$$Z \xrightarrow{i} X \xleftarrow{j} U$$

Given $[Z \to X]$ $\ni$ $j^*[Z \to X] = 0$

i.e. $\exists$.

$$\begin{tikzcd}
Z \arrow[r, hookleftarrow] \arrow[d] & Z_U \arrow[r] \arrow[d] & W \arrow[d, "h = (p, g)"] & \phi \arrow[l, hook'] \arrow[d] \\
X \arrow[r, hookleftarrow] & U \arrow[r] & U \times \mathbb{R} & U \arrow[l, hook']
\end{tikzcd}$$

$$g^{-1}(0) = \phi$$

$$g^{-1}(1) = Z_U$$

$$W = g^{-1}(-\infty, 0) \sqcup g^{-1}(0, \infty)$$

~~Also~~ both open hence closed also

throw away $g^{-1}(-\infty, 0)$ still h proper

Choose $f : U \to [0,1] \subset \mathbb{R}$ $\ni$ $f^{-1}\{0\} = Y$

$f(X - N) = 1$

Now consider the map

$$W \longrightarrow U \times \mathbb{R}$$

$$w \longmapsto \left(p(w), \frac{g(w)}{f(w)}\right)$$

claim proper — w_n $p(w_n) \to x$ $\dfrac{g(w_n)}{f(w_n)} \to r$

$x \in U$ $f(w_n) \to f(x) \neq 0$ so $g(w_n)$ conv. OKAY

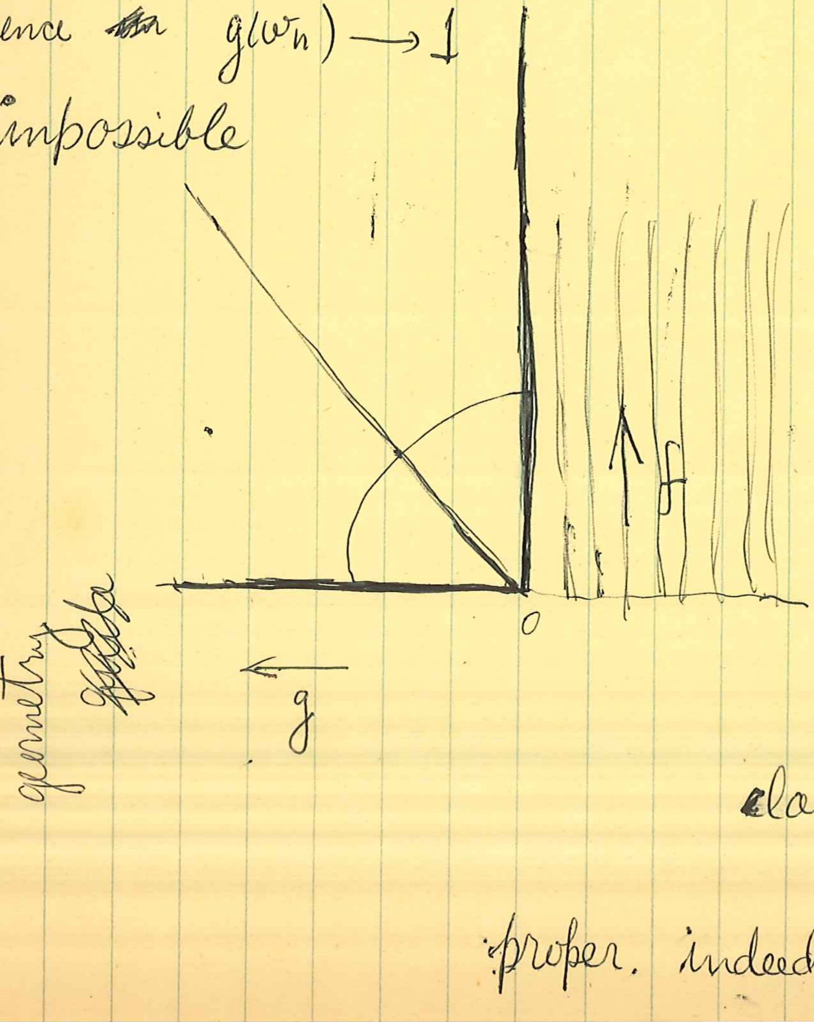
$x \in Y$ $f(w_n) \to 0$ so $g(w_n) \to 0$ again $w_{n_k} \to 0$ ✗.

if $p(\bar{\omega}_n) \longrightarrow u \in \mathcal{U}$
then OKAY

but if $p(\bar{\omega}_n) \longrightarrow y \in Y$
then $f(\bar{\omega}_n) \longrightarrow 0$

~~so $g(\bar{\omega}_n) \longrightarrow 0$~~

hence ~~so~~ $g(\bar{\omega}_n) \longrightarrow 1$
impossible



$W \xrightarrow[p]{} \mathcal{U} \times \mathbb{R}$

given $f : W \longrightarrow \cancel{\mathbb{R}_{>0}}\ (0,1)$ distance to Y in tub. nbd. then $\equiv 1$.

$g : W \longrightarrow \mathbb{R}$

basic data.

