}}(X,B)$ act on ξ.

The ultimate point of this nonsense is to take a more or less arbitrary $\mathcal{K}$ and show that it should map universally to a representable one. Thus you see that in the case of usual categories, the limiting criteria are fulfilled, i.e. given $k \longrightarrow h_{X_i}$ $i \in I$ then you get

$$k \longrightarrow \varprojlim h_{X_i} = h_{\varprojlim_i X_i}$$

when that limit exists.

What is a map from $\mathcal{K} \longrightarrow H_B$ to consist of? Clearly a cartesian functor. Check this the other way.

Take now $\mathcal{K}$ to be the category of virtual R-vector bundles. Obviously a whole treatise is needed to understand this construction and how to do it on the fibres of a fibred category.

It is most crucial for you to understand why, at least in the case of $\mathbb{F}_q$, a map from $\mathcal{K}$ to cohomology with coefficients in a field is the same as a map from $k = \pi_0 \mathcal{K}$ to such cohomology. Thus suppose that I am given a natural transformation from k to $H^q(\ , \mathbb{Z}/\ell)$.

Let $\theta : \widetilde{k\mathbb{F}_q}(G) \longrightarrow H^*(G, \mathbb{Z}/\ell)$ be a natural transformation. Quite generally I would like to know if the characteristic classes of representations are representable. Thus set

$$M^i(A) = \mathrm{Hom}\left(\widetilde{k\mathbb{F}_q}, H^i(\ , A)\right)$$

for any $\mathbb{Z}/\ell$ module A. Then $A \longmapsto M^i(A)$ is an additive functor compatible with inverse limits since

$$H^i(?, \varprojlim A_i) = \mathrm{Hom}\,(H_i(?), \varprojlim A_i)$$

Hence $\quad M^i(A) = \mathrm{Hom}\,(M_i, A) \quad$ for some M_i

More precisely, suppose given a functor F contravariant on a small category $\mathcal{C}$ to ~~modules~~ sets and $H.$ a covariant functor to $\mathbb{Z}/\ell$-modules. Set $H^{\cdot}(X, A) = \mathrm{Hom}(H.(X), A)$. Then

$$A \longmapsto \mathrm{Hom}\left(F, H^{\cdot}(\ , A)\right)$$

~~is~~ commutes with ~~[illegible]~~ inverse limits and hence is of the form $\mathrm{Hom}(M, A)$, where M is the cokernel

$$\bigoplus_{X \to Y \in \mathrm{Mor}\,\mathcal{C}} \mathbb{Z}/\ell\,[F(Y)] \otimes H.(X) \rightrightarrows \bigoplus_{X \in \mathrm{Ob}\,\mathcal{C}} \mathbb{Z}/\ell\,[F(X)] \otimes H.(X) \longrightarrow M \longrightarrow 0$$

If you call $M = H_*(F)$, then its clear that $H_*(F \times F) = H_*(F) \otimes H_*(F)$ and so with $F = \tilde{k}$ one gets a Hopf algebra.

May 15, 1970: Computation of some K_1's.

Let C be a category with a notion of direct sum or of exact sequence, permitting one to define a Grothendieck group functor $kC(X)$ or $kC(\pi)$ where π is a group. One can form then $\widetilde{kC}$ and consider natural transf. of it to $H^1(\ ,A)$. One shows that the set of such natural transformations, as a functor of A, is represented by a universal determinant from pairs (P,σ) with P in C and $\sigma \in \mathrm{Aut}(P)$

$$\det : \{(P,\sigma)\} \longrightarrow k_1 C$$

which adds for compositions of automorphisms and exact sequences.

I want to compute this in the case where C is the category of finite G sets where G is a fixed profinite group. Then $k_0 C$ is the free abelian group generated by the set of isomorphism classes of irreducible finite G-sets, equivalently conjugacy classes of open subgroups. The category C decomposes as a product of subcategories of isotypical G-sets. If S is isotypical with subgroup H, then the normalizer N of H acts freely on S^H and

$$G \times_N S^H \xrightarrow{\sim} S$$

as G-sets. Thus C becomes equivalent to the product of the categories of free finite N sets where H runs representatives up to conjugacy for the open subgroups.

To compute the determinant it is necessary to understand the determinant for the category of free finite $N/H = Q$ sets. Note that the group of autos of the Q set Q^n is the wreath product $\Sigma_n \tilde{\times} Q^n$ where the Q^n acts on the right. However it is clear that

$$(*) \qquad (\Sigma_n \tilde{\times} Q^n)_{ab} \xrightarrow{\sim} \mathbb{Z}_2 \times Q_{ab}$$

$$(\sigma, g_1, \ldots, g_n) \longmapsto (\mathrm{sign}(\sigma), \bar{g}_1 \cdots \bar{g}_n)$$

Indeed a map of $\Sigma_n \tilde{\times} Q^n$ to an abelian group A must first factor through $\Sigma_n \tilde{\times} Q^n_{ab}$, then through $\Sigma_n \times Q_{ab}$ since the sum map $Q^n_{ab} \to Q_{ab}$ gives the coinvariants for the Σ_n-action, and finally through $\mathbb{Z}_2 \times Q_{ab}$ since $(\Sigma_n)_{ab} = \mathbb{Z}_2$. Finally one notes that the above homomorphism $(*)$ is compatible with Whitney sum, hence is a determinant.

Conclusion: Let G a profinite group and let H_i $i \in I$ be a set of representatives for the conjugacy classes of open subgroups of G, and let N_i be the normalizer of H_i in G. Then

$$K_1(\text{cont. finite } G\text{-sets}) \cong \bigoplus_{i \in I} \left(\mathbb{Z}_2 \oplus (N_i/H_i)_{ab}\right)$$

Explicitly: If S is a continuous finite G-set, and θ is an automorphism of S, then one computes the determinant of θ as follows. First one decomposes

S into $\coprod_i S_i$ where $x \in S_i \Leftrightarrow$ the stabilizer of x is conjugate to H_i. Then one decomposes $S_i^{H_i}$ into orbits under N_i/H_i, say

$$S_i^{H_i} = \coprod_{j \in J_i} T_{ij}$$

and chooses a basepoint z_{ij} in T_{ij}, so that the action of θ becomes

$$\theta(z_{ij}) = q_{ij}\, z_{i(\sigma_i j)} \qquad \begin{array}{l} j \in J_i \\ q_{ij} \in N_i/H_i \end{array}$$

where σ_i is a permutation of J_i. Then one has

$$\det\theta = \sum_i \left(\det\sigma_i + \prod_{j \in J_i} \bar{q}_{ij} \right)$$

<u>Example:</u> Let G be a profinite group and let $\mathcal{C}$ be the category of representations of G over $\mathbb{F}_q$.

