March 17, 1969

Cobordism theory with supports.

We consider the category whose objects are pairs (X, U) X a C^∞ manifold of bdd. dimension, U an open subset of X and whose morphisms from (X, U) to (X', U') are C^∞-maps $f: X \to X'$ such that $fU \subset U'$. We consider contravariant functors F to (sets) on this category endowed with a Gysin homomorphism

$$f_* : F(X,U) \longrightarrow F(Y,V) \qquad (Y-V) = f(X-U)$$

for any proper-oriented map $f: X \to Y$, subject to the following axioms; where we write $F_{X-U}(X)$ instead of $F(X,U)$:

1. $g_* f_* = (gf)_*$ $\quad id_* = id$

(Base change axiom)

2. Given

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

U open in X, $A = X-U$

U' open in X', $A' = X'-U$

$f^{-1}U \subset U'$

diagram transversal cartesian

then

$$\begin{tikzcd} F_A(X) \arrow[r, "g'^*"] \arrow[d, "f_*"'] & F_{A'}(X') \arrow[d, "f'_*"] \\ F_{fA}(Y) \arrow[r, "g^*"] & F_{f'A'}(Y') \end{tikzcd}$$

commutes

3. $\underbrace{\text{If}}_{}$ (Excision Axiom) If A is closed in X, U is open in X, $A \subset U$, and $i: U \to X$ is the inclusion, then

$$i^* : F_A(X) \xrightarrow{\sim} F_A(U)$$

(Homotopy axiom)

4. If $f: (X, U) \longrightarrow (X', U')$ is a map such that $X \to X'$ and $U \to U'$ are homotopy equivalences, then

$$f^*: F(X', U') \xrightarrow{\sim} F(X, U).$$

<u>Proposition:</u> There is a ~~[illegible]~~ Gysin functor $\Omega(X, U)$ as above ~~[illegible]~~ endowed with an element $1 \in \Omega(pt, \phi)$ such that for any other (Gysin) functor F and $\alpha \in F(pt)$, there is a unique homomorphism of Gysin functors $\Omega \longrightarrow F$ sending 1 to α. ~~[illegible]~~

<u>Proof:</u> Definition of $\Omega(X, U)$:

March 19, 1969

Typical group laws:

Proposition: Let G be a formal group of dimension 1 over R and let c_0 be a typical coordinate. Then any other typical curve c in G may be uniquely written

$$c = \left(\sum_{n=0}^{\infty} V^n[a_n]\right) c_0 \qquad \text{with } a_n \in R.$$

Proof: $(V^n[a_n])\, c_0\, (t) = ([a_n]c_0)(t^{p^n}) = c_0(a_n t^{p^n}) = a_n t^{p^n} + \cdots$

It suffices to show that the degree of first non-zero coefficient of a typical curve is a power of p; in effect if $a_0, \cdots, a_{n-1}$ have been chosen so that $c - \sum_{i=0}^{n-1} V^i[a_i]\, c_0 \equiv 0$ mod degree $p^{n-1}+1$, then this curve will begin with $a_n t^{p^n}$ some $c_n \in R$ and we have to take $a_n = c_n$. Suppose therefore $c(t) = a t^n + \cdots$ typical and that $n = p^a k$ where $(p, k) = 1$, $k > 1$. Then

$$(F_k c)(t^k) = \sum_{\zeta^k=1}^{*} c(\zeta t) \equiv a\left(\sum_{\zeta^k=1} \zeta^n\right) t^n \qquad \text{mod deg } n+1$$

$$\equiv k a t^n \qquad "$$

But k is a unit and $F_k c = 0$ since c is typical, hence $a = 0$.

~~Corollary: If $\varphi(X)$ is a typical curve wrt typical formal group law with leading coefficient $a X^{p^h}$, then $\varphi(X)$ is a power series in X^{p^h}.~~

~~Proof:~~ $\varphi(X) \overset{\text{By preceding}}{=} \sum_{k=h}^{\infty *} (a_k X^{p^k}) = g(X^{p^h})$ ~~where~~

$g(Y) = \sum_{k=0}^{\infty *} a_{h+k} X^{p^k}$

~~Corollary: If $\varphi(X) = aX^{p^h} + \cdots$ is a typical curve wrt ~~to~~ a (not-necessarily typical) formal group law over a ring of characteristic p, then φ is a power series in X^{p^h}.~~

~~Proof: In effect by changing variable we know that $\exists$ power series $\xi(X) = X + \cdots$ with $\xi\{\varphi(X)\} = g(X^{p^h})$, hence $\varphi(X)$~~

Corollary 1: Let $F(X,Y)$ be a (not necessarily typical) group law over a $\mathbb{Z}_{(p)}$-algebra R. Then if $\varphi(X) = aX^{p^h} + \cdots$ is a typical curve, φ is a power series in X^{p^h}.

~~Proof: Let $\xi(X) = X + \cdots$ be a typical coordinate for F. Then by the preceding proposition~~

$$\varphi(X) = \sum_{n=0}^{\infty}{}^{*}\ \xi(a_n X^{p^n})$$

~~where the $a_n \in R$ are uniquely determined. Hence $a_0, \ldots, a_{h-1} = 0$ and $a_h = a$. So~~

$$\varphi(X) = \sum_{n\geq h}^{\infty}{}^{*}\ \xi(a_n \underset{\substack{\| \\ a_n (X^{p^h})^{p^{n-h}}}}{X^{p^{h+n-h}}})$$

$$= \sum_{m\geq 0}^{\infty}{}^{*}\ \xi(a_{h+m} (X^{p^h})^{p^m}) = g(X^{p^h})$$

~~where~~ $g(Y) = \sum_{m\geq 0}^{\infty}{}^{*}\ \xi(a_{h+m} Y^{p^m})$. ~~QED~~

Actually you are being stupid since

$$c = \sum_{n\geq h}^{\infty} V^n[a_n]\cdot c_0 = \underline{V^h}\cdot \underbrace{\sum_{n\geq h}^{\infty} V^{n-h}[a_n]\cdot c_0}_{g}$$

i.e. $\underline{c(X) = g(X^{p^h})}$.

<u>Corollary 2</u>: If $\varphi(X)$ is an endomorphism of a law $F(X,Y)$ over a ring of char p, then $\varphi(X) = g(X^{p^h})$ with $g'(0) \neq 0$ for some integer $h \geq 0$.

<u>Proof</u>: Let $\xi(X) = X + \cdots$ be change of coordinates such that $\xi * F$ is typical. Then $\psi = \xi * \varphi = \xi\circ\varphi\circ\xi^{-1}$ is an endo of $\xi * F$. This implies that ψ is typical wrt $\xi * F$ since

$$q \text{ prime} \neq p \qquad \sum_{\zeta^q=1}^{\circledast} \psi(\zeta X) \underset{\substack{\uparrow \\ \psi \text{ endo}}}{=} \psi\Big(\sum_{\zeta^q=1}^{\circledast} \zeta X\Big) \underset{\substack{\uparrow \\ \text{since } X \text{ typical for } \xi * F}}{=} 0$$

where $\circledast$ denotes composition for $\xi * F$. Thus

$$\xi\circ\varphi\circ\xi^{-1} = \eta(X^{p^h}) \qquad \eta'(0) \neq 0$$

$$\text{so} \qquad \varphi(X) = \xi^{-1}(\eta(\xi(X)^{p^h}))$$

$$= (\xi^{-1}\circ\eta)(\underset{\substack{\uparrow \\ \text{using that } pR=0.}}{\xi^{(p^h)}}(X^{p^h}))$$

$$= g(X^{p^h})$$

where $g'(0) = \underline{\eta'(0)} \neq 0.$ Q.E.D.

Let F be a typical law over a $\mathbb{Z}_{(p)}$-algebra R and let $c_0 : D \longrightarrow G$ be the associated formal group with coordinate. Following Cartier introduces elements $x_n(F)$ $n \geq 1$ $\in R$ by the formula

$$Fc_0 = \sum_{n \geq 0} V^n [x_{n+1}] c_0$$

i.e.

$$\sum^F_{\zeta^p = 1} (\zeta X) = \sum^F_{n \geq 1} (x_n X^{p^n})$$

Assume the following

Theorem (Cartier):

$$\begin{tikzcd} LT(R) \arrow[r] & R^{\mathbb{N}_+} \\ F \arrow[r, mapsto] & (x_n(F)) \end{tikzcd}$$

is bijective, or equivalently $LT = \mathbb{Z}_{(p)}[x_1(F_t), \ldots]$.

Proposition: The following are equivalent for a typical law F over a $\mathbb{Z}_{(p)}$-algebra R where h is an integer ≥ 0

(i) $x_1(F) = \cdots = x_{h-1}(F) = 0$

(ii) $F(X,Y) \equiv X + Y + \lambda C_{p^h}(X,Y)$ mod deg $p^h + 1$

(iii) For any R-algebra R' and element $a \in \mu_{p^h - 1}(R) = \{a \mid a^{p^h-1} = 1\}$, aX is an automorphism of F over R'.