$g^{-1}\{1\} = \phi$

and $g^{-1}\{0\} = Z_{\mathcal{U}}$

~~then~~ take

$$\frac{1 - g(\bar{\omega})}{f(\bar{\omega})} = h(\bar{\omega})$$

everywhere defined

claim

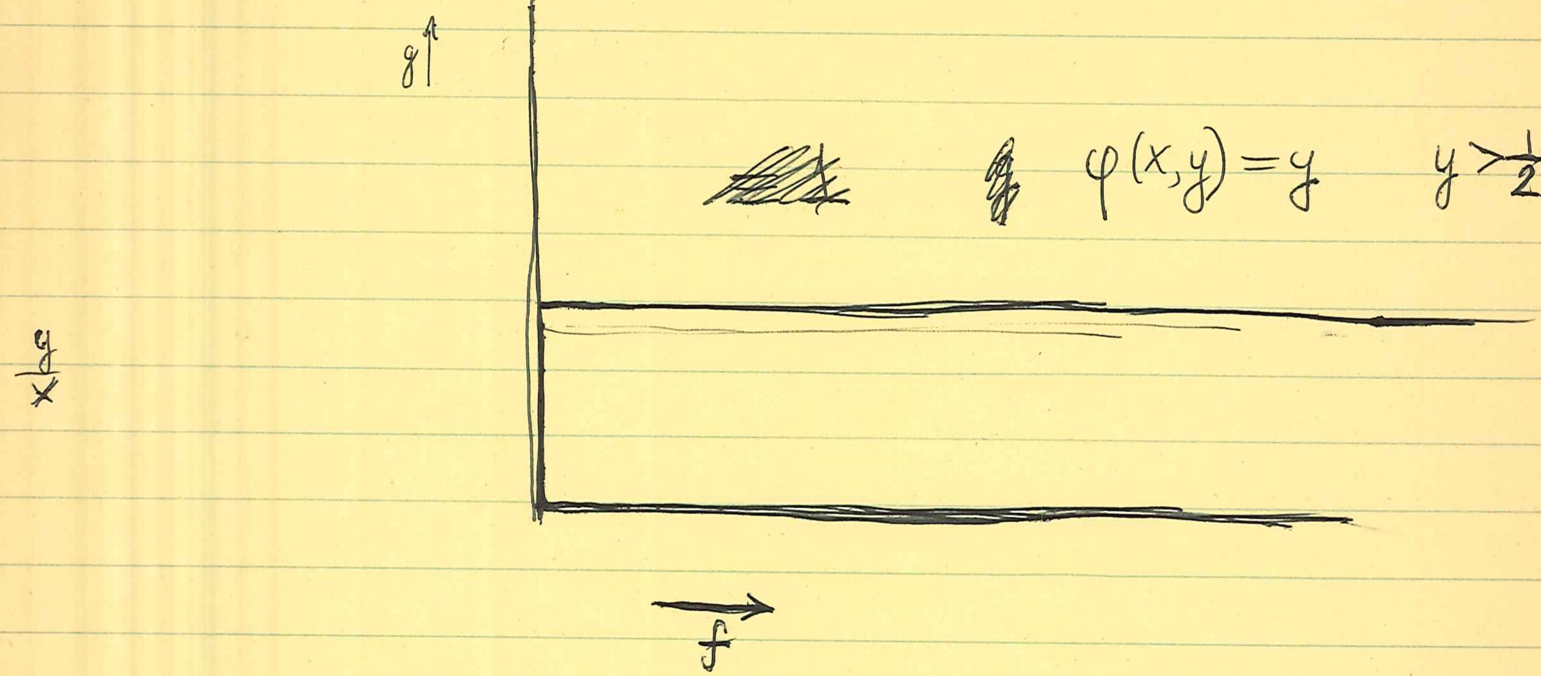
$W \longrightarrow X \times \mathbb{R}$

$\bar{\omega} \longmapsto (p(\bar{\omega}), h(\bar{\omega}))$

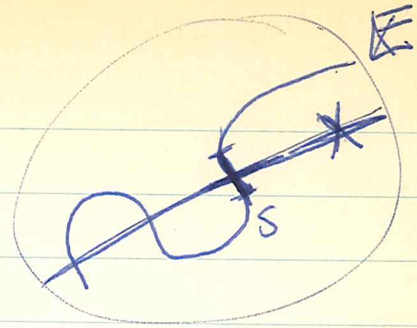
proper. indeed given $\bar{\omega}_n$ ~~[illegible]~~ $\ni$ $h(\bar{\omega}_n)$, $p(\bar{\omega}_n)$ converge choose a regular value of $\omega \mapsto \frac{g(\omega)}{f(\omega)}$

close to zero.

Choose a C^∞ fn on $\{(x, y) \mid x > 0,\ y \in \mathbb{R}\}$



$$
\begin{tikzcd}
\mathbb{P}(E|Y) \arrow[d] \arrow[dr, dash] & & \mathbb{P}E \arrow[d] \arrow[dr] & \\
Y \arrow[ddr, dash] & \tilde{X} \arrow[r] \arrow[dd, "f'"] & \mathcal{O}(-1) \arrow[r] \arrow[dd, "f"] & \mathbb{P}E \arrow[dd, "g"] \\
 & & X \arrow[d, "i"] & \\
 & X \arrow[r, "s"] & E \arrow[r] & X
\end{tikzcd}
$$



s transversal to zero

Then f' is a diffeomorphism over $s^{-1}\{0\}$, and over $\tilde{X}$ have that f^*E has a ~~section~~ subline bundle

Hironaka-Kleiman: E v.b. $/X$ then by blowing up submanifolds of X one obtains $\tilde{X}' \xrightarrow{f} X \ni f^*E$ is a sum of line bundles.

---

need to see why $\tilde{X}$ = blow up along Y.

and this is related to blow-up being independent

<u>Lemma</u>: If $Y \subset X$ submanifold and

$f: Z \to X$ transversal to Y

then
$$
\begin{tikzcd}
\tilde{Z} \arrow[r] \arrow[d] & \tilde{X} \arrow[d] \\
Z \arrow[r] & X
\end{tikzcd}
$$

is transversal cartesian.

$\boxed{\text{blowup}}$:

$$\begin{array}{ccc} E & & \mathbb{P}E \\ \downarrow & \text{v.b.} & \downarrow g \\ X & & X \end{array}$$

$$\underset{\mathbb{P}E}{\mathcal{O}}(-1) = \{(\ell, v) \mid \ell \text{ line in } E_{g(\ell)},\ v \in \ell\}.$$

$$\mathcal{O}(-1) \longrightarrow E$$

$$(\ell, v) \longmapsto v$$

diffeom over zero section

$$\begin{tikzcd} \mathbb{P}E \arrow[r, "j"] \arrow[d, "g"'] & \mathcal{O}(-1) \arrow[d, "f"] & X-E \arrow[l, hook'] \arrow[d, equal] \\ X \arrow[r, "i"] & E & X-E \arrow[l, hook'] \end{tikzcd}$$

So now given a closed submanifold $Y \subset X$, then

$$\tilde{X} = \tilde{U} \underset{U-Y}{\cup} X-Y$$

where $\tilde{U}$ is a tubular neighborhood of Y. Independent of choices.

$$\begin{tikzcd} & \tilde{X} \arrow[d] \\ Z \arrow[ur, dashed] \arrow[r, "\alpha"] & X \end{tikzcd}$$

