March 2$^{5}$, 1969: On cohomology operations

The problem is to determine all natural transformations $H^q(X,A) \longrightarrow H^r(X,A)$ where $(X,A)$ is a ringed topos over a fixed ringed topos $(P,k)$. Hopefully we can realize the Steenrod algebra (with $P^0 =$ Frobenius) directly as something associated to endomorphisms of the additive group as Atiyah-Hirzebruch suggest and also as Grothendieck hopes, to have ~~something~~ some kind of relative group scheme over $(P,k)$ which works even if $k = \mathbb{Z}$.

I. Reduction to a semi-simplicial problem:

1) Given $\Theta : H^q(X,A) \longrightarrow H^r(X,A)$ natural for all k-algebras ~~rings~~ in the topos $X$ we obtain a natural transf. for any sheaf $F$ of $k$-modules

$$\hat{\Theta} : H^q(X,F) \longrightarrow H^r(X, SF)$$

where $S$ is the symmetric algebra of $F$ over $k$, defined to be the composition

$$\underset{\substack{\text{induced by can map}\\ F \longrightarrow SF}}{H^q(X,F) \longrightarrow H^q(X,SF)} \xrightarrow{\Theta} H^r(X,SF)$$

Conversely given a nat. transf. $\hat{\Theta}$ one gets a $\Theta$ defined to be the composition

$$H^q(X,A) \xrightarrow{\hat{\Theta}} H^r(X,SA) \underset{\substack{\text{induced by}\\ \text{can. map}\\ SA \longrightarrow A}}{\longrightarrow} H^r(X,A)$$

The correspondence $\Theta \leftrightarrow \hat{\Theta}$ is clearly bijective. Also $\Theta$ is an additive homomorphism iff $\hat{\Theta}$ is. Unfortunately it doesn't seem to be possible to describe (easily) when $\Theta$ is a ring homomorphism in terms of $\hat{\Theta}$ ($\Theta$ and $\hat{\Theta}$ not being necessarily homogeneous)

2.) Let $(P, \mathcal{O}_P)$ be a ringed topos. By ~~[illegible]~~ a theorem of Dold, the category of cochain complexes of ~~[illegible]~~ $\mathcal{O}_P$-modules is equivalent to the category of co-simplicial $\mathcal{O}_P$-modules, permitting one to define the derived functors $\underline{R}T: D^{\geq 0}(P, \mathcal{O}_P) \longrightarrow D^{\geq}(P, \mathcal{O}_P)$ of a not-necessarily-additive functor $T$ on $\mathcal{O}_P$-modules. ~~[illegible]~~ If $q$ is an integer $\geq 0$ and $F$ is an $\mathcal{O}_P$-module, let ~~[illegible]~~ $F[q]$ denote the cosimplicial $\mathcal{O}_P$-module with only non-zero homology group in dimension $q$. Then

$$(*) \qquad \mathrm{Hom}_{D^{\geq 0}(P, \mathcal{O}_P)}(\mathcal{O}_P[q], F^{\cdot}) \simeq H^q(X, F^{\cdot})$$

($q$ th hyper-cohomology gp. of the complex $F^{\cdot}$)

<u>Proposition</u>: ~~[illegible]~~ The following abelian groups are canonically isomorphic

(i) The group of natural transformations $H^q(X, \mathcal{O}_X) \longrightarrow H^h(X, \mathcal{O}_X)$ defined on the category of ~~[illegible]~~ ringed topi over $(P, \mathcal{O}_P)$.

(ii) The group of natural transformations $H^q(X, F) \longrightarrow H^h(X, SF)$ defined on the category of pairs $(X, F)$, where $X$ is a topos over $P$ and $F$ is an ~~[illegible]~~ $\mathcal{O}_P$-module in $X$, and where $S$ is the symmetric algebra functor on $\mathcal{O}_P$-modules.

(iii) The group of natural transformations $H^q(P, F^{\cdot}) \longrightarrow H^h(P, S(F^{\cdot}))$ where $F^{\cdot}$ is a cochain complex of $\mathcal{O}_P$ modules in $P$.

(iv) The group $H^h(P, S(\mathcal{O}_P[q]))$.

<u>Proof</u>: (i) $\Leftrightarrow$ (ii) we've already done

(iii) $\Leftrightarrow$ (iv) clear by Yoneda's lemma and $(*)$.

(ii) $\longleftrightarrow$ (iii). Recall that ~~[illegible]~~ a cosimplicial object in a category

$\mathcal{C}$ is by definition a covariant functor $\Delta \to \mathcal{C}$; hence the category of cosimplicial objects in $P$ is the topos $\underline{\mathrm{Hom}}(\Delta, P)$. We may think of $\underline{\mathrm{Hom}}(\Delta, P) = \cos(P)$ as being sheaves on $P \times \Delta^{\circ}$ with the topology which is the product of the canonical topology of $P$ and the trivial topology of $\Delta^{\circ}$. There are two morphisms of topi corresponding to the projections

$$\cos(\text{sets}) \xleftarrow{\ g\ } \cos P \xrightarrow{\ f\ } P$$

$f^*(F) =$ constant cosimplicial object assoc. to $F$
$g^*(K) =$ simplicial constant sheaf.

$$f_*(F^\cdot) = \check{H}^0(F^\cdot)$$

$$R^q f_*(F^\cdot) = \check{H}^q(F^\cdot) \qquad \text{for } F^\circ \text{ abelian in } \cos(P).$$

I claim that $H^*(\cos P, F^\circ)$ is nothing but the hypercohomology $H^*(P, F^\cdot)$. To prove this I must find a canonical isomorphism

$$\underline{\underline{R}} f_*(F^\cdot[0]) \cong F^\circ$$

However if $I^{\cdot\cdot}$ is an injective resolution of $F^\cdot$, then we have

$F^\cdot \to \Delta I^{\cdot\cdot}$ quis, by spectral sequence $\quad H^v H^h(I^{\cdot\cdot}) \Longrightarrow H(\Delta I^{\cdot\cdot})$

$H^{0v} I^{\cdot\cdot} \to \Delta I^{\cdot\cdot}$ quis, —————— $\quad H^h H^v(I^{\cdot\cdot}) \Longrightarrow$ "

.. and fact that $I^{p\cdot}$ injective $\Longrightarrow H^{+v}(I^{p\cdot}) = 0$. Thus

$$\underline{R} f_* F^\cdot[0] \overset{\text{defn.}}{=} H^{0v}(I^{\cdot\cdot}) \simeq \Delta I^{\cdot\cdot} \simeq F^\cdot$$

Therefore from a transf. $H^*(X,F) \longrightarrow H^*(X,SF)$ for all $(X,F)$, $X$ topos over $P$, $F$ an $\mathcal{O}_P$-module in $X$ one deduces a transf.

$$H^q(\cos P, F^\cdot) \longrightarrow H^n(\cos P, SF^\cdot)$$

which is isomorphic to

$$H^q(P, F^\cdot) \longrightarrow H^n(P, SF^\cdot) \qquad \text{(hypercohomology).}$$

Thus (ii) $\longrightarrow$ (iii).

(iii) $\longrightarrow$ (ii): given $H^q(P,F^\cdot) \longrightarrow H^n(P,SF^\cdot)$ we get an operation $H^*(P,F) \longrightarrow H^*(P,SF)$ by taking $F^\cdot = F[0]$. But there is a map $H^n(P, S(\mathcal{O}_P[q])) \longrightarrow H^n(X, S(\mathcal{O}_X[q]))$ so one gets an operation for any $X$ over $P$.

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$$S^q(\mathcal{O}_P[q]) = \mathcal{O}_P \otimes_{\mathbb{Z}} S^{\mathbb{Z}}(\mathbb{Z}[q])$$

Thus

$$H^n(P, S^{\mathcal{O}_P}(\mathcal{O}_P[q])) = H^n(P, \mathcal{O}_P \otimes_{\mathbb{Z}} S^{\mathbb{Z}}(\mathbb{Z}[q]))$$

~~so we have from the universal coefficient theorem a short exact sequence~~

$$0 \longrightarrow \bigoplus_{i+j=n} H^i(P,\mathcal{O}_P) \otimes_{\mathbb{Z}} H^j(S^{\mathbb{Z}}\mathbb{Z}[q]) \longrightarrow H^n(P, S^{\mathcal{O}_P}(\mathcal{O}_P[q])),$$

$$\bigoplus_{i+j=n+1} \mathrm{Tor}_1^{\mathbb{Z}}(H^i(P,\mathcal{O}_P), H^j(S^{\mathbb{Z}}\mathbb{Z}[q])) \longrightarrow 0$$

~~which I believe splits non-canonically. (here one needs to know that $H^j(S^{\mathbb{Z}}\mathbb{Z}[q])$ is a finite type $\mathbb{Z}$ module; hence $q \geq 1$).~~

