\[ U_b^0 (S^{2n-1} \times \mathbb{Z}_p) \]
\[ U_b^0 \left( \frac{S^{2n-1} \times \mathbb{C}^*}{\mathbb{Z}_p} \right) \to U_b^0 \left( \frac{S^{2n-1} \times \mathbb{C}}{\mathbb{Z}_p} \right) \to U_b^0 \left( \frac{S^{2n-1} \times \mathbb{C}}{\mathbb{Z}_p} \right) \]
\[ U_b^{g+2} (L - X) \to U_b^{g+2} (L) \to U_b^{g+2} (L) \]
\[ U_b^{g+1} (\mathbb{S}L) \to U_b^{g} (X) \xrightarrow{\omega} U_b^{g+2} (X) \]

\[ \mathbb{S}L \to L - X \]
\[ X \to L \]

'properly homotopic'
November 29, 1969:

Consider the symmetric group $\Sigma(n)$. We wish to determine the minimal prime in $H^*(B\Sigma(n))$ mod $p$ cohomology ring. Such primes correspond to maximal elementary abelian $p$-subgroups of $\Sigma(n)$.

Theorem: Let $\Pi$ be a $p$-adic partition of $n$, that is, $\Pi$ is a decreasing sequence $p^\alpha \geq p^\beta \geq \cdots$ whose sum is $n$. Let $A_\Pi$ be the subgroup of $\Sigma(n)$ which is the product of $(\mathbb{Z}_p)^2$ acting on the
$S^{2n-1} \to S^n \to X$

Definition of $e$-invariant

$S^{2n-1} \to S^n$

$M^{n-1} \to pt$

$M^{n-1} = \partial X^n$

Todd $X^n \in \mathbb{Q}/\mathbb{Z}$.

$e$-invariant

I want the $H$.

Do same in unoriented theory.

$M^{n-1} = \partial X^n$

You consider the intersection of $M^{n-1}$ and $\partial X^n$.

Take something like the Whitney or maybe Wu genus.
Let $M$ be an oriented manifold and let $G$ be a finite group acting on $M$. Then $M/G$ is a rational homology manifold and has Pontryagin classes. Now

$$H^*(M/G) = H^*(M)^G = H^*_G(M).$$

Suppose $M$ is a complex manifold on which $G$ acts holomorphically. Then have a quotient manifold $M/G$ and the analogue of the index is $\chi(M/G)$, arithmetic genus. Idea is that sheaf of holomorphic functions $O_{M/G} \cong f_C^G$. Question: What are the Todd homology classes?

Same question for Stiefel-Whitney classes. According to Sullivan given a 4 polyhedron such that link has Euler char $= 0$ (2) one can define Stiefel-Whitney homology classes.
\[
\sum_{n \geq 0} t^n \text{ch}_n = \sum_{i=1}^{N} e^{tx_i} \frac{1}{n!} \sum_{i=1}^{N} x_i^n
\]

But by Newton formulas

\[
(-1)^n s_n = n! \text{ch}_n \mod \text{dec}.
\]

\[
\log \frac{1}{\prod (1-tx_i)} = \sum_{i=1}^{N} \sum_{k} \frac{(tx_i)^k}{k}
\]

\[
\log \frac{1}{1 + (-1)^{n-1} t^n c_n}
\]

\[
(-1)^{n-1} t^n c_n = \frac{1}{n} t^n s_n
\]

\[
\therefore s_n = (-1)^{n-1} n! c_n \mod \text{dec}.
\]

\[
\therefore \text{ch}_n = \frac{1}{n} (-1)^{n-1} n! c_n = \frac{(-1)^{n-1}}{(n-1)!} c_n \mod \text{dec}
\]

\[
\therefore \text{ch} = \sum_{n} \frac{(-1)^{n-1}}{(n-1)!} c_n \mod \text{dec}
\]
Thus it seems that the basic element \( x \) in \( K^1(\Sigma U(n)) \) goes under char into the \( e_1 \in H^4(U(n)) \) and so we should have that

\[
\psi^k x = kx
\]

i.e.

\[
\frac{\psi^k}{k}(x) = x.
\]

So I want to know what happens to \( ch \in H^{2*}(BU) \) under the map

\[
\tilde{H}(BU) \rightarrow H(U) \quad \text{suspension}
\]

What can I say about \( ch \) modulo decomposables?

\[
\Delta ch = ch \otimes c_i
\]

\[
\Delta c_n = \sum c_i \otimes c_j
\]

\[
ch = \sum a_x \cdot c^x
\]

Thus want to evaluate on the sphere \( S^{2n} \)

\[
ch \beta^n = 1
\]

want coeff of \( c_n \) in \( ch \).
need to understand signs

\[ (f \times f')_* = (f \times \text{id})_* \circ (\text{id} \times f')_* \]

**Definition:**
\[ O(f) \times O(f') \rightarrow O(f \times f') \] is the composition.

\[ O(f \times \text{id}) \times O(\text{id} \times f') \rightarrow O((f \times \text{id}) \circ (\text{id} \times f')) \]
\[
\frac{R[G][T]}{T^n - \lambda^{2g}(V)T^{n-1}} \rightarrow \frac{\mathbb{C}[T]}{\prod_{i=1}^{g} (T - \alpha_i)}
\]

\[R(G) \rightarrow \mathbb{C}\]

\[g \in G \text{ then } \psi^k \text{ acts for } k = 1 \quad (g) \]

\[\psi^k \text{ acts for } k = 1 \quad (g)\]

\[T = T^k \]

\[\alpha_i^k = \alpha_i\]

\[
\frac{\mathbb{C}[T]}{(T-\lambda)^n} \quad \psi^k(T) = T^k \quad \lambda^k = \lambda.
\]

\[T - \lambda \rightarrow T^k - \lambda^k = (T - \lambda) \left( T^{k-1} + T^{k-2} + \ldots + T^1 \right)
\]

\[\text{must set } (T - \lambda)^2 = 0 \text{ and this is } \lambda^{k-1} = \lambda^k.
\]

In other words, when there is nilpotence one gets eigenvalues of \( \lambda \).
Tom Dieck claims that
\( \mathbb{R}P^n \times \mathbb{R}P^n \) and \( \mathbb{C}P^n \)
are cobordant as \( \mathbb{R}_2 \)-manifolds.

So consider these as well in \( \mathbb{R}_2 \).
$B \otimes_A \Omega X \sim \to B \otimes_K K(X)$

$B \otimes_A R \to S$  $S$ augmented $B$-algebra

$B \otimes_A F_n R / F_{n+1} R \sim \to F_n S / F_{n+1} S$

$B \otimes_A gr_1 R \sim \to F_1 S / F_2 S$  $\text{free finite type over } B$

to calculate he

as $B/A$ is faithfully flat $gr_1 R$ must be proj. finite type

finite type: if $gr_1 R$ not f. t. has an inf. chain

of submodules $V_\alpha$ with union $gr_1 R$  same for $B \otimes_A V_\alpha$

proj. as above $gr_1 R$ is of fin. pres.  so

$B \otimes_A \text{Ext}^b_A (gr_1 R, M) = \text{Ext}^b_B (gr_1 B, M) = 0.$

It follows that $\Lambda gr_1 R \to R.$  $gr_1 R \mid A$-free?

\text{enough to compute action of Adams operations}

on $B \otimes_A gr_1 R$
So you get a map

\[ \gamma^0 \times \sum_i^{12n} \rightarrow MU \]

**GOAL:**
1. Prove tom Dieck's localization theorem
2. Generalize to localizing w.r.t. a subgroup \( H \).

\[ X \rightarrow X^H \]

- \( X^G \): \( G \)-bundle \( X \)
- \( X^H \): \( H \)-bundle

Also can consider \((X^H, v_X)\) as an \( N \) space + bundle.

\[ U_G(X) \rightarrow U_N(X^H) \]

Odd order \( \Rightarrow \) MSU(BG) \( \Rightarrow \) MU(BG)

\[ MSU(BG) \rightarrow MU(BG) \]

Somehow similar.

On the other hand, one has that all Euler classes lie in MSU(BG).
\[ u_G(X) \rightarrow u(X^0) \]

\[ y \rightarrow x \leftrightarrow (X - Y) \]

\[ P(E) \rightarrow X \rightarrow E \rightarrow Y \]

\[ x^0 \rightarrow \overrightarrow{X^0} \]

\[ \text{not exact. But it is both covariant + contravariant homotopy: } h_t : x \rightarrow y \text{ G maps} \]

\[ x^0 \rightarrow y^0 \]

Next problem is to put in more information.

\[ \text{bundle cobordism:} \]

get both a manifold and a v.b. over that manifold.

\[ (X, (x^0, v)) \]

\[ X \rightarrow (x^0, v) \]

\[ \text{not positive v. bundle, not virtual} \]
Instead of $K(X)$ we use $K(\mathbb{F}_p) \otimes U^*(X)$ which is known to be
$K(X)$ granted periodicity theorem.
Assume that

$$K(\mathbb{F}_p) \otimes U^*(X)$$

is a $\Lambda$-ring augmented over $H^*(X, \mathbb{Z})$ and introduce the associated Chern ring $\Lambda$. There is a good map

$$\text{Ch} \left( K(\mathbb{F}_p) \otimes U^*(X) \right) \xrightarrow{\alpha} \mathbb{H}(X)$$

and the hope is that one can prove that the map is a comm. diagram

$$\begin{align*}
U^*(X) & \\
\downarrow & \\
\text{Ch} \left( K(\mathbb{F}_p) \otimes U^*(X) \right) & \xrightarrow{\alpha} \mathbb{H}(X)
\end{align*}$$

since both $\alpha$. Then the map $\alpha$ must be surjective and so since $\text{Ch}$ has only negative degree components the same holds for $\mathbb{H}(X)$.