<u>Case 1</u>: $G = \{1\}$. One knows that the usual determinant

$$GL_n(\mathbb{F}_q)_{ab} \xrightarrow{\sim} \mathbb{F}_q^*$$

is an isomorphism, hence $K_1 = \mathbb{F}_q^*$.

<u>Case 2</u>: G a pro p-group so that every irreducible representation of G is trivial. Given a representation V with an automorphism θ, consider the socle of V, ie in this case the fixed subspace.

~~[illegible]~~ This must be stable under θ, so one sees that by replacing V by an associated graded representation for $\theta \times G$ one can make the G-action disappear. Hence only the determinant of θ on the underlying vector space matters, so $K_1 = \mathbb{F}_q^*$.

~~[illegible]~~ <u>General case:</u> Suppose C is an artinean abelian category. Then if M is an object of C together with an automorphism θ, one can break the pair down into irreducibles under the θ action, each of which must be isotypical ~~[illegible]~~ as an object of C. Thus if θ acts on M^n where M is a simple object one has $\theta \in GL_n(D)$ where $D = \mathrm{End}(M)$. Then the invariant is the Dieudonne determinant of θ

$$GL_n(D)_{ab} = D^*_{ab} \quad .$$

<u>Conclusion:</u> If C is an artinean abelian category, let $M_i, i \in I$ be ~~[illegible]~~ representatives for the isomorphism classes of simple objects of C and let $D_i = \mathrm{End}\, M_i$ be the skew-field of endos. Then

$$K_1(C) = \bigoplus_{i \in I} (D_i^*)_{ab}$$

May 22, 1970:

The following remark is negative in spirit and perhaps ~~[illegible]~~ shouldn't be taken too seriously. Let F be a field. According to a theorem of Tits ~~[illegible]~~ (Serre [ ], p. II-38) $G = GL_n(F)$ for $n \geq 3$ is the sum of ~~[illegible]~~ the subgroups

$$\Gamma_{k,n-k} = \begin{pmatrix} \boxed{k\times k} & \\ 0 & \boxed{(n-k)\times(n-k)} \end{pmatrix}$$

$1 \leq k < n$ amalgamated with respect to their intersections. However this doesn't imply that the cohomology of G can be computed from these subgroups. Indeed ~~[illegible]~~ take $l \geq 5$ and choose q so that $n =$ order of q in $(\mathbb{Z}/l)^*$ is ≥ 3. Then we know that $GL_n(\mathbb{F}_q)$ has cohomology mod l, since it contains $\mathbb{F}_{q^n-1}^*$. But the ~~subgroups~~ groups $\Gamma_{k,n-k}$ as well as its subgroups have no mod l cohomology since their orders are prime to l.

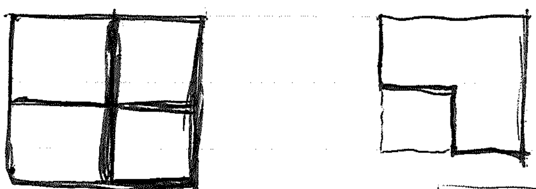
Suppose that A reg. $\Rightarrow$ $K_n(A) \xrightarrow{\sim} K_n(A[t])$. If true then we are in good condition to prove long exact sequence for $Y \subset X \supset U$ all regular.

note that $X \longrightarrow X \times \mathbb{P}^1 \longleftarrow X \times \mathbb{A}^1$
plus fact we know periodicity ~~at~~ would imply homotopy axiom.

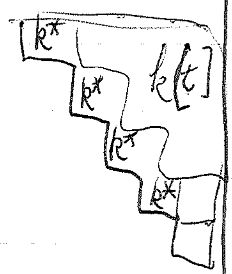
~~in any case, what's involved is to~~

So proj. bundle thm. + excision $\Longrightarrow$ homotopy axiom.

$GL_n(k[t])$



Take $B_n(k[t]) =$



$B(k) \longrightarrow GL_2(k)$
isom on homology mod p as the Sylow grp. is abelian

$$\Rightarrow \quad \begin{array}{ccc} B(k[t]) & \xrightarrow[\cong]{\text{isom on homology}} & GL_2(k[t]) \\ \uparrow & & \uparrow \\ B(k) & \xrightarrow{\cong} & GL_2(k) \end{array}$$

certainly not an isomorphism

$$GL_2(k[t]) = GL_2(k) *_{B(k)} B(k[t])$$

$k = \mathbb{F}_q$ and we consider $\boxed{\text{homology } \underline{\text{mod}}\ l.}$ Then

$$B(k) \xrightarrow{\cancel{\cong}} B(k[t]) \qquad \text{isom.}$$

and so $GL_2(k) \longrightarrow GL_2(k[t])$ isom mod l.

homology mod p. $GL_2(k[t])$. Seems unlikely somehow.

in fact $B(k) \longrightarrow GL_2(k)$ isomorphism on homology mod p. because Sylow subgp is abelian

Conjecture (K2)

$$0 \longrightarrow K_q(A) \longrightarrow K_q(K) \longrightarrow K_{q-1}(k) \longrightarrow 0$$
mod p-torsion

$\cancel{K_q(k)}$ $\quad$ A $\quad$ C^m $\quad$ $\boxed{C^m}$

$$[k(\mu_\ell) : k] = r$$

$\mu_\ell \in K^*$

$$\widehat{K(\mu_\ell)^*} = \widehat{K^*} = \hat{\mathbb{Z}} \times \boxed{A^*}$$

1. ~~Arithmetic Chern classes.~~

~~vector space over a field k~~

~~E a representation of G in a vector space of finite dimension over a field k. then have arithmetic Chern classes mod n~~

$$c_i(E) \in H^{2i}(\mathrm{Spec}(k), G; \mu_n^{\otimes i})$$

~~for n prime to the characteristic~~

$k = \mathbb{F}_q$ finite field with q elements

c_i

$$G \xrightarrow{f} S \qquad \underline{\underline{BG}}$$

Brauer-monodromy situation — S group scheme semi-simple

in case A is a d.v.r.

point of theory:

$$G(K) \longleftarrow G(A) \longrightarrow G(k)$$

the question is how to compute what K (p-adic nos.) ~~as a defintion we take~~ is. The problem is to determine something about the mod p cohomology