Suppose that $H^*(P, ?)$ commutes with countable direct sums. Then by the universal coefficient thm. there is a short exact sequence

$$0 \to \bigoplus_{i+j=n} H^i(P, \mathcal{O}_P) \otimes_{\mathbb{Z}} H^j(S^{\mathbb{Z}}\mathbb{Z}[q]) \longrightarrow H^n(P, \mathcal{O}_P \otimes_{\mathbb{Z}} S^{\mathbb{Z}}(\mathbb{Z}[q]))$$

$$\hookrightarrow \bigoplus_{i+j=n+1} \mathrm{Tor}_1^{\mathbb{Z}}\left( H^i(P, \mathcal{O}_P), H^j(S^{\mathbb{Z}}\mathbb{Z}[p])\right) \longrightarrow 0$$

which splits non-canonically. This seems difficult to work with so suppose that $\mathcal{O}_P$ is a $\mathbb{F}_p$-algebra and that $H^*(P, ?)$ commutes with direct sums; then

$$H^*(P, S\,\mathcal{O}_P[+q]) \simeq H^*(P, \mathcal{O}_P) \otimes H^*(S^{\mathbb{F}_p}(\mathbb{F}_p[q])).$$

This isomorphism has the following interpretation:

<u>Proposition</u>: Let $(P, \mathcal{O}_P)$ be a ringed topos of characteristic $p$ such that $H^*(P, ?)$ commutes with direct sums. Then any natural operation $\theta: H^q(X, \mathcal{O}_X) \longrightarrow H^p(X, \mathcal{O}_X)$ defined on the category of ringed topoi over $(P, \mathcal{O}_P)$ is uniquely expressible in the form

$$\theta(a) = \sum_i u_i \cdot \gamma_i a$$

where $\gamma_i$ is a basis for $H^*(S^{\mathbb{F}_p}(\mathbb{F}_p[q]))$ (see below) and the $u_i \in H^*(P, \mathcal{O}_P)$.

<u>Remark</u>: Let $P^0 : H^*(X, \mathcal{O}_X) \longrightarrow H^*(X, \mathcal{O}_X)$ be the operation induced by the map $x \longmapsto x^p$ on $\mathcal{O}_X$. From the determination of

$H^*(S^{\mathbb{F}_p}(\mathbb{F}_p[q]))$ we will know that for $q \geq 0$

$$H^q(S^{\mathbb{F}_p}(\mathbb{F}_p[q])) \simeq \bigoplus_{k=0}^{\infty} H^q(S^{\mathbb{F}_p}_{p^k}(\mathbb{F}_p[q]))$$

$$\simeq \bigoplus_{k=0}^{\infty} \mathbb{F}_p \cdot (P^0)^k \, \kappa_q = \boxed{\mathbb{F}_p[P^0] \cdot \kappa_q}$$

where $\kappa_q \in H^q(\mathbb{F}_p[q])$ is the canonical element, and that

$$H^r\left(S^{\mathbb{F}_p}(\mathbb{F}_p[q])\right) = \begin{cases} \mathbb{F}_p & r=0 \\ 0 & 0<r<q. \end{cases}$$

It is therefore necessary that $H^*(P, \mathcal{O}_p^{(N)}) \simeq H^*(P, \mathcal{O}_p)^{(N)}$ in order that the proposition be correct.

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## II) Determination of the stable operations over $\mathbb{F}_p$:

Recall that for a cosimplicial $k$-module $F$ there is an functorial exact sequence

$$0 \longrightarrow \Omega(F) \longrightarrow E(F) \longrightarrow F \longrightarrow 0$$

where $E(F)$ is homotopic to zero. Then

$$S(\Omega F) \longrightarrow S(EF) \longrightarrow SF$$

is the zero map so there is a canonical map in the homotopy category of simplicial $k$-modules

$$S(\Omega F) \longrightarrow \Omega SF$$

permitting us to define a suspension homomorphism

$$H^*(S(\Omega F)) \longrightarrow H^*(\Omega SF) \simeq H^{*-1}(SF).$$

In particular we get a map

$$H^*(SO[q]) \longrightarrow H^{*-1}(SO[q-1])$$

which may be interpreted as taking an operation $H^q(F) \xrightarrow{\theta} H^*(SF)$ into the operation

$$H^{q-1}(F) \simeq H^q(\Omega F) \xrightarrow{\theta} H^*(S\Omega F) \longrightarrow H^{*-1}(SF).$$

We define the stable operations to be elements of

$$\varprojlim_N H^{*+N}(SO[N]).$$

One knows that stable operations are always additive. In terms of an operation on $\mathcal{O}$-algebras $a$, an operation $\theta: H^*(P, a) \longrightarrow H^*(P, a)$ is stable if and only if given a surjection $\pi$ of $\mathcal{O}$-algebras

$$0 \longrightarrow J \longrightarrow a \xrightarrow{\pi} a'' \longrightarrow 0$$

and letting $a' = \mathcal{O} \oplus J$, the result of adjoining 1 to $J$, then

$$\begin{array}{ccc} H^*(a'') & \xrightarrow{\theta} & H^*(a'') \\ \downarrow \delta & & \downarrow \delta \\ H^{*+1}(J) & & H^{*+1}(J) \\ \downarrow & & \downarrow \\ H^{*+1}(a') & \xrightarrow{\theta} & H^{*+1}(a') \end{array}$$

is commutative.

$\theta = \mathbb{F}_p$, $P = (\text{Sets})$.

$$\varprojlim_N H^{*+N}(S\theta[N]) = \varprojlim_N \bigoplus_q H^{*+N}(S_q\theta[N]) = \bigoplus_q \varprojlim_N H^{*+N}(S_q\theta[N])$$

taking duals

$$\varprojlim_N H^{*+N}(S_q\theta[N])^\vee = \varinjlim_N H_{*+N}(\Gamma_q\theta[-N]).$$

or simply

$$H_*(\Gamma_q^\Delta \theta)$$

where $T^\Delta$ denotes the stabilized version of a functor $T$ on simplicial modules, given by:

$$T^\Delta X = \varinjlim_N \Omega^N T \Sigma^N X.$$

A basic fact is that if $T(X,Y)$ is a functor of two variables (such that $T(0,X) = T(X,0) = 0$), then $T^\Delta(X,Y) \sim 0$. Hence from the familiar Koszul sequences

$$0 \to \Lambda_n V \to \cdots \to S_{n-2}V \otimes \Lambda_2 V \to S_{n-1}V \otimes V \to S_n V \to 0$$

$$0 \to \Gamma_n V \to \cdots \to \Lambda_{n-2}V \otimes \Gamma_2 V \to \Lambda_{n-1}V \otimes V \to \Lambda_n V \to 0$$

one deduces canonical isomorphisms

$$\boxed{\begin{aligned} H_*(S_n^\Delta V) &\simeq H_{*-(n-1)}(\Lambda_n^\Delta V) \\ H_*(\Lambda_n^\Delta V) &\simeq H_{*-(n-1)}(\Gamma_n^\Delta V) \end{aligned}}$$

If $T$ is a functor on $\mathcal{O}$ modules $\ni$ $T(0)=0$ set

$$T_2(X,Y) = \mathrm{Ker}\,\{T(X\oplus Y) \longrightarrow T(X)+T(Y)\}$$

and define $T_a$ by

$$T_2(X,X) \longrightarrow T(X) \longrightarrow T_a(X) \longrightarrow 0$$

One sees that $T_a$ is the largest additive quotient functor of $T$. In effect $T_a$ is additive ~~functor~~ since

$$\begin{tikzcd}
T_2(X,X) \arrow[r] \arrow[d] \arrow[dr, bend left, "0"] & T(X) \arrow[r] \arrow[d] & T_a(X) \arrow[r] \arrow[d] & 0 \\
(T_a)_2(X,X) \arrow[r, "0"] & T_a(X) \arrow[r] & (T_a)_a(X) \arrow[r] & 0
\end{tikzcd}$$

$$\Rightarrow \quad T_a \simeq (T_a)_a$$

But $T = T_a \implies T_2(X,X) \xrightarrow{\nabla} T(X)$ is zero

$$\Rightarrow \quad T(\mu) = T(\pi_1) + T(\pi_2)$$

since

$$\begin{tikzcd}
0 \arrow[r] & T(X)+T(X) \arrow[r, "T(i_1)+T(i_2)"] & T(X+X) \arrow[r] \arrow[dr, "T(\mu)-T(\pi_1)-T(\pi_2)"'] & T_2(X,X) \arrow[r] \arrow[d, "\nabla"] & 0 \\
& & & T(X) &
\end{tikzcd}$$

$$\Longrightarrow \quad T \text{ additive} \quad \left( T(f+g) = T(\mu)\cdot T(f,g) = \begin{cases} T(\pi_1)T(f,g) \\ +T(\pi_2)T(f,g) \end{cases} = T(f)+T(g) \right)$$

Thus

$$\boxed{H_* T^{\Delta}(V) \simeq H_* T_a V}$$

since for additive functor $T \simeq T^{\Delta}$. Now

$$\Gamma_n(X \oplus Y) = \bigoplus_{i+j=n} \Gamma_i(X) \otimes \Gamma_j(Y)$$

and similarly for $\Lambda_n$, $S_n$. Thus

$$(\Lambda_n)_a = (S_n)_a = 0$$

and

$$(\Gamma_n)_a = \text{indecomposable part of } \Gamma_n .$$

Now assume $k = \mathbb{F}_p$.