Proof. (iii) is equivalent to $F(X,Y)$ having only terms of degree $n \equiv 1 \pmod{p^h - 1}$. In effect if $F(X,Y) = \sum F_n(X,Y)$ with

F_n of degree n, then $a^{-1}F_n(aX, aY) = a^{n-1}F_n(X,Y)$. Now $a^{n-1} = 1$ for all $a \in \mu_{p^h-1}(R')$ and R-algebras R' and this implies that $T^{n-1} = 1$ in $R[T]/(T^{p^h-1}-1)$. Now if $(n-1) = q(p^h-1)+r$ with $0 \le r < p^h-1$, then

$$(T^{n-1}-1) \equiv T^{(p^h-1)q} \cdot T^r - 1 \equiv T^r - 1 \quad \text{mod } T^{p^h}-1$$

$$\not\equiv 0 \quad \text{unless } r = 0.$$

Thus $n-1 \equiv 0 \mod (p^h-1)$ as claimed

So (iii) $\Rightarrow$ $F(X,Y) \equiv X+Y$ mod deg p^h, hence by Lazard that (ii) holds.

(ii) $\Rightarrow$ (i).

$$\sum_{\zeta^p=1}^{F} (\zeta X) \equiv 0 \quad \text{mod deg } p^h$$

hence $x_1, \ldots, x_{h-1}$ are zero by the uniqueness of a representation of a typical curve.

(i) $\Rightarrow$ (iii). It's enought to do this for the universal curve since we have to show only that terms of degree $\equiv 1$ mod p^h-1 occur. Thus we may assume R torsion-free and use the logarithm. Since $x_1, \ldots, x_{h-1}$ are zero we have

$$\sum_{\zeta^p=1} \ell(\zeta X) \equiv 0 \quad \text{mod deg } p^h$$

But $\ell(X) = X + a_1 \frac{X^p}{p} + \cdots$

thus we have that $a_1, \ldots, a_{h-1} = 0$, hence $\ell(\zeta X) = \zeta\, \ell(X)$ if

$\xi^{p^h-1} = 1$ in an extension ring of $R \otimes \mathbb{Q}$. Thus $F(\xi X, \xi Y) = \xi F(X,Y)$ if $\xi = \bar{T}$ in $R[T]/(T^{p^h}-1)$ so F has only terms of degree $\equiv 1 \mod p^h-1$ and so we are done.

Corollary: μ_{p-1} acts as autos of any typical law by $\xi \mapsto \xi X$.

This is the case $h=1$.

Consequences: 1.) Over $\mathbb{F}_p$, $(Vc_0)(X) = X^p$ is an endomorphism of any group law and also aX, $a \in \mathbb{F}_p$ is an endo of any typical law, hence any typical coordinate change $\xi(X) = \sum^F_{n \geq 0} a_n X^{p^n}$ $a_0 \in \mathbb{F}_p^*$ is an automorphism. It follows that over $\mathbb{F}_p$ distinct typicals laws are non-isomorphic (this due to Cartier).

2) In characteristic p one has that $VF = p$, hence the Cartier parameters $x_n(F)$ $n \geq 1$ may be read off from the ~~p~~ p map

$$[p]_F(X) = \sum^F_{n \geq 1} \left(x_n(F) \cdot X^{p^n} \right)$$

Consequently over $\mathbb{F}_p$ there is a ~~canonical~~ typical law of height h namely the one with $x_h = 1$ and all others zero. It is the unique typical law with

$$[p]_F = X^{p^h}$$

or equivalently the unique typical law with

$$F(X,Y) \equiv X+Y-C_{p^h}(X,Y) \quad (\text{mod deg } p^h+1).$$

3.) If R is p-complete, i.e. $R \xrightarrow{\sim} \varprojlim R/p^nR$, then $\mathbb{Z}_p^*$ acts as endos. of any group law F. Moreover there is a map $\mathbb{Z}_p \longrightarrow R$. Thus to $n \in \mathbb{Z}_p$ we have the maps

$$\left\{\begin{array}{l} G \xrightarrow{c_0^{-1}} D \xrightarrow{\;n\;} D \xrightarrow{c_0} G \\ G \xrightarrow{\;n\;} G \end{array}\right.$$

If $n \in \mu_{p-1}$ in $\mathbb{Z}_p$ and c_0 is typical, then the upper arrow is an automorphism of G. I claim these are the same automorphism. In effect by a universal argument one can suppose that $\mathbb{Z}_p^* =$ all the autos whence μ_{p-1} is the set of autos of finite order for p odd (for $p=2$, $\mu_{p-1}=1$). The above two autos, have same effect on the leading terms hence must coincide. Therefore if $pR=0$ and $u \in (\mathbb{F}_p)^*$, then the ~~endo~~ auto. uX of a typical law is the same as multiplication by the Teichmuller representative of u in $\mathbb{Z}_p^*$.

March 21, 1969

Some calculations of typical coordinate changes.

Suppose that we work over a torsion-free ring R and use the logarithm to determine the appropriate formulas.

Given a formal group G and a typical coordinate c_0. Let $\log : D \xrightarrow{\sim} G_a$ be the isomorphism with $(\log c_0)(X) = X + \cdots$;

$$(\log c_0)(x) = \sum_{n\geq 0} a_n \frac{x^{p^n}}{p^n} \qquad a_0 = 1$$

Let

$$c_1 = \sum_{m\geq 0} V^m[u_m]\cdot c_0 \qquad u_0 \in R^*$$

be a new typical coordinate. Then

$$(\log c_1)(x) = \sum_{m\geq 0} (\log c_0)(u_m x^{p^m})$$

$$= \sum_{m,n\geq 0} a_n \frac{1}{p^n}(u_m x^{p^m})^{p^n}$$

$$= \sum_{\nu\geq 0} \frac{x^{p^\nu}}{p^\nu} \sum_{m+n=\nu} p^m a_n (u_m)^{p^n}$$

Identifying c_0 with the sequence $\underline{a} = (a_0, a_1, \cdots)$ and $\underline{u} = (u_0, u_1, \cdots$ we have that c_1 is given by the sequence

$$\underline{b} = \underline{u}\cdot\underline{a}$$

where $$b_n = \sum_{h=0}^{n} p^{n-h} a_h (u_{n-h})^{p^h}$$

$$\begin{cases} b_0 = a_0 u_0 \\ b_1 = p a_0 u_1 + a_1 u_0^p \\ b_2 = p^2 a_0 u_2 + p a_1 u_1^p + a_2 u_0^{p^2} \end{cases}$$

Suppose $\underline{c} = \underline{v} \cdot \underline{b}$.

$$c_0 = b_0 v_0 = a_0 (u_0 v_0)$$

$$c_1 = p b_0 v_1 + b_1 v_0^p = p(a_0 u_0) v_1 + (p a_0 u_1 + a_1 u_0^p) v_0^p$$

$$= p a_0 (u_0 v_1 + u_1 v_0^p) + a_1 (u_0 v_0)^p$$

$$c_2 = p^2 b_0 v_2 + p b_1 v_1^p + b_2 v_0^{p^2}$$

$$= p^2 (a_0 u_0) v_2 + p(p a_0 u_1 + a_1 u_0^p) v_1^p + (p^2 a_0 u_2 + p a_1 u_1^p + a_2 u_0^{p^2}) v_0^{p^2}$$

$$= p^2 a_0 (u_0 v_2 + u_1 v_1^p + u_2 v_0^{p^2}) + p a_1 (u_0^p v_1^p + u_1^p v_0^{p^2}) + a_2 (u_0 v_0)$$

Now if $\underline{c} = \underline{v} \cdot (\underline{u} \cdot \underline{a}) = \underline{w} \cdot \underline{a}$ *we must have*

$$\boxed{W_0 = u_0 V_0}$$

$$\boxed{W_1 = u_0 V_1 + u_1 v_0^p}$$

$$= p^2 a_0 w_2 + p a_1 (u_0 v_1 + u_1 v_0)^p + a_2 (u_0 v_0)^{p^2}$$

$$w_2 = (u_0 v_2 + u_1 v_1^p + u_2 v_0^{p^2}) + \frac{a_1}{p a_0} \left[(u_0 v_1)^p + (u_1 v_0^p)^p - (u_0 v_1 + u_1 v_0^p)^p \right]$$

$$\boxed{w_2 = \left(u_0 V_2 + u_1 V_1^{p} + u_2 V_0^{p^2}\right) - \frac{a_1}{a_0} C_p\left(u_0 V_1, u_1 V_0^{p}\right)}$$

In the case that interests us $u_0 = V_0 = a_0 = 1$ so

$$\boxed{\begin{aligned} W_1 &= u_1 + v_1 \\ W_2 &= v_2 + u_1 v_1^{p} + u_2 - a_1 C_p(u_1, v_1) \end{aligned}}$$

An important thing to note is that not even in char. p is W_2 equal to $v_2 + u_1 v_1^{p} + u_2$ and hence independent of a_1.

Question: Does the polynomial $w_k = \Phi(u_i, v_i, a_i)$ have p-integral coefficients?

Answer: NO pursuing one step further to w_3 we find we need $\dfrac{a_2 - a_1^{p+1}}{p}$.

One might ask if it possible to find a group of typical coordinate changes analogous to the group G of power series under composition. In other words we would like a change of variables $\underline{u} = \varphi(\underline{x}, \underline{a})$ such that if we put $\underline{x} * \underline{a} = \varphi(\underline{x}, \underline{a}) \cdot \underline{a}$,

$$\underline{y} * (\underline{x} * \underline{a}) = \underline{z} * \underline{a}$$

then $\underline{z}$ depends only on y and x but not on a.

