Cobordism with supports

axioms

funct.

$$\begin{matrix} X & \xrightarrow{f} & Y \\ A & & B \end{matrix} \qquad f^{-1}B \subset A$$

$f^*: H_B(Y) \longrightarrow H_A(X)$

$f_*: H_A(X) \longrightarrow H_{fA}(Y)$.

cartesian axiom

$$\begin{matrix} X'^{\,A'} & \longrightarrow & X^{A} \\ \downarrow f' & & \downarrow f \\ Y' & \longrightarrow & Y \end{matrix}$$

$f^{-1}A \subset A'$

$H_A(X) \longrightarrow H_{g'^{-1}A}(X')$

($\Longrightarrow f_*$ comp. with increasing supports)

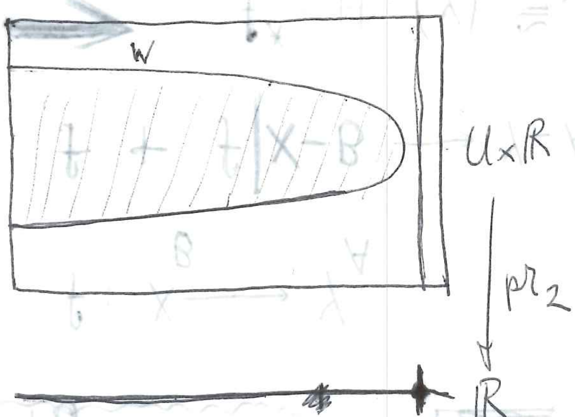
homotopy axiom

$$\begin{matrix} f: X & \longrightarrow & Y \\ B & & A \end{matrix}$$

f + $f|X-B \longrightarrow Y-A$ are heq's

$\Longrightarrow$ $f^*: H_A(Y) \xrightarrow{\cong} H_B(X)$

$$\boxed{H^{d-1}(U\times\mathbb{R}-W)} \xrightarrow{\delta} H^d_W(U\times\mathbb{R}) \longrightarrow H^d(U\times\mathbb{R}) \longrightarrow H^d(U\times\mathbb{R}-W)$$



$$\begin{array}{ccccc}
 & & H^d(Z) & & \\
 & & \downarrow & & \\
H^d(X, U\cup(X-Z)) & \to & H^d_Z(X) & \longrightarrow & H^d_{Z\cap U}(U) \\
 & & \downarrow & & \downarrow \\
H^d(X,U) & \longrightarrow & H^d(X) & \longrightarrow & H^d(U)^{\circ}
\end{array}$$

$(X, X-Z) \longrightarrow ($

$$\begin{array}{ccc}
H^d(X,U) & & H^d(X) \\
 & & \downarrow \\
H^d(X-Z, & & H^d(X-Z)
\end{array}$$

$$
\begin{array}{ccc}
H^0(Z_i) & \longleftarrow & \overset{1}{H^0(W)} \\
\downarrow s & & \downarrow s \\
H^d_{Z_i}(X) & \longleftarrow & H^d_W(X \times \mathbb{R}) \\
\downarrow & & \downarrow \\
H^d(X) & \overset{\sim}{\longleftarrow} & H^d(X \times \mathbb{R})
\end{array}
$$

Now suppose given U open in X and W joining $Z_1 - U$ to $Z_2 - U$ in $X - U$. Then one has

---

~~$H^0(Z_i)$~~

$Z_2 = \phi$.