<u>Lemma:</u> If $n$ is a positive integer which is not a power of $p$, then

$$H_*(S_n^{\Delta} V) = H_*(\Lambda_n^{\Delta} V) = H_*(\Gamma_n^{\Delta} V) = 0.$$

<u>Proof.</u> Write $n = p^a h$ where $h > 1$ and $(h,p) = 1$.

Then

$$\begin{array}{c} \Gamma_n(X) \xrightarrow{\Delta} \Gamma_n(X+X) \cong \bigoplus_{i+j=n} \Gamma_i(X) \otimes \Gamma_j(X) \xrightarrow{pr_{ij}} \Gamma_i(X) \otimes \Gamma_j(X) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \text{mult} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Gamma_n(X) \end{array}$$

carries

$$\gamma_n(x) \longmapsto \gamma_n((x,x)) \leftrightarrow \sum_{i+j=n} \gamma_i x \otimes \gamma_j x \longrightarrow \gamma_i x \otimes \gamma_j x \longrightarrow \frac{n!}{i!\,j!} \gamma_n x$$

If $n = p^a h$ ~~map~~

$$(1+X)^n \equiv (1+X^{p^a})^h = \sum_{i=0}^{h} \binom{h}{i} X^{p^a i} \qquad \text{mod } p$$

so
$$\frac{n!}{p^a!\,[p^a(h-1)]!} \not\equiv 0 \quad \text{mod } p.$$

Thus if $i = p^a$ we have that

$$\begin{tikzcd} \Gamma_n(X) \arrow[r] \arrow[rr, bend right, "\text{mult by } c \neq 0"'] & \Gamma_i(X) \otimes \Gamma_j(X) \arrow[r] & \Gamma_n(X) \end{tikzcd}$$

hence $\Gamma_n$ is a direct summand of a functor of two variables $T(X,Y)$ with $T(0,X) = T(X,0) = 0$. Hence

$$H_*(\Gamma_n^\Delta X) = 0$$

The other formulas may be deduced from the isos on page 10.

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<u>Case $n=p$:</u> Then have exact sequences

$$0 \longrightarrow V^{(p)} \xrightarrow{\sigma} S_p V \longrightarrow \bar{S}_p V \longrightarrow 0$$
$$0 \longrightarrow \bar{S}_p V \longrightarrow \Gamma_p V \xrightarrow{\tau} V^{(p)} \longrightarrow 0$$

where $\bar{S} V = SV/(v^p)$ is the restricted symmetric algebra functor and $\sigma v = v^p$, $\tau(\gamma_p v) = v^p$. ~~Actually~~ Here if $k$ is a ring of characteristic $p$ we denote by $V^{(p)}$ the module $k \overset{\bullet}{\otimes}_k V$ which is the base extension of $V$ by Frobenius $k \longrightarrow k$. Note that

$$\bar{S}V = \text{Image of canonical map } SV \longrightarrow \Gamma V.$$

These short exact sequences give rise to long exact sequences

$$\to H_*(V^{(\varphi)}) \longrightarrow H_*(S_p^\Delta V) \longrightarrow H_*(\bar{S}_p^\Delta(V)) \longrightarrow \cdots$$

$$\longrightarrow H_*(\bar{S}_p^\Delta V) \to H_*(\Gamma_p^\Delta V) \longrightarrow H_*(V^{(\varphi)}) \longrightarrow \cdots$$

I claim that the second sequence splits canonically. In effect as $k = \mathbb{F}_p$ we may identify $V^{(\varphi)}$ and $V$; then we have a non-additive section $\gamma_p : V \longrightarrow \Gamma_p V$ of $\tau$. Recalling that the ~~homology~~ homology of a simplicial module is the homotopy of the underlying simplicial set, it follows that $\gamma_p$ induces a section (additive) of $\tau_*$. Thus we have ~~that~~ a canonical isom.

$$\boxed{H_*(\Gamma_p^\Delta V) \simeq H_*(\bar{S}_p^\Delta V) \oplus \gamma_p H_*(V).}$$

Now take $V = k[0]$. Then $H_q(V) = 0 \quad q > 0$ and

$$\begin{array}{ccccccccccc} 0 \to & H_1(S_p^\Delta V) & \longrightarrow & H_1(\bar{S}_p^\Delta V) & \longrightarrow & H_0(V) & \longrightarrow & H_0(S_p^\Delta V) & \to & H_0(\bar{S}_p^\Delta V) & \to 0 \\ & \wr\wr & & & & \wr\wr & & \wr\wr & & \wr\wr & \\ & H_{1-p}(\Lambda_p^\Delta V) & & & & k & & 0 & & 0 & \\ & \wr\wr & & & & & & & & & \\ & 0 & & & & & & & & & \end{array}$$

where the zeros are either by dimensional reasons or because the additive quotients of the functors are zero. Thus

$$\boxed{\begin{array}{ll} H_q(S_p^\Delta V) \xrightarrow{\sim} H_q(\bar{S}_p^\Delta V) & q \geq 2 \\ H_1(\bar{S}_p^\Delta V) \simeq k, \quad H_0(\bar{S}_p^\Delta V) = H_1(S_p^\Delta V) = H_0(S_p^\Delta V) = 0 & \end{array}}$$

and so we see that

$$
\begin{aligned}
H_*(\Gamma_p^\Delta V) &\simeq H_*(\bar{S}_p^\Delta V) + \gamma_p H_*(V) \\
&\simeq H_*(S_p^\Delta V) + \gamma_p H_*(V) \qquad * \geq 2 \\
&\simeq H_{*-2(p-1)}(\Gamma_p^\Delta V) + \gamma_p H_*(V).
\end{aligned}
$$

Thus

$$
\left\{
\begin{array}{l}
H_q(\Gamma_p^\Delta k[0]) = \begin{cases} k & q \equiv 0, 1 \quad \text{mod } 2(p-1), \; q \geq 0 \\ 0 & \text{otherwise} \end{cases} \\[2ex]
H_q(\Lambda_p^\Delta [0]) = \begin{cases} k & q \equiv p-1, p \quad \text{mod } 2(p-1), \; q \geq 0 \\ 0 & \text{otherwise} \end{cases} \\[2ex]
H_q(S_p^\Delta [0]) = \begin{cases} k & q \equiv 0, 1 \quad \text{mod } 2(p-1) \\ & q \geq 2 \\ 0 & \text{otherwise} \end{cases}
\end{array}
\right.
$$

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Further analysis in characteristic 2. Here we replace the exact sequences on page 11 by the exact sequence

$$
\cdots \longrightarrow \Gamma V \otimes \Gamma_2 V \longrightarrow \Gamma V \otimes V \longrightarrow \Gamma V \longrightarrow \Gamma(V^{(2)}) \longrightarrow 0
$$

constructed in the following manner. Let $\delta$ be the derivation of $S(W) \otimes S(W)$ given by

$$
\begin{cases} \delta(w \otimes 1) = 1 \otimes w \\ \delta(1 \otimes w) = 0. \end{cases}
$$

One sees that $\delta^2 = 0$ and that if $W$ is 1-dimensional with basis $e$, then we obtain an sequence

$$S(W) \xrightarrow{\delta} S(W) \otimes W \xrightarrow{\delta} S(W) \otimes S_2 W \xrightarrow{\delta} \cdots$$

$$k[e] \longrightarrow k[e] \otimes e \longrightarrow k[e] \otimes e^2 \longrightarrow \cdots$$

where
$$\delta\, f(e) \otimes e^i = f'(e) \otimes e^{i+1}$$

or
$$\delta(e^j \otimes e^i) = j(e^{j-1} \otimes e^{i+1})$$

so that the sequence is acyclic and resolves $k[e^2]$. Thus by Künneth we have for any vector space $V$ over $k$ a resolution

$$0 \to S(W^{(2)}) \to S(W) \xrightarrow{\delta} S(W) \otimes W \xrightarrow{\delta} \cdots$$

which moreover transforms sums into tensor products. Taking duals one gets ~~the desired exact sequence of algebras~~ a differential bigebra $\Gamma(V) \otimes \Gamma_*(V)$ where $d = \delta^t$ is given by

$$d\big(\gamma_i(v) \otimes \gamma_j(v')\big) = \gamma_i(v)v' \otimes j\,\gamma_{j-1}(v')$$

Thus we have an exact sequence

$$0 \to \Gamma_2^a V \to V \otimes \Gamma_{2^a-1} V \longrightarrow \cdots \longrightarrow \Gamma_{2^a-1} V \otimes V \longrightarrow \Gamma_{2^a} V \xrightarrow{\tau} \Gamma_{2^{a-1}} V^{(2)} \to 0$$

again $\tau$ has a section given by $\gamma_2$ ~~where~~

Thus we get a canonical isomorphism

$$\boxed{H_q(\Gamma^{\Delta}_{2^a} V) \simeq H_{q-(2^a-1)}(\Gamma^{\Delta}_{2^a} V) \oplus H_q(\Gamma^{\Delta}_{2^{a-1}} V)}$$

from which it is possible to calculate $H_*(\Gamma_{2^a}^{a}V)$ additively by induction on $a$. It doesn't seem possible to get a hold on the composition structure on the dual in this way.