It seems reasonable to suppose φ compatible with the obvious filtration and isomorphism of the associated graded wit $\mathcal{B}_a$'s. Thus

$$\begin{cases} u_1 = x_1 \\ u_2 = x_2 + \varphi_1(x_1, a_1) \\ \quad \cdots \\ u_n = x_n + \varphi_n(x_1, \dots, x_{n-1}, a_1, \dots, a_{n-1}) \end{cases}$$

We now show this is impossible. In effect recall

$$\underline{x} * \underline{a} = \underline{u} \cdot \underline{a}$$
$$\underline{y} * \underline{b} = \underline{v} \cdot \underline{b} = \underline{w} \cdot \underline{a}$$

$$w_1 = u_1 + v_1 = x_1 + y_1 = z_1$$

$$\begin{cases} w_2 = v_2 + u_1 v_1^p + u_2 - a_1 C_p(x_1, y_1) \\ w_2 = z_2 + \varphi_1(z_1, a_1) \end{cases}$$

$$\begin{cases} v_2 = y_2 + \varphi_1(y_1, b_1) = y_2 + \varphi_1(y_1, pu_1 + a_1) \\ u_2 = x_2 + \varphi_1(x_1, a_1) \end{cases}$$

Thus we have

$$z_2 + \varphi_1(x_1 + y_1, a_1) = y_2 + \varphi_1(y_1, px_1 + a_1) + x_1 y_1^p$$
$$+ x_2 + \varphi_1(x_1, a_1) - a_1 C_p(x_1, y_1)$$

and we want z_2 to be independent of a_1. This means

$$\frac{\partial}{\partial a_1}\{\varphi_1(y_1, px_1 + a_1) + \varphi_1(x_1, a_1) - a_1 C_p(x_1, y_1) - \varphi_1(x_1 + y_1, a_1)\} \equiv 0$$

Now for convenience of notation drop all the one's and let φ' denote derivative with respect to the second argument. Then

$$\varphi'(y, px+a) + \varphi'(x,a) - C_p(x,y) - \varphi'(x+y, a) \equiv 0.$$

Setting $a=0$ and reducing modulo p we get

$$\varphi'(x,0) + \varphi'(y,0) - \varphi'(x+y, 0) - C_p(x,y) = 0 \quad \text{mod } p$$

which is false since $f(x)+f(y)-f(x+y)$ is a linear combination of $B_n(x,y)$.

March 23, 1969:

Relation between Chern classes in cobordism theory and geometric Chern classes:

Let E be a vector bundle of dimension n over X and let $V \subset \Gamma(X, E)$ be a subspace of sections which is ample in the sense that it induces an embedding of X into $\mathrm{Grass}_{N-n}(V) = \{ A \subset V \mid \dim A = N-n \}$, $N = \dim V$. Then one can prove: If W is a generic d-diml. subspace of V, then $X \longrightarrow \mathrm{Grass}_{N-n}(V)$ is transversal to the map

$$\tilde{Z}_W = \{ (\ell, A) \mid \ell \in \mathbb{P}W, \ \ell \subset A \} \xrightarrow{pr_2} \mathrm{Grass}_{N-n}(V)$$

and so we can form

$$X_W = \{ (\ell, x) \mid \ell \in \mathbb{P}W, \ x \in X, \ \ell \subset A_{(x)} = \mathrm{Ker}\{ V \xrightarrow{ev} E_{(x)} \} \}.$$

Then $f' : X_W \longrightarrow X$ (as an element of $\Omega(X)$) is the geometric Chern class of dimension $N-d+1$. ($f'(X_W)$ is where the sections of W^d become dependent so if $d = 1$ we get $c_n(E)$, while if $d = n$ we get something like $c_1(E) = c_1(\Lambda^n E)$). The problem is to relate the geometric Chern classes with the usual Chern classes. Thus we want to calculate the element of $\Omega(\mathrm{Grass})$ represented by the desingularized Shubert variety $\tilde{Z}_W$.

$$\begin{tikzcd}
\tilde{Z}_W \arrow[d, equal] & & \\
\{(\ell^1, A^{N-n}) \mid \ell' \subset W \cap A\} \arrow[r, "i'"] \arrow[d, "g'"] \arrow[dr, phantom, "\text{cart}"] & \{(\ell, A) \mid \ell' \subset A^{N-n} \subset V^N\} \arrow[r, "f"] \arrow[d, "g"] & \{A^{N-n} \subset V^N\} \\
\{\ell' \subset W^d\} \arrow[r, "i"'] & \{\ell' \subset V^n\} &
\end{tikzcd}$$

Let $G = \mathrm{Grass}_{N-n}(V^N)$ and let

$$0 \longrightarrow S^{N-n} \longrightarrow \pi^* V \longrightarrow E^n \longrightarrow 0$$

be the canonical sequence over G. Then the above diagram is

$$\begin{tikzcd}
\tilde{Z}_W \arrow[r, "i'"] \arrow[d, "g'"] & \mathbb{P}S \arrow[r, "f"] \arrow[d, "g"] & G \\
\mathbb{P}W \arrow[r, "i"] & \mathbb{P}V &
\end{tikzcd}$$

where f is the projective bundle of S and where g is the Grassman bundle of $\mathrm{Grass}_{N-n-1}(\pi^* V / \mathcal{O}(-1))$. We want to calculate $f_* i'_* 1 \in \Omega(G)$. We shall use

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Formula: Let E be a bundle over X of dim $n = r_1 + \cdots + r_k$ with $r_i > 0$. Let $D_{r_1, \ldots, r_k}(E)$ be the drapeau scheme of E classifying filtrations of E ~~of type with~~ of the type

$$0 \subset F_1 \subset \cdots \subset F_k = E \qquad \text{with} \quad \dim F_i / F_{i-1} = r_i .$$

Let π: D

<u>Formula</u>: Let E be a vector bundle over X of dim $n = r_1 + \cdots + r_k$ with $r_i > 0$ and let $D_{r_1, \ldots, r_k}(E)$ be the flag scheme of E endowed with a universal filtration

$$0 \subset F_1 \subset \cdots \subset F_k = \pi^* E \qquad \pi : D_{r_1, \ldots, r_k}(E) \longrightarrow X$$

with quotients

$$Q_i = F_i / F_{i-1}$$

of dimension r_i. Then $\Omega(D_{\underline{r}} E)$ is generated as an $\Omega(X)$-algebra by

$$c_j(Q_i) \qquad \begin{array}{l} 1 \leq i \leq k \\ 1 \leq j \leq r_i \end{array}$$

subject to the relations obtained by equating coefficients of

$$(*) \qquad c_T(\pi^* E) = \prod_{i=1}^{k} c_T(Q_i).$$

Thus

$$\Omega(\mathbb{P}S) = \Omega(pt)[c_1(\mathcal{O}(-1)), c_\cdot(S'), c_\cdot(E)]$$
$$\Omega(G) = \Omega(pt)[c_\cdot(S), c_\cdot(E)]$$
$$\Omega(\mathbb{P}V) = \Omega(pt)[c_1(\mathcal{O}(-1)), c_\cdot(S')]$$

subject to relations of the type $(*)$. Here S' is given by an exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow f^* S \longrightarrow S' \longrightarrow 0.$$

Now
$$f_* i'_* 1 = f_* g^* \underbrace{i_* 1}$$
$$i_* 1 = (c_1 \mathcal{O}(1))^{N-d}$$

so we have

$$\begin{array}{l}(n-d+1)\text{th geometric} \\ \text{Chern class of } E\end{array} = f_* \left[c_1\, \mathcal{O}(1)\right]^{N-d}$$

$$= \operatorname{res} \frac{Z^{N-d}\, \omega}{\prod_{j=1}^{N-n} F(Z, x_j)}$$

where the x_j are the phantom elts. such that

$$\prod_{j=1}^{N-n} (1 + T x_j) = c_T(S) = \frac{1}{c_T(E)}$$

Examples: ① Take integral cohomology. Then

$$\operatorname{res} \frac{Z^{N-d}\, dZ}{\prod_{j=1}^{N-n} (Z + x_j)} = \operatorname{res} \sum_{j \geq 0} Z^j c_{n-j}(E) \frac{dZ}{Z^d} = c_{n-d+1}(E)$$

since $$\prod_{j=1}^{N-n} (Z + x_j) \left(\sum_{j=0}^{n} Z^j c_{n-j}(E) \right) = Z^N$$

② K-theory: Then $F(X, Y) = X + Y - XY$

$$\omega = \frac{dZ}{F_2(Z, 0)} = \frac{dZ}{1-Z}$$

$$\operatorname{res} \frac{Z^{N-d}\, dZ}{(1-Z) \prod_{j=1}^{N-n} (Z + x_j - Zx_j)} = \operatorname{res} \frac{Z^{N-d}\, dZ}{(1-Z)\, Z^{N-n} \prod_{j=1}^{N-n} \left(1 + \left(\frac{1-Z}{Z}\right) x_j\right)}$$

$$= \text{res} \frac{dZ}{(1-Z)Z^{d-n}} \prod_{i=1}^{n}\left(1+\frac{1-Z}{Z}y_i\right)$$