~~Corollary~~ p-adic nos. $\mathbb{Z}_p$

I need somehow to be able to compute characteristic classes of $GL_n(\mathbb{Z}_p)$

$$GL_n(\mathbb{Z}_p) = \varprojlim_{\nu} GL_n(\mathbb{Z}_{p^\nu})$$

Cohomology

$\mathbb{Z}_p \qquad \mathbb{Q}_p$

A Dedekind domain

A d.v.r.

$$K_q(k) \longrightarrow K_q(A) \longrightarrow K_q(K)$$

a basic homomorphism which one ought to understand is the maps

$$K_q(K) \longrightarrow K_{q-1}(k)$$

for a local field

$$\begin{tikzcd}[column sep=small]
 & & & & & k^* & & & & \\
0 \arrow[r] & K_2(A) \arrow[r] & \mu_K \arrow[r] & k^* \arrow[r, "0"] \arrow[urr, "0"] & & A^* \arrow[u] \arrow[r, hook] & K^* \arrow[r] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

thus $K_2(A)$ is the p-primary of $K_2(K)$

I feel from the point of cohomology that the A and k are the same except for p torsion ie that

$$K_q(A) \xrightarrow{i^*} K_q(k)$$

is an isomorphism off the p-primary component. hence surjective. If so what is the ~~maps~~ composition

$$K_q(k) \xrightarrow{i_*} K_q(A) \underset{\cong}{\xrightarrow{i^*}} K_q(k)$$

should be zero

May 28, 1970: On algebraic K-theory:

In the following we work only with pointed spaces; $[X,Y]$ denotes the set of homotopy classes of basepoint-preserving maps from X to Y. Let $\mathcal{H}$ be the pointed homotopy category. If I stick to connected spaces it's the homotopy category of simplicial groups. So consider only conne spaces.

Lemma 1: If $H_1(X)=0$, then the functor

$$Z \longmapsto \{f \in [X,Z] \mid \pi_1(f)=0\}$$

is representable.

Proof: By attaching 2-cells to X we construct a map $X \longrightarrow X'$ which kills the fundamental group of X and X' has the same homology except that

$$0 \longrightarrow H_2(X) \longrightarrow H_2(X') \longrightarrow \begin{matrix}\text{free ab. gp. gen}\\ \text{by attached 2-cells}\end{matrix} \longrightarrow 0$$

As $H_1(X')=0$, all elements of $H_2(X')=\pi_2(X')$ are spherical, hence we can attach 3 cells to X' to kill the extra elements of $H_2(X')$ and so we obtain a map $X \xrightarrow{i} X''$ such that $\pi_1(X'')=0$ and $H_*(X) \xrightarrow{\sim} H_*(X')$.

I claim X'' and $X \xrightarrow{i} X''$ represent the functor in question. Indeed given an honest map of spaces $f: X \to Z$ such that $\pi_1(f)=0$, the obstructions to finding a lifting in

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Z \arrow[d] \\ X'' \arrow[r] \arrow[ur, dotted, "f^*"] & e \end{tikzcd}$$

lie in $H^{n+1}(X'', X; \pi_n(Z))$ which is zero. Similarly consider a homotopy $h: X \times I \longrightarrow Z$ ~~and two extensions~~ between f, g (both satisfying $\pi_1(f) = 0$) and two extensions f^*, g^*. Then to see that the extensions are homotopic over h consider

$$\begin{tikzcd} X \times I \cup X'' \times \dot{I} \arrow[r, "{(h, f^*, g^*)}"] \arrow[d, "j"] & Z \arrow[d] \\ X'' \times I \arrow[r] \arrow[ur, dotted] & e \end{tikzcd}$$

Now as $X \xrightarrow{i} X''$ is a homology isomorphism, so is j because its cofibre is the reduced suspension of X''/X. The top map (h, f^*, g^*) kills π_1, as one may see either by van Kampen (the π_1 is $\pi_1(X'') \underset{\pi_1(X)}{*} \pi_1(X'') = 0$, it appears that j is a homotopy equivalence!) or because h lifts to $\tilde{Z}$, the universal covering.

Thus we have the existence of the extension and its uniqueness up to homotopy proving the lemma.

<u>Notation:</u> $X \xrightarrow{i} X^+$ map of the lemma. If $f: X \to Z$ satisfies $\pi_1(f) = 0$, then the extension is to be denoted $f^+: X^+ \longrightarrow Z$.

<u>Remark</u>: The argument shows that the map of groupoids

$$\underline{\underline{\pi}}(X^+, Z) \xrightarrow{i^*} \underline{\underline{\pi}}(X, Z)$$

is fully faithful and has for image those $f \in [X, Z]$ with $\pi_1(f) = 0$.

Let $G = GL_\infty(A)$ and let $E = E_\infty(A)$ $= [E, E]$. Define BG^+ by the cocartesian square

$$\begin{tikzcd} BE \arrow[r, hook] \arrow[d, "i"] & BG \arrow[d, "i"] \\ BE^+ \arrow[r, hook] & BG^+ \end{tikzcd}$$

<u>Lemma 2</u>: $i : BG \longrightarrow BG^+$ represents the functor

$$Z \longmapsto \{ f \in [BG, Z] \mid \pi_1(f) \text{ kills } E \}.$$

<u>Proof</u>: Given a map $f : BG \longrightarrow Z$ of spaces (we must eventually distinguish maps + homotopy classes!), if $\pi_1(f)(E) = 0$, then $f|BE$ extends to BE^+, hence gives rise to a map $f^+ : BG^+ \longrightarrow Z$ with $f^+ i = f$.

Now suppose we have a homotopy $h : BG \times I \longrightarrow Z$ between f and g and extensions $f^+, g^+ :$ $BG^+ \longrightarrow Z$. Then by the proof of the preceding I know that I can form $\tilde{h}^+ : BE^+ \times I \longrightarrow Z$ joining f^+ and g^+ restricted to BE and $\ni$ $\tilde{h}^+ = h$ restricted to BE. Then $\tilde{h}^+$ and h define $h^+ : BG^+ \longrightarrow Z$ extending h and joining f^+ and g^+. This proves lemma.

Remark: Note that the last step uses that

$$
\begin{tikzcd}
BE \times I \cup BE^+ \times \dot{I} \arrow[r] \arrow[d] & BG \times I \cup BG^+ \times \dot{I} \arrow[d] \\
BE^+ \times I \arrow[r] & BG^+ \times I
\end{tikzcd}
$$

is cocartesian, hence the latter vertical map is a homotopy equivalence. Thus as in the remark following lemma 1 we have that

$$\underline{\underline{\Pi}}(BG^+, Z) \xrightarrow{i^*} \underline{\underline{\Pi}}(BG, Z)$$

is fully faithful with image those $f \in [X, Z]$ such that $\pi_1(f)$ kills E.