Question: $K(X)$ be a $\Lambda$-ring, hence $\bigoplus_{k \in \mathbb{Z}} K(X)$ does there exist a map

$$K(X) \longrightarrow \bigoplus_{n \geq 0} \mathbb{Z}(T_n) \otimes_{\mathbb{Z}} K(X)$$

ring hom.

$$\mathcal{L} \longmapsto \mathcal{T}(L) = \mathcal{L}^T = (1 + L - 1)^T$$

Yes assume $c_1(L)$ nilpotent.

$$= \sum_{n \geq 0} (T_n) (L-1)^n$$
Thus I can form a ring of the form
\[
\mathbb{Z}_2^*(4t) \times \mathbb{U}^*(4t)
\]
\[(a, b) + (a', b') = (a+a', b+b'+aa')?
\]
\[(a, b) \cdot (a', b') = (aa', a^2b+a^2b'-2bb')
\]

Then \[Q:
\]
\[a \ast b = a + b + ab?\]

\[
\begin{align*}
W_2(\mathbb{U}(4t)) & \rightarrow U_{\mathbb{Z}_2}^*(4t) \\
(a, b) & \rightarrow qa + b \xi \\
(a, b) + (a', b') & \rightarrow qa + b \xi + qa' + b \xi^2 \\
(a+a', b+b'-aa') & \rightarrow q(a+a') + (b+b'-aa') \xi \\
(a, b) \cdot (a', b') & \rightarrow (qa + b \xi)(qa' + b' \xi) \\
(aa', a^2b+a^2b'+2bb') & \rightarrow q(aa') + qa' \xi + qa' \xi + q^2bb'
\end{align*}
\]

...the basic identity is therefore that \[\xi^2 = \xi, u = \xi, r(u)\]

...question might be whether it is true that \[qa \cdot \xi = a^2 \xi \]
\[\xi^2 = 2 \xi \leftarrow \text{clear}\]
May 5, 1970: Here are some notes about computational problems in cohomology of groups.

1) mod $p$ cohomology of $GL_n(F_p)$. Sometimes ask Mumford what he knows. Milgram said for $q = p$, he could compute the cohomology of the group of triangular matrices by using the permutation representation on the points of the vector space $F_q^n$. The point was that the wreath product in the symmetric group is easy to see.

Observe that for $GL_3(F)$ the Sylow $p$-subgroup is the Heisenberg group: $x^p = y^p = z^p = 1$, $(x, z) = (y, z) = 1$, $(x, y) = z$, and this might lead to a solution of the case of extra-special $p$-groups.
May 5, 1970  Attempt to do $H^*(GL_n(F_2), 2|12)$ in
exceptional case by a similar method. How to handle the exceptional case $k=2$, $q \equiv 3 \text{ mod } 4$.
The idea I had last night was to use the subgroup

$$N = \left( \sum_m \times (\pi \times F_8^x)^m - (F_8^x)^e \right) \times (F_8^x)^{\pi \equiv g a l (F_8^2/F_8)}$$

of $GL_n(F_8)$, where $n = 2m + e$ $e = 0, 1$. Then
the orders are

$$|GL_n(F_8)| = q^{\frac{n(n-1)}{2}} \prod_{i=1}^{m} (q_i - 1) = q^{\frac{n(n-1)}{2}} \prod_{i=1}^{m} (q - 1)^{\text{even} i} \prod_{i=1}^{m} (q - 1)^{\text{odd} i}$$

$$|N| = m! (2(q^2 - 1)^m (q - 1)^e$$

And these have same power of 2 since $v_2(q^{odd} - 1) = 1$
and $v_2(q^{2i} - 1) = v_2(q^i) + v_2(q^2 - 1)$. To make the argument
work one needs to know whether $\pi \times C$ $C = F_8^2$ has its mod 2 cohomology detected by elem. ab. 2 gaps.
Its fiber 2 subgroup is

$$(\pi \times C)_2 = \langle y, x \rangle \quad \begin{cases} y^2 = 1 \\ x^{q-1} = 1 \\ yxy^{-1} = x^{-1} + 2^a \end{cases}$$

Here $y$ generates $\pi$ and $x$ is a
primitive $2^{a+1}$ root of 1 where $a = \frac{1}{2}(q + 1)$, $a + 1 = \frac{1}{2}(q^2 - 1)$,
and $y$ acts on $x$ by raising to the $q$th power and

$$q+1 \equiv 2^a (2^{a+1})$$
Let $G$ denote this group $\langle x, y \rangle$ and let's first determine its conjugacy classes of elementary abelian 2-subgroups. The elements of order 2 are

$$x^{2^i} y, \quad i = 0, 1, \ldots, 2^a - 1$$

$$\left( x^{2^i} y \right) \left( x^{2^j} y \right) \Rightarrow x^{2^i - 2^j}$$

hence $x^{2^i} y$ and $x^{2^j} y$ commute $\Leftrightarrow 2i \equiv 2j \mod 2^a$. Therefore the maximal elementary abelian 2-subgroups are

$$\langle x^{2^i}, x^{2^i} y \rangle \quad i = 0, 1, \ldots, 2^a - 1$$

Also we know that in the dihedral group $\langle x^2, y \rangle$ these subgroup fall into 2-conjugacy classes depending on whether $i$ is even or odd. In the whole group

$$x^{2^i} \langle x^{2^i}, x^{2^i} y \rangle x^{-2^i} = \langle x^{2^i}, x^{2^{i+2} + 2^j (1 - 2^{a-1})} y \rangle$$

since $x^{2^i} y x^{-2^i} = x^{2^i} x^{2^{i+2} + 2^j (1 - 2^{a-1})} y = x^{2^i} x^{2^j - 2^i} y = x^{2^j} x^{2^i} (1 - 2^{a-1}) y$

As $a \geq 2$ (since $q + 1 \equiv 0 (4)$), $1 - 2^{a-1}$ is a unit mod $2^c$, hence $2^j (1 - 2^{a-1})$ is any even number, and we conclude that all of these elementary 2-subgroups are conjugate.

Conclusion: Only one elementary abelian 2-subgroup of $G$ up to conjugacy and its of rank 2.

This has some implications when we try to compute the
G is an extension of one cyclic group by another and I think Wall has published computation in his article on extensions (see [5] for reference). Cohomology of G by means of the Hochschild-Serre spectral sequence associated to the extension

\[ 0 \rightarrow \langle x^2 \rangle \rightarrow \langle x, y \rangle \rightarrow \langle y \rangle \rightarrow 0 \]

With the notation of the Adams conjecture paper we know that \( d_2^* V = t_1^2 + t_2^2 \) since \( \mod 2 \) the 2^{nd} power drops out. If \( d_3 u = 0 \), then we would get same cohomology as dihedral, which isn't true since there would have to be two prime ideals. Hence \( d_3 u \neq 0 \). If \( d_3 u \) were a non-zero divisor in \( SF_{t_1 t_2}/(t_1^2 + t_2^2) \) then

\[ E_4^* = E_4^* \otimes S[u^2] \]

\( E_4^* \) would have Betti numbers 1, 2, 3, 1. Since it has rank 2, by Beyond 7. Hence of dimension 1, contradicting rank 2. Thus \( d_3 u \) is either divisible by \( t_1 \) or by \( t_1 + t_2 \) and one can check easy enough see that it is \( t_1 \) since \( \langle x^2, y \rangle \) acts on the plane.

Perhaps one can go on to show that \( u^2 \) is an infinite cycle and compute the cohomology.

It doesn't do me any good because the element \( t_1 \in \text{Hom}(G, Z/2) \) with \( t_1(x) = 1 \), \( t_1(y) = 0 \) dies on all the elementary abelian 2-subgroups.

Conclusion: Mod 2 cohomology of the Sylow 2-subgps of \( GL_2(F_q) \) \( q \equiv 3 \mod 4 \) is not detected by elementary abelian 2-subgroups.
May 7, 1970. On $H^*(GL_n(F_q), Z/p)$ as $F_q \to F_p$.

Suppose $n=2$, then the Sylow $p$-subgroup of $GL_2(F_q)$ is the group $U=\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in F_q$. Each conjugate corresponds to a line in $V=F_q^2$, i.e. to the line $L$ one associates $U^L$ those linear transfs inducing identity on $L$ and on $V/L$. The intersection of two conjugates must induces identity on two independent lines and so is the identity. Hence one has (coeff in $Z/p$)

$$H^*(GL_2(F_q)) \to H^*(U)^B$$

where $B=\text{Borel subgroup}$ $(^*^*)$ is the normalizer of $U$. (Actually this formula holds as the Sylow subgroup is abelian.)

Thus we wish to compute the invariants in

(p odd) $H^*(F_q, Z/p) = \Lambda[Hom(F_q, Z/p)] \otimes S[Hom(F_q, Z/p)]$

under the action of $F_q^*$. I am going to show that if $q = p^r$, then first non-zero invariant in $H^*(F_q)$ occurs in dimension $r(2p-3)$, extending the base to $F_q$ commutes with taking invariants and

$$Hom(F_q, F_q) = \bigoplus_{a=0}^{r-1} \frac{F_q}{Z_p} \otimes a$$

where $\sigma$ is Frobenius. If $x \in F_q^*$ then

$$\lambda^*(\sigma^a) = \sigma^a \circ \text{mult by } x \in F_q^*.$$
as trans. of \( F_q \). Thus if \( L \) denotes the standard representation of \( H_q^* \) on \( F_q \), we have

\[
\text{Hom}_{\mathbb{Z}_p}(F_q^*, F_q) = \bigoplus_{a=0}^{n-1} L^{\cdot 2^a}
\]

and the Poincaré series of \( H^*(F_q, F_q) \) as a representation of \( F_q^* \) is

\[
\prod_{a=0}^{n-1} (1 + tL^{\cdot 2^a} + t^2L^{\cdot 2^a} + t^3L^{\cdot 2^a} + \ldots)
\]

A typical term of this is of the form

\[
\bigotimes_{a=0}^{n-1} L^{\cdot 2a} \otimes L^{\cdot \varepsilon_a} = t^{\sum (2a - \varepsilon_a)} \otimes \sum n_a p^a
\]

where \( \varepsilon_a = 0, 1 \) and \( \varepsilon_a = 0 \) if \( n_a = 0 \). This will give an invariant if \( \sum n_a p^a \equiv 0 \pmod{p^{a-1}} \).

Now I wish to look for the first such invariant of positive degree. Suppose \( a_0 \) such that \( 0 < a_0 < r \) and \( n_{a_0} > p-1 \). If \( a_0 < r-1 \), then changing

\[
\begin{align*}
n_{a_0} &\rightarrow n_{a_0} - 1 \\
n_{a_0+1} &\rightarrow n_{a_0+1} + 1
\end{align*}
\]

\[
h_{a_0} p^{a_0} + n_{a_0+1} p^{a_0+1} = (n_{a_0} - 1) p^{a_0} + (n_{a_0+1} + 1) p^{a_0+1}
\]

but \( \sum 2h_{a_0} - \varepsilon_a \) changes by \( -2p + 2 - 1 < 0 \).
If \( a_0 = r - 1 \), then changing

\[
\begin{align*}
 n_{r-1} & \mapsto n_{r-1} - p \\
 \varepsilon_{r-1} & \mapsto 0 \\
 n_0 & \mapsto n_0 + 1
\end{align*}
\]

so

\[
n_{r-1} p^{r-1} + n_0 = (n_{r-1} - p) p^{r-1} + (n_0 + 1)
\]

stays same, but \( \Sigma 2n_a - \varepsilon_a \) changes by \(-2p + 2 + 1 < 0\). Therefore the minimal situation occurs with \( 0 \leq n_a < p \). However \( \Sigma n_a p^a = p^{r-1} \), plus uniqueness of \( \beta \)-adic expansion implies that \( n_a = p - 1 \) and hence all \( \varepsilon_a = 1 \) and so the first term is of degree

\[
\sum_{a=0}^{r-1} 2p - 1 = (2p - 3)h
\]

which proves our assertion.