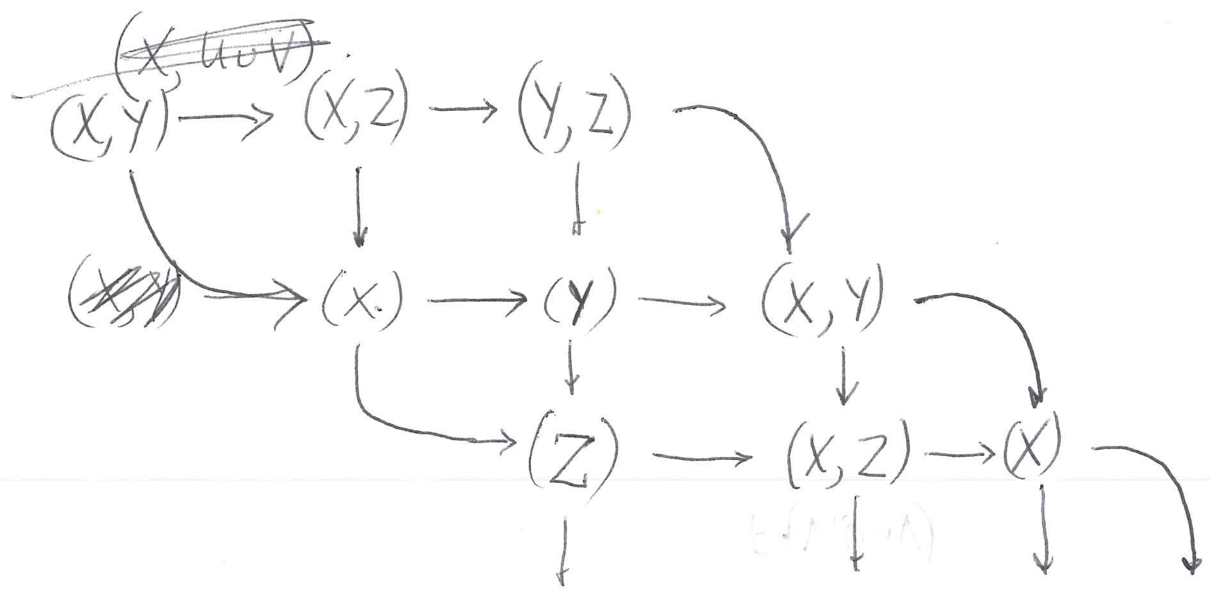
$$
\begin{array}{ccccc}
H^0(Z) & \longleftarrow & \overset{1}{H^0(W)} & \longrightarrow & H^0(\phi) \\
\downarrow s & & \downarrow s & & \downarrow \\
H^d_2(X) & \longleftarrow & H^d_W(X \times \mathbb{R}) & \longrightarrow & H^d_\phi(X) \\
\downarrow & & \downarrow & & \downarrow \\
H^d(X) & \overset{\sim}{\longleftarrow} & H^d(X \times \mathbb{R}) & \overset{\sim}{\longrightarrow} & H^d(X)
\end{array}
$$

$$F(X,U)$$

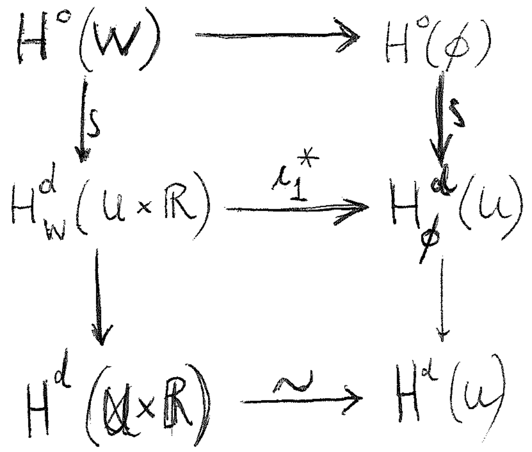
$$F^{\ell}(X,V) \qquad F^{\ell}(X) \qquad F^{\ell}(V)$$

