March 12, 1969

Decomposition of the Lazard scheme over $F_p$

Let $L_p = O(pt) \otimes F_p$ be the Lazard ring over $F_p$, let $F_0$
denote the universal group law. Let $\mathcal{X} = \text{Spec } L$ and let

$$\mathcal{X} = Z_1 \supset Z_2 \supset Z_3 \supset \cdots \supset Z_n$$

be the subschemes given by

$$Z_i(T) = \{\text{formal group laws over } T \text{ of height } \geq i\}$$

Let

$$X_i = Z_i - Z_{i+1}$$

so that

$$X_i(T) = \{\text{formal group laws over } T \text{ of height } = i\}$$

for any $F_p$-scheme $T$.

Fix an integer $h \geq 1$ and let $F_0$ be a group law of height
$h$ over $F_p$. It is clear that the group scheme $N$ of power
series $a_1 X + \cdots + a_h$ acts on $X_h$ and hence there is a
map

$$N \rightarrow X_h$$

$q \mapsto q \cdot F_0$

**Theorem.** The stabilizer $H_{F_0}$ of the law $F_0$ is an isomorphism of
profinite group scheme over $F_p$ and the above map induces an
isomorphism of schemes

$$N/H_{F_0} \sim X_h$$
Thus $X_h$ is a "homogeneous space" scheme under $N$.

**Proof.** Given an $F_p$-algebra $R$ and a formal group $F$ over $R$, consider the algebra $R' = R[a_1, a_1^{-1}, a_2, \ldots]$ obtained by adjoining elements $a_i$, $i > 1$, $a_i$ invertible, subject to the relations

$$(*) \quad \varphi F_0 = F,$$

where $\varphi(x) = \sum q_i x^i$.

Following Lazard we are going to analyze the equations $(*)$ and show that $R'$ is obtained from $R$ by adjoining roots of equations. Thus $R^* \rightarrow R'$ will be free and ind-etale. Applying this theorem $R = \text{coordinate ring of } X_h$ we obtain a map $X_h' \rightarrow X_h$ which is flat, integral, and pro-etale.

$X_h'(T) = \{(F, \varphi) \mid F \text{ is a formal group law of height } h \text{ and } \varphi \text{ is an isomorphism of } F \text{ with } F_0^j \text{ over } X_h \}$

is covering for the finite topology and so $N/H_F = X_h$. Also as $N \rightarrow X_h$ is etale, $H_F$ will be an pro-etale group scheme over $F_p$.

A. If $q$ is not a power of $p$, then $a_2$ is a polynomial in $a_1^2, \ldots, a_{q-1}$ in $R$.

In effect let $\varphi_{q-1}(x) = \sum_{i=0}^{q-1} a_i x^i$ so that

$\varphi_{q-1} \ast F_0 = F + c B_q \mod \text{degree } q+1$
where \( x \in R[\alpha_1, \ldots, \alpha_{g-1}] \). One then has that \( \alpha_g = -c \).

B. If \( g = 1 \), then \( \alpha_1 \) satisfies the root of the equation
\[
\lambda^h \lambda_0 - \lambda_0 = \lambda^h
\]
where
\[
[p]_F^e (x) = \lambda_0 x^p + \ldots
\]
\[
[p]_F (x) = \lambda x^p + \ldots
\]
\((\lambda_0, \lambda \in R^* \text{ since the law has height } h)\).

C. If \( g = p^k \), \( k \geq 1 \), then \( \alpha_g \) has a root of the equation
\[
\lambda^h \alpha_g^p - \lambda^p \alpha_g = \mu_k
\]
where \( \mu_k \in R[\alpha_1, \ldots, \alpha_{g-1}] \).

In effect, one has \( \psi_{g+1} * F_0 \equiv F \mod \deg g+1 \) hence
\[
\psi_{g-1} * [p]_F^e \equiv [p]_F^e + \mu_k x^{p^k} \mod \deg p^h(p^{k+1})
\]
and also that \( \psi_g (x) = \psi_{g-1} (x) + a_g x^g \) so
\[
\psi_g * [p]_F^e \equiv \psi_{g-1} * [p]_F^e + (\lambda^h \alpha_g - \lambda \alpha_g^p) x^{g^h} \mod \deg p^h(p^{k+1})
\]
so \( \alpha_g \) satisfies the boxed equation.