It seems unlikely that \( H^*(U)^{\beta} \) is a polynomial ring when \( q = p^r \) and \( r \geq 2 \). Thus if \( p = 2 \) we want the invariants in just the symmetric algebra i.e.

\[
\sum t \Sigma n_a \otimes \Sigma n_a 2^n
\]

which is

\[
\sum t^{n_0 + n_1} = \sum t^{3m_0 + 3m_1} \{ 1 + t^2 + t^4 \}
\]

So if \( r = 2 \) the Poincare series of the invariants is

\[
\sum t^{n_0 + n_1} = \sum t^{3m_0 + 3m_1} \{ 1 + t^2 + t^4 \}
\]
\[
\frac{1 + t^2 + t^4}{(1 - t^3)^2}
\]
which is not the Poincare series of a polynomial ring with 2 generators. This calculation tends to suggest great difficulty exists in finding a closed formula for \( H^*(\text{GL}_n(F_p)) \) even for \( n = 2 \).

Corollary of this calculation

\[
\tilde{H}^*(\text{GL}_2(F_p), \mathbb{Z}/p) = 0 \quad \text{if} \quad a < r(2p-3).
\]

\[
\tilde{H}^*(\text{GL}_2(F_p), \mathbb{Z}/p) = 0
\]

**Question:**

\[
\tilde{H}^*(\text{GL}_n(F_p), \mathbb{Z}/p) = 0 \quad ?
\]

**Proof:** By induction on \( n \). Let \( \text{GL}_n(F_p) \) act on the matrices.

**Remark:** We show that the Borel subgroup of upper triangular matrices has no mod \( p \) cohomology and we use induction on \( n \). So \( B \) acts on a vector ope
V of dimension n and possesses a flag. Let B be the subgroup inducing the identity on the bottom line of the flag, so that $B = \mathbb{F}_p^* \times B'$. The cohomology of B injects into that of B' and is invariant under the action of $\mathbb{F}_p^*$. But $B' = B_{n-1} \times \mathbb{F}_p$ and $B_{n-1}$ has no cohomology mod p, hence none in that of $\mathbb{F}_p^{n-1}$ as one sees by using a composition series. So finally one reduces to show that there is no cohomology of $\mathbb{F}_p^{n-1}$ invariant under the multiplication by $\mathbb{F}_p^*$.

To be more precise use homology, whence the homology mod L of $V = \mathbb{F}_p^{n-1}$ as an abelian group is $\mathbb{F}(V) \otimes \Lambda(V)$, $\Lambda$ and $\mathbb{F}$ being taken over $\mathbb{Z}/p$. I want to show there are no co-invariants for the multiplication action of $\mathbb{F}_p^*$.

Can extend the base $V \otimes_{\mathbb{F}_p} \mathbb{F}_p = \mathbb{F}_p^* \otimes_{\mathbb{F}_p} (\mathbb{F}_p \otimes_{\mathbb{F}_p} \mathbb{F}_p)$ and $\mathbb{F}_p \otimes_{\mathbb{F}_p} \mathbb{F}_p \sim \text{Hom}(\mathbb{Z}/p, \mathbb{F}_p)$.

$\lambda \otimes v \rightarrow (v \rightarrow \lambda \cdot v)$

and $\mathbb{F}_p$ acts on a function by $(\lambda f)(x) = \lambda^x f(x)$. 
So I go back to calculation in the case $n = 1$ and I'm now looking for a minimal $\sum 2n_{ai} - e_{ai}$ with $\sum n_{ai} p^a = 0$ ($p^{n-1}$).

where $a = 0, \ldots, r-1$ and $i = 1, \ldots, n-1$. But if the first sum is less than $2N$, then each $n_{ai} \leq N$ and so

$$\sum n_{ai} p^a \leq N(n-1) \frac{p^{r-1}}{p-1}$$

which shows that there should be trouble with $n = p$ or $p+1$. 
Part II. Last section

May 11, 1970

Heuristic calculation part of paper:

The cohomological calculations of the preceding sections suggest some natural conjectures concerning algebraic K theory and what the groups $K_i(F_q)$, suitably defined for all $i \geq 0$, ought to be.

Let $R$ be a ring which need not be commutative and let $X$ be a space. By an $R$-vector bundle over $X$ I mean a locally-trivial fibre bundle whose fibers are projective finitely-generated $R$-modules. If $X$ is connected with basepoint $x$ and locally-simply connected so that $\pi_1(X,x)$ exists, then the category of $R$-vector bundles over $X$ is equivalent to the category of projective finitely-generated $R$-modules endowed with an action of $\pi_1(X,x)$, i.e., representations of $\pi_1(X,x)$ over $R$. Let $kR(X)$ be the Grothendieck group of the category of $R$-vector bundles over $X$. Then $kR(X)$ is a contravariant functor on the homotopy category (of CW complexes say). Consider the following statement.

Statement: There is a universal morphism of functors in the homotopy category $kR \rightarrow \mathcal{B}_\ast$. 

of $kR$ to a representable functor
Since $KR$ admits a decomposition

$$KR(X) = [X, K_0(R)] 	imes \tilde{KR}(X)$$

it follows that $B_u$ admits a product decomposition

$$B_u = K_0(R) \times \mathbb{B}.$$

The groups $K_i(R)$ are defined by

$$K_i(R) = \pi_i(B_u) \quad i \geq 0.$$

Suppose $R$ is commutative. Then given $i$, one has the $i$th arithmetic Chern class

$$c_i : \tilde{KR}(X) \rightarrow H^{2i}(X, \text{Spec} R; \mu_n^{\otimes i})$$

where the cohomology involves the étale cohomology of $\text{Spec} R$. Let $I^*$ be an injective resolution of $\mu_n^{\otimes i}$ in the category of sheaves for the étale topology on $\text{Spec} R$, and set

$$C^*(R, \mu_n^{\otimes i}) = I^*(\text{Spec} R)$$

Then one knows that

$$H^g(X, \text{Spec} R; \mu_n^{\otimes i}) = H^g(X, C^*) \quad \text{(hypercohomology)}$$

$$= [X, K(g, C^*)]$$
where $K(g, C^i)$ denotes the simplicial abelian group associated to the chain complex $S^g C^i$, $C^i$ shifted up $g$ units.

By the universal property we get a map

$$B^i \rightarrow K(2i, C^i(\text{Spec } R, \mu_{n^{2i}}))$$

and taking homotopy groups, we get maps

$$c^i_\# : K_a(R) \rightarrow H^{2i-q}(\text{Spec } R, \mu_{n^{2i}}).$$

Problem: work out ring properties; one would expect lots of $(i-1)!$ factors. The above homomorphisms might generalize the maps found by Bass-Tate in low dimensions.

Now suppose $R = F_q$. As the Brauer map

$$\tilde{K}^P F_q(X) \rightarrow [X, E\mathbb{P}^3]$$

is a map to a representable functor, there is by the universal property a map

$$B \rightarrow E\mathbb{P}^3$$

By the universal property, an element of $H^0(B, A)$ is an abelian group, is the same thing as a natural transformation from $\tilde{K}F_*$ to $H^0(B, A)$ and we have seen this is the
same as a cohomology class of $E_{\infty}^8$ if $A = \mathbb{Z}/l$. Thus the map induces an isomorphism on $H^*$ with coefficients in $\mathbb{Z}/l$ for all primes $l$. This makes it reasonable to conjecture that

\[(*) \quad \tilde{\pi}_i(B) \cong \tilde{\pi}_i(E_{\infty}^8).\]

Now the latter can be computed using the homotopy long exact sequence of the fibration

$$E_{\infty}^8 \rightarrow BU \xrightarrow{\Phi^8} BU$$

and one finds

$$\tilde{\pi}_{2i}(B) = 0 \quad i \geq 0$$

$$0 \rightarrow \tilde{\pi}_{2i}(BU) \xrightarrow{q^i-1} \tilde{\pi}_{2i}(BU) \rightarrow \tilde{\pi}_{2i-1}(E_{\infty}^8) \rightarrow 0$$

so that $\tilde{\pi}_{2i-1}(B)$ is cyclic of order $q^i-1$.

Now the isomorphism $\ast$ depends on the lifting $\phi: \mathbb{F}_q^* \rightarrow C^*$, and it would be nice to have a functorial formula for $K_i(C)$ independent of $\phi$. $\phi$ is a generator of the dual

$$\text{Hom}(\mathbb{F}_q^*, C^*) = \lim_{\rightarrow} C_n$$
which is non-canonically isomorphic to

\[ \lim_{(n,p) \to 1} \mathbb{Z}/n\mathbb{Z} \cong \prod_{l \neq p} \mathbb{Z}/l^k \]

We wish to define an action of the group of units of this ring on \( E \mathbb{F}_8 \). Given an integer \( k \) because \( \pi_1 \) commutes with \( \pi_0 \) it induces a well-defined map \( \pi^k : E \mathbb{F}_8 \to E \mathbb{F}_8 \), which is the identity of course, if \( k \) is a power of \( \pi \). On the homotopy groups

\[ \pi^k = \text{mult. by } k \text{ on } \pi_2 \mathbb{F} \]

hence if \( k \) and \( k' \) are congruent modulo some large integer prime to \( p \) they induce the same transformation of a given Postnikov part of \( E \mathbb{F}_8 \). Thus we get an action of the above profinite \( \mathbb{F} \) ring on \( E \mathbb{F}_8 \). Clearly then if \( \varphi \in \text{that ring then } \)

\[ k \varphi \]

is commutative, because one can use a density argument.
Let $\phi_{F_8}$ be the unique element of $\mathbb{F}_8^*$ such that

$$\phi(\phi_{F_8}) = \exp \frac{2\pi i}{8-1}$$

Then we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}/8-1 & \xrightarrow{\phi} & \mathbb{Z}/8-1 \\
\downarrow & & \downarrow \\
\mu_{8-1} & \xrightarrow{\otimes i} & \mu_{8-1}
\end{array}$$

for $k$ a unit in $\mathbb{Z}/8-1$

$$\phi^k(\phi_{F_8}^k) = \phi(\phi_{F_8}) = \exp \frac{2\pi i}{8-1}$$

One concludes that there is a canonical isomorphism

$$\mathbb{Z}_{2i-1}(B) = K_{2i-1}(F_8) \xrightarrow{\sim} \mu_{8-1}$$

independent of the choice of $\phi$.