To generalize this calculation to odd characteristic we need a generalization of the exact sequence $\boxtimes$ which we ~~try~~ fail now to derive. Let $V$ be a module over a ring $k$ which is an $\mathbb{F}_p$-algebra, $p$ odd. Endow the algebra $\Gamma V \otimes \Lambda V$ with the derivation $d$ given by

$$d\, \gamma_n(x) \otimes 1 = 0$$
$$d\,(1 \otimes x) = x \otimes 1.$$

Then

$$\begin{tikzcd}
 & & 0 \\
 & & V^{(p)} \arrow[u] \\
\Gamma_{p-2}V \otimes \Lambda_2 V \arrow[r, "d"] & \Gamma_{p-1}V \otimes V \arrow[r, "d"] & \Gamma_p V \arrow[u] \\
S_{p-2}V \otimes \Lambda_2 V \arrow[r] \arrow[u, "s"] & S_{p-1}V \otimes V \arrow[r] \arrow[u, "s"] & S_p V \arrow[r] \arrow[u] & 0 \\
 & & V^{(p)} \arrow[u] \\
 & & 0 \arrow[u]
\end{tikzcd}$$

exact if $V$ flat which we assume

exact

so that the homology of the ~~exact~~ upper sequence is $V^{(p)}$ in the middle.

Let $K = \mathrm{Ker}\{\Gamma_{p-1}V \otimes V \xrightarrow{d} \Gamma_p V\}$

so that there is an exact sequence

$$\Gamma_{p-2}V \otimes \Lambda_2 V \xrightarrow{d} K \xrightarrow{\pi} V^{(p)} \longrightarrow 0,$$

~~If one has a section s of π, then one can form the algebra~~

where $\pi$ can be describe a little more explicitly as follows. Suppose $V$ is free with base $e_i$. Then the elements $e_i^{p-1} \otimes e_i$ of $S_{p-1}V \otimes V$ under the multiplication $d$ go to $e_i^p$ which is a base for $V^{(p)}$. Hence the elements $\gamma_{p-1}(e_i) \otimes e_i$ span a complement of $\operatorname{Im} d$ in $K$. Thus we have a diagram

$$\begin{tikzcd}
\Gamma_{p-1}V \otimes V \arrow[r, "d"] \arrow[d, "-h"] & \Gamma_p V \arrow[r] \arrow[d, "h"] & V^{(p)} \arrow[r] \arrow[d, "\wr\ id"] & 0 \\
\Gamma_{p-2}V \otimes \Lambda_2 V \arrow[r, "d"] & K \arrow[r, "\pi"] & V^{(p)} \arrow[r] & 0.
\end{tikzcd}$$

where $h$ denotes the derivation of $\Gamma V \otimes \Lambda V$ of degree $+1$ such that $h(\gamma_n(x) \otimes y) = \gamma_{n-1} x \otimes x \wedge y$. Note that $dh + hd$ is the derivation of degree $0$ of $\Gamma V \otimes \Lambda_\cdot V$ given by $(dh+hd)\, z = kz$ if $z \in \Gamma_i V \otimes \Lambda_j V$ with $i+j = k$.

Note form a cocartesian square of algebras

$$\begin{tikzcd}
\Gamma(\Gamma_{p-1}V \otimes \Lambda_2 V) \arrow[r] \arrow[d, "\Gamma(-h)"] & \Gamma(\Gamma_p V) \arrow[d] \\
\Gamma V \otimes \Lambda V \arrow[r] & Q(V)
\end{tikzcd}$$

Note that

$$\Gamma_p V \xrightarrow{h} \Gamma_{p-1} V \otimes V \xrightarrow{d} \Gamma_{p-2} V \otimes \Lambda^2 V$$

is exact as it's the dual of the Koszul sequence. Thus we have

$$\begin{tikzcd}
\Gamma_{p-1}V\otimes V/\Gamma_p V \arrow[r, "\bar{d}"] \arrow[d, "\bar{h}"] & \Gamma_p V \arrow[r] \arrow[d, "h"] & V^{(p)} \arrow[r] \arrow[d] & 0 \\
\Gamma_{p-2}V\otimes \Lambda_2 V \arrow[r, "d"] \arrow[d] & K \arrow[r] & V^{(p)} \arrow[r] & 0 \\
{} & & &
\end{tikzcd}$$

Problem: Does $\exists$ an extension of $\Gamma_{p-2} V \otimes \Lambda_2 V$ by $V^{(p)}$ mapping onto $K$? ?

$$\begin{tikzcd}
\Gamma_{p-3}V\otimes\Lambda_3 V \arrow[r] \arrow[dr] & \Gamma_{p-2}V\otimes\Lambda_2 V \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & V^{(p)} & \\
& V^{\otimes(p-2)}\otimes V^{\otimes 2} \arrow[r] & L \arrow[r] & V^{(p)} \arrow[r] & 0
\end{tikzcd}$$

~~what you're trying to do is probably impossible since otherwise you would have a functorial resolution~~

$$\boxed{\mathrm{Ext}^1\left((?)^{(p)}, \Gamma_{p-2}V\otimes\Lambda_2 V\right)}$$

~~There are maps you haven't considered e.g. the~~

$$\mathrm{Ext}^1\left(V^{(p)}, \Gamma_{p-2}V\otimes\Lambda_2 V\right)$$

Let $V$ be a vector space of finite dimension over $k$ and consider the algebra $\bar{S}(V^\vee)$. Then we wish to calculate

$$\mathrm{Ext}^*_{\bar{S}(V^\vee)}(k,k) = F^*(V)$$

as a functor of $V$. It is an anti-commutative algebra functor transforming sums into tensor product. ~~There are two cases.~~

~~Case 1: char $k = 2$. Then~~

$$\cancel{F^*(V) = S_*(V)}$$

~~Case 2: char $k = p$ odd. Take $V$ to be one dimensional with base $e$. Then~~

Then take $V$ to be 1 dimensional with basis element $e$.

Case 1: char $k = 2$. Then $\bar{S}(V^\vee) = k + k\check{e}$ $\quad \check{e}^2 = 0$ so using the bar resolution to calculate the Ext we find that

$$F(ke) = k[e]$$

$$\therefore \quad \boxed{F^*(V) = S_*(V)}$$

Case 2: char $k = p$, odd. $\quad \bar{S}(V^\vee) = k[x]/(x^p)$.

$$F(ke) = k[e, f] \qquad \begin{array}{l} \deg e = 1 \quad e^2 = 0 \\ \deg f = 2 \end{array}$$

and where $f$ is probably $\beta e$, $\beta$ = Bockstein, since changing $e$ to $\lambda e$ changes $f$ to $\lambda^p f$. If true, then there is a functorial ism

$$\boxed{F^*(V) \cong \Lambda(V) \otimes S(\beta V)}.$$

**Lemma A:** $\operatorname{res}\begin{bmatrix} f\omega \\ fg \end{bmatrix} = \operatorname{res}\begin{bmatrix} \omega \\ g \end{bmatrix}$

*Proof:*

$$\begin{tikzcd}
 & & A/f \arrow[r, equal] & A/f & \\
0 \arrow[r] & A/fg \arrow[r, "\overline{fg}"] \arrow[d, equal] & A/f^2g^2 \arrow[r, "\pi"] & A/fg \arrow[r] \arrow[l, bend right, "{[b,h]}"'] & 0 \\
0 \arrow[r] & A/fg \arrow[r, "\bar{g}"] \arrow[d, "p"] & A/f g^2 \arrow[r] \arrow[u, "\cdot f"] \arrow[d] & A/g \arrow[r] \arrow[u, "f"] \arrow[d, equal] \arrow[l, bend right, "h'"'] & 0 \\
0 \arrow[r] & A/g \arrow[r, "\cdot\bar{g}"] & A/g^2 \arrow[r, "h''"] & A/g \arrow[r] & 0
\end{tikzcd}$$

Claim $h$ induces $h'$. $\left( \begin{aligned} & h(fx) - f\cdot h(x) \equiv fg\,(f^2g^2A) \\ & \Rightarrow h(\operatorname{Im} f) \subset (\operatorname{Im} f). \end{aligned} \right)$

$$\operatorname{res}\begin{bmatrix} f\omega \\ fg \end{bmatrix} = \operatorname{tr}_{A/fg}\left\{ fa\left(\overline{fg}\right)^{-1}[b,h] \right\} = \operatorname{tr}_{fA/fg}\left\{ fa\left(\overline{fg}\right)^{-1}[b,h] \right\}$$

$\theta : A/g \xrightarrow{\sim} fA/fg$
$$= \operatorname{tr}_{A/g}\Big\{ \underbrace{\theta^{-1} fa\left(\overline{fg}\right)^{-1}}_{\pi a(\bar{g})^{-1}[b,h']} \underbrace{[b,h]\,\theta}_{\theta\,[b,h']} \Big\}$$

$$= \operatorname{tr}_{A/g}\left\{ a(\bar{g})^{-1}[b,h''] \right\}$$

$$= \operatorname{res}\begin{bmatrix} \omega \\ g \end{bmatrix}.$$

---

Compatibility with base extension $k \to k'$.