$$F(U,U\cap V) \longrightarrow F(U) \longrightarrow F(U\cap V)$$

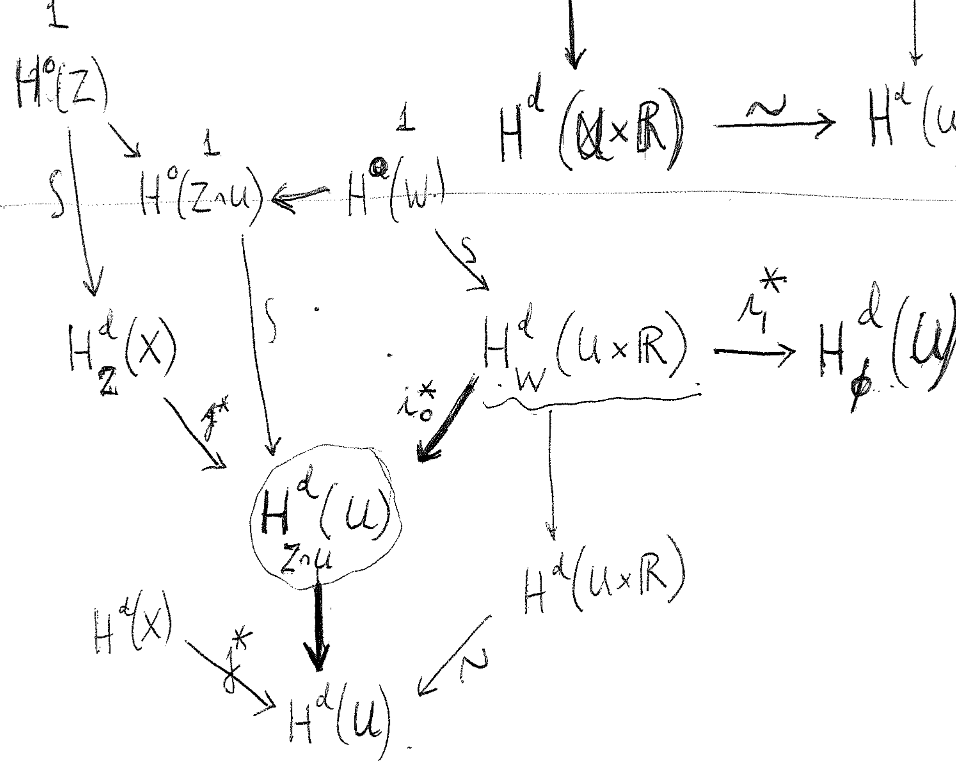
$$F(V,U\cap V)$$



(X,Y)
(X,Z)
(Y,Z)
(X.)
(Y)
(X,Y)
(Z)
(X,Z)
(X)

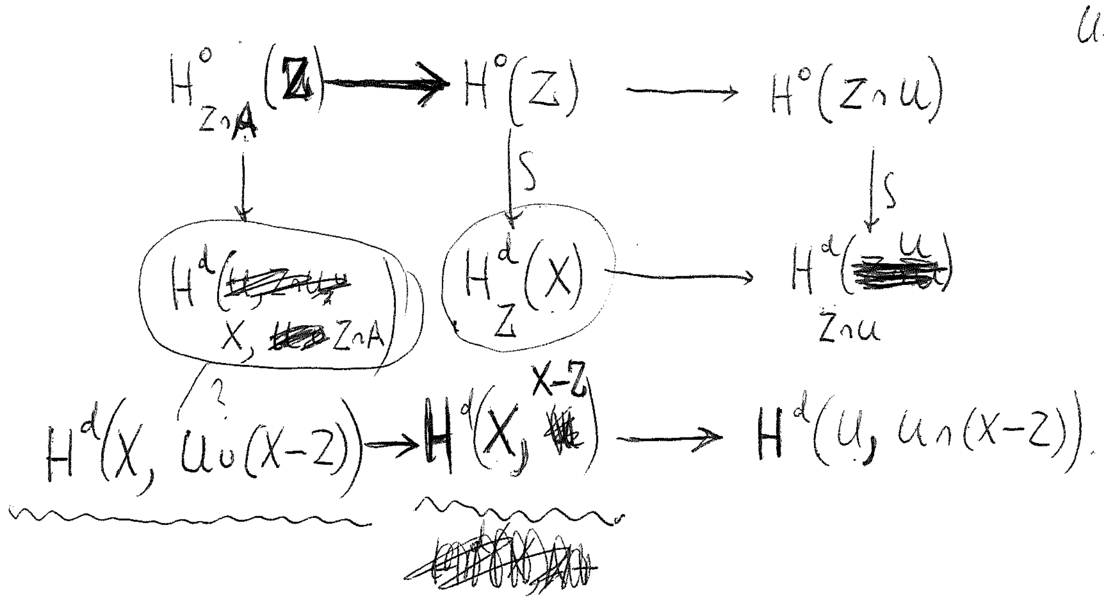


H°(W)
H°(∅)
H^d_W(U × ℝ)
H^d_∅(U)
H^d(U × ℝ)
H^d(U)

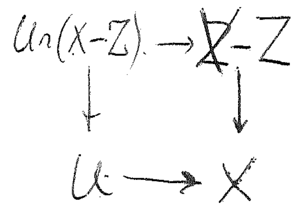


H°(Z)
H°(Z∩U)
H°(W)
H^d_Z(X)
H^d_W(U × ℝ)
H^d_∅(U)
H^d_{Z∩U}(U)
H^d(U × ℝ)
H^d(X)
H^d(U)

Z, Z ∩ U



H°_{Z∩A}
H°(Z)
H°(Z∩U)
H^d(X, Z∩A)
H^d_Z(X)
H^d_{Z∩U}(U)
H^d(X, U∪(X−Z))
H^d(X, X−Z)
H^d(U, U∩(X−Z))



U∩(X−Z)
X−Z
U
X

$W \quad \omega \in H^{q-1}(U, U - Z \cap U)$

$\delta\omega \in H^q$

(X, V)

$H^d_Z(X)$

$W \quad H^{q-1}(U - Z \cap U)$ ← open set.