Our next problem is to see how the groups $K_i(F_8)$ vary. To consider an extension.
\[ \sim \left( \prod x^{d-1} + \cdots + \prod x^1 \right) + \prod \phi \]

So, \( \beta \) fits into a morphism of fibrations.

Using these morphisms of fibrations, we see that

\[ K_{2i-1}(\mathbb{F}_q) \xrightarrow{\phi} \mathbb{Z}/q^{d-1} \xrightarrow{\phi_{q^d}} \mathbb{M}_{q^{d-1}} \]

commutes since

\[ \phi(\phi_{q^d}) = \exp \frac{2\pi i}{q^{d-1}} \]

Also

\[ K_{2i-1}(\mathbb{F}_q) \xrightarrow{\phi} \mathbb{Z}/q^{d-1} \xrightarrow{\phi_{q^d}} \mathbb{M}_{q^{d-1}} \xrightarrow{\phi_{q^d}} \mathbb{M}_{q^{d-1}} \]
commutes. So we conclude that

\[ K_{2i-1}(\mathbb{F}_q) \xrightarrow{\sim} \mu_{q^{i-1}} \]

\[ f^* \downarrow f^*_x \]

\[ K_{2i-1}(\mathbb{F}_q^{d-1}) \xrightarrow{\sim} \mu_{q^{di-1}} \]

where \( \alpha \) is the unique map such that

\[ \alpha \left( \text{Norm}^i \right) = (q^{d-1}i + \cdots + q^i + 1) \]

\[ \xi = \text{a primitive } (q^{di-1})^\text{th} \text{ root of 1. Perhaps a better way of putting it is to say that} \]

\[ f^* f^*_x z = \sum_{\sigma \in \text{Gal}(\mathbb{F}_q^{d-1}/\mathbb{F}_q)} \xi^z \]

or

\[ f^*(f^*_x \xi^i) = (\sum_{a=0}^{d-1} q^a \xi^i) \xi^i = \frac{q^{di-1} - 1}{q^i - 1} \xi^i \]

Note that

\[ f^*_x f^*_x = d \cdot x \]

since \( f^* \) is injective on passage to the limit we obtain non-canonical isomorphisms

\[ K_{2i}(\mathbb{F}_q) \cong \bigoplus_{\ell \mid p} \mathbb{Q}_\ell / \mathbb{Z}_\ell \]

\[ K_{2i}(\mathbb{F}_q) = 0 \quad i > 0 \]
The last thing we want to compute is the integral arithmetic Chern class map

$$c_i : K_a(F_q) \to \lim_{\pi \to 1} H^{2i-2}(\text{Gal}(F_{q^i}/F_q), \mu_{q^i})$$

where $\pi = \text{Gal}(F_{q^i}/F_q)$.

Now one computes easily that

$$\lim_{\pi \to 1} H^0(\text{Gal}(F_{q^i}/F_q), \mu_{q^i}) = \begin{cases} 0 & i \neq 1 \\ \mu^{\otimes i}_{q^{i-1}} & i = 1. \end{cases}$$

(Recall how this goes: $\pi = \mathbb{Z}$ with Frobenius $q$ for generator and $q$ acts on $\mu_n$ by multiplying by $q$. Now

$$H^1(\pi, M) = M/(q-1)M$$

is the functorial isomorphism. Hence

$$H^1(\pi, \mu_{q^i}) \cong \mu_{q^i}/(q^i-1)\mu_{q^i} \to \mu_{q^{i-1}}$$

provided $q^{i-1}$ divides $n$. This is compatible as $n \to \infty$.

What we want to know therefore is that

$$kF_b(X) \xrightarrow{\phi} EY^b(X)$$

and

$$H^{2i}(X, \mu_{q^i}) \cong H^{2i-2}(X, Z/q^{i-1})$$

isomorphism induced by $\phi$. 

10
commutes. If we suppose this is true, then we can compute the map
\[ c_i^\# : K_{2i-1}(\mathbb{F}_8) \rightarrow \mu_{8^{i-1}} \]
because it is isomorphic to the map
\[ \text{(x)} \quad \tilde{\pi}_{2i-1}(E\mathbb{F}_8) \rightarrow \mathbb{Z}/8^{i-1} \]
which is obtained by evaluating \( E(c_i) \) on a spherical homology class. Actually it is better to see this by writing the diagram
\[
\begin{array}{c}
\mathbb{F}_8(X) \\
\phi \\
\downarrow \\
[X, B_{\mathbb{F}_8}] \\
\downarrow \\
H^{2i-1}(X, \mathbb{Z}/8^{i-1}) \\
\phi \\
\end{array}
\]
and this gives a square by putting \( X = S^{2i-1} \).
To compute the map $\Phi$ is easy because the generator of $\Pi_{2i-1}(\mathbb{F}^8)$ comes from the generator of $\Pi_{2i}(BU) = \Pi_{2i-1}(U)$.

One knows $j^* \Phi(c_i) = \text{suspension of } c_i \in H^{2i}(BU, \mathbb{Z}/2^{i-1})$. This is naturality of $\Phi$ with respect to the morphism of squares

\[
\begin{array}{ccc}
U & \longrightarrow & BU^I \\
\downarrow & & \downarrow \\
E_{\mathbb{F}^8} & \longrightarrow & BU^I \\
\downarrow & & \downarrow \Delta \\
BU \times BU & \longrightarrow & BU \times BU
\end{array}
\]

so $\mathbf{\text{we conclude that}}$

\[
\langle j^* (\text{gen. of } \Pi_{2i-1}(U)), \Phi(c_i) \rangle
= \langle \text{gen. of } \Pi_{2i-1}(U), \text{susp. } c_i \rangle
= \langle \text{gen. of } \Pi_{2i}(BU), c_i \rangle = (-1)^{i-1}(2-i)!
\]

Conclusion (conjectural): Both groups $K_{2i-1}(\mathbb{F}_8)$ and $H^3(\mathbb{F}, \mu_{2i})$ are isomorphic via the map

\[
c^# : K_{2i-1}(\mathbb{F}_8) \longrightarrow \lim_{(n, p) = 1} H^3(\mathbb{F}, \mu_{n^i})
\]
Crazy idea: barycentric coordinates

\[ \sum_{i=0}^{n} t_i = 1 \]

Think of \( t_i \) as the energy of the \( i \)-th vertex. Is there any point in introducing

Better: Think of the \( \{t_i\} \) as a probability measure on the vertices.

Is there any point in introducing a wave function, i.e., a complex valued function on the vertices so that \( |\psi(i)|^2 = t_i \)?
May 14, 1970

To understand fibre-bundle theory as it occurs in homotopy theory with the goal of finding the correct formulation of representing a fibre-bundle theory, I want to start with the simplest examples: principal $G$-bundles where $G$ is a discrete group. For each space $X$ let $\mathcal{K}(X)$ be the category of principal $G$-bundles over $X$. Then $\mathcal{K}$ is a fibred category over the category $\mathcal{I}$ of topological spaces.

Next I want to consider $\mathcal{I}$ as a 2-category where a 2-morphism is a homotopy class of homotopies. Then the pull-back should give rise to a functor

$$\mathcal{K}(Y) \times \text{Hom}(X,Y) \rightarrow \mathcal{K}(X)$$

$$f \rightarrow f^*(\xi)$$

in some way to be made precise. Now although $f^*(\xi)$ is not determined it is determined up to canonical isomorphism in $\mathcal{K}(X)$, and similarly if $h: f \Rightarrow g$ is a homotopy, then there is a definite isomorphism $f^*(\xi) \Rightarrow g^*(\xi)$ associated to it. I recall that the good way to think of $h$ is as coming from inverting a homotopy equivalence, i.e.

$$X \times I \xrightarrow{h} Y$$

$$\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow p_1 & & \downarrow p_2 \\
X & \xrightarrow{} & X
\end{array}$$
So let's try only to use that \( f^* \) is an equivalence of categories when \( f \) is a hom or more precisely that if \( f \) is a homotopy equivalence then there are equivalences of categories

\[
\begin{align*}
K(X) \quad \text{source} & \quad \overset{\text{in}_K}{\longrightarrow} \quad \overset{\text{target}}{\text{Arrows}/f} & \quad \overset{\text{target}}{\longrightarrow} \quad K(Y) \\
\end{align*}
\]

(Note that for a general fibred category one has an equivalence of categories

\[
\{ \text{Cart arrows in } K/f \} \quad \overset{\text{target}}{\longrightarrow} \quad K(Y)
\]

which on choosing an inverse gives \( f^* \). For the \( K \) under consideration all arrows are cartesian.)

\[
\begin{align*}
K(X \times I) \quad \overset{\text{target}}{\longrightarrow} \quad \text{Cart}/1_0 & \quad \overset{\text{source}}{\longrightarrow} \quad K(X) \\
\overset{\text{source}}{\longrightarrow} \quad \text{Cart}/i_1 & \quad \overset{\text{target}}{\longrightarrow} \quad K(X)
\end{align*}
\]

(seems to be a mess.)

Next let's see what it means to represent this bundle theory by \( BG \). First of all there is a universal bundle \( ? \) over \( BG \) and so for any \( X \) we can consider the category \( K(X)/? \)

\[
\begin{align*}
K(X) \quad \overset{\text{source}}{\longrightarrow} \quad K(X)/? \\
\end{align*}
\]
(To get any bearing, suppose $K$ is a discrete fibred category, hence $\text{Ob } K(X) = \{ * \in F(X) \}$, only identity morphisms. Then for $x_1$ over $B$ to represent $F$ means that $x_1 \in \text{Ob } K$ is the final object of $K$ and hence

$$K(X) \xrightarrow{\sim} K(X)/x_1$$

is an equivalence of categories.)

Back to principal $G$-bundles. An object of $K(X)/x_1$ is a square

$$\eta \quad \xrightarrow{x_1} \quad \exists$$

$$\xymatrix{ X \ar[r] & B }$$

and a morphism is a map of squares inducing the identity on $X, B, \exists$. Now given two maps $\eta \to \exists$ and $\eta' \to \exists$ over the same $f : X \to B$, one knows that there can be only one $\sim$ map from $\eta$ to $\eta'$ and vice versa. Thus the category $K(X)/x_1$ is discrete.