March 7-9, 1969

~~[illegible]~~

## Review of formulas for $\Omega(\mathbb{P}\check{E})$:

$$\Omega(\mathbb{P}E^\vee) = \Omega(X)[\xi] \Big/ \left(\xi^n - c_1(E)\xi^{n-1} + \cdots + (-1)^n c_n(E^*)\right)$$

where $n = \dim E$

$\xi = c_1(\mathcal{O}(1))$.

If $P(Z) \in \Omega(X)[[Z]]$, then

$$\boxed{f_* P(\xi) = \operatorname{res}\left\{\frac{P(Z)\,\omega}{\prod_{j=1}^{n} F(Z, I x_j)}\right\}}$$

$= \mathrm{Norm}_{\Omega(\mathbb{P}\check{E})[[Z]] \to \Omega(X)[[Z]]}\left(F(Z, I\xi)\right)$

where $\omega = \dfrac{dZ}{F_x(0, Z)}$ is the invariant differential on the formal group, where $F$ is the formal group law coming from the way Chern class behaves under $\otimes$ of line bundles, where $I$ is the inversion belonging to $F$, and where the $x_j$ are the phantom elements such that

$$c_t(E) = \prod_{j=1}^{n}(1 + t x_j)$$

Problems: 1. prove above formula giving a clear meaning to res. (perhaps as a trace or norm). (done March 8

2. Find a God-given $P$ such that $f_* P(\xi) = 1$ (something close to $c_{n-1}(f^*\check{E}/\mathcal{O}(-1))$.

March 7, 1969:

## Residues in dimension 1:

Let $A$ be a ring, let $B$ be an $A$-algebra, and let $f$ be a non-zero-divisor in $B$ such that $B/fB$ is a projective finitely generated $A$-modules. Following ~~on~~ Cartier (unpublished, see however Tate) we define a residue homomorphism

$$\Omega_{B/A} \longrightarrow A$$

$$\omega \longmapsto \mathrm{res}\,\frac{\omega}{f} \qquad = \left(\text{Grothendieck's residue symbol } \mathrm{res}\left[\tfrac{\omega}{f}\right]\right)$$

as follows. Let $\pi : B \longrightarrow B/fB$ be the canonical map; ~~and~~ as $B/fB$ is a projective $A$-module there exists a $A$-linear homomorphism $h : B/fB \longrightarrow B$ such that $\pi h = \mathrm{id}$. ~~Let~~ $b \in B$; then there is a unique $A$-linear ~~endomorphism of B/fB which we denote by~~ ~~$\pi \frac{x}{f}[b,h]$ such that~~ map from $B/fB$ to $B$, ~~which we shall~~, denoted by $f^{-1}[b,h]$, such that

(for a better defn. see page 6)

$$f\big((f^{-1}[b,h])\,u\big) = b\,h(u) - h(\pi(b)\,u)$$

for all $u \in B/fB$. If $x \in B$, then $u \longmapsto \pi(x\, f^{-1}[b,h]\,u)$ is an $A$-linear endomorphism of $B/fB$ ~~and~~ so has a trace. Thus we obtain a map

$$B \xrightarrow{\ D\ } \mathrm{Hom}_A(B, A)$$

$$b \longmapsto \big(\cancel{x} \longmapsto \mathrm{tr}\ \pi(\cancel{x} f^{-1}[b,h]\,u)\big)$$

One sees immediately that $D$ is a derivation of the $A$-algebra $B$ with values in $\mathrm{Hom}_A(B,A)$ given the $B$-module structure $(x\varphi)(y) = \varphi(xy)$:

$$\begin{aligned} D(b_1 b_2)(x) &= \mathrm{tr}\ \pi\left(x f^{-1}[b_1 b_2, h]\right) \\ &= \mathrm{tr}\ \pi\left(x f^{-1}(b_1[b_2,h] + [b_1,h]b_2)\right) \\ &= \mathrm{tr}\ \pi\left(b_1 x f^{-1}[b_2,h]\right) + \mathrm{tr}\ \pi\left(b_2 x f^{-1}[b_1,h]\right) \\ &= (b_1 Db_2 + b_2 Db_1)(x). \end{aligned}$$

$D$ gives rise to a unique homomorphism of $B$-modules

$$\Omega_{B/A} \xrightarrow{\Theta} \mathrm{Hom}_A(B,A)$$

such that $\Theta d = D$, and composing this with the ~~evaluation~~ map

$$\mathrm{Hom}_A(B,A) \longrightarrow A$$

given by evaluation at 1 one obtains the map

$$\mathrm{res}\left(\frac{\ }{f}\right) : \Omega_{B/A} \longrightarrow A$$

characterized by formula

$$\boxed{\mathrm{res}\ \frac{x\,db}{f} = \mathrm{tr}\ \pi(x f^{-1}[b,h])}$$

The residue map is independent of the choice of $h$; indeed given another $A$-linear section $h'$ of $\pi$ we have that $h - h' = f\varphi$

where $\varphi \in \mathrm{Hom}_A(B/fB, B)$, and

$$\begin{aligned} tr\,\pi(x f^{-1}[b, fh]) &= tr\,\pi(fx f^{-1}[b, h]) \\ &\quad + tr\,\pi(x[b,h]) \\ &= 0. \end{aligned}$$

For the applications we have in mind, $B = A[[Z]]$ and $\Omega_{B/A}$ will be ~~too~~ big. Instead we shall want $\mathrm{res}\frac{\omega}{f}$ defined for $\omega \in \hat{\Omega}_{B/A} \simeq B\,dZ$. ~~$\hat{\Omega}_{B/A}$ We recall that $\hat{\Omega}_{B/A}$ is characterized $\mathrm{Hom}_B(\hat{\Omega}_{B/A}, M)$ by the formula~~

~~$\mathrm{Hom}_B(\hat{\Omega}_{B/A}, M) \cong \mathrm{Der}_A(B, M)$~~

We recall that $\hat{\Omega}_{B/A}$ is the smallest quotient module of $\Omega_{B/A}$ such that

$$\mathrm{Hom}_B(\hat{\Omega}_{B/A}, M) \cong \mathrm{Der}_A(B, M)$$

for all $B$ modules killed by a power of the augmentation ideal $ZB$.

Suppose $B = A[[Z]]$ and $f$ is an element of $B$ admitting a factorization of Weierstrass type

$$f(Z) = u(Z)\,g(Z)$$

$$g(Z) = Z^n - c_1 Z^{n-1} + \cdots + (-1)^n c_n$$

where $u$ is a unit in $B$ and the $c_i$ are nilpotent elements of $A$. Then

$$B/fB \simeq B/gB = \bigoplus_{i=0}^{n-1} A\,\xi^i$$

where $g(\xi) = 0$.

Moreover
$$\xi^n = c_1 \xi^{n-1} + \cdots\cdots$$
is nilpotent and hence $\xi$ is nilpotent, i.e. $\exists N$ with
$$Z^N = g(Z) f(Z)$$
in $B$. Therefore with the notations as above, the image of $D$ is contained in $\mathrm{Hom}_A(B/fB, B)$ ~~[illegible]~~ identfied with a submodule of $\mathrm{Hom}_A(B,A)$ via $\pi$; as this is killed by $f$ it is killed by a power of $Z$, hence the residue map induces a map
$$\mathrm{res}\ \frac{\quad}{f} : \hat{\Omega}_{B/A} \longrightarrow A.$$

I). *Example*: (to check signs). Suppose $f(Z) = Z^n$. Then we may take $h: B/fB \cong A[[Z]]/(Z^n) \longrightarrow A[[Z]]$ to be
$$h(\xi^i) = Z^i \qquad 0 \le i < n.$$
Then if $b = Z$ we have
$$bh(\xi^i) = Z^{i+1} \qquad 0 \le i < n$$
$$h(\pi(b)\xi^i) = h(\xi^{i+1}) = \begin{cases} Z^{i+1} & 0 \le i < n-1 \\ 0 & i = n-1 \end{cases}$$
$$\therefore\ [b,h](\xi^i) = \begin{cases} 0 & 0 \le i < n-1 \\ Z^n & i = n-1 \end{cases}$$

so

$$\frac{1}{f}[b,h](\xi^i) = \begin{cases} 0 & 0 \le i < n-1 \\ 1 & i = n-1 \end{cases}.$$

Thus

$$\operatorname{res}\left(\frac{Z^j\,dZ}{Z^n}\right) = $$

$$= \operatorname{tr}\left(\xi^j \cdot \pi\left(\tfrac{1}{f}[b,h]\right)\right)$$

$$= \begin{cases} 0 & j \ne n-1,\ j \ge 0 \\ 1 & j = n-1 \end{cases}$$

which is of course the way it should be.