(Before going on I want to check this last assertion carefully. Thus I want to prove that

$$\pi_1 \operatorname{Hom}(BG^+, Z; f^+, g^+) \xrightarrow{i^*} \pi_1 \operatorname{Hom}(BG, Z; f, g)$$

is bijective. Surjectivity results from the fact that if $h : BG \times I \longrightarrow Z$ represents an element of the latter the it comes from

$$\begin{tikzcd}
(BG^+ \times \dot{I}) \cup (BG \times I) \arrow[r, "{(f^+, g^+, h)}"] \arrow[d] & Z \arrow[d] \\
BG^+ \times I \arrow[r] \arrow[ur, dotted] & e
\end{tikzcd}$$

as we have already seen. For injectivity suppose I have $h^+_a, h^+_b : (BG^+ \times I) \longrightarrow Z$ joining f^+ and g^+. Suppose I know that $h^+_a \iota$ and $h^+_b \iota$ are homotopic keeping f, g fixed. Then I have a homotopy of the horizontal arrow furnished by the constant homotopies of f^+, g^+ and the homotopy joining $h^+_a \iota$ and $h^+_b \iota$, and this homotopy is compatible with the extensions h^+_a and h^+_b, so as I know well one gets a big homotopy of the dotted arrow, proving injectivity of ι^*.)

Remark: The above argument shows that one gets a universal map $i : X \longrightarrow X^+$ killing any subgroup E of $\pi_1(X)$ such that $E = [E, E]$, and that

$$\underline{\underline{\Pi}}(X^+, Z) \xrightarrow{\ i^*\ } \underline{\underline{\Pi}}(X, Z)$$

is fully faithful with image the $f \in [X, Z]$ such that $\pi_1(f)(E) = 0$. In particular i induces an isomorphism on homology.

May 29, 1970

Theorem: Let X be a (pointed connected) space (CW cx. is understood) and let $E \subset \pi_1(X)$ be a normal subgroup such that $[E,E] = E$. Then in the pointed homotopy category there is a morphism $f: X \longrightarrow Y$ with the following properties.

1) f induces an isomorphism $\pi_1(X)/E \xrightarrow{\sim} \pi_1(Y)$ and $H_*(X,L) \simeq H_*(Y,L)$ for all $\pi_1(Y)$-modules L.

2) For any Z, $f^*: [Y,Z] \longrightarrow [X,Z]$ is injective with image the set of ~~[illegible]~~ α such that $\pi_1(\alpha)(E) = 0$.

Before taking up the proof suppose $f: H_*(X,L) \xrightarrow{\sim} H_*(Y,L)$ for all $\pi_1(Y)$-modules L. Then for $* = 0$

$$L \underset{\pi_1(X)}{\otimes} \mathbb{Z} \xrightarrow{\sim} L \underset{\pi_1(Y)}{\otimes} \mathbb{Z}$$

for all L, so taking $L = \mathbb{Z}[\pi_1(Y)]$ we have

$$\mathbb{Z}[\pi_1(Y)/f_*\pi_1(X)] \xrightarrow{\sim} \mathbb{Z}$$

and hence $f_*: \pi_1(X) \longrightarrow \pi_1(Y)$ is onto. For $* = 1$ we have

$$\begin{array}{ccc} H_1(X,L) & \xrightarrow{\sim} & H_1(Y,L) \\ \downarrow \wr & & \downarrow \wr \\ H_1(\pi_1(X),L) & \longrightarrow & H_1(\pi_1(Y),L) \\ \| & & \\ \mathrm{Tor}_1^{\mathbb{Z}[\pi_1 X]}(\mathbb{Z},L). & & \end{array}$$

Computing the Tor by the sequence $0 \to \bar{\mathbb{Z}}[\pi_1 X] \to \mathbb{Z}[\pi_1 X] \to \mathbb{Z} \to 0$

$$\begin{tikzcd}
0 \arrow[r] & H_1(\pi_1(X), L) \arrow[r] \arrow[d, "\cong"] & \mathbb{Z}[\pi_1 X] \otimes_{\pi_1 X} L \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, equal] & \mathbb{Z} \otimes_{\pi_1 X} L \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & H_1(\pi_1(Y), L) \arrow[r] & \mathbb{Z}[\pi_1 Y] \otimes_{\pi_1 Y} L \arrow[r] & L \arrow[r] & \mathbb{Z} \otimes_{\pi_1 Y} L \arrow[r] & 0
\end{tikzcd}$$

for all L, so taking $L = \mathbb{Z}[\pi_1 Y]$ and using the exact sequence

$$0 \longrightarrow E_{ab} \longrightarrow \mathbb{Z}[\pi_1 X] \otimes_{\pi_1 X} \mathbb{Z}[\pi_1 Y] \xrightarrow{\cong} I[\pi_1 Y] \longrightarrow 0$$

{where $E = \operatorname{Ker}\{\pi_1(X) \to \pi_1(Y)\}$}
we have $E_{ab} = 0$, i.e. $E = [E, E]$. Hence the hypotheses of the theorem are best possible.

Next we show 1) $\Longrightarrow$ 2). The best statement is that

$$f^* : \underline{\mathrm{Hom}}(Y, Z) \longrightarrow \underline{\mathrm{Hom}}(X, Z)$$

(space of basepoint preserving maps) is a homotopy equivalence with the union of the components of $\underline{\mathrm{Hom}}(X,Z)$ corresponding to $\alpha \in [X, Z] \ni (\pi_1\alpha)(E) = 0$. May assume f a cofibration, whence f^* is a fibration. It's enough to show (i) any $g : X \to Z \ni \pi_1(g)(E) = 0$ factors through f and (ii) $(X \times I) \cup (Y \times I^\circ) \longrightarrow Y \times I$ is a homotopy equivalence. (Indeed (ii) $\Rightarrow$ f^* induces a homotopy equivalence of the path spaces of $\underline{\mathrm{Hom}}(Y, Z)$ at g^+ and of $\underline{\mathrm{Hom}}(X, Z)$ at g^+f. For (ii) note that by van Kampen

$$\pi_1((X \times I) \cup (Y \times I^\circ)) = \pi_1(X) *_{\pi_1(X)} \pi_1(Y) = \pi_1(Y)$$

and Mayer-Vietoris $\cdot$ $H_*((X \times I) \cup (Y \times I^\circ), L) \xrightarrow{\sim} H_*(Y \times I, L)$

for any $\pi_1(Y)$-module L, hence the map in question is a homotopy equivalence by the Whitehead theorem in the form used by Artin–Mazur. For (i) use obstruction theory.