So what's really going on is that whereas in the discrete case one got $K(X)$ by letting the discrete category $\text{Hom}(X, B)$ act on $x_1$, here we get the full groupoid $K(X)$ by letting the groupoid $\text{Hom}(X, B)$ act on $x_1$. 
The ultimate point of this nonsense is to take a more or less arbitrary $K$ and show that it should map universally to a representable one. Thus you see that in the case of usual categories, the limiting criteria are fulfilled, i.e. given $k \to h_{x_i}$ i.e. I then you get

$$k \to \varinjlim h_{x_i} = h_{\varinjlim x_i}$$

where that limit exists.

What is a map from $K \to H^*_B$ to consist of? Clearly a cartesian functor. Check this the other way.

Take now $K$ to be the category of virtual $R$-vector bundles. Obviously a whole treatise is needed to understand this construction and how to do it on the fibres of a fibred category.

It is most crucial for you to understand why, at least in the case of $F_2$, a map from $K$ to cohomology with coefficients in a field is the same as a map from $k = \Omega_0^0 K$ to such cohomology. Thus suppose that I am given a natural transformation from $k$ to $H^8(\_\_\_, \Z/2)$. 

Let \( \Theta: \tilde{\mathbb{F}}_p (G) \rightarrow H^* (G, \mathbb{Z}/l) \) be a natural transformation. Quite generally I would like to know if the characteristic classes of representations are representable. Thus set

\[
\tilde{M}_0^i (A) = \text{Hom} (\tilde{\mathbb{F}}_p, H^i (\cdot, A))
\]

for any \( \mathbb{Z}/l \) module \( A \). Then \( A \mapsto \tilde{M}_0^i (A) \) is an additive functor compatible with inverse limits since

\[
H^i (\cdot, \varprojlim A_i) = \text{Hom} (H_i (\cdot), \varprojlim A_i)
\]

Hence

\[
\tilde{M}_0^i (A) = \text{Hom} (M_0^i A) \quad \text{for some } M_i
\]

More precisely, suppose given a functor \( F \) contravariant on a category \( C \) and \( H \), a covariant functor to \( \mathbb{Z}/l \)-modules. Set \( H^* (X, A) = \text{Hom}(H(X), A) \). Then

\[
A \mapsto \text{Hom} (F, H^* (\cdot, A))
\]

commutes with inverse limits and hence is of the form \( \text{Hom}(M, A) \), where \( M \) is the cokernel

\[
\bigoplus \mathbb{Z} [F(Y)] \otimes H^* (X) \rightarrow \bigoplus \mathbb{Z} [F(X)] \otimes H^* (X) \rightarrow M \rightarrow 0
\]

X→Y ∈ Mor C\xrightarrow{X∈Ob C}

If you call \( M = H_*(F) \), then it's clear that \( H_*(F \times F) = H_*(F) \otimes H_*(F) \) and so with \( F = \tilde{\mathbb{F}}_p \) one gets a Hopf algebra.
May 15, 1970: Computation of some \( K_1 \)'s.

Let \( C \) be a category with a notion of direct sum or of exact sequence, permitting one to define a Grothendieck group functor \( k^C(X) \) or \( k^C(\pi) \) where \( \pi \) is a group. One can form then \( k^C \) and consider natural trans. of it to \( H^1(\pi, A) \). One shows that the set of such natural transformations, as a functor of \( \pi \), is represented by a universal determinant from pairs \( (P, \sigma) \) with \( P \) in \( C \) and \( \sigma \in Aut(P) \)

\[
\text{det}: \{ (P, \sigma) \} \rightarrow k^C
\]

which adds for compositions of automorphisms and exact sequences.

I want to compute this in the case where \( C \) is the category of finite \( G \)-sets where \( G \) is a fixed profinite group. Then \( k^C \) is the free abelian group generated by the set of isomorphism classes of irreducible finite \( G \)-sets, equivalently conjugacy classes of subgroups. The category \( C \) decomposes as a product of subcategories of isotypical \( G \)-sets. If \( S \) is isotypical with subgroup \( H \), then the normalizer \( N \) of \( H \) acts freely on \( S^H \) and

\[
G \times_N S^H \rightarrow S
\]

as \( G \)-sets. Thus \( C \) becomes equivalent to the product of the categories of free finite \( N \)-sets where \( H \) runs representatives up to conjugacy for the open subgroups.
To compute the determinant it is necessary to understand the determinant for the category of free \( N/H = Q \) sets. Note that the group of automorphisms \( \Sigma_n \times Q^n \) where the \( Q^n \) acts on the right. However it is clear that

\[
(\Sigma_n \times Q^n)_{ab} \rightarrow \mathbb{Z}_2 \times Q_{ab}
\]

\[
(\tau_1, 
\begin{array}{c}
\vdots
\end{array}, 
\tau_n) \mapsto (\text{sign}(\tau), \bar{\tau}_1 \cdot \bar{\tau}_n)
\]

Indeed a map of \( \Sigma_n \times Q^n \) to an abelian group \( A \) must first factor through \( \Sigma_n \times Q^n_{\text{ab}} \), then through \( \Sigma_n \times Q_{\text{ab}} \) since the sum map \( Q_n^{\text{ab}} \rightarrow Q_{\text{ab}} \) gives the coinvariants for the \( \Sigma_n \)-action, and finally through \( \mathbb{Z}_2 \times Q_{\text{ab}} \) since \( (\Sigma_n)_{\text{ab}} = \mathbb{Z}_2 \). Finally one notes that the above homomorphism \((\tau)\) is compatible with Whitney sum, hence is a determinant.

**Conclusion:** Let \( G \) a profinite group and let \( H_i \) \( i \in I \) be a set of representatives for the conjugacy classes of open subgroups of \( G \). Let \( N_i \) be the normalizer of \( H_i \) in \( G \). Then

\[
K_i((\text{finite } G\text{-sets})) = \bigoplus_{i \in I} \left( \mathbb{Z}_2 \otimes (N_i/H_i)_{\text{ab}} \right)
\]

Explicitly: If \( S \) is a continuous finite \( G \)-set and \( \Theta \) is an automorphism of \( S \), then one computes the determinant of \( \Theta \) as follows. First one decomposes
$S$ into $\bigsqcup S_i$, where $\{ x \in S_i \}$ is the stabilizer of $x$ \( \Leftrightarrow \) the stabilizer of $x$ is conjugate to $H_i$. Then one decomposes $S_i$ into orbits under $N_i/H_i$, say

$$S_i^{H_i} = \bigsqcup_{j \in T_i} T_i^{ij}$$

and chooses a basepoint $z_{ij}$ in $T_i^{ij}$, so that the action of $\Theta$ becomes

$$\Theta(z_{ij}) = \prod_{j \in T_i} q_{ij} z_i(s_{ij}) \quad j \in T_i$$

where $s_i$ is a permutation of $T_i$. Then one has

$$\det \Theta = \sum_i \left( \det s_i + \prod_{j \in T_i} q_{ij} \right)$$

Example: Let $G$ be a profinite group and let $C$ be the category of representations of $G$ over $F_q$.

Case 1: $G = \{1\}$. One knows that the usual determinant

$$\text{GL}_n(F_q) \rightarrow F_q^*$$

is an isomorphism, hence $K_1 = F_q^*$. 

Case 2: $G$ a pro-$p$-group so that every irreducible representation of $G$ is trivial. Given a representation $V$ with an automorphism $\Theta$, consider the socle of $V$, i.e., in this case, the fixed subspace.
This must be stable under $\Theta$, so one sees that by replacing $V$ by an associated graded representation for $\Theta \times \mathfrak{g}$, one can make the $G$-action disappear. Hence only the determinant of $\Theta$ on the underlying vector space matters, so $K_1 = F_g^*$. 

**General cases:** Suppose $C$ is an artinian abelian category. Then if $M\Theta$ is an object of $C$ together with an automorphism, one can break the pair down into irreducibles under the $\Theta$ action, each of which must be isotypical as an object of $C$. Thus if $\Theta$ acts on $M^*$ where $M$ is a simple object one has $\Theta \in \text{GL}_n(D)$ where $D = \text{End}(M)$. Thus the invariant is the Dieudonné determinant of $\Theta$

$$\text{GL}_n(D)_{ab} = D_{ab}^*.$$ 

**Conclusion:** If $C$ is an artinian abelian category, let $M_i, i \in I$ be representatives for the isomorphism classes of simple objects of $C$ and let $D_i = \text{End} M_i$ be the skew-field of endos. Then

$$K_1(C) = \bigoplus_{i \in I} (D_i^*)_{ab}$$
May 22, 1970:

The following remark is negative in spirit and perhaps shouldn't be taken too seriously. Let $F$ be a field. According to a theorem of Tits (see [I], p II-38) $G = \text{GL}_n(F)$ for $n \geq 3$ is the sum of the subgroups

$$\Gamma_{k,n-k} = \begin{pmatrix} k \times k & \mathbb{O} \\ \mathbb{O} & (n-k) \times (n-k) \end{pmatrix}$$

$1 \leq k < n$ amalgamated with respect to their intersections. However, this doesn't imply that the cohomology of $G$ can be computed from these subgroups. Indeed, take $l = 5$ and choose $g$ so that $n = \text{order of } g \text{ in } (\mathbb{Z}/5\mathbb{Z})^*$ is $\geq 3$. Then we know that $\text{GL}_n(F_g)$ has cohomology mod $l$, since it contains $F_l^{*}$. But the subgroups $\Gamma_{k,n-k}$ as well as subgroups have no mod $l$ cohomology since their orders are prime to $l$. 
Suppose that \( A \text{ reg.} \Rightarrow \text{K}_n(A) \cong \text{K}_n(A[t]). \) If true then we are in good condition to prove long exact sequence for \( Y \subset X = U \) all regular.

Note that \( X \longrightarrow X \times \mathbb{P}^1 \leftarrow X \times \mathbb{A}^1 \) plus fact we know periodicity would imply homotopy axiom.

In any case, what involved is to do proj. bundle thm. + excision \( \Rightarrow \) homotopy axiom.

\[
\text{GL}_n(k[t])
\]

Take \( \text{B}_n(k[t]) \equiv \begin{array}{c}
\text{K} \vspace{0.5cm} \\
\text{K} \vspace{0.5cm} \\
\text{K} \vspace{0.5cm} \\
\text{K} \end{array} \)

\[
\text{GL}_2(k[t]) = \text{GL}_2(k) \times_{B(k)} B(k[t])
\]

\( k = \frac{\mathbb{F}_7}{7} \)

and we consider homology mod \( l \). Then \( \text{B}(k) \longrightarrow \text{B}(k[t]) \) isom.

and so \( \text{GL}_2(k) \to \text{GL}_2(k[t]) \) isom mod \( l \).