~~at each point z with $\alpha(z) \in Y$ what z specifies a tangent line~~

~~I_Y sheaf of fns. vanishing along Y~~

~~then $\alpha^* I_Y \longrightarrow \mathcal{L} \longrightarrow 0$~~

Some basic geometry. If

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is an exact sequence of vector bundles, then

$$\begin{tikzcd}
\mathbb{P}E' \arrow[r, "i"] \arrow[dr, "f'"'] & \mathbb{P}E \arrow[d, "f"] & \mathbb{P}E - \mathbb{P}E' \arrow[l] \arrow[d, "\text{affine bundle for } \mathcal{O}(1)\otimes f''^{*}E'"] \\
& X & \mathbb{P}E'' \arrow[l, "f''"']
\end{tikzcd}$$

and there is a transversal cartesian square

$$\begin{tikzcd}
\mathbb{P}E' \arrow[r, "i"] \arrow[d, "i"] & \mathbb{P}E \arrow[d, "s"] \\
\mathbb{P}E \arrow[r, "0"] & \mathcal{O}(1)\otimes f^{*}E'
\end{tikzcd}$$

The point is that there is a canonical section s of $\mathcal{O}(1)\otimes f^{*}E''$ which associates to a line $l \subset E_x$, the map $l \longrightarrow E_x \longrightarrow E''_x$ viewed as an element of $(\mathcal{O}(1)\otimes f^{*}E'')_l$. This section is transversal to zero and vanishes on $\mathbb{P}E'$.

$$\boxed{i_* 1 = e(\mathcal{O}(1)\otimes f^{*}E')}$$

So the situation is that there is an exact sequence

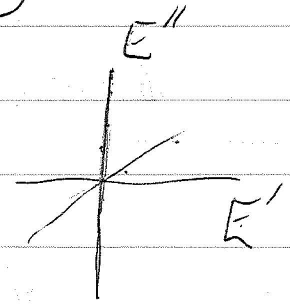
$$0 \longrightarrow U^*(\mathbb{P}E') \xrightarrow{i_*} U^*(\mathbb{P}E) \longrightarrow U^*(\mathbb{P}E'') \longrightarrow 0$$

and one can work out the algebra structure.

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

exact sequence o

$$E = E' \oplus E'' \quad \text{over } X$$



$$\begin{tikzcd}
\mathbb{P}E' \arrow[r, "i"] & \mathbb{P}E \arrow[d] & \underline{\mathrm{Hom}}(\mathcal{O}(-1)_{\mathbb{P}W}, g^*E') \arrow[l, "j"'] \arrow[d] \\
& X & \mathbb{P}\cancel{W}E'' \arrow[l, "g"]
\end{tikzcd}$$

get straight.

$$\begin{tikzcd}
\mathbb{P}E' \arrow[r, "\iota"] \arrow[dr, "f'"'] & \mathbb{P}E \arrow[d, "f"] & U \arrow[l, "j"'] \arrow[d] & \simeq \underline{\mathrm{Hom}}(\mathcal{O}_{\mathbb{P}E'}(-1), f''^*E') \\
& X & \mathbb{P}E'' \arrow[l, "f''"] \arrow[ul] \arrow[u, bend right]
\end{tikzcd}$$