*(since
\[
\psi_{g-1} * [p]_F^e \equiv [p]_F^e \mod \deg g+1
\]
one has that \( \psi \psi_{g-1} * F_0 \equiv F \mod g+1 \), lemma 6).
Therefore one has

\[ R \rightarrow R[a_1] \rightarrow R[a_1, \ldots, a_{p-1}] \rightarrow R[a_1, \ldots, a_p] = R[a_1, \ldots, a_{p-1}] \rightarrow R[a_1, \ldots, a_p] \]

- etale
e of group
\[ \mu_{p^{p-1}} \]
- etale
e of degree
\[ p^k \]
- etale
e degree
\[ p^h \]

Questions: Over \( \mathbb{F}_p \), suppose \( [p]_p(X) = a_1x^1 + \cdots + a_{p-1}x^{p-1} + (X^p + \cdots + X^{p^2}) + \cdots \) are the coefficients of degree not a power of \( p \) zero when the law is typical? Can the above proof of Lazard be simplified using a reduction to a typical law?
1. Let $F$ be a group law over $\mathbb{F}_p$ with $[p]_F(x) = x^p.\$

Let $A$ be the scheme of automorphisms of $F$, and let $A_1$ be the subgroup scheme of automorphisms $\equiv \text{id} \mod \deg 2$. Calculate the cohomology with coefficients $\mathbb{F}_p$ of $A$ and $A_1$ over $\mathbb{F}_p$. Let $E$ be the group scheme associated to the finite group $E^*$, where $E$ is the maximal order in the division algebra over $\mathbb{Q}_p$ with invariant $\frac{1}{h}$. $E^*$ is a semi-direct product $(\mathbb{F}_p^*)^* \times E_1$ where $E_1$ is a group of automorphisms $\equiv \text{id} \mod \deg 2$. (In fact Lubin claimed that one could assume $F$ had only terms of degree divisible by $p^h$ so that $(\mathbb{F}_p^*)^*$ acts as endos in the obvious way e.g. $\alpha X$. Then any endo is uniquely expressible as an infinite sum $\sum_{n=0}^{\infty} \alpha_n X^n$, where $\alpha_n \in (\mathbb{F}_p^*)^*$, where $F$ is the Frobenius (endo since $F$ defined over $\mathbb{F}_p$) and where the sum is taking in the sense of endos. $E_1$ consists of those autors. $E_1$ is a $p$-adic Lie group of dimension $h^2$, and as it is torsion free its cohomology should have Poincaré duality of dimension $h^2$. It is unlikely that the cohomology algebra be an exterior algebra except if $h < p-1$. In effect the logarithm log: $E^* \to D$ doesn’t have a matching exponential except for valuation $\geq \frac{1}{p-1}$ and one has the element $\varphi$ in $E_1$ with valuation $\frac{1}{h}$ since $\varphi^h = \text{the series } X^{p^h} = [p]_F$. To what extent does the cohomology of the
scheme of autos of a group law of height $h$ over $\mathbb{F}_p$, approach the cohomology of the steenrod algebra as $h \to \infty$

2. Can one find a cohomology theory given by a $\mu$-algebra spectrum (convergent) having a group law of a given height such that the endos of the formal group law give rise to Adams operations in the cohomology theory.

3. Operations in BP theory corresponding to the operators $1 + a_1 V + a_2 V^2 + \cdots$ on typical coordinates.

4. Lifting laws from char $p$ to char 0. Using Lazard we have the canonical group law on $X_1$ becomes canonically isomorphic to $\hat{\mathbb{G}}_m$ over $\mathbb{N}$.

\begin{matrix}
\text{Spec } L[\mathbb{P}^{-1}] \otimes \mathbb{F}_p & \text{Spec } L[\mathbb{P}^{-1}] \otimes \mathbb{Z}_p \\
\text{represents laws of ht 2} & \text{laws of ht 1 over } \mathbb{Z}_p \\
\text{on char. } p & \\
\end{matrix}

should exist by then an equivalence of etale topology of $Y_1$ and $X_1$.