Homology mod \( p \). \( \text{GL}_2(k[t]) \) seems unlikely somehow.

In fact \( \text{B}(k) \longrightarrow \text{GL}_2(k) \) isomorphism on homology mod \( p \). Because below auto is abelian.
Conjecture:

\[ 0 \rightarrow K_0(A) \rightarrow K_0(K) \rightarrow K_{-1}(k) \rightarrow 0 \]

mod $p$-torsion

\[ [k(\mu_p) : k] = n \]

\[ \mu_p \in K^* \]

\[ K(\mu_p)^* = \hat{K}^* = \hat{\mathbb{Z}} \times \mathbb{A}^* \]
The question is how to compute what $K(p\text{-adic}\text{,mod } k)$ is. The problem is to determine something about the mod $p$ cohomology.

I need somehow to be able to compute characteristic classes of $\text{GL}_n(\mathbb{Z}_p)$.
Cohomology

\[ \mathbb{Z}_p \rightarrow \mathbb{Q}_p \]

A Dedekind domain

A d.v.r.

\[ K_0(k) \rightarrow K_0(A) \rightarrow K_0(K) \]

a basic homomorphism which one ought to understand

is the map

\[ K_0(K) \rightarrow K_{-1}(k) \]

for a local field

\[ 0 \rightarrow K_2(A) \rightarrow \mu_K \rightarrow k^* \rightarrow A^* \rightarrow K^* \rightarrow \mathbb{Z} \]

thus \( K_2(A) \) is the \( p \)-primary of \( K_2(K) \)

I feel from the point of cohomology that

the \( A \) and \( k \) are the same except for \( p \) torsion i.e.

\[ K_0(A) \xrightarrow{i^*} K_0(K) \]

hence surjective.

is an isomorphism off the \( p \)-primary component. If

so what is the composition

\[ K_0(k) \xrightarrow{i^*} K_0(A) \xrightarrow{i^*} K_0(K) \]

should be zero.
May 28, 1970: On algebraic K-theory:

In the following we work only with pointed spaces: \([XY]\) denotes the set of homotopy classes of basepoint-preserving maps from \(X\) to \(Y\). Let \(\mathcal{H}\) be the pointed homotopy category. If I stick to connected spaces it's the homotopy category of simplicial groups. To consider only connected spaces.

**Lemma 1:** If \(H_1(X) = 0\), then the functor

\[\mathbb{Z} \rightarrow \{f \in [X, Z] \mid \pi_1(f) = 0\}\]

is representable.

**Proof:** By attaching 2-cells to \(X\) we construct a map \(X \rightarrow X'\) which kills the fundamental group of \(X\) and \(X'\) has the same homology except that

\[0 \rightarrow H_2(X) \rightarrow H_2(X') \xrightarrow{\text{free grp gen by attached 2-cells}} 0\]

As \(H_1(X') = 0\), all elements of \(H_2(X') = \pi_2(X')\) are spherical, hence we can attach 3 cells to \(X'\) to kill the extra elements of \(H_2(X')\) and so we obtain a map \(X' \rightarrow X''\) such that \(\pi_1(X'') = 0\) and \(H_1(X) \rightarrow H_1(X')\).

I claim \(X''\) and \(X' \rightarrow X''\) represent the functor in question. Indeed given an honest map of spaces \(f : X \rightarrow Z\) such that \(\pi_1(f) = 0\), the obstructions to finding a lifting in
lie in $H^{n+1}(X', X; \pi_1(Z))$ which is zero. Similarly consider a homotopy $h: X \times I \to Z$ between $f, g$ (both satisfying $\pi_1(f) = 0$) and two extensions $f^*, g^*$. Then to see that the extensions are homotopic over $h$ consider

$$X \times I \cup X' \times I \xrightarrow{(h, f^*, g^*)} Z$$

Now as $X \overset{i}{\to} X'$ is a homology isomorphism, so is $j$ because its cofibre is the reduced suspension of $X''/X$. The top map $(h, f^*, g^*)$ kills $\pi_1$, as one may see either by van Kampen (the $\pi_1$ is $\pi_1(X') \ast \pi_1(X)$ $= 0$, it appears that $j$ is a homotopy equivalence!) or because $h$ lifts to $\tilde{Z}$, the universal covering.

Thus we have the existence of the extension and its uniqueness up to homotopy proving the lemma.

Notation: $X \overset{i}{\to} X'$ map of the lemma. If $f: X \to Z$ satisfies $\pi_1(f) = 0$, then the extension $i$ to be denoted $f^*: X' \to Z$. 
Remark: The argument shows that the map of groupoids

\[
\pi_1(X^+, Z) \xrightarrow{i^*} \pi_1(X, Z)
\]

is fully faithful and has for image those \( f \in [X, Z] \) with \( \pi_1(f) = 0 \).

Let \( G = GL_\infty(A) \) and let \( E = E_\infty(A) = [E, E] \). Define \( BG^+ \) by the cocartesian square

\[
\begin{array}{ccc}
BE & \xrightarrow{i} & BG \\
\downarrow{} & & \downarrow{i} \\
BE^+ & \xrightarrow{i} & BG^+
\end{array}
\]

Lemma 2: \( i: BG \rightarrow BG^+ \) represents the functor

\[ Z \rightarrow \{ f \in [BG, Z] \mid \pi_1(f) \text{ kills } E \} \]

Proof: Given a map \( f: BG \rightarrow Z \) of spaces (we must eventually distinguish maps + homotopy classes), if \( \pi_1(f)(E) = 0 \), then \( f|BE \) extends to \( BE^+ \), hence gives rise to a map \( f^+: BG^+ \rightarrow Z \) with \( fi = f \).

Now suppose we have a homotopy \( h: BG \times I \rightarrow Z \) between \( f \) and \( g \) and extensions \( f^+, g^+: BG^+ \rightarrow Z \). Then by the proof of the preceding I know
that I can form $\tilde{h}^+: BE^I \to Z$ joining $f^+$ and $g^+$ restricted to $BE$ and $\Rightarrow h^+ = h$ restricted to $BE$. Then $\tilde{h}^+$ and $h$ define $h^+: BG^+ \to Z$ extending $h$ and joining $f^+$ and $g^+$. This proves lemma.

Note that the last step uses that

\[ BE \times I \cup BE^+ \times I' \to BG \times I \cup BG^+ \times I' \]

is cocartesian, hence the latter vertical map is a homotopy equivalence. Thus, as in the remark following lemma 1, we have that

\[ \pi_1(BG^+, Z) \to \pi_1(BG, Z) \]

is fully faithful with image those $f \in \pi_1(X, Z)$ such that $\pi_1(f)$ kills $E$.

Before going on I want to check this last assertion carefully. Thus I want to prove that

\[ \pi_1(\text{Hom}(BG^+, Z; f^+, g^+)) \to \pi_1(\text{Hom}(BG, Z; f, g)) \]

is bijective. Surjectivity results from the fact that if $h: BG \times I \to Z$ represents an element of the latter, then it comes from
as we have already seen. For injectivity suppose I have $h^+_a, h^+_b : (BG^+ \times I) \to Z$ joining $f^+$ and $g^+$. Suppose I know that $h^+_a \cdot \xi$ and $h^+_b \cdot \xi$ are homotopic keeping $f, g$ fixed. Then I have a homotopy of the horizontal arrows furnished by the constant homotopies of $f^+, g^+$ and the homotopy joining $h^+_a \cdot \xi$ and $h^+_b \cdot \xi$, and this homotopy is compatible with the extensions $h^+_a$ and $h^+_b$, so as I know well one gets a big homotopy of the dotted arrow, proving injectivity of $i^*$.

Remark: The above argument shows that one gets a map $i : X \to X^+$ killing any subgroup $E$ of $\pi_1(X)$ such that $E = [E, E]$, and that

$$
\pi_1(X^+, Z) \xrightarrow{i^*} \pi_1(X, Z)
$$

is fully faithful with images the $f \in [X, Z]$ such that $\pi_1(f)(E) = 0$. In particular $i$ induces an isomorphism on homology.
Theorem: Let $X$ be a (pointed connected) space ($CW$ or is understood) and let $E \subseteq \pi_1(X)$ be a normal subgroup such that $[E,E] = E$. Then in the pointed homotopy category there is a morphism $f: X \to Y$ with the following properties:

1) $f$ induces an isomorphism $\frac{\pi_1(X)}{E} \xrightarrow{\cong} \pi_1(Y)$ and $H_*(X, L) \cong H_*(Y, L)$ for all $\pi_1(Y)$-modules $L$.

2) For any $Z$, $f^*: [Y, Z] \to [X, Z]$ is injective with image the set of $\alpha$ such that $\pi_1(X)(E) = 0$.

Before taking up the proof, suppose $f: H_*(X, L) \xrightarrow{\cong} H_*(Y, L)$ for all $\pi_1(Y)$-modules $L$. Then for $*=0$

\[
\pi_1(X) \xrightarrow{f_*} \pi_1(Y) \xrightarrow{\otimes Z} L \xrightarrow{\pi_1(X)} L \xrightarrow{\pi_1(Y)} Z
\]

for all $L$, so taking $L = Z[\pi_1(Y)]$ we have

\[
Z[\pi_1(Y)/f_*\pi_1(X)] \xrightarrow{\cong} Z
\]

and hence $f_*: \pi_1(X) \to \pi_1(Y)$ is onto. For $*=1$ we have

\[
\begin{array}{ccc}
H_1(X, L) & \xrightarrow{\cong} & H_1(Y, L) \\
\downarrow \mathbb{S} & & \downarrow \mathbb{S} \\
H_1(\pi_1(X), L) & \xrightarrow{\cong} & H_1(\pi_1(Y), L)
\end{array}
\]

Computing the $\text{Tor}_1$ by the sequence $0 \to Z[\pi_1X] \to Z[\pi_1X] \to Z \to 0$. 
\[
0 \rightarrow H_1(\pi_1(X), L) \rightarrow \mathbb{Z}[\pi_1 X] \otimes L \rightarrow L \rightarrow \mathbb{Z} \otimes L \rightarrow 0
\]

\[
0 \rightarrow H_1(\pi_1(Y), L) \rightarrow \mathbb{Z}[\pi_1 Y] \otimes L \rightarrow L \rightarrow \mathbb{Z} \otimes L \rightarrow 0
\]

for all \( L \), so taking \( L = \mathbb{Z}[\pi_1 Y] \) and using the exact sequence,

\[
0 \rightarrow E_{ab} \rightarrow \mathbb{Z}[\pi_1 X] \otimes \mathbb{Z}[\pi_1 Y] \rightarrow I[\pi_1 Y] \rightarrow 0
\]

where \( E = \text{Ker}\{\pi_1(X) \rightarrow \pi_1(Y)\} \), we have \( E_{ab} = 0 \), i.e. \( E = [E, E] \). Hence the hypotheses of the theorem are best possible.