Claim

$$U \simeq \mathcal{O}_{\mathbb{P}E'}(1) \otimes (f'')^* E'$$

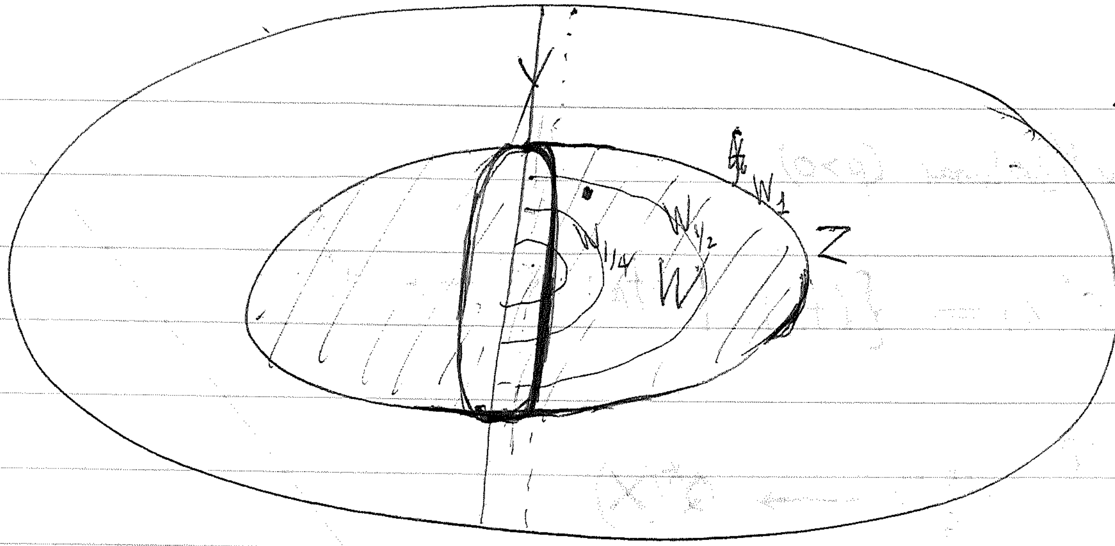
a line $\ell \subset E_x$ not in E'_x

projects into E''_x and $p(\ell) \in \mathbb{P}(E''_x)$

and then ℓ

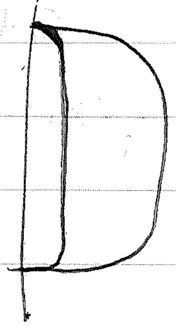
ℓ $\quad E_x = E'_x \oplus E''_x$

$\ell \quad p(\ell)$



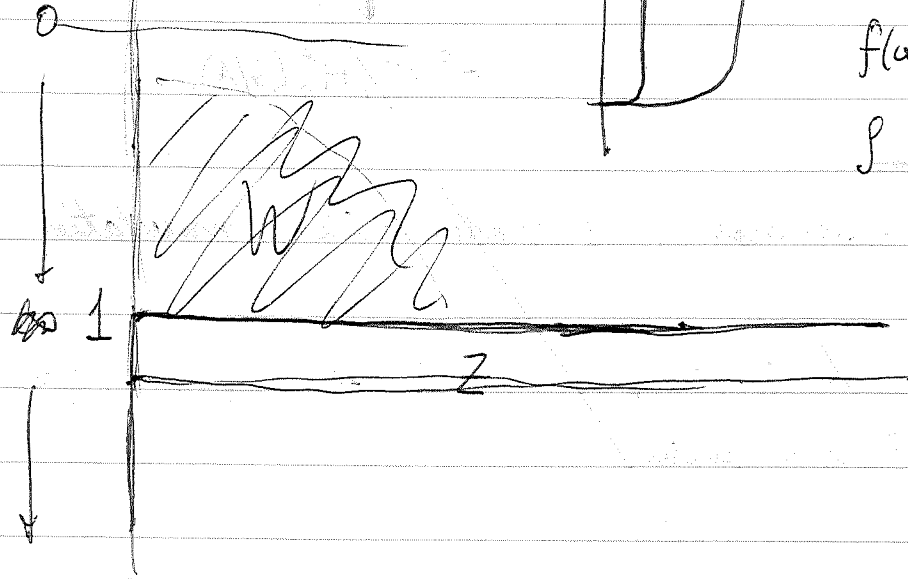
$\rho(Y)=1$
$\rho(X-N)=0$

$0 \leq f \leq 1$
$0 \leq \rho \leq 1$

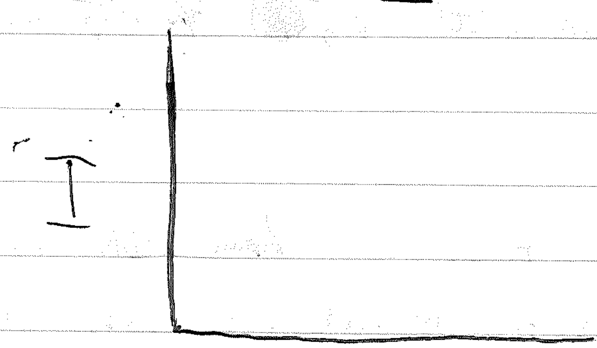


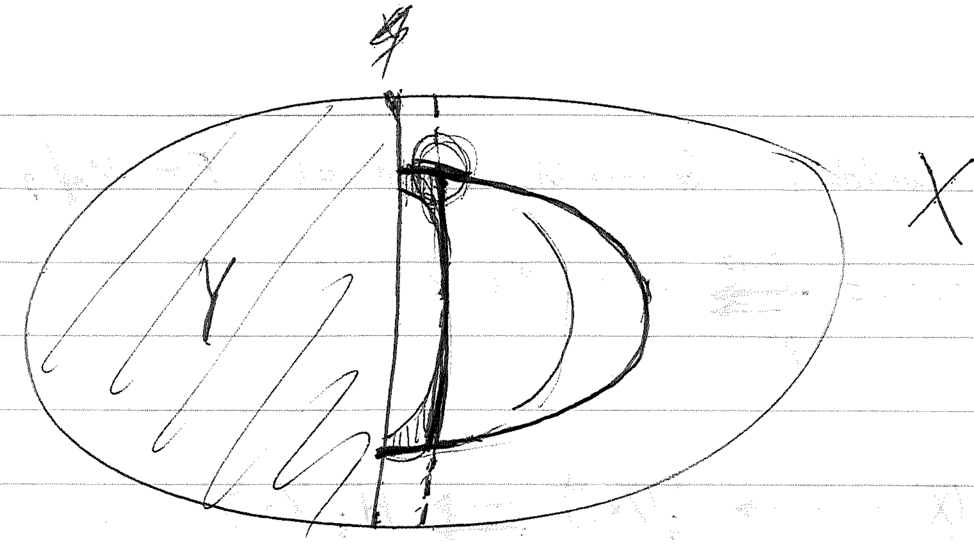
$\underline{1 = f(\omega)\,\rho(\omega,\pi^{-1}y)}$

then clearly have
$f(\omega) = 1$
$\rho = 0.$

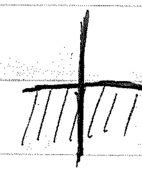


Consider $W \times I$





$$\partial(Z\cap N) = Z\cap\partial N$$



$$W \longrightarrow U\times\mathbb{R}$$

want $\tilde{W} \longrightarrow X\times\mathbb{R}$

new function

along the intersection take product nbd.

$$Z\cap N\times I$$

$$Z_- \cup_{Z\cap\partial N} W\cap\partial N$$

~~Hiller~~

We basically understand ~~Hiller~~ Todd classes for X/G, namely

$$\begin{array}{ccccc} K(X/G) & \longrightarrow & K_G(X) & \xrightarrow{\int_X} & R_G(pt) \\ \Big\downarrow{\scriptstyle ch} & & & & \Big\downarrow{\scriptstyle \int_G} \\ H^{ev}(X/G) & & \xrightarrow[\langle \mathrm{Todd}(X/G),\, ch\, x\rangle]{} & & \mathbb{Z} \end{array}$$

reason worked is because

$$E \simeq (f_* f^* E)^G \qquad E \text{ free over } X/G$$

so

$$\begin{aligned} H^*(X/G, E) &= H^*(X/G, (f_* f^*E)^G) \\ &= H^*(X/G, f_* f^*E)^G \qquad \text{comp. red.} \\ &= H^*(X, f^*E)^G \qquad f \text{ finite} \end{aligned}$$

Is there a similar interpretation for whitney classes.

X manifold, then basic formula is

$$\chi(X) = \int_X w(\tau_X) \qquad \text{mod } 2.$$

and more generally if $i: Y \subset X$ is an embedding

$$\chi(Y) = \int_X i_*(w(\tau_Y)) \qquad \tau_Y \to i^*\tau_X \to \nu_i$$

$$i_*$$

$$= \int_X \left(i_* \{w^{-1}(\nu_i)\}\right) w(\tau_X)$$

Outline of book on cobordism theory

1. manifolds
   category Man
   products
   example of Grassmannian
2. vector bundles, homomorphisms over f, exact sequences.
   direct sum, inverse image
   algebraic operations $\otimes$
   can. bundles over Grassmannian. & universal property.