Using Lubin-Tate it should be true that the canon law on $Y_1$ becomes canonically isomorphic to $\hat{\mathbb{G}}_m$ over $\mathbb{M}$.\)
5) Let $E$ be the endomorphism of the group law of height $h$ over $\mathbb{F}_p$, but $D = \text{quotient field of } E$, $\mathcal{O} = \text{maximal ideal of } E$, so that $E/\mathcal{O} \simeq \mathbb{F}_p$. We want to calculate the cohomology of $A_1 = 1 + \mathcal{O}$ with coefficients in $\mathbb{Z}/p\mathbb{Z}$. As a first approximation (?), we calculate the cohomology

$$\lim_{\rightarrow} H^*(U, \mathbb{Q}_p)$$

where $U$ runs over the open subgroups of $A_1$. By a theorem of Lazard this is the same as the Lie algebra cohomology $H^*(\mathfrak{g}, \mathbb{Q}_p)$ where $\mathfrak{g}$ is the Lie algebra of $A_1$. Here one has an exponential-logarithm correspondence between $h_{-1} + E \otimes \mathbb{Q}^\ast$ and $E \otimes \mathbb{Q}^\ast$, whereas $\mathfrak{g} = \mathfrak{a} \otimes \mathbb{Q}^\ast$. Let $v = \text{normalized valuation}$, $\mathbb{F} = \text{Frobenius}$ so that $\mathbb{F}^h = p$ and $v(\mathbb{F}) = \frac{t}{h}$; hence a must be an integer $\geq \frac{h}{p-1}$.

In any case the Lie algebra of $A_1$ as a $p$-adic analytic group is from $\mathbb{Q}_p$ to $K$ a splitting field $D$ under bracket. After base extension $D$ becomes isomorphic to $\text{Hom}_K(V, V)$ where $\dim_K V = h$ and one knows that

$$H^*(\text{gl}(V), K) = \text{an exterior algebra with generators of degree } 1, 3, \ldots, 2h-1.$$ 

Hence the same formula will hold before base extension as $D$ being reductive will have its cohomology an exterior algebra on the primitive elements (Koszul). Thus

$$H^*(A_1, \mathbb{Z}_p) \simeq \text{exterior algebra on generators of degree } 1, 3, \ldots, 2h-1 \mod p\text{-torsion.}$$

for a large. Actually this formula should hold for
$a = 1$ as the action of $A_n/A_{n+1}$ on $H^*(A_{n+1}, \mathbb{Z}_p)$ should be trivial as it coincides with an inner auto. of the Lie algebra.

Note that in Fröhlich, prop. 3, page 80, $E/pE$ is isomorphic to the ring of endos. $\sum_{i=0}^{h-1} \xi^i x^i$ of the additive group over $\mathbb{F}_p$, hence $A/A_n^* = E^*/1 + pE$ is isomorphic to the group of auto. of the additive group truncated at $\leq$ order $h$ over $\mathbb{F}_p$. This suggests as $h \to \infty$ that $A/A_n$ tends to the group of points of the group scheme given by the dual of the Steenrod algebra mod Bockstein. Hopefully this means that the cohomology of $A$ converges to that of the Steenrod algebra as $h \to \infty$. 
Lemma: Let $X = Z_0 \supset Z_1 \supset Z_2 \supset \cdots$ be a topological space filtered by closed subsets and let $F$ be a sheaf on $X$. Then there is a spectral sequence

$$E_1^{p,q} = H_{Z^p/Z^p+1}^q(X, F) \Rightarrow H^{p+q}(X, F).$$

Proof: Recall that if $Y, Z$ are closed subsets of $X$ with $Y \subset Z$, then we have a long exact sequence

$$\delta : H^p_y(X, F) \to H^p_Z(X, F) \to H^p_{Z^p/Z^p+1}(X, F) \to H^{p+1}_y(X, F).$$

It is the long exact sequence of $\text{Ext}$ resulting from applying $\text{Ext}^p(-, X, F)$ to

$$0 \to \mathbb{Z}_y \to \mathbb{Z}_Z \to \mathbb{Z}_y \to 0.$$

So one gets an exact couple

$$\begin{array}{cccc}
H^{p+q} & \to & H^{p+q} & \to \delta \\
\downarrow & & \downarrow & \\
H^{p+q} & \to & H^{p+q} & \to \delta \\
\downarrow & & \downarrow & \\
H^{p+q} & \to & H^{p+q} & \to \delta \\
\downarrow & & \downarrow & \\
& & \to & \\
\end{array}$$

Thus if $E_1^{p,q} = H_{Z^p/Z^p+1}^q(X, F)$, we have

$$d_1 : E_1^{p,q} \to E_1^{p+1,q}.$$

which is correct.