Next we show 1) \( \Rightarrow \) 2). The best statement is that

\[ f^* : \text{Hom}(Y, Z) \rightarrow \text{Hom}(X, Z) \]

(space of basepoint preserving maps) is a homotopy equivalence with the union of the components of \( \text{Hom}(X, Z) \) corresponding to \( \emptyset \in [X, Z] \). Assume \( \pi_1(Y)(E) = 0 \). May assume \( f \) a cofibration, whence \( f^* \) is a fibration. It's enough to show (i) any \( g : X \rightarrow Z \rightarrow \pi_1(Y)(E) = 0 \) factors through \( f \) and (ii) \( (X \times I) \cup (Y \times I^*) \rightarrow Y \times I \) is a homotopy equivalence.

Indeed (ii) \( \Rightarrow f^* \) induces a homotopy equivalence of the path spaces of \( \text{Hom}(Y, Z) \) at \( g^+ \) and of \( \text{Hom}(X, Z) \) at \( g^f \).

For (ii) note by van Kampen

\[
\pi_1((X \times I) \cup (Y \times I^*)) = \pi_1(Y) \ast_{\pi_1(X)} \pi_1(Y) = \pi_1(Y)
\]

and Mayer-Vietoris

\[
H_*((X \times I) \cup (Y \times I^*), L) \rightarrow H_*((Y \times I^*), L)
\]
for any $\pi_1(Y)$-module $L$, hence the map in question is a homotopy equivalence by the Whitehead theorem in the form used by Artin-Mazur. For (i) use obstruction theory.

\[
\begin{array}{c}
X \\
\downarrow \\
Y
\end{array} \rightarrow 
\begin{array}{c}
Z_n \\
\downarrow \\
Z_{n-1}
\end{array} \rightarrow 
\begin{array}{c}
P(\pi_1(Z)) \times_{\pi_1(Z)} K(\pi_n Z, n) \\
\downarrow \\
P(\pi_1(Z)) \times_{\pi_1(Z)} K(\pi_n Z, n^*)
\end{array}
\]

The obstructions to producing a lifting are in

\[H^{n+1}(Y, X; \pi_n(Z)) \quad n \geq 2\]

where one gets started, i.e. $E$

\[
\begin{array}{c}
X \\
\downarrow \\
Y
\end{array} \rightarrow 
\begin{array}{c}
K(\pi_1 Z, 1) \\
\downarrow \\
E
\end{array}
\]

because $\pi_1(g)(E) = 0$ and $\pi_1(X)/E \approx \pi_1(Y)$.

Conclude: 1) $\Rightarrow$ 2) is consequence of obstruction theory.

As 2) gives a universal property for $Y$ and the map $f : X \rightarrow Y$, it follows that the map $f$ is unique up to canonical isomorphism in the pointed homotopy category.
Existence of \( f: X \to Y \): Let \( p: \tilde{X} \to X \) be the covering space associated to the subgroup \( E \), i.e., \( \pi_1(p): \pi_1(\tilde{X}) \to \pi_1(X) \) is an isomorphism with \( E \). Then \( H_1(\tilde{X}) = E \cdot ab = 0 \), so by attaching 2 and 3 cells to \( \tilde{X} \) we may construct a map \( \tilde{f} \)

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X}^+ \\
p & & p \\
X & \xrightarrow{f} & Y
\end{array}
\]

inducing isomorphisms on homology and killing \( E \). Define \( Y \) by the cocartesian square.

By van Kampen

\[
\pi_1(Y) = \pi_1(X) \times \pi_1(\tilde{X}^+) = \pi_1(X)/E
\]

and if \( L \) is a \( \pi_1(Y) \)-module, then one has long exact sequences

\[
\ldots \to H_b(\tilde{X}, L) \xrightarrow{\sim} H_b(\tilde{X}^+, L) \to H_b(\tilde{X}, \tilde{X}^+, \tilde{X}^+, \tilde{X}^+, \text{L}) \xrightarrow{\text{L}} H_b(\tilde{X}^+, \text{L})
\]

\[
\ldots \to H_b(X, L) \xrightarrow{\sim} H_b(Y, L) \to H_b(Y, X, L)
\]

showing that \( H_*(X, L) \cong H_*(Y, L) \). The theorem is proved.

Corollary: Let \( \tilde{Y} \to Y \) be the covering of \( Y \).
Then the obvious map (obtained from * page 9) is a homotopy equivalence.

**Proof:** As both $\tilde{X}^+$ and $\tilde{Y}$ are 1-connected it's enough to check the map induces an isomorphism on homology. But

$$H_*(\tilde{X}, Z) \xrightarrow{\sim} H_*(\tilde{Y}, Z)$$

induced by hypothesis.

So $\tilde{X} \to \tilde{X}^+ \to \tilde{Y}$ all induce isomorphisms on homology.

**Pleasant consequences:** Denote by $G = GL_\infty(A)$, $E = E_\infty(A)$ and let

$$E \xrightarrow{BE} E^+ \xrightarrow{BE^+}$$

be the square in which the horizontal maps are those
constructed in the theorem using the subgroup $E$. Thus one attaches $2 + 3$ cells (in fact one of each will do!) to get $BE^+$ and then forms the cocartesian square. By the corollary $BE^+$ is the universal covering (up to homotopy type) of $BG^+$. Thus

$$\pi_i(BE^+) = \pi_i(BG^+) \quad i \geq 2$$

But as $BE^+$ is simply-connected

$$\pi_2(BE^+) = H_2(BE^+) = H_2(BE)$$

proving that the $K_2$ agrees with Milnor's!
H-space structure of $BG^+$:

Let $X \to X^+$ be a map such as in the theorem: $\pi_1(X) \to \pi_1(X^+)$ and passing to the coverings with group $\pi_1(X^+)$ gives a homology isomorphism $\tilde{X} \to \tilde{X^+}$. Observe that the family of these maps is closed under composition and with change. (In effect if

\[
\begin{array}{c}
X \to X^+ \\
\downarrow \\
Z \to Z^+
\end{array}
\]

is cocartesian, then for any $\pi_1(Z^+)$ module we have by long exact sequences that $H_\ast(Z, L) \cong H_\ast(Z^+, L)$, and taking products with a fixed space (in effect the covering of $X \times Z \to X^+ \times Z$ is $\tilde{X} \times \tilde{Z} \to \tilde{X^+} \times \tilde{Z}$, $\tilde{Z}$ and $\tilde{X}$ universal coverings, and this map is a homology isomorphism).

Conclusion: The class of maps we are interested in are those $X \to X^+$ such that on taking the coverings with group $\pi_1(X^+)$ we get a homology isomorphism $\tilde{X} \to \tilde{X^+}$ (= universal covering of $X^+$). (Note this implies $\pi_1(X) \to \pi_1(X^*)$ otherwise $\tilde{X}$ won't be connected).

Corollary: $(X^\ast \times Y)^+ = X^+ \times Y^+$. 
So applying the universal property of the theorem we know that if we choose an isomorphism \( A^\infty \oplus A^\infty \cong A^\infty \) it gives \( \mu \) which extends

\[
\begin{align*}
\text{BG} \times \text{BG} & \xrightarrow{\mu} \text{BG} \\
\downarrow & \\
\text{BG}^+ \times \text{BG}^+ & \xrightarrow{\mu^+} \text{BG}^+
\end{align*}
\]

in an essentially unique way to \( \mu^+ \). I want to show that \( \mu^+ \) is an H-space structure on \( \text{BG}^+ \) and hence must prove that

\[
\begin{align*}
\text{BG}^+ & \\
\downarrow{(id,0)} & \\
\text{BG}^+ \times \text{BG}^+ & \xrightarrow{\mu^+} \text{BG}^+
\end{align*}
\]

is homotopic to the identity. This is the induced map associated to the embedding \( G \rightarrow G \) produced by the group \( G \) acting on the first factor and the isomorphism \( A^\infty \oplus A^\infty \cong A^\infty \). For \( \mu^+ \) to be homotopy associative means that the two maps \( G^3 \rightarrow G \) induced by the isomorphisms

\[
\begin{align*}
A^\infty & \cong A^\infty \oplus A^\infty \cong (A^\infty \oplus A^\infty) \oplus A^\infty \\
& \cong A^\infty \oplus (A^\infty \oplus A^\infty)
\end{align*}
\]

have the same effect on \( \text{BG}^+ \) note that the two isos. are conjugate by an automorphism of \( A^\infty \). Hence \( \mu^+ \) will give an H-space.
Proposition: Let \( u : G \rightarrow G \) be an embedding induced by an isomorphism \( A^n \oplus A^{n+1} \cong A^n \) whenever \( 0 < n < \infty \).

Then \( \text{BG} \xrightarrow{u} \text{BG} \)

\( \text{BG} \xrightarrow{u} \text{BG} \)

\( \text{BG} \xrightarrow{u} \text{BG} \)

is homotopy commutative, i.e. up to homotopy \( u \) induces the identity on \( \text{BG}^+ \).

Admit for a moment the

Lemma: Let \( G_n = \text{GL}_n(A) \) be regarded as a subgroup of \( G = \text{GL}_\infty(A) \) in the standard way and suppose that \( \Theta \in G_\infty \) that \( I_n \oplus \Theta \in G \), centralizes \( G_n \). Let \( f : \text{BG} \rightarrow \text{BG}^+ \)

be the canonical map, and let \( \Theta \) denote the automorphism of conjugation by \( \Theta \) on \( G \) as well as \( \text{BG} \). Then there is a homotopy joining \( f(I_n \oplus \Theta)^\# \) to \( f \) which restricts to the constant homotopy of \( f|_{\text{BG}_n} \).

\( \text{BG} \xrightarrow{f} \text{BG} \)

\( \text{BG} \xrightarrow{f} \text{BG} \)

\( \text{BG} \xrightarrow{f} \text{BG} \)

(Or more precise language the first triangle commutes, etc.)
perhaps better, comes with an essentially unique homotopy since $I_n \otimes \Theta$ centralizes $G_n$, and the lemma asserts the second triangle may be made to commute by a homotopy whose effect as maps from $BG_n$ to $BG^+$ is the constant homotopy.

Proof of the proposition: I perceive that I haven't been careful enough because an isomorphism $A_n \otimes A_n \cong A^n$ doesn't induce an embedding $G \to G$, since there is no reason why the linear transformation should be equal almost everywhere (relative to the standard basis) to the identity. So therefore I shall only consider isomorphisms $\phi$ of $(\otimes^n \oplus \otimes^n) \cong A^n$ resulting from dividing up $N$ into two disjoint sets of $\infty$ and $n$ elements and ordering these sets.