II). Suppose $f(Z) = Z^n - c_1 Z^{n-1} + \cdots + (-1)^n c_n$, $c_i$ nilpotent. Again taking same $b, h$ as in I, we find

$$bh(\xi^i) = Z^{i+1}$$

$$hb(\xi^i) = h(\xi^{i+1}) = \begin{cases} Z^{i+1} & 0 \le i < n-1 \\ c_1 Z^{n-1} - \cdots & i = n-1 \end{cases}$$

$$\therefore [b,h](\xi^i) = \begin{cases} 0 & i < n-1 \\ Z^n - c_1 Z^{n-1} + \cdots & i = n-1 \end{cases}$$

$$\frac{1}{f}[b,h](\xi^i) = \begin{cases} 0 & 0 \le i < n-1 \\ 1 & i = n-1 \end{cases}$$

$$\therefore \operatorname{res}\frac{Z^j\,dZ}{Z^n - c_1 Z^{n-1} + \cdots} = \begin{cases} 0 & 0 \le j < n-1 \\ 1 & j = n-1 \end{cases}$$

March 8, 1969:

## Residues in dimension 1 again:

Let $A$ be a ring, let $B$ be an $A$-algebra, and let $f$ be a non-zero divisor in $B$ such that $B/fB$ is a projective finitely generated $A$-module. We propose to define the residue symbol

$$\Omega_{B/A} \longrightarrow A$$

$$\omega \longmapsto \operatorname{res}\begin{bmatrix} \omega \\ f \end{bmatrix}$$

where $\Omega_{B/A}$ is the module of differentials of the $A$-algebra $B$. As $f$ is a non-zero divisor, there is an ~~Consider the~~ exact sequence

$$0 \longrightarrow B/fB \xrightarrow{i^{(f)}} B/f^2B \xrightarrow{\pi} B/fB \longrightarrow 0,$$

where $i^{(f)}(z+fB) = fz + f^2B$ and $\pi(b+f^2B) = b+fB$. Since $B/fB$ is projective as an $A$-module, there is an $A$-linear section $h$ of $\pi$. If $b \in B$, let $(i^{(f)})^{-1}[b,h] \in \operatorname{End}_A(B/fB)$ be given by

$$i^{(f)}\left((i^{(f)})^{-1}[b,h]\,v\right) = b\,h(v) - h(bv)$$

~~Then~~ As $B/fB$ is projective and finitely generated as an $A$-module there is a ~~[illegible]~~ trace homomorphism

$$\operatorname{tr}: \operatorname{End}_A(B/fB) \longrightarrow B,$$

~~Endowing~~

~~let $\operatorname{Hom}_A(B,A)$ be symbols with the $B$-module structure~~

and we obtain a map

$$D : B \longrightarrow \mathrm{Hom}_A(B, A)$$

$$(Db)(x) = \mathrm{tr}\,(x\, {\overset{(f)}{i}}{}^{-1}[b,h]).$$

One verifies that $D$ is a derivation of the $A$-algebra $B$ with values in $\mathrm{Hom}_A(B,A)$, regarded as a $B$-module via the rule $(x\varphi)(y) = \varphi(xy)$, hence $D$ gives rise to a homomorphism of $B$-modules

$$\Omega_{B/A} \xrightarrow{\ \theta\ } \mathrm{Hom}_A(B,A)$$

such that $\theta \circ d = D$. Composing with "evalution at $1 \in B$" one obtains ~~a map~~ the residue map

$$\Omega_{B/A} \longrightarrow A$$

$$\omega \longmapsto \mathrm{res}\begin{bmatrix}\omega \\ f\end{bmatrix},$$

characterized by the formula

$$\mathrm{res}\begin{bmatrix} x\,db \\ f \end{bmatrix} = \mathrm{tr}\,(x\, {\overset{(f)}{i}}{}^{-1}[b,h]).$$

One sees easily that the residue is independent of the choice of $h$.

<u>Proposition:</u> Let $a, f$ be non-zero divisors in an $A$-algebra $B$ such that $B/fB$ and $B/afB$ are projective finitely generated $A$-modules. Then if $\omega \in \Omega_{B/A}$

$$\mathrm{res}\begin{bmatrix} a\omega \\ af \end{bmatrix} = \mathrm{res}\begin{bmatrix} \omega \\ f \end{bmatrix}.$$

*Proof:* May assume $\omega = x\,db$. ~~In the~~ Consider the map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & B/af^2B \arrow[r, "(a)_1"] \arrow[d, "\pi_2"] & B/a^2f^2B \arrow[r] \arrow[d, "\pi_1"] & B/aB \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & B/fB \arrow[r, "(a)_2"] & B/afB \arrow[r] & B/aB \arrow[r] & 0
\end{tikzcd}$$

where all the maps are the obvious projections except the maps $(a)_1, (a)_2$ induced by multiplication by $a$. Then one sees that an $A$-linear section $h_1$ of $\pi_1$ covers the identity of $B/aB$ and hence induces a section $h_2$ of $\pi_2$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & B/afB \arrow[r, "(af)"] & B/a^2f^2B \arrow[r, "\pi_1"'] & B/afB \arrow[r] \arrow[l, bend right, "h_1"'] & 0 \\
0 \arrow[r] & B/afB \arrow[r, "(f)"] \arrow[u, "id"] \arrow[d, "\pi_5"] & B/af^2B \arrow[r, "\pi_2"'] \arrow[u, "(a)_1"] \arrow[d, "\pi_4"] & B/fB \arrow[r] \arrow[l, bend right, "h_2"'] \arrow[u, "(a)_2"] \arrow[d, "id"] & 0 \\
0 \arrow[r] & B/fB \arrow[r, "(f)"] & B/f^2B \arrow[r, "\pi_3"'] & B/fB \arrow[r] \arrow[l, bend right, "h_3"'] & 0
\end{tikzcd}$$

where a map labelled $\pi$ is a canonical projection and where $(u)$ denotes a map induced by multiplication by an element $u$. Note that $h_2$ induces a section $h_3$ of $\pi_3$. Then

~~Proof. We may assume that $\omega = x\,db$. As $a$ and $f$ are non zero divisors~~

$$\operatorname{res}\begin{bmatrix} a\,x\,db \\ af \end{bmatrix} = \operatorname{tr}_{B/afB}\left(ax(af)^{-1}[b,h_1]\right)$$

But the map of which the trace is to be taken has its image in $aB/afB$, hence the residue equals

$$\operatorname{tr}_{aB/afB}\left(ax(af)^{-1}[b,h_1]\right)$$

Using the isomorphism $(a)_2 : B/fB \longrightarrow aB/afB$ this becomes

$$\operatorname{tr}_{B/fB}\left((a)_2^{-1}\,ax(af)^{-1}[b,h_1](a)_2\right)$$

$$= \operatorname{tr}_{B/fB}\left(\pi_5\, x\,(af)^{-1}[b,h_1](a)_2\right)$$

$$= \operatorname{tr}_{B/fB}\left(x\,(f)^{-1}\,\pi_4\,(a)_1^{-1}[b,h_1](a)_2\right) \qquad \text{since } \pi_5(af)^{-1} = (f)^{-1}\pi_4(a)_1$$

$$= \operatorname{tr}_{B/fB}\left(x\,(f)^{-1}\,\pi_4\,[b,h_2]\right)$$

$$= \operatorname{tr}_{B/fB}\left(x\,(f)^{-1}[b,h_3]\right)$$

$$= \operatorname{res}\begin{bmatrix} x\,db \\ f \end{bmatrix}$$

by definition. <u>QED</u>.

For Adams.

Proof that $\operatorname{res}\begin{bmatrix} a\omega \\ af \end{bmatrix} = \operatorname{res}\begin{bmatrix} \omega \\ f \end{bmatrix}$ if $a, f$ are non-zero divisors in a $k$-algebra $A$ such that $A/fA$ and $A/aA$ are finitely gen. projective $k$-modules.