In addition one has

$U \cap V$

$\downarrow$

$H^q(X, (X - Z) \cap V)$.

$X, X - Z$

$\downarrow$

$X, X - V \cap U$

$Z \cup A$

$$H^{q-1}(U \cap V) \xrightarrow{\delta} H^q(U, V \cap U) \xleftarrow{} H^q(X, V)$$

$$\begin{array}{ccc} & & H^q_Z(X) \\ & & \downarrow \\ H^{q-1}(U \cap V) & \rightarrow & H^q_{Z \cup A}(X) \\ & & \downarrow \\ & & H^q_{Z \cap U}(U) \end{array}$$

example

$$\Omega(U) \xrightarrow{\delta} \Omega_{*}(X, U)$$

given by $W \longrightarrow U$ $\quad \partial W = \phi$.

~~how to define~~ It seems that one must have

$$H(U) \xrightarrow{\delta} H(X, U)$$

at one's disposal.

$$H^q_{A\cap Z}(X) = H^q(X, U \cup (X-Z)) \qquad H^q_Z(X)$$

$$\begin{array}{ccc} & & H^q_Z(X) \\ & & \downarrow \\ H^q(X,U) & \longrightarrow & H^q(X) \end{array}$$

$Z = \phi$

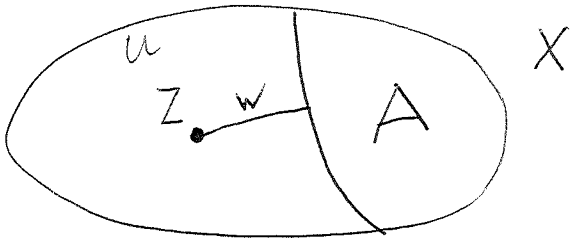
$$\begin{array}{ccccc} H^q_{A\cap Z}(X) & \longrightarrow & H^q_Z(X) & \longrightarrow & H^q_{Z\cap U}(U) \\ \downarrow & & \downarrow & & \downarrow \\ H^q(X, U) & \longrightarrow & H^q(X) & \longrightarrow & H^q(U) \end{array}$$

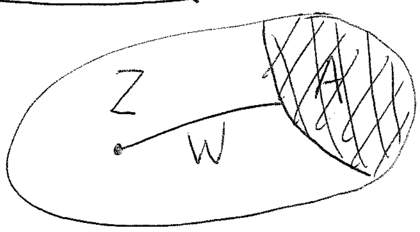
Claim W defines an element of

$$H^{q-1}(U - Z \cap U) = H^{q-1}((X-Z) \cap U)$$

$$= 1$$

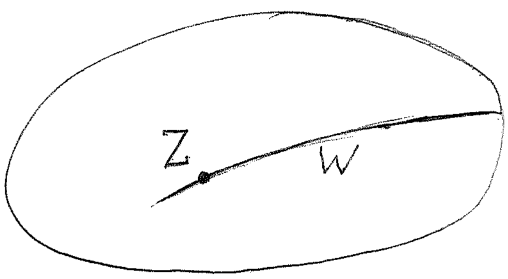
$H^*(X, U)$

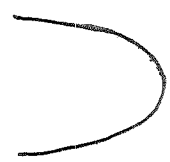




$$H^0(Z) \xrightarrow{\sim} H^d_Z(X) \longrightarrow H^d(X)$$

suppose $U = X$. Then have $Z = \partial W$ where W is of codimension $d-1$. To show that $\iota_* 1 = 0$.





Method form

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \\ \mathbb{R} & & \end{array}$$

$$W \xrightarrow[\text{proper}]{} X \times \mathbb{R} \qquad \begin{array}{ccc} Z & \longrightarrow & W \\ \downarrow & & \downarrow \text{proper} \\ X & \xrightarrow{i_0} & X \times \mathbb{R} \end{array}$$

Given $Z \xrightarrow{f} X$ proper oriented

$+ \quad \varphi : Z \simeq \partial W \qquad W \xrightarrow{g} U$ proper-or.