3. tangent bundle, df,
   tangent bundle of Grassmannian
4. implicit function theorem - local description of immersions & submersions.
   transversal maps + existence of fibred product

5. theorem of Sard-Brown

6. Whitney-type theorems
   existence of C^∞ functions
   sections for v.b.
   embedding & immersion theorems

7. tubular neighborhood thm.

8. transversality theorem.

§. Transversality theorem:

Lemma: Given a transversal cartesian square, & a map h

$$\begin{tikzcd} & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Z \arrow[r, "h"] & Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

then h is transversal to f' iff gh is transversal to f.

Proof: Given $z \in Z$, $x' \in X'$ $\ni$ $h(z) = f'(x') = y'$, have

$$\begin{tikzcd} & T_{x'}(X') \arrow[r] \arrow[d] & T_x(X) \arrow[d] \\ T_z(Z) \arrow[r] & T_{y'}(Y') \arrow[r] \arrow[d] & T_y(Y) \arrow[d] \\ & C_1 \arrow[r, "\sim"] & C_2 \arrow[d] \\ & 0 & 0 \end{tikzcd}$$

so $T_z(Z) \to C_1$ is surjective $\iff$ $T_z(Z) \to C_2$ surj.

<u>transversality lemma:</u> Given

$$\begin{tikzcd} & Z \arrow[d, "f"] \\ X \times S \arrow[r, "h"] & Y \end{tikzcd}$$

such that h and f are transversal, then for almost all $s \in S$, $h_s : X \longrightarrow Y$, $h_s(x) = h(x, s)$, is transveral to f.

<u>Proof:</u> Form W

$$\begin{tikzcd} & W \arrow[r] \arrow[d] & Z \arrow[d, "f"] \\ X \arrow[r, "\iota_s"] \arrow[d] & X \times S \arrow[r, "h"] \arrow[d, "p_2"] & Y \\ pt \arrow[r, "\varepsilon_s"] & S & \end{tikzcd}$$

By Sard for almost all s, ~~$pt \to S$ is a transversal~~ $pt \longrightarrow S$ and $W \longrightarrow S$ are transversal. As ~~[illegible]~~ ε_s and p_2 are trans. $\overset{\text{lemma}}{\Longrightarrow}$ ι_s transversal to $W \longrightarrow X \times S$. But h, f transversal $\overset{\text{lemma}}{\Longrightarrow}$ $h_s = h\iota_s$ trans. to f.

Thom's transversality theorem: Given

$$\begin{array}{ccc} & & Z \\ & & \downarrow g \\ X & \xrightarrow[f]{} & Y \end{array}$$

and a closed subset $A \subset X$ such that f and g are transversal over A (transversality hold at each (x, z), $f(x) = g(z)$ with $x \in A$). Then f^{ε} is homotopic to f' where f' trans. to g and $f' = f$ on A.

Proof: If we factor f: $X \xrightarrow{i} U \xrightarrow{p} Y$ where p is transversal to g, then its enough to make i transversal to $g': U \times_Y Z \longrightarrow U$. Hence take the factorization $X \longrightarrow X \times Y \longrightarrow Y$ given by graph can suppose f (closed) embedding. Next we can factor $X \xrightarrow{i} U \xrightarrow{p} Y$ where U is a open tubular nbd of X in Y; p being a submersion we may therefore suppose Y is a v.b. over X and f is the zero section.

Let $S \subset \Gamma(Y \to X)$ be a finite-diml. sub-space of sections which span the fibre over each point. Let $\rho: X \to \mathbb{R}$ be a C^∞ fn. $\ni$ $\rho^{-1}(0) = A$. Then take

$$h: X \times S \longrightarrow Y \qquad \text{map of v.b.}$$
$$(x, s) \longmapsto \rho(x) s(x)$$

Claim h transversal to g. OKAY at points (x,s) $x \in A$ since already $T_x(X)$ goes onto the cokernel of dg. At points $x \notin A$, $\rho(x) \neq 0$, so h is a surjective map of v.b. over X, hence h is a submersion, so OKAY.

By transversality lemma $\exists\, s \ni f' = h_s$ tranversal to g and then clearly $f' = f$ on A $(\rho(A) = 0)$ and f' homotopic to f (S vector space).

We have used

tubular nbd. thm.,

existence of fibred products transversally

existence of finite diml. space of sections of a v.b. (hence need bdd. dimension) __transversality lemma:__ Given $Z \xrightarrow{f} X$ and a family $h:$ $Y \times S \longrightarrow X$ of maps from Y to X parameterized by S. If h is transversal to f, then for almost all $s \in S$, h_s is transversal to f.

__Proof:__

$$\begin{tikzcd}
W_s \arrow[r] \arrow[d] \arrow[dr, phantom, "3"] & W \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & Z \arrow[d, "f"] \\
Y \arrow[r, "\iota_s"] \arrow[d] \arrow[dr, phantom, "2"] & Y \times S \arrow[r, "h"'] \arrow[d] & X \\
pt \arrow[r, "\varepsilon_s"'] & S &
\end{tikzcd}$$

h + f trans $\Longrightarrow$ can form ①. For allmost all s (Sard-Brown) ε_s is trans. to $W \longrightarrow S$, ~~the~~ + can form W_s. As ② trans cart + ~~...~~ ② + ③ is $\Longrightarrow$ ③ is tr. cart $\Longrightarrow$ (as ① is) ③ + ① is tr cart, i.e. h_s trans to f.