Key diagram is

$$\begin{tikzcd}
0 \arrow[r] & A/af^2 \arrow[r, "\bar a"] \arrow[d, shift right] & A/a^2f^2 \arrow[r] \arrow[d, shift right] & A/a \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & A/f \arrow[r, "\bar a"] \arrow[u, dashed, shift right, bend right, "h_2"'] & A/af \arrow[r] \arrow[u, dashed, shift right, bend right, "h_1"'] & A/a \arrow[r] & 0
\end{tikzcd}$$

Note: $\bar a$ denotes the map induced by multiplication by $a$.

for it shows that a $k$-linear section $h_1$ will cover the identity map of $A/a$ and hence induce a section $h_2$. Thus we will get the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A/af \arrow[r, "\overline{af}"] & A/a^2f^2 \arrow[r] & A/af \arrow[r] \arrow[l, dashed, bend right, "h_1"'] & 0 \\
0 \arrow[r] & A/af \arrow[r, "\bar f"] \arrow[u, "id"] \arrow[d, "\pi"] & A/af^2 \arrow[r] \arrow[u, "\bar a"] \arrow[d] & A/f \arrow[r] \arrow[u, "\bar a"] \arrow[d, "id"] \arrow[l, dashed, bend right, "h_2"'] & 0 \\
0 \arrow[r] & A/f \arrow[r, "\bar f"] & A/f^2 \arrow[r] & A/f \arrow[r] \arrow[l, dashed, bend right, "h_3"'] & 0
\end{tikzcd}$$

where $\pi$ is the natural projection and where <u>all</u> squares are commutative. Now set $\omega = x\,dy \in \Omega_{A/k}$. Then

$$\operatorname{res}\begin{bmatrix} a\omega \\ af \end{bmatrix} \overset{\text{defn}}{=} \operatorname{trace}_{A/af}\left(ax\,(\overline{af})^{-1}[y, h_1]\right)$$

$$= \operatorname{trace}_{aA/af}\left(ax\,(\overline{af})^{-1}[y, h_1]\right)$$ $(\mathrm{tr}_V\theta = \mathrm{tr}_{\theta V}\theta)$

$$= \operatorname{trace}_{A/f}\left(\pi \times (\overline{af})^{-1}[y, h_1]\,\bar{a}\right)$$ (using the isomorphism $\bar{a} : A/f \xrightarrow{\sim} aA/af$)

$$= \operatorname{trace}_{A/f}\left(x\,(\bar{f})^{-1}[y, h_3]\right)$$ (~~using~~ diagram chasing)

$$\overset{\text{defn}}{=} \operatorname{res}\begin{bmatrix}\omega \\ f\end{bmatrix}$$

Send Quillen copy.

## $f_*$ for a projective bundle:

Let $E$ be a complex vector bundle over $X$ of dimension $n$, let $f: \mathbb{P}\check{E} \to X$ be the projective bundle of lines in $E$. One knows that $\Omega(\mathbb{P}\check{E})$ is a free module over $\Omega(X)$ with basis $1, \xi, \ldots, \xi^{n-1}$ where $\xi = c_1(\mathcal{O}(1))$ and where

$$\xi^n - f^*c_1(E)\,\xi^{n-1} + \cdots + (-1)^n f^*c_n(E) = 0$$

Let $F(X,Y) \in \Omega(pt)[[X,Y]]$ be the formal group law ~~expressing the tensor product expressing~~ giving the behavior of the first Chern class with respect to tensor product of line bundles, let $I(X) \in \Omega(pt)[[X]]$ be the inverse for the formal group

$$F(X, IX) = 0,$$

and let $\omega$ be the invariant differential form on the formal group

$$\omega = \frac{dZ}{F_Y(Z,0)} \quad (= d\,l(Z) \text{ if a logarithm exists e.g. over } \mathbb{Q})$$

<u>Theorem</u>: If $a(Z) \in \Omega(X)[[Z]]$, then

$$\boxed{f_*\, a(\xi) = \operatorname{res}\left[\frac{a(Z)\,\omega}{\operatorname{Norm}_{\Omega(\mathbb{P}\check{E})[[Z]] \to \Omega(X)[[Z]]} F(Z, I\xi)}\right]}$$

*Proof*: We begin by showing that the residue symbol is well-defined. Write

$$F(X,Y) = X + Y + XYG(X,Y)$$

and make the substitution

$$Y = F(Z, IX) \quad \text{or} \quad Z = F(X,Y).$$

Then we obtain the identity

$$\boxed{Z - X = F(Z,IX)\{1 + XG(X, F(Z,IX))\}}$$

Thus

$$Z - \xi = F(Z, I\xi)\{1 + \xi G(\xi, F(Z,I\xi))\}$$

~~where~~ in $\Omega(\mathbb{P}\check{E})[[Z]]$, where the expression in brackets is a unit since $\xi$ is nilpotent. Thus taking norms from $\Omega(\mathbb{P}\check{E})[[Z]]$ to $\Omega(X)[[Z]]$ we have

$$\begin{aligned} \text{Norm } F(Z,I\xi) &= (\text{unit}) \text{ Norm}(Z-\xi) \\ &= (\text{unit}) \left(Z^n - f^*c_1(E) Z^{n-1} + \cdots\right) \end{aligned}$$

Hence if $Q = \text{Norm } F(Z, I\xi)$ we have that $Q$ is a non-zero divisor and that $\Omega(X)[[Z]]/(Q) \cong \Omega(\mathbb{P}\check{E})$ is finitely generated and free over $\Omega(X)$. Thus the residue is defined.

As both ~~sides~~ sides of the formula are compatible with base change we may suppose that $E = L_1 + \cdots + L_n$. Letting $x_i = c_1(L_i) \in \Omega(X)$ we have

$$\mathrm{Norm}\, F(Z, I\xi) = \prod_{j=1}^{n} F(Z, Ix_j).$$

To prove the formula ~~we~~ we use induction on $n$. Suppose $n=1$. Then $\mathbb{P}E \cong X$ so $f_* a(\xi) = a(x)$. But $x$ is nilpotent so

$$\mathrm{res}\begin{bmatrix} a(Z)\,\omega \\ F(Z, Ix) \end{bmatrix} = \mathrm{res}\begin{bmatrix} (1 + x\,G(x, F(Z, Ix)))\,a(Z)\,\omega \\ Z - x \end{bmatrix}$$

$$= \frac{(1 + x\,G(x, F(x, Ix)))}{F_y(x, 0)}\, a(x)$$

$$= \frac{1 + x\,G(x, 0)}{F_y(x, 0)}\, a(x) = a(x)$$

since

$$F_y(X, 0) = 1 + X\,G(X, 0).$$

Thus the formula is proved.

Now suppose $n > 1$, ~~and~~ let $F = L_1 + \cdots + L_{n-1}$ and let

$$\mathbb{P}\check{L}_n \xrightarrow{i} \mathbb{P}\check{E} \xleftarrow{j} \mathbb{P}\check{F}$$

be the canonical inclusions. Then

$$i_* 1 = c_{n-1}(\mathcal{O}(1) \otimes f^*\check{F}) = \prod_{j=1}^{n-1} F(\xi, Ix_j)$$

$$j_* 1 = c_1(\mathcal{O}(1) \otimes f^* \check{L}_n) = F(\xi, I x_n)$$

where for convenience we regard $\Omega(X)$ as being included in $\Omega(\mathbb{P}\check{E})$ via $f^*$. I claim that the formula to be proved holds if $a(Z) = \alpha(Z) \cdot F(Z, I x_n)$. In effect

$$f_* \{\alpha(\xi) F(\xi, I x_n)\} = f_* \{\alpha(\xi) j_* 1\}$$

$$= f_* j_* \alpha(j^* \xi) = g_* \alpha(\xi')$$

$$\operatorname{res} \begin{bmatrix} \alpha(Z) F(Z, I x_n)\, \omega \\ \prod_{j=1}^{n} F(Z, I x_j) \end{bmatrix} = \operatorname{res} \begin{bmatrix} \alpha(Z)\, \omega \\ \prod_{j<n} F(Z, I x_j) \end{bmatrix}$$

where $g : \mathbb{P}\check{F} \longrightarrow X$ is the canonical map and $\xi' = c_1(\mathcal{O}(1))$ on $\mathbb{P}\check{F}$. The equality of these two results by the induction hypothesis. As

$$Z - x_n = F(Z, I x_n) \cdot (\text{unit})$$

the formula holds if $a(Z) = \alpha(Z) \cdot (Z - x_n)$. Similarly one proves that the formula holds if $a(Z) = \alpha(Z) \cdot \prod_{j<n} (Z - x_j)$. By the division algorithm

$$\prod_{j<n} (Z - x_j) = q(Z)(Z - x_n) + \prod_{j<n} (x_n - x_j)$$

It follows that the formula holds if $a(Z)$ is a multiple of

$\prod_{j<n} (x_n - x_j)$ and hence

$$\prod_{j<n} (x_n - x_j) \left\{ f_* \, a(\xi) - \operatorname{res} \left[ \frac{a(Z)\,\omega}{\prod_{j=1}^{n} F(Z, Ix_j)} \right] \right\} = 0 .$$

But now we can argue universally: we may assume that $X_N = (\mathbb{P}^N) \times \cdots \times (\mathbb{P}^N)$ $n$ times and that $L_i$ is $\mathcal{O}(1)$ on the $i$th factor. Taking $a(Z) = Z^{\xi}$ and passing to the inverse limit as $N \to \infty$ we obtain an equation

$$\prod_{j<n} (x_n - x_j) \cdot u = 0$$

in the ring $\Omega(pt)[[x_1, \cdots, x_n]]$. As $x_n - x_j$ is a non-zero divisor it follows that $u = 0$; hence the theorem is proved.