Let the embedding $\iota: G \to G$ be obtained by such a partition $N = N' \sqcup N''$ and isomorphisms of $N'_+ = G$ and $N''_+ = G$.

Let $i_n: G_n \to G_n$ denote the inclusion and $\beta_n = \phi_n \circ G_n: G_n \to G$ and let $\alpha_n: G_n \to G$ be standard embeddings. Choose a permutation matrix $\Theta_n \in G$ such that

$$\beta_n = \Theta_n \alpha_n$$

($\#$ denotes conjugation).

and let $\phi_n$ be defined by $\Theta_n \phi_n = \Theta_n$. Then

$$\Theta_n \alpha_{n-1} = \beta_{n-1} = \phi_n \iota = \Theta_{n-1} \circ \phi_n \circ \alpha_{n-1}$$

so $\phi_n \phi_{n-1} = \alpha_{n-1}$, i.e. $\phi_n$ is a permutation leaving the numbers $\{1, \ldots, n-1\}$ fixed, i.e. of the form
By the lemma, there exists a homotopy
\[ \varepsilon_n : p \Rightarrow p \varphi_n^\# \]
such that
\[ \varepsilon_n \times \alpha_{n-1} : p \alpha_{n-1} \Rightarrow p \varphi_{n-1}^\# \alpha_{n-1} = p \alpha_{n-1} \]
is the constant homotopy (we assume that we use a simplicial model so that \( B\Gamma_n \to B\Gamma_{n+1} \to \cdots \) are cofibrations.)

Define the homotopies
\[ \gamma_n : p \Rightarrow p \theta_n^\# \]
recursively by
\[ p \overset{\varepsilon_n}{\Rightarrow} p \varphi_n^\# \overset{\gamma_{n-1} \times \varphi_n^\#}{\Rightarrow} p \theta_{n-1}^\# \theta_n^\# = p \theta_n^\# \]

Let
\[ \delta_n = \gamma_n \times \alpha_n : p \alpha_n \Rightarrow p \theta_n^\# \alpha_n = p \beta_n \]
and \( \delta_n \times \iota_n = \delta_{n-1} \) because
\[ p \alpha_{n-1} \overset{\varepsilon_n \times \alpha_{n-1}}{\Rightarrow} p \varphi_{n-1}^\# \alpha_{n-1} \overset{\gamma_{n-1} \times \varphi_{n-1}^\# \alpha_{n-1}}{\Rightarrow} p \theta_{n-1}^\# \varphi_{n-1}^\# \alpha_{n-1} = p \theta_{n-1}^\# \alpha_{n-1} \]

Thus the \( \delta_n \) give a compatible family of homotopies from \( p \alpha_n \) to \( p \beta_n \) and so \( p \) is homotopic to \( p \) proving the proposition.
(Remark: Note that we must adjust for composition of a trivial homotopy; ugh; this hopefully becomes intelligible in 2-category language.)

Now we prove the lemma. Let \( \oplus : G_n \times G_n \to G \) be the direct sum map. Then we have a commutative square of spaces (where \( BG_n = \overline{\mathcal{W}(G_n)} \))

\[
\begin{array}{ccc}
BG_n & \xrightarrow{\oplus} & BG \\
\downarrow{id \times \varphi^#} & & \downarrow{(1_n \oplus \varphi)^#} \\
BG_n \times BG & \xrightarrow{\oplus} & BG \\
\downarrow{\text{id} \times \varphi^#} & & \downarrow{(1_n \oplus \varphi)^#} \\
BG_n \times BG^+ & \to & BG^+ \\
\end{array}
\]

The idea of the proof consists in showing that given a homotopy \( p \Rightarrow p \varphi \) there is another \( p \Rightarrow p(1_n \oplus \varphi)^# \) compatible with it.

First step: understand the homotopy \( p \Rightarrow p \varphi^# \). Quite generally one has a map

\[
\begin{array}{ccc}
[X, B] \times \pi_1(B) & \to & [X, B] \\
\downarrow{\text{hom}(X, B)} & & \downarrow{\text{hom}(Xu e, B)} \\
\text{hom}(Xu e, B) & \to & B
\end{array}
\]

which results from the fibration
If we map $B \to B'$, this gives a map of fibrations, and if $r \in \pi_1(B)$ goes to zero, the effect of $r$ on $[X, B]$ becomes trivial in $[X, B']$. So we start with $\varphi$ and represent it by $\lambda : S' \to BG$. Then form

$$BG \times O \cup (e \times I) \xrightarrow{id + \lambda} BG$$

So we get $H_t : BG \to BG$ such that $H_0 = id$ and $H_1 = \varphi^#$. And $t \mapsto H_t(e)$ is $\lambda$. Now compose with $p : BG \to BG^+$ and use the fact that $p \lambda$ contracts to a point. Covering

It seems that you have

$$BG_n \times BG \xrightarrow{\oplus} BG$$

$$id \times p \xrightarrow{=} id \times p \varphi^#$$

and

$$BG_n \times BG^+ \xrightarrow{\oplus^+} BG^+$$

and you want to extend the homotopy.

This homotopy moves $p \varphi^#$ to $p$. The picture to draw is:
\[ \text{and now you contract } p\lambda : S^1 \to BG^+ \text{ to basepoint} \]
\[ \text{and this the basepoint-preserving homotopy of} \]
\[ p \Rightarrow p(I_n \oplus \varphi)^\# \Theta \]

But if \( \Theta \) is a cofibration as you may assume, then \( \Theta^* \) is a fibration so you can lift this homotopy to one

\[ p \Rightarrow p(I_n \oplus \varphi)^\# . \]

Thus you get the compatible family of homotopies you need.

The above is probably correct although impossible to understand.
critical problem is that there is no formula for 
\[ c_t^*(f^* x) \]
without denominators.

Example: Suppose \( L \) is a line bundle over \( X \) and we consider \( f = \text{id} : X \rightarrow L \)
\[ i_* : K(X) \longrightarrow K_{\text{lin}}(L) \]
for \( G(x) \) with
\[ c_t^* i_*(x) = 1 + i_* G^t(x) \]
\[ \Rightarrow i^* c_t^*(i_*(x)) = c_t^* (i^* i_* x) = 1 + i^* i_* G(x) \]
Now \( i^* i_* 1 = 1 - L^{-1} \) in \( K \)
and \( i^* i_* 1 = c_1(L) \) in \( U \) so this gives

\[ G(x) = \frac{c_t(x)}{c_t(x L^{-1})} - 1 \]

In other words you write universally

\[ \frac{c_t(x)}{c_t(x L^{-1})} = 1 + a_1 c_1(L) + a_2 c_1(L)^2 + \ldots \]

and

\[ G(x) = a_1 + a_2 c_1(L) + \ldots \]

Now calculate this for the case where \( c_1(L) = 0 \)

Suppose \( x = \sum L_i \) note that in general

\[ G(x + y) = Gx + Gy + c_1(L) \cdot Gx \cdot Gy \]
so that when $c_1(L) = 0$ $G$ is additive. For a line bundle $x = \mathcal{L}$ get

$$\frac{c_1(x)}{c_1(x,L^{-1})} = \frac{1 + tc_1(x)}{1 + tF(c_1x, L)c_1L}$$

can now suppose $(c_1L)^2 = 0$ for calculation of $a$,

$$F(c_1x, L) = c_1x + F_2(c_1x, 0) \frac{L}{c_1L}$$

$$= c_1x \frac{L}{c_1L} + F_2(c_1x, 0) c_1L$$

$$\frac{c_1(x) - 1}{c_1(x,L^{-1})} = \frac{1 + tx}{1 + t_1x - tF_2(c_1x, 0) c_1L} - 1$$

$$= \frac{tF_2(c_1x, 0) c_1L}{1 + tc_1x - tF_2(c_1x, 0) c_1L} = c_1L G(x).$$

for a line bundle $x$ and $(c_1L)^2 = 0$ have

$$G(x) = \frac{tF_2(c_1x, 0)}{1 + tc_1x - tF_2(c_1x, 0) c_1L}$$

so if $c_1L = 0$ get

$$G(x) = \frac{tF_2(c_1x, 0)}{1 + tc_1(x)}$$

now for

$$F(x, y) = x + y - xy \quad F_2(x, 0) = 1 - x$$
\[ y = L \]

\[ c_t \left( \chi_v(L) \right) \]

\[ c_t \left( (1-v^t) \cdot L \right) = \sum_{n=0}^{\infty} c_1(l)^n \left( \sum_{n=0}^{\infty} F(c_{1^n}, c, L)^n \right) \]

\[ c_t \left[ (1-v^t) \cdot L \right] = \frac{c_t(L)}{c_t(v^{-t} \cdot L)} = \frac{\sum_{n=0}^{\infty} c_1(l)^n}{\sum_{n=0}^{\infty} F(c_{1^n}, c, L)^n} \]

Curiously:

\[ U(X) \xrightarrow{\phi} U(X \times C^+) \]

and

\[ K(X) \xrightarrow{\sim} K(X \times C^+) \]

The point is that

\[ c_t(x \cdot x) = G(c(x)) \]
Summary of calculations made.

1. \[ u^*(x) \xrightarrow{\Delta_1} K_*(mu) \otimes_{K_*(\mu^t)} K_*(x) \]
   \[ \hat{\phi}_a(c_1^u(l)) = \sum_{n \geq 0} a_n c_1^u(l)^{n+1} \]

2. \[ K_*(x) \xrightarrow{\Delta_2} K_*(Bu) \otimes_{K_*(\mu^t)} K_*(x) \]
   \[ \psi^T(\beta) = T\beta \]
   \[ \psi^T(L) = L^T = \sum_{n \geq 0} (T_n)(L-1)^n \]

3. \[ K_*(mu) \xrightarrow{\Delta_3} K_*(mu) \otimes_{K_*(\mu^t)} K_*(Bu) \]
   \[ Z[a_{ij} \ldots] \otimes_{Z^{K*(\mu^t)}} Z[a_{ij} \ldots] \otimes_{Z^{\Gamma \otimes Z^{K_*(\mu^t)}}} \]
   \[ \sum_{n \geq 0} a_n x^{n+1} \quad \rightarrow \quad \sum_{n \geq 0} a_n x^{n+1} \circ \sum_{n \geq 0} \frac{1}{n+1} (T_{n+1})(-\beta)^n x^{n+1} \]
N. Jacobson, Schur's theorems on commutative matrices, B.A.M.S. 50 (1944) 431-436