$c_T(E) = \prod(1+Ty_i)$

$$= \text{res} \sum_{i=0}^{n} c_i(E) \frac{dZ}{(1-Z)Z^{d-n}} \left(\frac{1-Z}{Z}\right)^{i}$$

$$= \sum_{i=0}^{n} c_i(E) \ \text{res} \frac{(1-Z)^{i-1}dZ}{Z^{d-n+i}}$$

but

$$\frac{(1-Z)^{i-1}}{Z^{d-n+i}}dZ = \sum_{j=0} \binom{i-1}{j}(-1)^j Z^{j-d+n-i}\,dZ$$

whose **residue** occurs when $j-d+n-i=-1$ and is

$$\binom{i-1}{d-n+i-1}(-1)^{d-n+i-1}.$$

Recall that we are calculating the rth geometric class where $r = n-d+1$, so we get

$$\boxed{\begin{matrix}r\text{th geometric Chern} \\ \text{class in K-theory of } E\end{matrix} = \sum_{i=0}^{n} (-1)^{i-r}\binom{i-1}{i-r} c_i^K(E)}$$

$$= \sum_{j\geq 0} (-1)^j \binom{j+r-1}{r-1} c_{j+r}^K(E) \qquad r\geq 1$$

③ Connected K-theory. $F(X,Y) = X+Y-aXY$

$$\omega = \frac{dZ}{F_2(Z,0)} = \frac{dZ}{1-aZ}$$

$r = n-d+1$

$$\text{geom } c_r(E) = \text{res} \frac{Z^{N-d}\, dZ}{(1-aZ)\prod_{j=1}^{N-n}(Z+x_j-aZx_j)} = \text{res} \frac{Z^{N-d} dZ}{(1-aZ)\cancel{Z}^{N-n}\prod\left(1+\frac{1-aZ}{Z}x_j\right)}$$

$$= \text{res} \frac{dZ}{(1-aZ)Z^{d-n}} \sum_{i=0}^{n} c_i(E) \cdot \left(\frac{1-aZ}{Z}\right)^i$$

$$= \sum_{i=0}^{n} c_i(E) \text{ res } \frac{dZ}{Z^{d-n+i}} \sum_{j=0}^{i-1} \binom{i-1}{j}(-a)^j Z^j$$

contributes when
$j = d-n+i-1 = i-r$

$$= \sum_{i=0}^{n} c_i(E) \binom{i-1}{i-r} (-a)^{i-r}$$

$$\therefore \boxed{\text{geom } c_r(E) = \sum_{i \geq r}^{n} (-a)^{i-r} \binom{i-1}{i-r} c_i(E))}$$

Generalize these formulas to determine in terms of the formal group law the elements in the cobordism ring of a homogeneous projective variety G/P (G reductive P parabolic) represented by the (canonical) desingularizations of the Shubert varieties (= closure of orbits of N.

March 27, 1969.

Operations in generalized cohomology theories

Let h^* and h_* be the generalized cohomology and homology theory associated to a homotopy commutative and associative ring spectrum E:

$$h^*(X) = \{X, E\}$$
$$h_*(X) = \{S, E \wedge X\}$$

Then one can form a cosimplicial ring spectrum

$$\begin{tikzcd} E \arrow[r, shift left] \arrow[r, shift right] & E \wedge E \arrow[l, bend right] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & E \wedge E \wedge E \end{tikzcd}$$

which one might call the "Amitsur complex" of E over S. Observe that this is not just a co-semi-simplicial ring spectrum for it has an operator $\Lambda^p E \longrightarrow \Lambda^q E$ for any map $[1,p] \rightarrow [1,q]$. Taking homotopy one obtains a cosimplicial commutative graded ring

( ) $$\begin{tikzcd} h_*(pt) \arrow[r, shift left] \arrow[r, shift right] & h_*(E) \arrow[l, bend right] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & h_*(E \wedge E) \cdots \end{tikzcd}$$

Following Adams we make the following assumption:

( ) $h_*(E)$ is a flat left $h_*(pt)$ module.

Then one knows that for any spectrum Y there is an isomorphism

( ) $$h_*(E \wedge Y) \xleftarrow{\sim} h_*(E) \otimes_{h_*(pt)} h_*(Y)$$

by means of which the cosimplicial ring may be written

$$\begin{tikzcd}
(\,) & h_*(pt) \arrow[r, shift left, "\eta_l"] \arrow[r, shift right, "\eta_r"'] & h_*(E) \arrow[l, bend right, "\varepsilon"'] \arrow[r, shift left=2, "1 \otimes id"] \arrow[r, "\Delta"] \arrow[r, shift right=2, "id \otimes 1"'] & h_*(E) \otimes_{h_*(pt)} h_*(E).
\end{tikzcd}$$

There are various identities satisfied by these maps for which the reader is referred to Adams' lectures [ ]. **These identities** say that ( ) is a groupoid object in the opposed category of the category of graded (anti-) commutative rings. In effect the cosimplicial ring ( ) gives rise to a covariant functor from rings to simplicial sets; by ( ) the resulting simplicial set as a functor from the category of finite sets [n] to sets ~~is compatible with~~ transforms amalgamated sums into fiber products and **so** by Grothendieck is isomorphic to the simplicial set associated to a groupoid. We shall refer to the system ( ) of rings as a <u>groupoid scheme</u> (abuse of terminology natural in topology).

Using the cohomology we can also define a covariant functor C from rings to (small) categories as follows.

$$Ob\, C(R) = Hom_{(rings)}\left(h(pt), R\right)$$

$Hom_{C(R)}(\varphi_1, \varphi_2) =$ stable natural transformations compatible with products from $h^*_{\varphi_1}$ to $h^*_{\varphi_2}$

where $h^*_{\varphi_*}(X) = R_{\overset{\nwarrow}{\varphi}} \otimes_{h(pt)} h(X)$

We now wish to show these category schemes are the same and show that the h(pt)-module stable operations $h \longrightarrow M \otimes_{h(pt)} h$, where M is an $h(pt)$-module is $\mathrm{Hom}_{h(pt)}(h_*(E), M)$. It will be necessary to make the following hypotheses.

Hypothesis:

h^* is generated by finite complexes X satisfying the Kunneth theorem i.e.

$$h^*(X) \otimes_{h(pt)} h^*(Y) \xrightarrow{\sim} h^*(X \times Y)$$

for all finite complexes Y, where generated means that $\forall$ complex U and element $u \in h(U)$ $\exists$ a map $f: U \longrightarrow X$ where X is a finite complex satisfying the Kunneth theorem such that $u \in \mathrm{Im} f^*: h(X) \longrightarrow h(U)$.

Remarks:

a) $h^*(X)$ flat over $h(pt)$ $\Longrightarrow$ Kunneth theorem holds

b) If X is a finite complex, then Kunneth for X $\Longleftrightarrow$ $h^*(X)$ is a finitely generated projective $h(pt)$ module.

Proof (assuming properties of Spanier-Whitehead duality that I haven't checked): Choose an embedding $X \longrightarrow S^N$ with N large and let DX be S^N with the complement of an open regular nbd shrunk to a point. Then there are maps

$$\varphi: S^N \longrightarrow X \wedge DX$$

$$\psi: DX \wedge X \longrightarrow S^N$$

playing the role of the identity transf. (resp trace) if one thinks of DX as a dual of X. Thus the composite

$$(\ )\qquad S^N \wedge X \longrightarrow X \wedge DX \wedge X \longrightarrow X \wedge S^N$$

is the interchange map. Let $\sum u_i \otimes v_i \in h^*(DX \wedge X) \cong h^*(DX) \otimes_{h(pt)} h^*(X)$ by Künneth be the image of 1 under the composition

$$\overset{1}{h(pt)} \simeq h(S^N) \xrightarrow{\varphi^*} h(D^*\cancel{X} \wedge X).$$

If $\lambda \in h^*(DX)$ and $x \in h(X)$, let $\langle \lambda, x \rangle =$ image of $x \otimes \lambda$ under

$$h^*(\cancel{X}) \underset{h(pt)}{\otimes} h^*(DX) \longrightarrow h(X \wedge DX) \xrightarrow{\varphi^*} h(S^N) \simeq h(pt)$$

Than as ( ) is interchange we have for $x \in h(X)$

~~$$x \longmapsto u \otimes x \longmapsto \sum u_i \otimes v_i \otimes x$$~~

$$x \longmapsto x \otimes [S^N] \longmapsto \sum x \otimes u_i \otimes v_i \longmapsto \sum_i \langle x, u_i \rangle v_i$$

Thus

$$\boxed{\sum_{i=1}^{n} \langle x, u_i \rangle v_i = x}$$

which implies that $h(X)$ is a direct summand of $h(pt)^n$.

Interchanging X and DX one sees that

$$\boxed{\lambda = \sum_{i=1}^{n} u_i \langle v_i, \lambda \rangle}$$

These equations show that the pairing $\langle , \rangle$ is a perfect duality between $h^*(X)$ and $h^*(DX)$. Moreover as $h^*(DX) \simeq h_{n-*}(X)$ we get the formula

$$\boxed{h^*(X) = \operatorname{Hom}_{h(pt)} (h_*(X), h(pt))}$$

or that the Kronecker product

$$h^*(X) \times h_*(X) \longrightarrow h(pt)$$

is a perfect duality.