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Z_n \arrow[r] \arrow[d] & P(\pi_1(Z)) \times_{\pi_1(Z)} K(\pi_n Z, n) \arrow[d] \\
Y \arrow[r] \arrow[ur, dotted] & Z_{n-1} \arrow[r] & P(\pi_1(Z)) \times_{\pi_1(Z)} K(\pi_n Z, n+1)
\end{tikzcd}$$

The obstructions to producing a lifting are in

$$H^{n+1}(Y, X; \pi_n(Z)) \qquad n \geq 2$$

where one gets started, i.e. $\exists$

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & K(\pi_1 Z, 1) \arrow[d] \\
Y \arrow[r] \arrow[ur, dotted] & e
\end{tikzcd}$$

because $\pi_1(g)(E) = 0$ and $\pi_1(X)/E \simeq \pi_1(Y)$.

Conclude: 1) $\Rightarrow$ 2) is a consequence of obstruction theory.

As 2) gives a universal property for the Y and the map $f: X \to Y$, it follows that the map f is unique up to canonical isomorphism in the pointed homotopy category.

Existence of $f: X \to Y$: Let $p: \tilde{X} \to X$ be the covering space associated to the subgroup E, i.e. $\pi_1(p): \pi_1(\tilde{X}) \to \pi_1(X)$ is an isomorphism with E. Then $H_1(\tilde{X}) = E_{ab} = 0$, so by attaching 2 and 3 cells to $\tilde{X}$ we may construct a map j

$$(*) \qquad \begin{tikzcd} \tilde{X} \arrow[r, "j"] \arrow[d, "p"'] & \tilde{X}^+ \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

inducing isomorphisms on homology and killing E. Define Y by the cocartesian square.

By van Kampen

$$\pi_1(Y) = \pi_1(X) \underset{\pi_1(\tilde{X})}{*} \pi_1(\tilde{X}^+) = \pi_1(X)/E$$

and if L is a $\pi_1(Y)$-module, then one has long exact sequences

$$\begin{tikzcd} \arrow[r] & H_q(\tilde{X}, L) \arrow[r, "\sim"] \arrow[d] & H_q(\tilde{X}^+, L) \arrow[r] \arrow[d] & H_q(\tilde{X}^+, \tilde{X}; L) \arrow[d, phantom, "\wr\wr"] \\ \arrow[r] & H_q(X, L) \arrow[r, "\sim"] & H_q(Y, L) \arrow[r] & H_q(Y, X; L) \end{tikzcd}$$

showing that $H_*(X, L) \simeq H_*(Y, L)$. The theorem is proved.

Corollary: Let $\tilde{Y} \to Y$ be the universal covering of Y.

Then the obvious map $\tilde{X}^+ \longrightarrow Y$ (obtained from (*) page 9) is a homotopy equivalence.

<u>Proof:</u> As both $\tilde{X}^+$ and Y are 1-connected it's enough to check ~~they~~ the map induces an isom. on homology. But

$$\begin{tikzcd}
H_*(\tilde{X}, \mathbb{Z}) \arrow[r] \arrow[d, phantom, "\wr|"] & H_*(\tilde{Y}, \mathbb{Z}) \arrow[d, phantom, "\wr|"] \\
H_*(X, \mathbb{Z}[\pi_1 Y]) \arrow[r, "\sim", "\text{by hyp.}"'] & H_*(Y, \mathbb{Z}[\pi_1 Y])
\end{tikzcd}$$

(Serre spect. sequence)

so $\tilde{X} \longrightarrow \tilde{X}^+ \longrightarrow \tilde{Y}$ all induce isos. on homology

---

~~Pleasant consequences: Denote by~~

$$\cancel{BG \longrightarrow BG^+}$$
$$\cancel{BE \longrightarrow BE^+}$$

~~the~~ map

<u>Pleasant consequences:</u> Denote by $G = GL_\infty(A)$ $E = E_\infty(A)$ and let

$$\begin{tikzcd}
BE \arrow[r] \arrow[d] & BE^+ \arrow[d] \\
BG \arrow[r] & BG^+
\end{tikzcd}$$

be the square in which the horizontal maps are those constructed in the theorem using the subgroup E. Thus one attaches 2 & 3 cells (in fact one of each will do!) to get BE^+ and then forms the cocartesian square. By the corollary BE^+ is the universal covering (up to homotopy type) of BG^+. Thus

$$\boxed{\pi_i(BE^+) = \pi_i(BG^+) \qquad i \geq 2}$$

But as BE^+ is simply-connected

$$\pi_2(BE^+) = H_2(BE^+) = H_2(BE)$$

proving that the K_2 agrees with Milnor's!

H-space structure of BG^+:

Let $X \longrightarrow X^+$ be a map such as in the theorem: $\pi_1(X) \twoheadrightarrow \pi_1(X^+)$ and passing to the coverings with group $\pi_1(X^+)$ gives a homology isomorphism $\tilde{X} \longrightarrow \tilde{X}^+$. Observe that the family of these maps is closed under composition and cobase change (in effect if

$$\begin{array}{ccc} X & \longrightarrow & X^+ \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z^+ \end{array}$$

is cocartesian, then for any $\pi_1(Z^+)$ module we have by long exact sequences that $H_*(Z, L) \simeq H_*(Z^+, L)$.) and taking products with a fixed space (in effect the covering of $X \times Z \longrightarrow X^+ \times Z$ is $\tilde{X} \times \tilde{Z} \longrightarrow \tilde{X^+} \times \tilde{Z}$, $\tilde{Z}$ and $\tilde{X^+}$ universal coverings, and this map is a homology isomorphism.)

__Conclusion:__ The class of maps we are interested in are those $X \longrightarrow X^+$ such that on taking the coverings with group $\pi_1(X^+)$ we get a homology isomorphism $\tilde{X} \longrightarrow \tilde{X^+}$ (= universal covering of X^+). (Note this implies $\pi_1(\tilde{X}) \twoheadrightarrow \pi_1(X^+)$ otherwise $\tilde{X}$ won't be connected).