Define an element of [redacted] $\boxed{H_*(X, A)}$

~~Stage 1~~

$Z \hookrightarrow X \times \mathbb{R}^n$ assume $n = 0$

$$\begin{array}{ccc} Z & \hookrightarrow & X \\ \uparrow & & \uparrow \\ Z \cap U & \longrightarrow & U \\ \downarrow & & \downarrow i_0 \\ W & \hookrightarrow & U \times I \end{array}$$

$H^0(Z) \xrightarrow{\simeq} H^d_Z(X) \longrightarrow H^d(X)$

$H^0(Z \cap U) \xrightarrow{\simeq} H^d_{Z \cap U}(U) \longrightarrow H^d(U)$

$H^d(U \times I)$

$H^q(X) \qquad H^q(U)$

$H^q(X, U) \longrightarrow H^q(X) \longrightarrow H^q(U)$

[redacted] $H^q(X) \longrightarrow H^q(U)$

$H^q(Mi, U) \nearrow H^q(X)$

homology: given a map

$$
\begin{array}{ccc}
X'^{\,A'} & \longrightarrow & X^{A} \\
\downarrow & & \downarrow f \\
Y'^{\,B'} & \longrightarrow & Y^{B}
\end{array}
\qquad\qquad
\begin{array}{ccc}
H_{A'}(X') & \xleftarrow{\;g'^*\;} & H_A(X) \\
\downarrow f'_* & & \downarrow f_* \\
H_{B'}(Y') & \xleftarrow{\;g^*\;} & H_B(Y)
\end{array}
$$

These things are to coincide in the following cases:

~~enough to consider~~ ~~$B = fA$~~ ~~$A' = g^{-1}A$~~ ~~$B' = f'g'^{-1}A$~~ ~~$g^{-1}fA$~~

I want (this) to be cartesian and transversal in some neighborhood of A' in X'.

$$
\begin{array}{ccc}
\emptyset & \qquad & W, w \\
 & & \downarrow \\
X_A & & X_A
\end{array}
$$

$W \xrightarrow{\;f\;} X$

and $fW \subset A \implies H(W) \xrightarrow{\;f_*\;} H_A(X)$.

suppose given W over $X \times \mathbb{R}$

proper over $U \times \mathbb{R}$

proper over a nbd of $X \times 0$

empty over

Given W proper over a nbd of $(U \times \mathbb{R}) \cup (X \times 0)$ in $X \times \mathbb{R}$ and emptly over $U \times 1$.

Define an element of $H^*(X, U)$.

Idea somehow is that this neighborhood N ~~defines a map~~ plays the role of M_j.

~~$H^*(X, N)$~~

$(1+\varepsilon, \infty)$

$\therefore$ have element of $H^*(M_j, U \times 1)$

$$\begin{array}{c} H^*(M_j, U \times 1) \\ \uparrow \wr \quad \text{homotopy axiom} \\ H^*(X, U) \end{array}$$

---

homotopy axiom is to be formulated: ~~the way how~~

~~Hard method is~~ $f^*: H(X, U) \xrightarrow{\sim} H(Y, V)$

if a heq of ~~pairs~~. X with Y and U with V.

---

if W proper of X and situated over A it defines an element of $H^*(X, U)$:

$$\begin{array}{ccc} & & W \\ & & \downarrow \\ X_{,A} & \longrightarrow & X \end{array}$$

$(M_j, U) \longrightarrow (X, U)$

$Z \mapsto$ submanifold of X
$W \mapsto$ ——— $U \times I$

$\therefore Z \cup W \mapsto$ "submanifold" of $M_j = X \cup_{U \times 0} U \times I$

hence $\exists$ a fundamental class in $H^d(M_j, U \times 1)$

now use homotopy axiom somehow to show that

$$\boxed{H^d(M_j, U \times 1) \simeq H^d(X, U)}$$

C^0 style. $\quad$ C^∞ style

---

basic result is that any element of $\Omega_A(X)$ is representable as a manifold over M_j not meeting $U \times 1$

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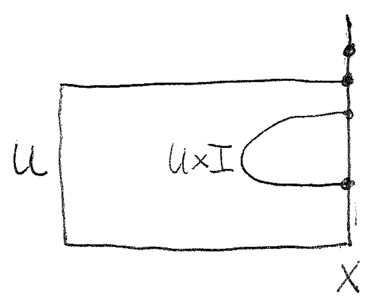
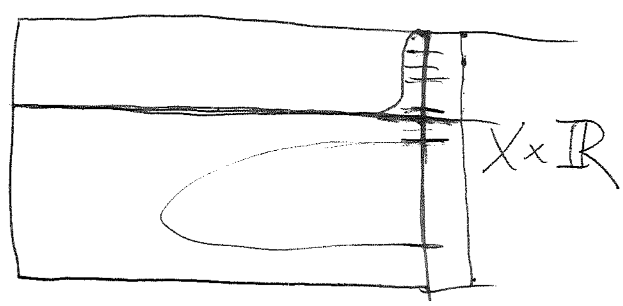
basic thm.