__trans. thm:__ Suppose given $Z \xrightarrow{f} X$ and $g: Y \to X$ and F closed $\subset$ Y $\ni$ g trans to f on F. Then $g \sim g_0$ $\quad g = g_0$ on F $\quad g_0 \pitchfork f$.

first factor g into $Y \hookrightarrow X \times V \xrightarrow{\text{sub.}} X$.

$$\begin{tikzcd}
& Z \times V \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{tr cart.}"] & Z \arrow[d] \\
Y \arrow[r] & X \times V \arrow[r] & X
\end{tikzcd}$$

so can assume g embedding, in fact can take X to be a tubular nbd of ~~X~~, so can assume X is a vector bundle over Y & g is the inclusion of zero section. Now let W be a finite diml. space of generating sections for this vector bundle. ~~Let ...~~

tub. nbd. thm.

gen. sections

~~...~~ Choose $\rho : Y \longrightarrow [0,1] \subset \mathbb{R}$ $\ni \rho^{-1}(0) = F$ ~~$\rho(F) = 0$ ... $\rho(Y - U) = 1$~~ Then take h to be the map of v.b. over Y given by

$$h : Y \times W \longrightarrow X$$

$$(y, w) \longmapsto \rho(y) w(y)$$

note that if ~~...~~ $\rho(y) \neq 0$, then h is surjective, hence a submersion, so h transversal to f over $(Y - F) \times W$. But also trans. over $U \times W$. ~~So~~ take a generic element of W.

Lemma: If F closed $\subset X$, then $\exists$ C^∞ fn. $\rho : X \rightarrow [0,\infty) \subset \mathbb{R} \ni \rho^{-1}(0) = F$.

Proof: Local question so can assume X open in $\mathbb{R}^n$ so

KK.

to each $\omega \in W$ have $s_\omega \in \Gamma(X, E)$

$$s : W \longrightarrow \Gamma(X, E).$$

embedding theorem

X dim $\leq d$

$f : X \longrightarrow V$ embedding on F

dim $V \geq 2d+1$

Then f can be modified without change on F to be an embedding (closed).

$X = \bigcup K_n$

---

compact case first.

choose a covering $X = \bigcup^n U_i$ and a finer covering $\bigcup V_i$, $\bar{V}_i \subset U_i$. and ~~a~~ maps

$$f_i : X \longrightarrow \mathbb{R}^n$$

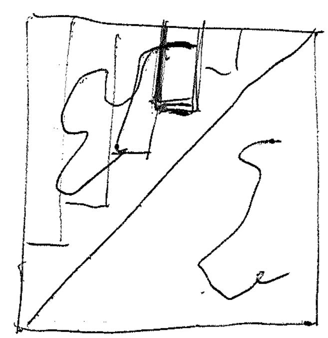
which are embeddings on V_i. Then

$$\prod f_i : X \longrightarrow V$$

is an embedding (roughly).

trivial

now want ~~line field~~



non-compact case:

$$\begin{array}{ccc} f : F & \longrightarrow & f(F) \quad \text{homeo} \\ \cap \text{ closed} & & \cap \text{ closed} \\ X & \longrightarrow & V \end{array}$$

f immersion on F

hence f immersion in a nbd U of F.

$U \longrightarrow$

course:

basic prerequisites + notation

implicit fn. thm. to prove fibre product exists

tubular nbd. thm.

~~any~~ embedding thm. + any bundle has a finite family of sections which generate.

openness of transversality

~~Proof~~ <u>Remark</u>: The above proof shows that ~~the~~ if
~~~~ $\mathcal{T} \hookrightarrow \Gamma(X,E)$ is a subspace of sections
such that

(i) given F can find enough t vanishing on F
to span E over $X-F$

(ii) given t_n $\varinjlim t_n$ exists in $\mathcal{T}$.

Then can take $W \subset T$.

<u>Example</u>: $T = \Gamma(X, \theta) \xrightarrow{d} \Gamma(X, T)$

hence get

Any <u>X^n immerses in $\mathbb{R}^{2n}$.</u>
~~~~

X C^∞ manif countable dimension n.
construct a proper fn. $f: X \to \cancel{\mathbb{R}}.[0,\infty)$
assume $\mathbb{N}$ regular values.

To prove X embeds in $\mathbb{R}^{2n+1}$.
Idea is that $f^{-1}(k) = X_k$ embeds in $\mathbb{R}^{2n-1}$ and that we can join the embeddings in $\mathbb{R}^{2n+1}$ maybe even $\mathbb{R}^{2n}$

Check dimensions: Thus ~~$\mathbb{R}$~~ given ~~$f^{-1}[k,k+1$~~

$$\begin{tikzcd}
f^{-1}(k) \cup f^{-1}(k+1) \arrow[r, hook, "h"] \arrow[d] & \mathbb{R}^{2n-1} \times \{0,1\} \arrow[d] \\
f^{-1}[k,k+1] \arrow[r, "\bar{h} \leftarrow \text{average}"] & \mathbb{R}^{2n-1} \times [0,1]
\end{tikzcd}$$

can be extended to $f^{-1}[k,k+1]$ in $\mathbb{R}^N$. more precisely, ~~there is a ...~~ we can find a map

$$\varphi: f^{-1}[k,k+1] \longrightarrow W = \mathbb{R}^m$$

vanishing on the ends such that

$$f^{-1}[k,k+1] \xrightarrow{(\varphi, \bar{h})} W \times \mathbb{R}^{2n-1} \times [0,1]$$

is an embedding.

Now I seek a projection from $W \times \mathbb{R}^{2n-1} \times [0,1] \longrightarrow \mathbb{R}^{2n-1} \times [0,1]$ keeping things fixed at the ends.

<u>Prelimaries</u>

differentiable manifold + tangent space + morphism
example of Grassmanian Gauss map for an immersion

~~[illegible]~~ inverse fn. theorem., ~~[illegible]~~ local description of immersions, submersions.

Whitney theorem + stability for vector bundles.

tubular neighborhood theorem

transversality
  existence of fibred product
  • transversality theorem.

---

cobordism (unoriented). the definition requires only the transversality theorem.
Review proof that $(g : Y \longrightarrow X)$ ~~$g^*(Z \xrightarrow{f} X)$~~

$$[f : Z \longrightarrow X] \in N^q(X)$$

$$g^*[f : Z \longrightarrow X] \in N^q(Y)$$

to show depends only on the cobordism class.

So you must show that first independent of motion of g. So if g_0, g_1 are homotopic to g + both transversal to f we can ~~make them~~ join them by a smooth map $h : Y \times \mathbb{R} \longrightarrow X$ $h(y,0) = g_0(y)$
$h(y,1) = g_1(y)$

so we must make h trans. to

Whitney theorem for vector bundles: Let E be a vector bundle of rank $\leq r$ over a manifold X of dimension $\leq d$. ~~which generates over the open subset U~~ Suppose W is a f.d. v.s. and $\varphi : X \times W \longrightarrow E$ is a hom. of vector bundles which is surjective over the closed set F. ~~Let F be a closed subset of U.~~ If $\dim(W) \geq d+r$ (= total dim E) then φ may be modified without being changed over F so as to be surjective on all of X.