__Corollary:__ $(X \times Y)^+ = X^+ \times Y^+$.

So applying the universal property of the theorem p. 15 we know that if we choose an isomorphism $A^\infty \oplus A^\infty \cong A^\infty$ it gives μ which extends

$$\begin{tikzcd} BG \times BG \arrow[r, "\mu"] \arrow[d] & BG \arrow[d] \\ BG^+ \times BG^+ \arrow[r, "\mu^+"] & BG^+ \end{tikzcd}$$

in an essentially unique way to μ^+. I want to show that μ^+ is an H-space structure on BG^+ and hence must prove that

$$\begin{tikzcd} BG^+ \arrow[d, "(id,0)"] & \\ BG^+ \times BG^+ \arrow[r, "\mu^+"] & BG^+ \end{tikzcd}$$

is homotopic to the **identity**. This is the induced map associated to the embedding $G \longrightarrow G$ produced by the group G acting on the first factor and the isomorphism $A^\infty \oplus A^\infty \cong A^\infty$. For μ^+ to be homotopy associative means that the two map $G^3 \longrightarrow G$ induced by the isomorphisms

$$A^\infty \cong A^\infty \oplus A^\infty \cong (A^\infty \oplus A^\infty) \oplus A^\infty$$
$$\cong A^\infty \oplus (A^\infty \oplus A^\infty)$$

have the same effect on BG^+; note that the two isos. are conjugate by an automorphism of A^∞. Hence μ^+ will give an H-space structure once we prove:

p.15 <u>Proposition</u>: Let $u : G \to G$ be an embedding induced by an isomorphism $A^\infty \oplus A^n \simeq A^\infty$ where $0 \leq n \leq \infty$.
Then

$$\begin{tikzcd} BG \arrow[rr, "u"] \arrow[dr] & & BG \arrow[dl] \\ & BG^+ & \end{tikzcd}$$

is homotopy commutative, i.e. up to homotopy u induces the identity on BG^+.

Admit for a moment the

<u>Lemma</u>: Let $G_n = GL_n(A)$ be regarded as a subgroup of $G = GL_\infty(A)$ in the standard way and suppose that $\theta \in G$ so that $I_n \oplus \theta \in G$ centralizes G_n. Let $f : BG \to BG^+$ be the canonical map, and ~~[illegible]~~ let $\theta^\#$ denote the automorphism of conjugation by θ on G as well as BG. Then there is a homotopy joining $f(I_n \oplus \theta)^\#$ to f which restricts to the constant homotopy of $f|BG_n$.

$$\begin{tikzcd} & BG \arrow[dd, "(I_n\oplus\theta)^\#"] \arrow[dr, "f"] \arrow[dd, phantom, shift left=6ex, "\Downarrow"] & \\ BG_n \arrow[ur, hook] \arrow[dr, hook] & & BG^+ \\ & BG \arrow[ur, "f"'] & \end{tikzcd}$$

(In more precise language the first triangle commutes, or perhaps better, comes with an essentially unique (commuting homotopy since $I_n \oplus \theta$ centralizes G_n, and the lemma asserts the second triangle may be made to commute by a homotopy whose effect as maps from BG_n to BG^+ is the constant homotopy.)

Proof of the proposition: I perceive that I haven't been careful enough because an isomorphism $A^\infty \oplus A^\infty \cong A^\infty$ doesn't induce an embedding $G \longrightarrow G$, since there is no reason why the linear transformation should be equal almost everywhere (relative to the standard basis) to the identity. So therefore I shall only consider isomorphisms ρ of $A^\infty \oplus A^n \simeq A^\infty$ resulting from dividing up $\mathbb{N}_+$ into two disjoint sets $(N' \amalg N'')$ of ∞ and n elements respectively and ordering these sets.

Let the embedding $u: G \longrightarrow G$ be obtained by such a partition $N_+ = N' \amalg N''$ and an isom. of $\mathbb{N}_+$ and N'. Let $i_n: G_{n-1} \longrightarrow G_n$ denote the inclusion and $\beta_n = u \restriction G_n : G_n \longrightarrow G$ and let $\alpha_n : G_n \longrightarrow G$ be standard embedding. Choose a permutation matrix $\theta_n \in G$ such that

$$\beta_n = \theta_n^\# \alpha_n \qquad (\#\text{ denotes conjugation}).$$

and let φ_n be defined by $\theta_{n-1}\varphi_n = \theta_n$. Then

$$\theta_{n-1}^\# \alpha_{n-1} = \beta_{n-1} = \beta_n i = \theta_n^\# \alpha_n i = \theta_{n-1}^\# \varphi_n^\# \alpha_{n-1}$$

so $\varphi_n^\# \alpha_{n-1} = \alpha_{n-1}$, i.e. φ_n is a permutation leaving the numbers $\{1, \ldots, n-1\}$ fixed, i.e. of the form

$(I_{n-1} \oplus \varphi_n')$. By the lemma there exists a homotopy

$$\varepsilon_n : p \Rightarrow p\varphi_n^{\#}$$

such that

$$\varepsilon_n * \alpha_{n-1} : p\alpha_{n-1} \Rightarrow p\varphi_n^{\#}\alpha_{n-1} = p\alpha_{n-1}$$

is the constant homotopy (we assume that we use a simplicial model so that $BG_n \to BG_{n+1} \to \cdots$ are cofibrations.)

Define homotopies

$$\gamma_n : p \Rightarrow p\theta_n^{\#}$$

recursively by

$$p \xRightarrow{\varepsilon_n} p\varphi_n^{\#} \xRightarrow{\gamma_{n-1} * \varphi_n^{\#}} p\theta_{n-1}^{\#}\varphi_n^{\#} = p\theta_n^{\#}$$

Sets

$$\delta_n = \gamma_n * \alpha_n : p\alpha_n \Rightarrow p\theta_n^{\#}\alpha_n = p\beta_n$$

and $\delta_n * \iota_n = \delta_{n-1}$ because

$$\begin{tikzcd}
p\alpha_{n-1} \arrow[r, Rightarrow, "\varepsilon_n * \alpha_{n-1}"] \arrow[dr, Rightarrow, "\text{trivial homotopy}"'] & p\varphi_n^{\#}\alpha_{n-1} \arrow[r, Rightarrow, "\gamma_{n-1} * \varphi_n^{\#}\alpha_{n-1}"] \arrow[d, equal] & p\theta_{n-1}^{\#}\varphi_n^{\#}\alpha_{n-1} = p\theta_n^{\#}\alpha_{n-1} \arrow[d, equal] \\
& p\alpha_{n-1} \arrow[r, Rightarrow, "\delta_{n-1} = \gamma_{n-1} * \alpha_{n-1}"] & p\theta_{n-1}^{\#}\alpha_{n-1}
\end{tikzcd}$$

Thus the δ_n give a compatible family of homotopies from $p\alpha_n$ to $p\beta_n$ and so p is homotopic to pu proving the proposition.

(Remark: Note that we must adjust for composition of a trivial homotopy; ugh; this hopefully becomes intelligible in 2-category language.)