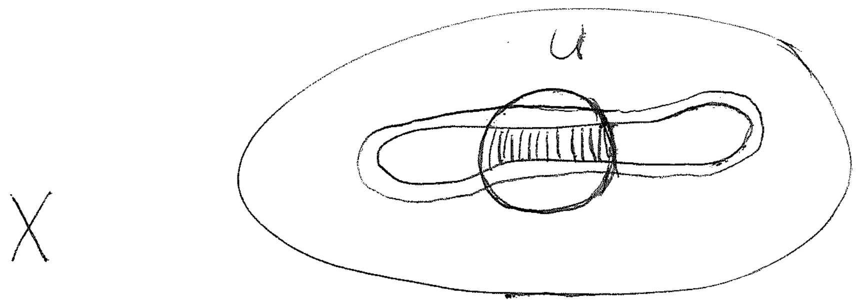




given some $V \subset X \times \mathbb{R}$ containing $U \times I$

given $W \cup Z \times (-\varepsilon, \varepsilon)$ submanifold over $X \times \mathbb{R}$ proper over ~~$0 \times \mathbb{R}$~~ $X \times 0 \cup U \times I$

$$\Omega(X, U) = ~~[\ldots]~~ \quad [X, U; M] \downarrow [X, U; K]$$

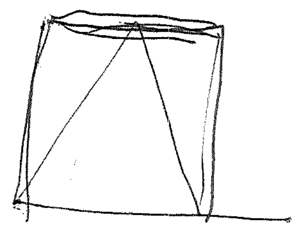


Choose a normal tube to W and to Z

The normal tube to Z

Have $j : U \to X$

$$C(j) \longrightarrow MU(d).$$

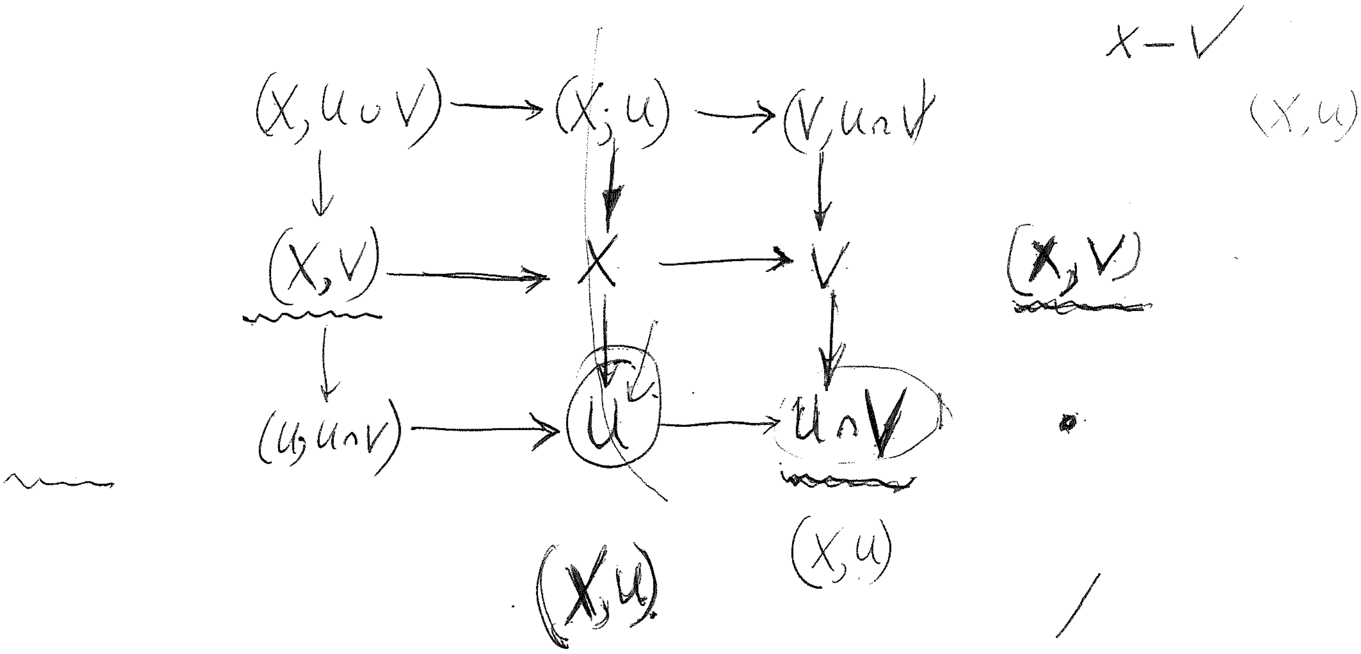


$$H^d(C(j), U) \longleftarrow H^d(MU(d), pt)$$

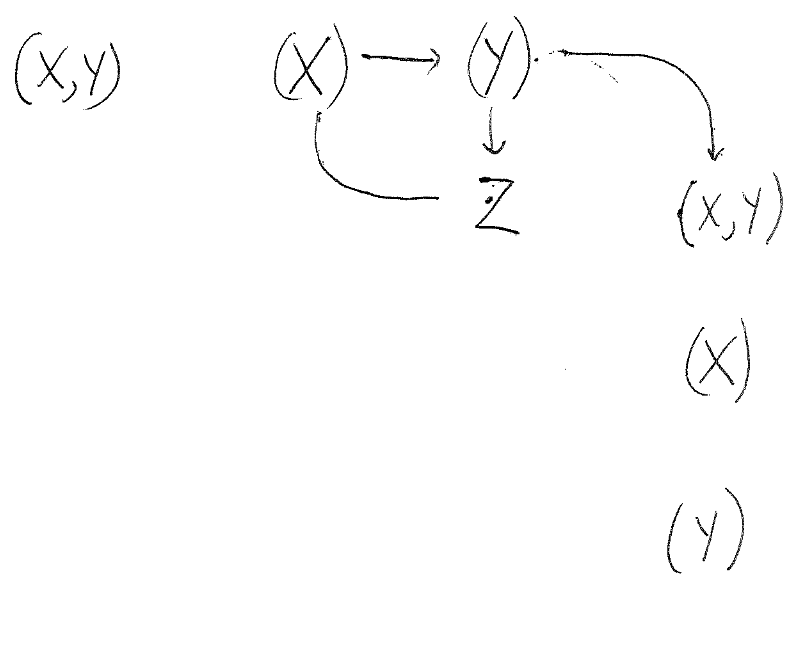
$$H^d(X, U)$$

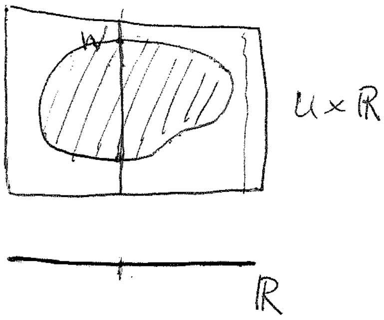
W gives $H^{q-1}(U-(Z\cap U))$

Z gives something $H^{q}(X, X-Z)$



braid triad $X \supset Y \supset Z$