According to the hypothesis we may represent the spectrum E as follows. Consider the suspension category of finite complexes (an object is a pair (X, n)), and consider the category I whose objects are triples (X, n, u) where $u \in h^0(\Sigma^n X)$ with X satisfying the Keenneth thm. Then

$$h^0(Z) = \varinjlim_{I} \{Z, \Sigma^n X\}$$

and the category I is filtering. We write E_i $i \in I$ for the inductive system indexed by I so that

$$h^0(X) = \varinjlim \{X, E_i\} .$$

The stable operations are given by

$$\text{Homst}(h, h \otimes_{h(pt)} M) = \varprojlim_i h^*(E_i) \otimes_{h(pt)} M$$

$$= \varprojlim_i \text{Hom}_{h(pt)}(h_*(E_i), M)$$

$$= \text{Hom}_{h(pt)}(\varinjlim_i h_*(E_i), M)$$

$$\boxed{\text{Homst}(h, h \otimes_{h(pt)} M) = \text{Hom}_{h(pt)}(h_*(E), M).}$$

Let R be an $h(pt)$-algebra, ~~and~~ let $\theta: h \to R \otimes_{h(pt)} h$ be a stable operation and let $\varphi: h_*(E) \to R$ be the ~~corresponding~~ $h(pt)$ linear map corresponding to θ under this isomorphism. We now show that θ is a ring homomorphism iff φ is a ring homomorphism. In effect for θ to be a ring homomorphism means that for objects X, Y of the suspension category of finite cxs. we have

$$\begin{tikzcd}
h(X) \otimes_{h(pt)} h(Y) \arrow[r, "\theta \otimes \theta"] \arrow[d, "\boxtimes"] & h_R(X) \otimes_R h_R(Y) \arrow[d, "\boxtimes"] \\
h(X \wedge Y) \arrow[r, "\theta"] & h_R(X \wedge Y)
\end{tikzcd}$$

commutes. ~~So take $X = E_i$, $Y = E_j$ and one has~~

$$\begin{tikzcd}
\text{Hom}(h_*(E_i), h(pt)) \otimes_{h(pt)} \text{Hom}(h_*(E_j), h_*(pt)) \arrow[r, "\theta \otimes \theta"] \arrow[d, "\boxtimes"] & \text{Hom}(h(E_i), R) \otimes \text{Hom}(h(E_j), R) \arrow[d, "\boxtimes"] \\
\text{Hom}(h_*(E_i \wedge E_j), h(pt)) \arrow[r, "\theta"] & \text{Hom}(h(E_i \wedge E_j), R)
\end{tikzcd}$$

It is enough to check this for $X = E_i$, $Y = E_j$ and for the canonical elements $u_i \in h^0(E_i)$, $u_j \in h^0(E_j)$. Let $\mu : E_i \wedge E_j \longrightarrow E_k$ be the map $\ni$ $\mu^* u_k = u_i \boxtimes u_j \in h^0(E_i \wedge E_j)$
To show that

$$\theta u_i \boxtimes \theta u_j = \mu^* \theta u_k$$

Recall how θ of an element is computed: given $\alpha \in h(X)$ find $f : X \longrightarrow E_i$, $f^* u_i = \alpha$, and $\theta\alpha = f^*(\theta u_i)$. Now

$$\varepsilon_i : h_*(E_i) \longrightarrow h_*(E)$$

yields by duality

$$\begin{tikzcd} h^*(E_i) & h^*(E) \arrow[l, "\varepsilon_i^t"'] \\ \varphi_i & \varphi \arrow[l, mapsto] \end{tikzcd}$$

$$\varepsilon_i^t(\varphi) = \varphi_i \in h^*(E_i)$$ and in fact $\varphi_i = \theta u_i$.

So to show that

$$\theta u_i \boxtimes \theta u_j \overset{?}{=} \mu^* \theta u_k \text{ [illegible]}$$

$$\overset{\shortparallel}{\varphi_i} \boxtimes \varphi_j \overset{?}{=} \mu^* \varphi_k \quad ?$$

$$\overset{\shortparallel}{\varepsilon_i^t\varphi} \boxtimes \varepsilon_j^t\varphi \overset{?}{=} \underline{\mu^* \varepsilon_k^t \varphi} \qquad = (\varepsilon_i^* \boxtimes \varepsilon_j^*)\mu^* \varphi$$

$$\begin{tikzcd} h_*(E_i \wedge E_j) \arrow[r, "\mu"] & h_*(E_k) \arrow[r, "\varepsilon_k"] & h_*(E) \\ & h_*(E_i) \otimes h_*(E_j) \arrow[ul, "\cong"] \arrow[dl, dash, "\varepsilon_i \otimes \varepsilon_j"'] & \\ h_*(E) \otimes h_*(E) \arrow[r, "\boxtimes"] & h_*(E \wedge E) \arrow[uur, bend right, "\mu_*"'] & \end{tikzcd}$$

~~$\varepsilon_i \otimes \varepsilon_j$~~

Therefore want $\mu_*^{\cancel{*}}\varphi \overset{?}{=} \varphi \boxtimes \varphi$.

[illegible]

$$\begin{tikzcd}
h_*(E)\otimes h_*(E) \arrow[d, "S"', "\boxtimes"] & \\
h_*(E\wedge E) \arrow[r, "\mu_*"'] & \boxed{h_*(E)}\ \cancel{\varphi}
\end{tikzcd}$$

Thus want

$$\begin{tikzcd}
h_*(E)\otimes h_*(E) \arrow[r] \arrow[d, "\varphi\otimes\varphi"] & h_*(E\wedge E) \arrow[r] & h_*(E) \arrow[d, "\varphi"] \\
R\otimes R \arrow[rr, Rightarrow, "\text{mult. in } R"] & & R
\end{tikzcd}$$

to commute, or ~~in fact~~ for $\varphi : h_*(E) \longrightarrow R$ to be a ring homomorphism where ring structure is as above.

$$\begin{tikzcd}
h_*(X)\otimes_{h_*(pt)} h_*(Y) \arrow[r, phantom, "="] \arrow[ddd] & \pi(E\wedge X)\otimes_{\pi(E)}\pi(E\wedge Y) \arrow[d] \\
& \pi(E\wedge X\wedge E\wedge Y) \arrow[d, phantom, "\shortparallel"] \\
& \pi(E\wedge E\wedge X\wedge Y) \arrow[d] \\
h(X\wedge Y) & \pi(E\wedge X\wedge Y)
\end{tikzcd}$$

so now if $X, Y = E$

$$\begin{tikzcd}
h_*(E) \otimes h_*(E) \arrow[dd] & = & \pi(E\wedge E) \otimes \pi(E\wedge E) \arrow[d] \\
 & & \pi(E\wedge E\wedge E\wedge E) \arrow[d, "id \wedge T \wedge id"] \\
h_*(E) & & \pi(E\wedge E\wedge E\wedge E) \arrow[d, "\mu\wedge\mu"] \\
 & & \pi(E\wedge E)
\end{tikzcd}$$

Thus we have shown that $\Theta : h^* \longrightarrow h^*_R$ is a ring hom $\Leftrightarrow$ $\varphi : h_*(E) \longrightarrow R$ is.

$\Theta : h$ [illegible] $\longrightarrow h \otimes_{h(pt)\nearrow_\varphi} R$ ring homomorphism

Apply to $h(pt) \longrightarrow h(pt) \otimes_{h(pt)\nearrow_\varphi} R \xrightarrow{\sim} R$

ones gets a new ring homomorphism from $h(pt)$ to R.

<u>Claim</u> this is the same as

$$h(pt) \xrightarrow{\eta_r} h_*(E) \xrightarrow{\hat{\theta}} R$$

where η_r is the ~~composition~~ map

$$\pi(E) \longrightarrow \pi(E\wedge E)$$

$$e \qquad\qquad 1\otimes e$$

)

definitely not $h(pt)$-linear in general.

<u>Proof</u>. ~~How is $\hat{\theta}$ defined?~~ ~~Recall that~~ First we show

that $\hat{\Theta}$ determines Θ as follows: For any X, ~~$h^*(X) = h_*(DX)$~~ is an $h_*(E)$-comodule by

$h_*(X) = h^*(X)^*$

$$\begin{tikzcd}
 & & h_*(E \wedge X) \arrow[d, phantom, "\cong"] \\
h_*(X) \arrow[r, phantom, "="] \arrow[urr, bend left, "y \mapsto 1\wedge y"] \arrow[drr, bend right, "\Delta"'] & \pi_*(E \wedge X) \arrow[r, "e\wedge y \mapsto e\wedge 1\wedge y"'] & \pi_*(E \wedge E \wedge X) \\
 & & h_*(E) \otimes_{h(pt)} h_*(X) \arrow[u, "\cong"]
\end{tikzcd}$$

and therefore given $h_*(E) \xrightarrow{\hat{\Theta}} h(pt)$ one can form

$$\Theta = m(\hat{\Theta} \otimes id) \circ \Delta.$$

Now why is this the same as the initial Θ? ~~[illegible] We therefore need to know a lot about basic Spanier Whitehead duality.~~

$$\text{"}h^*(E)\text{"} = \varprojlim_i \operatorname{Hom}_{h(pt)}(h_*(E_i), h(pt))$$

So we are given $\underbrace{\Theta u_i}_{} \in h^*(E_i) = \operatorname{Hom}_{h(pt)}(h_*(E_i), h(pt))$

~~[illegible]~~

So consider $u_i : h_*(E_i) \longrightarrow h(pt)$ and to determine

$$\begin{tikzcd}
h_*(E_i) \arrow[r, "\Delta"] \arrow[rr, bend right, "\Theta u_i"'] & h_*(E) \otimes_{h(pt)} h_*(E_i) \arrow[r, "\hat{\Theta} \cdot u_i"] \arrow[d, phantom, "\shortparallel ?"] & h(pt) \\
 & {} &
\end{tikzcd}$$

Why are these the same?

$$\begin{tikzcd}
h_*(E_i) \arrow[r] \arrow[rd] & h_*(E \wedge E_i) \\
 & h_*(E) \otimes_{h(pt)} h_*(E_i) \arrow[u, "\cong"', "\boxtimes"]
\end{tikzcd}$$

$$\begin{tikzcd}
h^*(E_i) = h^*(SE_i) & h^*(E \wedge E_i) \arrow[l] \\
 & \theta 1 \boxtimes u_i \\
 & \hat{\theta} \cdot u_i \arrow[u, mapsto]
\end{tikzcd}$$

Thus we have the wrong definition of Δ. Instead we must take

$$h_*(X) \longrightarrow h_*(E \wedge X) \xleftarrow{\sim} h_*(E)_r \otimes_{h(pt)} h_*(X)$$

in other words you $\overset{\text{first}}{\text{must twist}}$ $\hat{\theta}$ under $\overset{\text{the}}{\text{inversion}}$

$$\underline{h_*(E)} \xrightarrow{i} h_*(E)$$

after which it seems impossible to keep track of $\hat{\theta}$ since i is not $h(pt)$-linear. Let

?