QED
Lemma. If \( Y, Z \) are closed subsets of \( X \), with \( Y \subseteq Z \), then
\[
H^*_Z(Y, F) \cong H^*_Z(X - Y, F).
\]

Proof: If \( I \) is injective, we have
\[
0 \to \Gamma_Y(X, I) \to \Gamma(X, I) \to \Gamma(X - Y, I) \to 0.
\]
By exactness, we obtain an exact sequence
\[
0 \to \Gamma(Y, I) \to \Gamma_Z(X, I) \to \Gamma^*_Z(X - Y, I) \to 0.
\]
This yields an exact sequence
\[
0 \to \Gamma_Y(X, I) \to \Gamma_Z(X, I) \to \Gamma^*_Z(X - Y, I) \to 0.
\]
and hence an isomorphism
\[
\Gamma^*_Z(Y, I^*) \cong \Gamma^*_Z(X - Y, I^*).
\]
for any injective complex. \( \Box \).

Suppose now that we want to calculate the cohomology of the Zariski scheme \( X = \text{Spec}(\mathbb{Z}[t]) \otimes \mathbb{F}_p \) with values in \( \mathcal{O}_X \), where the group is \( N_1 \), using the decomposition of \( X \) into orbits under \( N \). The problem is to relate the cohomology of \( X, Y, Z \) where \( Z \) is defined by \( f = 0 \), \( f \) a non-zero divisor in \( A(X) \) which we can even assume invariant under \( f \). All I can say is that...
there is a long exact sequence

\[ \rightarrow H^*(A(N_1), A(x)) \rightarrow H^*(A(N_1), A(\mathbb{L})) \rightarrow \lim_{n \rightarrow \infty} H^*(A(N_1), A(x)/t^nA(x)) \]

(= \( H^*(x) \)) \hspace{1cm} (= H^*(\mathbb{L}) \) \hspace{1cm} (= H^{* -1}_Z(x) \)

and a Bockstein spectral sequence for calculating

\[ H^*(A(N_1), A(x)/t^nA(x)) \]

in terms of \[ H^*(A(N_1), A(x)/t^nA(x)) \]

(= \( H^*(\mathbb{L}) \)).

Not much help. Possibly one can get to a position where the other cohomology theories than \( H^*(x, \mathcal{O}_X) \) are used.
March 1, 1967

Review of local cohomology

Let $X$ be a manifold, $A$ a closed subset of $X$.

(i) $H^*_A(X) \cong H^*(X, X-A)$ elements are represented by $\{Z \to X, g: W \to X-A\}$ where $\alpha: Z/X-A \cong \partial W$.

(ii) $H^*(A) = \lim_{\mu \to A} H^*(\mu)$ exact rep. by $\{W \to X\}$ pro.or.

(iii) $H^*(X, A) = H^*_{\mu//X}(X-A)$ exact sequences

\[ H^*_A(X) \to H^*(X) \to H^*(X-A) \to \ldots \]
\[ H^*_{\mu//X}(X-A) \to H^*(X) \to H^*(A) \to \ldots \]

Excision

$H^*_A(X) \cong H^*_A(U)$ where $U$ open $\supset A$.

Gysin isom.

$H^*(A) \cong H^*_{\text{oriented}}(X)$ if $A \subset X$ oriented submanifold of codim $d$.

(iv) $H_*(X, X-A)$ exact rep. by $\{W \to X\}$ pro.or.

(v) $H_*(X, A)$ exact rep. by $\{Z \to X-A\}$ $Z$ oriented; $\partial Z = \phi$.

(f) $H_*(A)$ exact rep. by $\{Z \to X\}$ $Z$ comp.or $W$ orientable $\{Z|X-A = \partial W, f: W \to X-A \ni f^{-1}F$ comp. for all $F$ closed in $X$ $F \cap A = \phi\}$.
exact sequences:

$$H_\ast(A) \to H_\ast(X) \to H_\ast(X, A) \to \cdots$$

$$H_\ast(X-A) \to H_\ast(X) \to H_\ast(X, X-A) \to \cdots$$

excision:

$$H_\ast(X, X-A) = H_\ast(U, U-A) \quad \text{if} \quad A \subset U \text{ open}$$

$$H_\ast(X, A) = H_\ast(X-F, A-F) \quad \text{if} \quad F \text{ closed and } F \subset \text{Int } A.$$