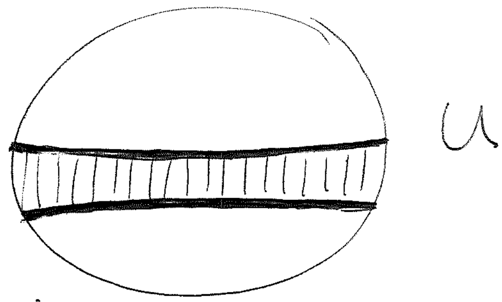




so W should define an element of $H^{d-1}(U\times\mathbb{R}-W) \xrightarrow{\delta} H^d_W(U\times\mathbb{R})$
with boundary W

$$\begin{array}{ccc} H^{d-1}(U\times\mathbb{R}-W) & \xrightarrow{\delta} & H^d_W(U\times\mathbb{R}) \\ \downarrow \iota_0^* & & \downarrow \iota_0^* \\ H^{d-1}(U-Z\cap U) & \xrightarrow{\delta} & H^d_{Z\cap U}(U) \end{array}$$

$\therefore\ \iota_0^* W = \delta$



I now have a class in $H^{d-1}(U - Z\cap U) \xrightarrow{\delta} H^d(Z\cap U)$

hence a class in $H^d(X-Z, U-(Z\cap U))$

~~hence a class in $H^d(X, U)$~~

2nd attempt:

You have defined $\Omega_A(X)$. On the other hand one has defined $H_A^*(X)$ by sheaf theory. Problem: Define natural transf

$$\Omega_A(X) \longrightarrow H_A(X)$$

functorial + compatible with exact sequences.