Now we prove the lemmas. Let $\oplus : G_n \times G_\infty \longrightarrow G$ be the direct sum map. Then we have a commutative square of spaces (where $BG_n = |\overline{W}(G_n)|$)

$$\begin{tikzcd}
 & BG_n \times BG \arrow[r, "\oplus"] \arrow[d, "id \times \varphi^\#"] & BG \arrow[d, "(I_n \oplus \varphi)^\#"] \\
BG_n \arrow[ur] \arrow[r] & BG_n \times BG \arrow[r, "\oplus"] \arrow[d] & BG \arrow[d] \\
 & BG_n \times BG^+ \arrow[r] & BG^+
\end{tikzcd}$$

The idea of the proof consists in showing that given a homotopy $p \Rightarrow p\varphi^\#$ there is another $p \Longrightarrow p(I_n \oplus \varphi)^\#$ compatible with it.

First step: understand the homotopy $p \Rightarrow p\varphi^\#$. Quite generally one has a map

$$[X, B] \times \pi_1(B) \longrightarrow [X, B]$$

which results from the fibration

$$\begin{array}{c} \underline{\mathrm{Hom}}(X, B) \\ \downarrow \\ \underline{\mathrm{Hom}}(X \cup e, B) \\ \downarrow \\ B \end{array}$$

If we map $B \longrightarrow B'$, this gives a map of fibrations and if $\gamma \in \pi_1(B)$ goes to zero, the effect of γ on on $[X,B]$ becomes trivial in $[X,B']$.

So we start with φ and represent it by $\lambda : S^1 \longrightarrow BG$. Then form

$$\begin{tikzcd}
BG \times O \cup (e \times I) \arrow[r, "id + \lambda"] \arrow[d] & BG \\
BG \arrow[ur, "H"] & H_1 = \varphi^{\#}.
\end{tikzcd}$$

So we get $H_t : BG \longrightarrow BG \ni H_0 = id$ & $H_1 = \varphi^{\#}$ and $t \longmapsto H_t(e)$ is λ. Now compose with $p: BG \to BG^+$ and use the fact that $p\lambda$ contracts to a point. Covering

It seems that you have

$$\begin{tikzcd}
BG_n \times BG \arrow[rr, "\oplus"] \arrow[d, shift right=2, "id \times p"'] \arrow[d, shift left=2, "id \times p\varphi^{\#}"] \arrow[d, phantom, "\Rightarrow"] & & BG \arrow[d, shift right=2, "p"'] \arrow[d, shift left=2, "p(I_n \oplus \varphi)^{\#}"] \\
BG_n \times BG^+ \arrow[rr, "\oplus^+"] & & BG^+
\end{tikzcd}$$

and you want to extend the homotopy.

this homotopy moves $p\varphi^{\#}$ to p.

The picture to draw is:

$$\begin{tikzcd}
p \\
\mathrm{Hom}(BG, BG^+) \arrow[rr, "\oplus^*"] \arrow[d, "p(I_n\oplus\varphi)^{\#}"'] & & \mathrm{Hom}(BG_n \times BG, BG^+) \arrow[d, "\oplus^+(id \times p\varphi^{\#}) = p(I_n\oplus\varphi)^{\#}\oplus"] \\
p \arrow[rr, "\oplus^*"] \arrow[d] & & \oplus^+(id \times p) = p\oplus \arrow[d] \\
BG^+ \arrow[rr] & & BG^+
\end{tikzcd}$$

$$\oplus^+(0, p\lambda) \quad = \quad$$

λ path of $\varphi \in G$ $\qquad$ $p \circ$ path of $I_n \oplus \varphi$ in G

and now you contract $p\lambda : S^1 \longrightarrow BG^+$ to basepoint and this the basepoint-preserving homotopy of

$$p\oplus \Longrightarrow p(I_n \oplus \varphi)^{\#}\oplus$$

but if $\oplus$ is a cofibration as you may assume, then $\oplus^*$ is a fibration so you can lift this homotopy to one

$$p \Longrightarrow p(I_n \oplus \varphi)^{\#}.$$

Thus you get the compatible family of homotopies you need.

<u>The above is probably correct although impossible to understand.</u>

critical problem is that there is no formula for

$$c_t(f_* x)$$

without denominators.

example: Suppose L is a line bundle over X and we consider $f = i : X \to L$

$$i_* : K(X) \xrightarrow{\simeq} K_{pr/X}(L)$$

$\exists\ G(x)$ with

$$c_t\, i_*(x) = 1 + i_*\, G(x)$$

$$\Rightarrow \quad i^* c_t(i_*(x)) = c_t(i^* i_* 1 \cdot x) = 1 + i^* i_* 1 \cdot G(x)$$

Now

$$i^* i_* 1 = 1 - L^{-1} \text{ in } K \text{ ~~so this gives~~}$$

and $i^* i_* 1 = c_1(L)$ in U so this gives

$$G(x) = \frac{\dfrac{c_t(x)}{c_t(x \cdot L^{-1})} - 1}{c_1(L)}$$

In other words you write universally

$$\frac{c_t(x)}{c_t(x \cdot L^{-1})} = 1 + a_1 c_1(L) + a_2 c_1(L)^2 + \cdots$$

and

$$G(x) = a_1 + a_2 c_1(L) + \cdots$$

Now calculate this for the case where $c_1(L) = 0$

suppose $x = \sum L_i$ note that in general

$$G(x+y) = Gx + Gy + c_1 L \cdot Gx\, Gy$$

so that when $c_1(L)=0$ G is additive. For a line bundle ~~...~~ $x = M$ I get

$$\frac{c_t(x)}{c_t(x \cdot L^{-1})} = \frac{1+tc_1(x)}{1+t\,F(c_1x, Ic_1L)}$$

now can suppose $(c_1L)^2=0$ for calculation of a_1

$$F(c_1x, Ic_1L) = c_1x + F_2(c_1x,0)\,Ic_1L$$

$$= c_1x - F_2(c_1x,0)\,c_1L$$

$$\frac{c_t(x)}{c_t(x.L^{-1})} - 1 = \frac{1+tx}{1+tc_1x - tF_2(c_1x,0)\,c_1L} - 1$$