Gysin isomorphism:

$$H_\ast_{oriented}(X, X-A) \cong H_\ast(A) \quad \text{A oriented submanifold}$$

continuity:

$$H_\ast(A) = \lim_{K \subset A} H_\ast(K) \quad \text{K compact}$$

$$H_\ast(X, A) = \lim_{K \subset A} H_\ast(X, K)$$

$$H_\ast(X, X-A) = \lim_{K \subset X-A} H_\ast(X, K)$$

It seems reasonable therefore if L is locally closed in X to set

$$H_\ast(X, L) = \lim_{K \subset A} H_\ast(X, K)$$

$$H_\ast(L) = \lim_{K \subset L} H_\ast(K)$$
where one has a single exact sequence

$$H^*_\ast(L) \rightarrow H^*_\ast(X) \rightarrow H^*_\ast(X, L) \rightarrow \cdots$$

and a single excision

$$H^*_\ast(X, L) = H^*_\ast(X - F, L - F)$$

if $F$ closed $\subset$ Int $L$. 

**Duality results:**

If $X$ is oriented, then

$$H^*_K(X) \cong H_{n-\ast}(K)$$

$$H^*(K) \cong H_{n-\ast}(X, X-K)$$

If $X$ is compact and oriented, then we also have

$$H^*(X-A) \cong H_{n-\ast}(X, A)$$

$$H^*(X, A) \cong H_{n-\ast}(X-A)$$

Note explicitly the compactness condition [if $X - A$ is relatively compact in $X$.] The basic duality results are

$$H^*_K(X) \cong H_{n-\ast}(K)$$

$K$ compact

$$H^*(X-A) \cong H_{n-\ast}(X-A)$$

$X-A$ rel. compact in $X$

for they imply the others by passage to the limit.
Must work supports into the axioms

basic object is $H^*_A(X)$ $A$ closed.

because then can define

$$H^*_\Delta(X) = \lim_{A \in \mathcal{A}} H^*_A(X).$$

basic properties:

excision $H^*_A(X) = H^*_A(U)$ $U$ open $\cap A$.

exact sequence

$$\delta : H^*_A(X) \rightarrow H^*_A(U) \rightarrow H^*_A(V) \rightarrow H^*_A(X - A) \rightarrow \delta \ldots$$

functionality: if $f : X \rightarrow Y$

then get $f^* : H^*_B(Y) \rightarrow H^*_A(X) (\rightarrow H^*_A(U))$

and $f^*_A : H^*_A(X) \rightarrow H^*_A(U) (\rightarrow H^*_B(Y))$

need $A = f^* B$ and $A \subset B$.

and $f^*_A : H^*_A(X) \rightarrow H^*_A(U) (\rightarrow H^*_B(Y))$

closed since $f$ proper $+ A$ closed.

Note that the map

$$H^*_A(X) \rightarrow H^*_A(X)$$

if $A \subset A'$

is a special case of any $f^*_A$. 


\( f^* : H^*_B(Y) \rightarrow H^*_A(X) \)
\( f^* : H^*_A(X) \rightarrow H^*_B(Y) \)

\((X, A)\) objects

two kinds of morphisms, ordinary map \( f^{-1}B \subset A \)
\( f : (X, A) \rightarrow (Y, B) \)

want \( f^*_B : H^*_B(Y) \rightarrow H^*_A(X) \).

proper oriented map
\( f : (X, A) \rightarrow (Y, B) \) \( \supset fA \subset B \)

Then one wants
\( f^*_A : H^*_A(X) \rightarrow H^*_B(Y) \)
b) compatibility axiom

\[(X, A) \quad (X, A')\]

suppose that \( A \subset A' \). Then you have two

maps

\[\Omega_A(X) \xrightarrow{\text{id}_X} \Omega_A'(X)\]

this follows from b)

\[\begin{array}{ccc}
A & \xrightarrow{\text{id}} & A \\
\downarrow & & \downarrow \\
\Omega & \xrightarrow{\text{id}} & \Omega
\end{array}\]

\[\Omega' \]

\[\text{excision axiom}\]

Point is to prove that is an isomorphism

\[\text{Han}(K, \mathbb{Q}^+ \xrightarrow{\beta} \text{Han}^\otimes \{\Omega, \mathbb{Q}\} \xrightarrow{\lambda} \]

given a theory you can twist it by a character class \(\beta\):

\[\beta: \Omega \rightarrow \mathbb{Q} \quad \text{when does it come from an } x\]

Begin by defining

\[\overline{\beta}(E) = i_x^* \beta \cdot 1\]

to know \[A\] independent of choice of \(i\)