Let F_t be the universal typical group law over LT so that by Cartier $LT = \mathbb{Z}_{(p)}[x_1, x_2, \cdots\cdots]$ where the $x_i \in LT$ are defined by the formula

$$\sum_{\zeta^p=1}^{F} \zeta X = \sum_{n\geq 1}^{F} x_n X^{p^n}$$

Then there are polynomials $\Phi_i(u_1, \cdots, u_i, v_1, \cdots, v_i, x_1, \cdots, x_i)$ such that

$$\sum_{n,m\geq 0}^{F_t} u_m v_m^{p^n} X^{p^{n+m}} = \sum_{n\geq 0}^{F_t} \Phi_n(\underline{u}, \underline{v}, \underline{x}) X^{p^n}$$

and we have the following unpleasant structure theorem.

Theorem: Let

$$\begin{tikzcd} LT \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A \otimes_{LT} A \end{tikzcd}$$

be the category scheme associating to any $\mathbb{Z}_{(p)}$-algebra its category of typical formal group laws. Then

$$A = LT[u_1, u_2, \cdots\cdots]$$ as an η_ℓ algebra

where η_r is given by the group low $(F_t)_{\underline{u}}$ over $LT[\underline{u}]$.

Moreover

$$\begin{tikzcd} \Delta: LT[\underline{u}] \arrow[r] \arrow[dr] & LT[\underline{u}] \otimes_{LT} LT[\underline{V}] \arrow[d, phantom, "\cong"] \\ & LT[\underline{u}, \underline{V}] \end{tikzcd}$$

is the ~~Sends~~ left LT algebra map sending u_n into $\Phi_n(\underline{u}, \underline{V}, \underline{x})$.

Let R be a ring over $\mathbb{Z}_{(p)}$, F formal group law over R and $a_1, a_2, \cdots$ elements of R. Then define

$$\varphi_{\underline{u},F}(X) = \sum_{n\geq 0}^{F} u_n X^{p^n} \qquad u_0 = 1$$

and consider the new group law

$$\varphi_{\underline{a}}^{-1} * F = F_{\underline{u}}.$$

Now given $\underline{v} = (v_1, v_2, \cdots)$ another sequence, we have

$$\varphi_{\underline{v},F_{\underline{u}}}(X) = \sum_{n\geq 0}^{\varphi_{\underline{u}}^{-1}*F} v_n X^{p^n}$$

$$= \varphi_{\underline{u}}^{-1} \sum_{n\geq 0}^{F} \varphi_{\underline{u}}(v_n X^{p^n})$$

$$= \varphi_{\underline{u}}^{-1} \sum_{n\geq 0}^{F} \sum_{m\geq 0}^{F} u_m (v_n X^{p^n})^{p^m}$$

$$= \varphi_{\underline{u}}^{-1}\left(\sum_{n,m\geq 0}^{F} (u_m v_n^{p^m}) X^{p^{n+m}} \right).$$

Therefore

$$(F_{\underline{u}})_{\underline{v}} = \varphi_{\underline{v},F_{\underline{u}}}^{-1} * \varphi_{\underline{u}}^{-1} * F$$

$$= \gamma_{\underline{u},\underline{v},F}^{-1} * F$$

where one knows that $\gamma_{\underline{u},\underline{v},F}(X) = \sum_{n,m\geq 0}^{F} u_m v_n^{p^m} X^{p^{m+n}}$ can be put in the form $\sum_{n\geq 0}^{F} w_n X^{p^n}$

Now pass to operations in BP– theory.

$$\begin{tikzcd}
\Omega \arrow[d, "\pi"'] \arrow[dr, "\theta\pi = \hat{\varphi}"] & \\
BP \arrow[r, "\theta"'] & BP \otimes_{LT} R
\end{tikzcd}$$

θ ring hom – stable

The operation $\theta\pi$ is given by a power series $\bar{\varphi}(X) = X + \cdots$ in $R[[X]]$ ~~[illegible]~~. Such a power series comes from a θ iff $\hat{\varphi} F^{\Omega}$ is typical. But recall

$$\hat{\varphi}(c_1^{\Omega}(L)) = \bar{\varphi}(c_1^{BP}(L))$$

so

$$\boxed{\hat{\varphi} F^{\Omega} = \bar{\varphi} * F^{BP}}$$

Conclusion:

$$\mathrm{Homst}^{\otimes}(BP, BP \otimes_{LT} R) = \{ \bar{\varphi} \in R[[X]] \mid \bar{\varphi} * F \text{ typical} \}$$

where F = group law given by $LT \to R$.

$$= \mathrm{Hom}_{\underset{\sim}{LT}(R)}(F, \text{something})$$

~~Hence given a ring R with~~ [illegible]

$$= \{ \text{sequences } (u_1, u_2, \cdots) \text{ in } R \}.$$

So take $R = LT[u_1, u_2, \cdots]$ and let

$$\underline{s_{\underline{u}}} : BP \longrightarrow BP \otimes_{LT} LT[\underline{u}]$$

be the operation given by the power series $\varphi_{\underline{u}, F_t}^{-1}$ (or perhaps $\varphi_{\underline{u}, F_t}$?)

Then we can define $s_\alpha : BP \longrightarrow BP$ stable operations by

$$\sum_\alpha s_\alpha u^\alpha = s_u$$

and as s_u is a ring homomorphism we do have

$$s_\alpha(xy) = \sum_{\beta+\gamma=\alpha} s_\beta(x)\, s_\gamma(y).$$

$$\begin{tikzcd} BP \arrow[r, "s_u"] & BP \otimes_{LT} LT[\underline{u}] \arrow[r, "s_v \otimes id"] & BP \otimes_{LT} LT[\underline{v}] \otimes_{LT} LT[\underline{u}] \end{tikzcd}$$

$$\therefore \boxed{s_v \circ s_u = s_{\Phi(u,v,x)}} \quad ?$$

Problem: Prove that any stable operation $BP \longrightarrow BP$ can be uniquely expressed in the form $\sum q_\alpha s_\alpha$ inf sum where $q_\alpha \in LT$.

Program.

Equivariant cobordism theory

Definition: – calculation using bdries

free action

$\Omega_G(G \times_H X) = \Omega_H(X)$.

exact sequence of a pair

Mayer-Vietoris

Calculation of $\Omega_G(PE)$.

Determinations of $\Omega_G(pt.)$

Localization thm of Segal (same as Segal)

$\Omega(X_G) = \Omega_G(X)\hat{}$?

Atiyah-Bott-Singer-fixpoint thm.

Reality + Power Operations

where one allows the group to have an orientation $G \longrightarrow \{\pm 1\}$ which acts on the orientation of the manifold in question!

Program: Ideas (i) Determination of $\mathcal{N}^*(pt)$

(ii) try to determine $\Omega(pt)$

~~(iii)~~ using the same method over $\mathbb{Z}_{(p)}$

Define BP using typical ~~curve~~ parameterization of Cartier!

(iii) ~~Generically the~~ If $\operatorname{Spec} k \xrightarrow{spec} \Omega(pt)$ is a "generic" geometric point of $\Omega(pt)$, then the height of the induced formal group over k should be 1. Thus there should exist ~~even in~~ char. p a generic changes of coordinates to ~~that~~ $\mathbb{G}_m$, possibly incorporating ~~involving~~ the Galois group of the algebraic closure of the fraction field of $\Omega(pt) \otimes \mathbb{F}_p$.

Somehow says that generically Ω is same as K-theory

↑

(iv) Cartier's parameters for typical laws.

(V) Adams operations are somehow related to endomorphisms of the formal group law. Hence ~~they should~~ in a theory with a group law of large height there should be lots of Adams operations hence a big **e** invariant!!! Need a theory with torsion-free coefficients (so that Thom classes for sphere bundles over spheres alway exist) with lots of Adams operations

Basic questions:

~~Does there exist over $\mathbb{Z}/p^2$ a universal family of laws of height h?~~

For X torsion-free finite cx. is it true that

$$\mathrm{Chern}(X) \xrightarrow{\cong} \Omega(X)$$

NO

Can you embed framed cobordism in a theory having somekind of Chern classes? *Example:* Let X be a finite complex. Then a real vector bundle over X of dim n is the same as a ~~complex~~ bundle of dimension n endowed with a conjugation action. A kind of an Equivariant cobordism. First Chern class lies where?