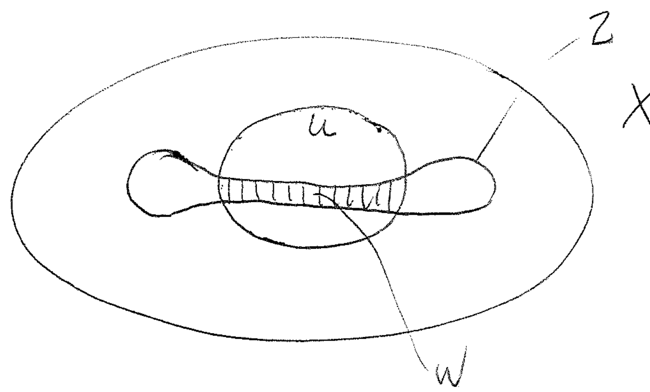
---

given $\quad U$ open in X

$Z \hookrightarrow X$ submanifold

$W \hookrightarrow U$ submanifold with ∂

$\exists \quad \partial W = Z \cap U$.



define the class of Z as an element of $H(X, U)$

$$H^{d-1}(U) \xrightarrow{\delta} H^d(X, U) \longrightarrow H^d(X) \longrightarrow H^d(U)$$

$$
\begin{array}{ccccc}
(X,(X-Z)\cup U) & \longrightarrow & (X,U) & & \\
\downarrow & & \downarrow & & \\
(X, X-Z) & \longrightarrow & (X) & \longrightarrow & (X-Z) \\
\downarrow & & \downarrow & & \\
((X-Z)\cup U, X-Z) & & (U) & &
\end{array}
$$

$$
\boxed{
\begin{array}{ccccc}
(X, U\cup V) & \longrightarrow & (X,U) & & \\
\downarrow & & \downarrow & & \\
(X,V) & \longrightarrow & X & \longrightarrow & V \\
\downarrow & & \downarrow & & \downarrow \\
\cdot & & U & \longrightarrow & U\cap V
\end{array}
}
$$

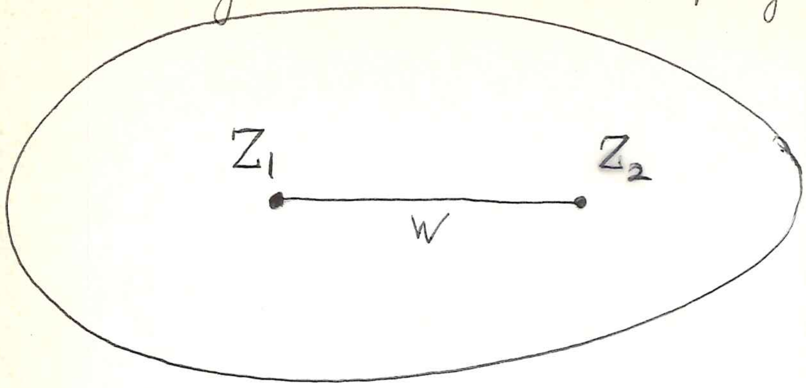
---

attempt

$$
\begin{array}{ccc}
H^0(W) & & \\
\downarrow & & \\
H^d_W(U\times\mathbb{R}) & \xrightarrow{i_1^*} & H^d_\phi(U) \\
\downarrow & & \\
H^d(U\times\mathbb{R}) & &
\end{array}
$$

$$
\begin{array}{ccccc}
H^d_{(A\times\mathbb{R})\cup W}(X\times\mathbb{R}) & \longrightarrow & \overset{?}{H^d_W(X\times\mathbb{R})} & \longrightarrow & H^d_W(U\times\mathbb{R}) \\
H^d(X\times\mathbb{R}, U\times\mathbb{R}) & \longrightarrow & H^d(X\times\mathbb{R}) & \longrightarrow & H^d(U\times\mathbb{R})
\end{array}
$$

~~ultimately comes down to a projection~~



$$H^d_{Z_1}(X) \longrightarrow H^d(X) \longleftarrow H^d_{Z_2}(X)$$

$$\begin{array}{ccccc} Z_{i+1} & \longrightarrow & W & \longrightarrow & pt \\ \downarrow & & \downarrow & & \\ X \times \{i\} & \longrightarrow & X \times [0,1] & & \end{array}$$

Thus we have W inside of $X \times [0,1]$

enough to have W proper over $X \times \mathbb{R}$.

$$H^0(W)$$