$$= \frac{t\,F_2(c_1x,0)\,c_1L}{1+tc_1x - tF_2(c_1x,0)\,c_1L} = c_1L\;G(x).$$

for a line bundle x and $(c_1L)^2=0$ have

$$G(x) = \frac{tF_2(c_1x,0)}{1+tc_1x - tF_2(c_1x,0)\cdot c_1L}$$

so if $c_1L=0$ get

$$\boxed{G(x) = \frac{tF_2(c_1x,0)}{1+tc_1(x)}}$$

now for $F(X,Y) = X+Y-XY$ $\quad F_2(X,0) = 1-X$

$$y = L$$

ν = line bundl

$$c_{\underline{t}}\left(\lambda_{-1}(\nu^*)\cdot L\right)$$

$$c_{\underline{t}}\left((1-\nu^{-1})\cdot L\right) \cancel{=}$$

$$c_{\underline{t}}\left[(1-\nu^{-1})\cdot L\right] = \frac{c_{\underline{t}}(L)}{c_{\underline{t}}(\nu^{-1}\cdot L)} = \frac{\sum t_n\, c_1(L)^n}{\sum t_n\, F(c_1\nu^{-1}, c_1 L)^n}$$

$$\frac{c_{\underline{t}}\left[(1-\nu^{-1})\cdot L\right] - 1}{c_1(\nu)}$$

$$\frac{\sum t_n\, c_1(L)^n - \sum t_n\, \overbrace{F(c_1\nu^{-1}, c_1 L)^n}^{c_1 L +}}{c_1(\nu)\quad \sum t_n\, F(c_1\nu^{-1}, c_1 L)^n}$$

---

curiously:

$$U(X) \xrightarrow{\cong} U(X \times \mathbb{C}^+)$$

and

$$K(X) \xrightarrow{\cong} K(X \times \mathbb{C}^+)$$

and if the inversion of the formula doesn't exist.

the point is that

$$c_t(i_* x) = G(c(x))$$

Summary of calculations made.

①
$$\begin{tikzcd} U^*(X) \arrow[r, "\Delta_1"] \arrow[dr, bend right, "\hat{\varphi}_a"'] & K_*(MU) \otimes_{K^*(pt)} K^*(X) \arrow[d, phantom, "\cong" rotate=90] \\ & \mathbb{Z}[a_1, \cdots] \otimes_{\mathbb{Z}} K^*(X) \end{tikzcd}$$

$$\hat{\varphi}_a(c_1^U(L)) = \sum_{n \geq 0} a_n \, c_1^K(L)^{n+1}$$

②
$$\begin{tikzcd} K^*(X) \arrow[r, "\Delta_2"] \arrow[dr, bend right, "\psi^T"'] & K_*(\underset{\sim}{BU}) \otimes_{K^*(pt)} K^*(X) \arrow[d, phantom, "\cong" rotate=90] \\ & \Gamma \otimes_{\mathbb{Z}} K^*(X) \end{tikzcd}$$

$$\begin{cases} \psi^T(\beta) = T\beta \\ \psi^T(L) = \text{“} L^T \text{”} = \sum_{n \geq 0} \binom{T}{n} (L-1)^n \end{cases}$$

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③
$$\begin{tikzcd} K_*(MU) \arrow[r, "\Delta_3"] & K_*(MU) \otimes_{K^*(pt)} K_*(\underset{\sim}{BU}) \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{Z}[a_1, \cdots] \otimes_{\mathbb{Z}} K^*(pt) \arrow[r] & \mathbb{Z}[a_1, \cdots] \otimes_{\mathbb{Z}} \Gamma \otimes_{\mathbb{Z}} K^*(pt) \end{tikzcd}$$

$$\sum_{n \geq 0} a_n X^{n+1} \longmapsto \sum_{n \geq 0} a_n X^{n+1} \circ \sum_{n \geq 0} \frac{1}{T} \binom{T}{n+1} (-\beta)^n X^{n+1}$$

(4)
$$\begin{array}{ccc}
K_*(\underset{\sim}{BU}) & \longrightarrow & K_*(\underset{\sim}{BU}) \underset{K^*(pt)}{\otimes} K_*(\underset{\sim}{BU}) \\
\wr\!| & & \wr\!| \\
\Gamma \otimes_{\mathbb{Z}} K^*(pt) & \longrightarrow & \Gamma \otimes_{\mathbb{Z}} \Gamma \otimes_{\mathbb{Z}} K^*(pt)
\end{array}$$

$$\beta \longmapsto \beta$$
$$T \longmapsto T' \cdot T''$$

$T' = T \otimes 1 \otimes 1$
$T'' = 1 \otimes T \otimes 1$

Now it's clear how to form the basic sequence.

$$U^*(X) \longrightarrow K_*(MU) \underset{K^*(pt)}{\otimes} K^*(X) \rightrightarrows K_*(MU) \underset{K^*(pt)}{\otimes} K_*(\underline{BU}) \underset{K^*(pt)}{\otimes} K^*(X) \rightrightarrows$$

$$U^*(X) \longrightarrow \mathbb{Z}[a] \otimes_{\mathbb{Z}} K^*(X) \rightrightarrows \mathbb{Z}[a] \otimes_{\mathbb{Z}} \Gamma \otimes_{\mathbb{Z}} K^*(X) \rightrightarrows$$

Take $X = pt$
$$\begin{cases} \sum a_n X^{n+1} \longmapsto \sum a_n X^{n+1} \\ \beta \longmapsto T\beta \end{cases}$$

$$U^*(pt) \longrightarrow \mathbb{Z}[\underline{a}] \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}] \rightrightarrows \mathbb{Z}[a] \otimes \Gamma \otimes \mathbb{Z}[\beta, \beta^{-1}]$$

$$\begin{cases} \sum a_n X^{n+1} \longmapsto \sum a_n X^{n+1} \circ \sum \frac{1}{T} \binom{T}{n+1} (-\beta)^n X^{n+1} \\ \beta \longmapsto \beta \end{cases}$$

Change variable by ~~$\mathbb{Z}[\alpha_0^{-1}, \alpha_0, \alpha_1, \ldots]$~~ setting $\sum \alpha_n X^{n+1} = \sum a_n X^{n+1} \circ (-\beta)^{-1} X$ in the middle & $\sum \alpha_n X^{n+1} = \sum a_n X^{n+1} \circ (-T\beta)^{-1} X$ on right and it becomes

$$U^*(pt) \longrightarrow \mathbb{Z}[\alpha_0^{-1}, \alpha_0, \alpha_1, \ldots] \rightrightarrows \mathbb{Z}[\alpha_0^{-1}, \ldots] \otimes_{\mathbb{Z}} \Gamma$$

$$F^U \longmapsto \sum \alpha_n X^{n+1} * (X + Y + XY) \qquad \sum \alpha_n X^{n+1} \longmapsto \sum \alpha_n X^{n+1}$$
$$\longmapsto \sum \alpha_n X^{n+1} \circ \sum_{n \geq 1} \binom{T}{n} X^n$$

N. Jacobson, Schur's theorems on commutative matrices, B.A.M.S. 50 (1944), 431-436