L has $\mathbb{Z}/2$ acting anti-linearly. We know that the classification is $H^1(X, \mathbb{Z}/2)$.

$$0 \longrightarrow \mathbb{Z}^{sg} \xrightarrow{2\pi i} \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

thus the usual first Chern class lands in

$H^2(X, \mathbb{Z}/2; \mathbb{Z}^{sg})$ where $\mathbb{Z}^{sg}$ is the sign representation on $\mathbb{Z}$.

and so it is probably true that

$$H^2(X, \mathbb{Z}/2; \mathbb{Z}^{sg}) \cong H^1(X, \mathbb{Z}/2);$$

Note that one then gets a theory of Chern classes for real vector bundles with

$$c_g(E) \in H^{2g}(X, \mathbb{Z}/2; (\mathbb{Z}^{sg})^{\otimes g})$$

Ingredients

Weil's calculation of the ζ-function

Fredholm theory

Step A: Interpret ζ as ~~the~~ $\boxed{\text{Fredholm determinant}}$ of a suitable operator

---

$$\zeta(s) = \prod_{x \text{ closed}} \frac{1}{1 - \frac{1}{(Nx)^s}} \qquad X$$

$$Nx = \text{card } k(x)$$

$$= q^{\deg x} \qquad\qquad z = q^{-s}$$

$$\log \zeta(s) = -\sum_x \log(1 - z^{\deg x})$$

$$= \sum_x \sum_{n \geq 1} \frac{z^{\deg x \cdot n}}{n}$$

each x gives rise to $\deg x$ geometric points in fact the

$$\sum_{m \geq 1} \frac{z^m}{m} \underbrace{\sum_{x \ \deg x | m} \deg x}_{\text{no of fix pts of } \Phi^m}$$

Weil, Boubaki 312, June 66

Tate – analytic continuation of ζ fns.

André Weil talk.

G group acting on X (adelic manifold)

to determine distributions Δ on X such that

$$g \cdot \Delta = \omega(g)^{-1} \Delta$$

where ω representation of G in Y.

and Δ dist on X values in Y.

$X = K$ local field

$G = K^\times$

$X = A_k$ adèles of a global field k

$G = A_k^\times$ idèles de k

$\omega : A_k^\times / k^\times \longrightarrow \mathbb{C}^*$.

determine distributions Δ on X values in $\mathbb{C}$ $\ni$

$$g \Delta = \omega(g)^{-1} \Delta.$$

contention:

$$\Delta(f) = \int_{A_k^\times} f(x)\, \omega(x)\, d^\times x.$$

formulas for $\Omega(PE)$ ✓

meaning of the residue

back to R-R.

The van Est spectral sequence.

de Rham cohomology for BP theory.

$\Omega_A(X)$.

---

crystalline cohomology – perhaps the simplicity of the Steenrod operations can be put to use. Thus ~~[illegible]~~ over char p one might have Lie theory of some sort.

---

x ~~$\gamma_n(x+x$~~

$\gamma_i(x)\otimes\gamma_j(x) \longrightarrow$

Morse theory for the ∂-operator

~~X complex manifold dim n.~~

Let X be a C^∞ manifold. $\varphi : X \longrightarrow \mathbb{R}$ proper with ~~non-degenerate critical~~ non-degenerate critical points

Then ~~compose~~ we have Morse decomposition of X and Morse inequalities

$$\sum_{q\leq n} (-1)^q \dim H_q(X) \leq \sum_{q\leq n} (-1)^q \operatorname{Crit}_q(X,\varphi)$$

where $\operatorname{Crit}_q(X,\varphi)$ = number of critical points of index q.

Idea: For a complex manifold and the $\bar\partial$ operator we have (Hörmander) a similar theorem.

generalization is as follows: Let E be a ~~complex~~ holom. vector bundle on as complex manifold X. Then let φ be a ~~non~~ ~~function~~ hermitian metric

Question: How can we use this information.

Problem: Given X ~~[illegible]~~ , E can you find Morse-R-R inequalities

$$\sum_{q\leq n} (-1)^q \dim H^q(X,E) \leq$$

φ has non-degenerate critical points $\Longleftrightarrow$ $(\operatorname{grad}\varphi) : X \longrightarrow T_X$ is transversal to zero. How is the index of a critical point related to the eigenvalues of Hess φ. likely that $\operatorname{index}^{\operatorname{grad}\varphi} = (-1)^{\operatorname{index}\varphi}$

Yes!!

Supports:

Recall that $\Omega_A(X)$ $\begin{matrix} Z \\ \downarrow \\ X \end{matrix}$ + trivialization over $X-A$.

Given $Q_A(X)$ theory need to define

$$\Omega_A(X) \longrightarrow Q_A(X).$$

f:

$\varphi(\gamma_1 t) * \cdots * \varphi(\gamma_n t)$

$= \varphi(\gamma_1 t * \cdots * \gamma_n t)$

Thus given

$$\begin{tikzcd} Z & Z|X\text{-}A \arrow[l, hook'] \arrow[d] & W \arrow[l, "\approx"'] \arrow[dl, bend left] \\ X \arrow[from=1-1, d] & X\text{-}A \arrow[l, hook'] & \end{tikzcd}$$

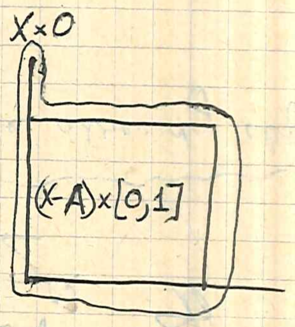
idea was that $W \longrightarrow (X-A) \times \mathbb{R}$

$$Z \longrightarrow X \times 0.$$

Thus I get a manifold ~~obs~~ ~~$X \times \mathbb{R}$~~ proper $\overset{(+\text{oriented})}{}$ over a nbd. of ~~X~~

$$(X-A) \times I \cup X \times 0$$

in $X \times \mathbb{R}$

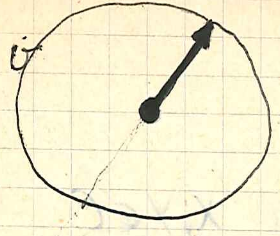


and empty over $X \times 1$.

$$U \times \mathbb{R} \cup X \times 0$$

Spanier - Whitehead duality

Weinstein told me that if you take a vector



can consider its projection onto the hypersurface and then ^we^ get a vector field on the ~~sphere~~ ^X^ hypersurface

one knows that

$$\chi(H) = \text{no. of zeros of the vector field } X \text{ counted correctly!}$$

Now this vector field vanishes whenever the normal vector is up or down

---

Choose H ~~to~~ to be ~~[illegible]~~ ^even^ diml. Then $\chi(H)$ should be non-zero ^(in general)^ hence the ~~[illegible]~~ signs at opposite ends are different! But now ?

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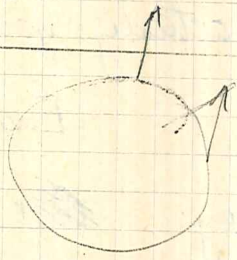
need the degree of



$$H \longrightarrow S^{N-1}$$

idea is to take N odd so H even diml.

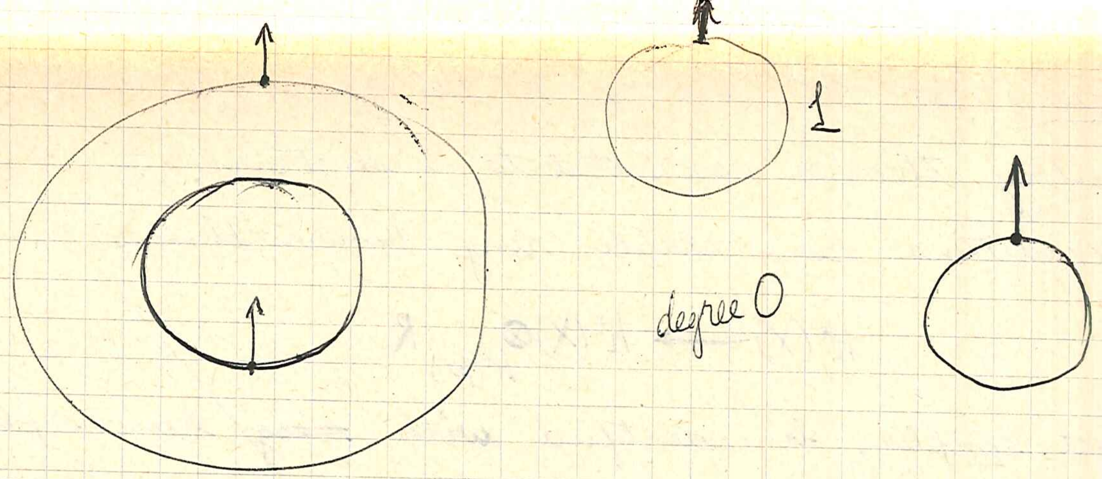
~~then take vector field on~~

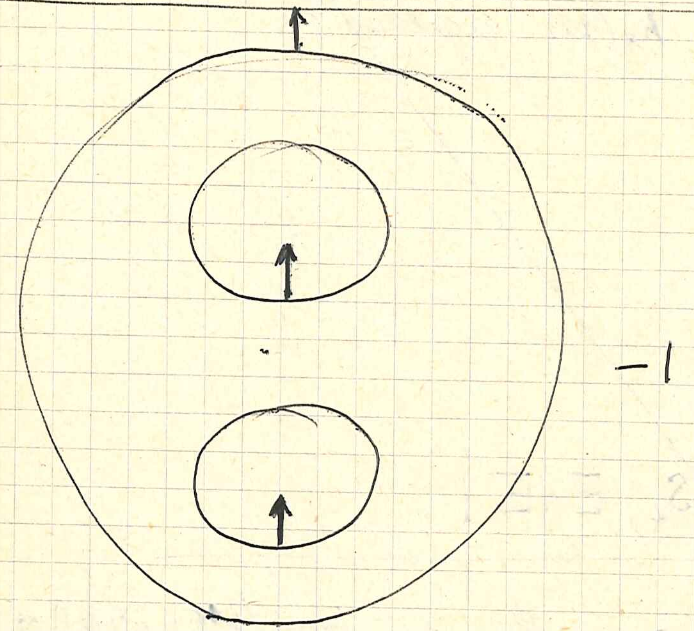
$\chi(t+aX)$? ~~3~~

idea would be that this way one can show that

$$2\cdot\deg \underset{\uparrow}{\cancel{=}} \underline{\chi(H)} = \chi(X^n)\cdot\chi(S^{N-n-1})$$

unlikely





$$\begin{tikzcd} pt \arrow[r] & X^{\nu} \arrow[r] & pt \\ S^{N} & X^{\nu} \arrow[l] & S^{N} \arrow[l] \end{tikzcd}$$