~~Proof: Let $X = \bigcup K_n$, K_n compact $K_n \subset \operatorname{Int} K_{n+1}$ be an exhaustion. It is enough to prove that given a compact K it~~

~~Proof: It is enough to show that φ can be modified so as to be surjective on $F \cup K$ where K is any comp~~

Proof: It is enough to show that given any compact set K, φ can be modified without change over F so that φ is surjective over $F \cup K$. Indeed if $X = \bigcup K_n$ is an exhaustion, then one constructs φ_n surj on $F \cup K_n$ $\quad \varphi_n | F \cup K_{n-1} = \varphi_{n-1} | F \cup K_{n-1}$ and puts $\varphi = \lim \varphi_n$.

Let U be open $\ni$ φ surjective, and covers $F \cup K$ by U and open sets where E is trivial not meeting F. Then one can find a map $X \times V \longrightarrow E$ surjective on $K - U$ and zero on F. Then have $X \times (W \oplus V) \longrightarrow E$ surjective on $F \cup K$.

Claim $\exists \quad W' \subset W \oplus V \ni \dim W' = \dim W$ + $W' \cap V = 0$ such that $X \times W' \longrightarrow E$ is surj over $F \cup K$. Let U' be the open subset on which $X \times (W \oplus V) \longrightarrow E$ surjective. Then set Z of W' is a manifold of dim

$$(1) \qquad (\dim W)(\dim V).$$

Let Z' be the manifold whose points are triples (x, H, W') where $x \in U'$, $H \subset E(x)$ is a hyperplane and W' is a plane in $Z \ni$ the image of $W' \longrightarrow W \oplus V \longrightarrow E$ is contained in H. Then $\dim Z'$ is

$$(2) \qquad \dim X + (\operatorname{rang} E - 1) + (\dim W)(\dim V - 1)$$

and the bad set of W' is the image of the map $Z \to Z'$ $(x, H, W') \longmapsto W'$. Thus not all W' are bad if $(2) < (1)$ i.e.

$$\dim X + \operatorname{rank}(E) \leq \dim W$$

which is the hypothesis.

Clear that the required modification is the composite ~~Remark / The above proof~~

$$W \xleftarrow{\sim} W' \longrightarrow W \oplus V \longrightarrow \Gamma(X, E)$$

Notes for lecture1 Goal: to set up the appropriate machinery about the category of manifolds so that the determination of the ~~xxxxxx~~ unoriented cobordism ting can be carried through.

Definition of differentiable manifold as a space with a countable basis for the open sets endowed with a atlas of differentiable charts. Morphism of differentiable mans.

Tangent bundle, df, immersion, submmersion, etale map , submanifold (= subspace Y of X with a differentiable manifold structure on Y such that the inclusion i: Y X is an immersion)

Implicit function theorem (An etale map is a local isomorphism). Applications: description of submersions and immersions locally.

Transversality! Defn: maps f and g are transversal. existence of the fibred product.

$$\begin{array}{ccc} X' & & X \\ & & \\ Y' & & Y \end{array}$$

sorites on transversal cartesian squares. Thom transversality theorem in Abraham form.

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before the above perhaps should put ~~proofxof~~ discussion of vector bundles, the Whitney type theorems, tubular neighborhood theorem

Proof of the Whitney theorems: Uses weak Sard-Brown theorem that if f:X Y is a smooth map and dim X is less than dim Y, then f(X) is of measure zero. which has a simple proof based on the Jacobian estimate.

Now suppose E is a vector bundle over X compact. Let V be a finite dimensional space of generating sections; V exists bb compactness. Then we search for a subspace of V of dimension dim X + dim E =+ 1 which still generates. Let dim X = d, dime E = r and dim V = N. Then the ~~xxx~~ set of W of dim w is a manifold of dim w(N-w). The bad W are the image of the Grassman bundle of w-1 palnes in ~~xxxxxxxxx~~ V/L where L is a line in the kernel of V E(x) for some x. Then the dimension of this is

~~dimxXxxxdimx(PE)~~

dim X + dim of lines in V-- E(x) + dim of W/L in V/L

d + N-r-1 +(N-w)(w-1) ?less than? (N-w)w

~~thus need d-r-1 less than w XX or w at least d~~ d+w -r-1 $\angle$ or w$\angle$ d-r

So there is little

Remarks on course

definition of differentiable manifold by charts (U, L, ϕ).
category of differentiable manifolds.

tangent bundle.

Grassmannian. $\dim V = N$

$$Gr_q(V) = \{ A \subset V \mid \dim A = q \}.$$

$$B \in Gr_{N-q}(V) \qquad U_B = \{ A \in Gr_q(V) \mid A \oplus B = V \}.$$

$$\phi_{AB} : U_B \xrightarrow{\sim} \mathrm{Hom}(A, B)$$

$$A' \longmapsto u \qquad\qquad A' = \{ a + u(a) \mid u(a) \in B \}.$$

Claim that given $A \oplus B = V = A' \oplus B'$. Then have to show

$$\phi_{AB}(U_B \cap U_{B'}) \quad \text{open in} \quad \mathrm{Hom}(A, B)$$

$$\phi_{A'B'}(U_B \cap U_{B'}) \quad \text{"} \quad \text{"} \quad \mathrm{Hom}(A', B')$$

and that

$$\phi_{A'B'} \circ \phi_{AB}^{-1} : \phi_{AB}(U_B \cap U_{B'}) \longrightarrow \phi_{A'B'}(U_B \cap U_{B'})$$

is a C^∞-map.

First step: from A, B to A', B. Then from A', B to A, B'.

Whitney embedding thm. $f: X \longrightarrow \mathbb{R}^{2n+1}$ proper, then arbitrarily close to f lies an embedding.

suppose f embedding over a closed set F
i.e. $df : T(X)|F$ is injective and $f: F \longrightarrow f(F)$ homeo

suppose $K \subset X$ modify f near K.