∴ thus the degree seems to be

$$\boxed{\frac{1}{2}\chi(H)}$$

∴ But $\chi(H) = \chi(X)\,\chi(S^{N-n-1})$

hence the degree of

$$S^{N} \longrightarrow X^{\nu} \longrightarrow S^{N}$$

appears to be $\chi(X)$

$$\begin{tikzcd} S \arrow[r] \arrow[rr, bend right, "\text{degree} = \chi(X)"'] & X \wedge DX \arrow[r] & S \end{tikzcd}$$

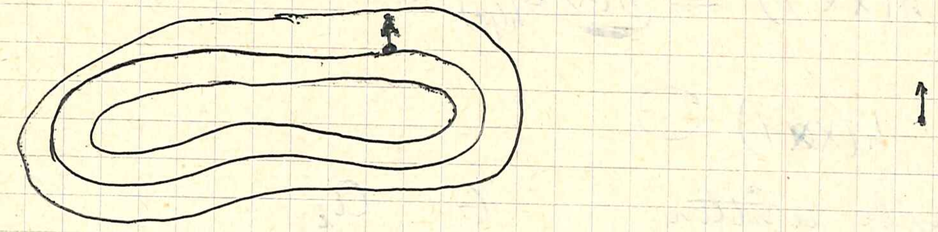
Case of a manifold

X compact manifold

$X \hookrightarrow \mathbb{R}^N$ embedding, normal bundle ν

$$\boxed{\overset{\text{open}}{\mathcal{U}} \text{ tubular nbd. of } X \simeq \nu \quad \text{via exp.}}$$

map $\vec{\mathcal{U}} \longrightarrow \mathring{D}^N$ by the Gauss map sending a point in $\mathcal{U}$ into the normal vector ending at that point.



defines a map $X^\nu = \mathcal{U} \cup \{\infty\} \longrightarrow S^N$ which is (hopefully) the dual to the map of a point to X. Thus the top coh class of X^ν should be cospherical

Consider $S^N \longrightarrow X^\nu \longrightarrow S^N$. This map has degree 0? ~~[illegible]~~

DX finite cx with basepoint

$$S^N \longrightarrow X \wedge DX$$

$$DX \wedge X \longrightarrow S^N$$

∃ ~~$X \wedge S^N$~~

$$\begin{tikzcd} S^N \wedge X \arrow[r] & (X \wedge DX) \wedge X \arrow[d] \\ & X \wedge S^N \end{tikzcd}$$

induces identity map on cohomology.

~~DX~~

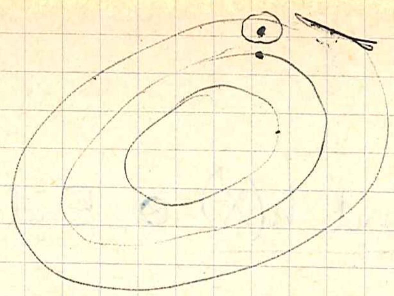
$$\begin{tikzcd} S^N \arrow[r] & \dot{U} \arrow[r] & S^N \\ S^N \arrow[r] & DX \arrow[ur] & \end{tikzcd}$$

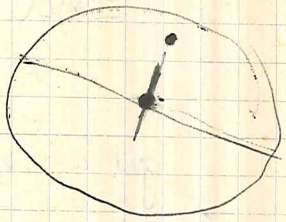
degree $\chi(X)$

~~euler class~~

$$\begin{tikzcd} H(S^N) & H^N(DX) \arrow[l] & H^N(S^N) \arrow[l] \arrow[dll, dash] \\ & H^n(X) \arrow[u, "\wr", "c"'] \arrow[ul, "\int"'] & H^0(pt) \arrow[l, "\text{euler class for a manifold !!}"'] \arrow[u] \end{tikzcd}$$

$$\int e = \chi(X)$$





$$S^N \longrightarrow S^N$$

take a point $x \in S^N$ and consider all the things mapping to it

to take generic plane and look at the tangenCy points on the spherical nbd.

given an oriented hypersurface in $\mathbb{R}^N$

$$H \subset \mathbb{R}^N$$

consider Gauss map

$$H \longrightarrow S^{N-1}$$

what is it's degree?

thus how many times does one get a given tangency.

~~$X \wedge DX \longrightarrow S$~~

~~$DX \wedge X$~~

$$DX \wedge X \longrightarrow S$$

$$X \hookrightarrow S^N$$ embeds in

$$\begin{tikzcd} \boxed{U \times X} \arrow[d] \\ {} \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[d] & \\ X \wedge DX \arrow[r] \arrow[d] & S \\ \boxed{DX} & \end{tikzcd}$$

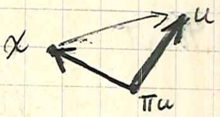
<u>retraction</u>

$$\begin{tikzcd} X \wedge DX \arrow[r] \arrow[d] & S \\ DX & ? \end{tikzcd}$$

---

compact manifold have

$$X \longrightarrow DX$$



$$X \wedge X^{\nu}$$
$$\cup$$
$$\overset{x}{X} \times \overset{u}{U} \cong (\overset{x}{X} \times \overset{\pi u}{X}) \times_X \overset{u}{U}$$
$$\cup$$
$$\overset{x-\pi u}{\tau} \times_X \overset{u-\pi u}{\nu} \overset{\longmapsto (\pi u,\ u-x)}{\cong} X \times \mathbb{R}^N$$

passing to compactifications we get

~~so get~~ $$X \wedge X^{\nu} \longrightarrow X \wedge S^N \longrightarrow S^N.$$

$$X \wedge DX$$
$$\cup$$
$$X \times U \longrightarrow$$

$f \sim g \Rightarrow f^* = g^*$

$$\begin{tikzcd} & W \arrow[d] \\ X \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] & X \times \mathbb{R} \end{tikzcd}$$

$\iota_0 \sim \iota_1$

$$\begin{tikzcd} Z \arrow[d] \\ X \arrow[d, "f"] \\ Y \end{tikzcd}$$

consequence is that

$\boxed{f \sim g} \Rightarrow f_* = g_*$

means that

$$X \times \mathbb{R} \xrightarrow{h} Y$$

(h proper)

$$\begin{tikzcd} X \arrow[r, shift left, "h_0"] \arrow[r, shift right, "h_1"'] \arrow[d, "\iota_0{,}\iota_1"] & Y \arrow[d, "\iota_0{,}\iota_1"] \\ X \times \mathbb{R} \arrow[r, "orient", "h"'] & Y \times \mathbb{R} \end{tikzcd}$$

proper

suppose h_0 oriented

$$\begin{tikzcd} X \arrow[d] \\ X \times \mathbb{R} \arrow[r, "\nu"'] & {} \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "\lambda_{h_0}"] \arrow[d, "\iota_0"] & A \arrow[d, "\Phi"] \\ X \times \mathbb{R} \arrow[r, "\nu_h"'] \arrow[ur, dashed] & \mathbb{Z} \times BO \end{tikzcd}$$

YOU MUST MOVE A ~~PROPER~~ PROPER ORIENTED MAP TO PROVE $f'_* g'^* = g^* f^*$

If u, v, w are three reps. ~~I would~~

$$\begin{tikzcd}
\boxed{D_u \times D_v \times D_W} \arrow[r] \arrow[d] & D_u \times D_{v\otimes W} \arrow[d] \\
D_{u\otimes v} \times D_W \arrow[r] & \underline{D_{u\otimes v\otimes W}}
\end{tikzcd}$$

should commute.

$$\underline{R\{X, Y, Z\}} \qquad F(X, F(Y, Z))$$

$$F_{u,v} : D_u \times D_v \longrightarrow D_{u\otimes v}$$

$$\mathrm{Hom}(D_u \times D_v, D_{u\otimes v}) = \mathrm{Hom}\left(R[X]/\rho_{u\otimes v}(X), \; R[X]/\rho_u \otimes R[X]/\rho_v\right)$$

I want an elt. :

$$= \mathrm{Hom}($$

$$\longrightarrow \overset{\xi}{\underline{R[X]/\rho_u \otimes R[X]/\rho_v}} \longrightarrow \overset{\rho_{u\otimes v}(\xi)}{R[x]/\rho_u \otimes R[X]/\rho_v}$$