~~Wwe~~ choose functions $h: \cancel{X} \longrightarrow V$ zero on K_ε
embedding on

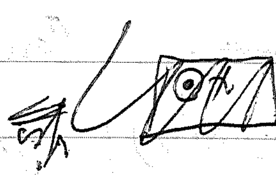
$$1 - \begin{bmatrix} n \\ 1 \end{bmatrix} + p \begin{bmatrix} n \\ 2 \end{bmatrix} - \cdots$$

Tits theory

$$\sum \frac{q^n - 1 \cdots q - 1}{q^i - 1 \cdots q^j - 1 \cdots}$$

$$\frac{q^3 - q \quad q^2 - 1}{q - 1}$$

$$\frac{\begin{matrix} q^2 + q + 1 \\ q^2 + 1 \end{matrix}}{q^3 + q + q^4 + 2q^2 + 1}$$



constant situation

constant orbit types

$G \times_N X^H = X$

Define: immersion, submersion, submanifold

Whitney embedding thm: Let $f: X \longrightarrow V$ be a proper map which is a 1-1 immersion on a closed set F. Then if $\dim V \geq 2(\dim X)+1$, can modify f outside of F so as to be an embedding.

Proof: Enough to show that can enlarge from F to $F \cup K$, where K is compact. Do it this way: choose an exhaustion K_n' of V, and let $K_n = f^{-1}K_n'$ an exhaustion of X. Now we shall ~~modify f so as to be an embedding on $F \cup K_n$~~ construct $f_n \ni f_n = f_{n-1}$ on $K_{n-1} \cup F$ and $\ni$ f_n 1-1 immersion on $K_n \cup F$. Then $f = f_n$?

Point is to keep f_n proper, $f_n = f_{n-1}$ on $K_{n-1} \cup F$ f_n 1-1 immersion on $K_n \cup F$. Also you want to have that f_n carries the complement of K_{n+1} into complement of K_{n+1}'.

$K_n = f^{-1}K_n'$ Is it possible to keep f same outs

$$f(K_n) \subset K_n' \subset \mathrm{Int}(K_{n+1}')$$

modification will be such that

$$\bar{f}(K_n) \subset K_{n+1}'.$$

Assume K_n' chosen $K_n \ni f(K_n) \subset \mathrm{Int}\, K_n'$.

construct $f_n \ni \begin{cases} f_n = f & \text{outside of } K_n \cdot F \cup (X - K_n) \\ f_n = f_{n-1} & \text{on } K_{n-1} \cup F \end{cases}$

topological transversality still unproved

$\Rightarrow$ existence of missing top 4-manifold almost parallelizable of index 8, since $\pi_4(G/PL) \to \pi_4$

$\Rightarrow$ annulus conjecture in all dimensions.

~~Top spaces~~ G top. gp. hence gives a group in the gross topos of all top. spaces + etale maps. It also gives a group in the small topos of etale spaces over a pt e.g. sets. Should be a map

$$\text{Gross topos}/pt \xrightarrow{f} \text{Sets}$$

$$G \longmapsto f_*(G) = G_d \text{ regarded as a discrete gp.}$$

Now maybe you get comparison between BG & BG_d?

§ 9. Example: the symmetric group

Let ~~$\Sigma(n)$~~ S_n be the symmetric group on n-letters

dimension, minimal primes,

Infinite symmetric group? $\quad \Sigma_\infty = \varinjlim S_n$

$$S_k \times S_\ell \longrightarrow S_{k+\ell}$$

$$H_*(BS_k) \otimes H_*(BS_\ell) \longrightarrow H_*(BS_{k+\ell})$$

$H_*(\infty) = \varinjlim H_*(k)$ is a Hopf algebra by Nakioka

---

Take $\overset{\text{real}}{}$ regular representation of Σ_n call it ρ_n. Then ~~accord~~ clearly $w_t(\rho_n)|\Sigma_{n-1} = w_t(\rho_{n-1})$, and according to Milgram these Whitney classes generate $H^*(S_n)$

<u>variations on the Whitney classes</u>: Let k be a ramified discrete valuation field. and G be a group. Then characters

$$G \longrightarrow k^*$$

systematically relate

Basic fact geometrically is that

$$H_i(S_n) \xrightarrow{\sim} H^{dn-i}((S^d)^n/S_n)$$

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & W \arrow[r, "w \mapsto \lambda"] \arrow[d, "w \otimes p^{\nu}"'] & k \arrow[r] \arrow[d, "id"] & 0 \\
K_V \arrow[r] & S_{p^\nu} W \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

$$f(g) = g\lambda - \lambda$$

question is whether $f(g)^{p^\nu}$ might be a boundary?

$$f(g)^{p^\nu} = (g\lambda)^{p^\nu} - \lambda^{p^\nu} = g^{\nu}\sigma^{p^\nu} - \sigma^{p^\nu}$$

not too encouraging.

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but maybe in $S_{p^\nu} V$ it becomes a boundary. i.e.

$$\prod_{g \in G} g\lambda - \lambda$$

$$\prod_{g \in G} g\lambda \qquad \text{basic invariant}$$

---

yes $\prod_{g \in G} (g\lambda - \lambda)$ $\qquad \prod_{g \in G} f(g)$

~~consider the~~

Steenrod alg

f invariant polynomial under G

$$f(t_0 v + t_1 v^g + \cdots + t_h v^{g^h})$$

take v to be a generic vector of V_Ω

so that $\quad t_0 v + t_1 v^g + \cdots + t_{h-1} v^{g^{h-1}}$

give every vector

G commutes with $\mathcal{a}$

So I want to form a ring

$$\dot{\prod_{k\geq 1}} H^{2kq}(B\Sigma_k) = \left\{ (a_k)_{k\geq 1} \;\middle|\; \operatorname{ind}_{\Sigma_k\times\Sigma_e}^{\Sigma_{k+e}} a_k \otimes a_e = a_{k+e} \right\}$$

and the point is to describe homomorphisms of this ring into another. ~~Next idea might be to describe the basic product map!!~~

Example: K theory $\quad R(\Sigma_k)$ has basis π_α induced from trivial representation of $\Sigma_{\alpha_1}\times\cdots\times\Sigma_{\alpha_n}$.