---

~~Recall that~~ An application:

Let $P_n$ denote the ~~class of~~ element $f_* 1$ in $\Omega(pt)$ where $f : \mathbb{P}^n \to pt$. Then applying the theorem we see that

$$P_n = \operatorname{res} \left[ \frac{\omega}{Z^{n+1}} \right] \qquad n \geq 0$$

hence

$$\boxed{\omega = \sum_{n=0}^{\infty} P_n Z^n \, dZ}$$

Over $\Omega(pt) \otimes \mathbb{Q}$ there exists a unique logarithm series $\ell(X) = X + \cdots \ni$

$$\cancel{\ell(X)} \quad \ell(F(X,Y)) = \ell(X) + \ell(Y)$$

It is known that

$$\boxed{d\,l(z) = \omega}$$

hence

$$\boxed{l(Z) = \sum_{n=0}^{\infty} p_n \frac{Z^{n+1}}{n+1} \qquad \text{formula of Myshenko}}$$

March 10, 69 Stong-Hattori from rigidity of type 1 laws

these are the restricted analytic functions on $\mathbb{Z}_p \times \cdots \times \cdots$

tells me if I am willing to work over this bases instead of $\Omega(pt)_{(p)}[P_{p-1}^{-1}]$, then after an etale extension can make law $= G_m$..

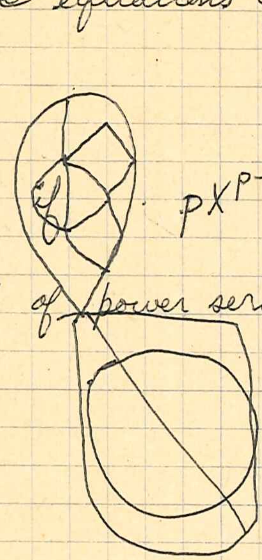
---

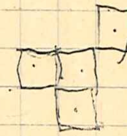
Point is that you have to solve equations of the form

$$\boxed{X^p - X - a}.$$

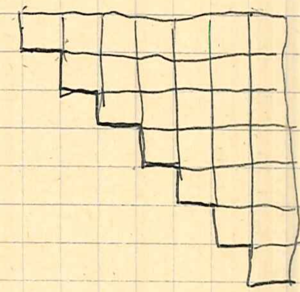
which are unramified if $pX^{p-1} - 1$ doesn't vanish

N is what kind of power series?





$$\varprojlim \mathbb{Z}/p^n\mathbb{Z}[b_0, b_1, b_2, \cdots][b_0^{-1}]$$



$$\sum a_\alpha b^\alpha \quad \ni \quad a_\alpha \longrightarrow 0.$$

thus this sum makes sense in any algebra $R$ over $\mathbb{Z}_p$ which is p-complete e.g. $\varprojlim_n R/p^n R \xleftarrow{\sim} R$

---

Thm: Any theory $\Gamma$ such that ~~[illegible]~~ $\Gamma(pt)$ is p-complete and $\ni$ $P_{p-1}$ is a ~~[illegible]~~ unit is equivalent after an integral ind-etale bases change to K-theory.

Proof: $1 \to \mathbb{Z}_p^* \longrightarrow N \longrightarrow X_1 \xrightarrow{\ \iota\ } \mathbb{Z}$.

same argument drawn out with algebras to be convincing!

__Theorems:__

$$\begin{tikzcd} A_K \arrow[r, hook, "\text{open}"] \arrow[d] & \overset{\text{Néron}}{A} \arrow[d] \\ \operatorname{Spec} K \arrow[r, hook, "\text{open}"'] & \operatorname{Spec} R \end{tikzcd} \qquad 0 \to T \to A_k \to B_k \to 0$$

$$\operatorname{Spec} k$$

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & A \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & \varprojlim T_n \arrow[r] \arrow[u, "\substack{\text{tate}\\ \text{exponential}}"'] & \varprojlim A_n \arrow[r] \arrow[u] & \varprojlim B_n \arrow[r] \arrow[u, "\wr"'] & 0 \end{tikzcd}$$

torus over R. — formal scheme over R not alg. — abelian variety over R

A rigid analytic space!!!

---

$$\begin{tikzcd} \operatorname{Spec}\{L[P_{p-1}] \otimes \mathbb{Z}/p^n\mathbb{Z}\} & \operatorname{Spec}\{L[P_{p-1}^{-1}] \otimes \mathbb{F}_p\}^{\text{"}X_1} \arrow[l] \\ N_{\mathbb{Z}/p^n\mathbb{Z}} \arrow[u] & N_{\mathbb{F}_p} \arrow[u, "\text{"etale" covering"}'] \\ \mathbb{Z}/p^* \arrow[u] & \mathbb{Z}_p^* \arrow[u] \end{tikzcd}$$

---

should be true in the limit which is restricted ~~power series~~ over $\mathbb{Z}_p$.

~~These are restricted power series~~

Restr $\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}[Q_1, Q_2, \ldots]$

i.e. given any $n$ $a_\alpha \in p^n\mathbb{Z}$ for all but a finite no of $a_\alpha$.

$\sum a_\alpha Q^\alpha$ where $a_\alpha \to 0$ strongly

## formal category schemes.

A ring, $Th(A)$ cat. of maps $R \longrightarrow A$ surjective with nilpotent kernel. Then a formal category is particular kind of ~~extension~~ category object in the category $\mathrm{Hom}(Th(A)^{o}, \text{sets})$. Think of it as

$$X_2 \rightrightarrows X_1 \rightrightarrows X_0 \qquad X_1 \times_{X_0} X_1 \longrightarrow X_1$$

Proposition: Let $C$ be a category object in a topos $\mathcal{T}$. Then the homotopic derived category of the following topoi coincide

- ⓐ Functors on $C$ with values in $\mathcal{T}$
- ⓑ $C^{o}$
- ⓒ simplicial objects over $\mathrm{Sing}\, C$ $\mathcal{T}$
- ⓓ cosimplicial objects of $\mathcal{T}$ over $\mathrm{Sing}\, C$ ?

The basic idea is to consider the map

---

**PETRIE:**

$X \longmapsto K(X) \otimes (\mathbb{Z}/p\mathbb{Z})$ universal theory with the $G_m$ group law with values in $\mathbb{Z}/p\mathbb{Z}$ algebras

$K(X) \otimes \mathbb{F}_{p^\infty}$ universal theory with $G_m$ law with values in $k = \mathbb{Z}/p\mathbb{Z}$ algs and with a $k$-linear Gysin homomorphism. + with $G_m$ law

$\Omega(X) \otimes \mathbb{F}_{p^\infty}$ universal for $k$ linear cohomology theories

~~Letter to Cartier & outline of final draft of paper.~~

$K(X) \otimes k$ : $\Omega(X) \otimes k\,[1/P_{p-1}]$

have equivalent group laws? **NO.** You must work somewhere else!

$\cancel{K(X) \otimes \mathbb{F}_p}$

So you start with the law $F^{\Omega}$ over $(\Omega \otimes \mathbb{F}_p)[1/P_{p-1}]$ which is of height 1. This means that if

$$\Omega(pt)[1/P_{p-1}] \longrightarrow B$$

is the ind-~~etale~~ covering ~~~~ given by the $\mathbb{G}_m$ law where $B$ is the affine ring of the ~~~~ (algebraic) group scheme of power series under composition, then

$$\boxed{\Omega \otimes_{\Omega(pt)} B}$$

is a ~~~~ universal theory with values in $B$-algs.

I would like to conclude that the theory

$\Omega \otimes_{\Omega(pt)} B$ is equivalent as a cohomology theory to

$$K \otimes_{K(pt)} B$$

~~and consequently that~~ The idea is to show that $\exists$ can. isom

$$\Theta : \boxed{K(X)} \otimes_{K(pt)} B \simeq \Omega(X) \otimes_{\Omega(pt)} B$$

of contravariant functors with values in B-algs.

If true for all $X$, then by Atiyah resolutions it's true that

$$0 \leftarrow \boxed{K(X)} \leftarrow \boxed{K(X_0)} \leftarrow K(X_1) \leftarrow K(X_2) \leftarrow \cdots$$

$MU \xrightarrow{p} MU \to MU/p$

generalized coh. theory

$$X_1 \leftarrow X_0 \leftarrow X$$

$$Y_2 \leftarrow X_1 \leftarrow Y_1$$

so you have these triangles

$$\begin{tikzcd}
K(X)\otimes_{K(pt)} B_* \arrow[d] & & \Omega(X) \arrow[ll] \arrow[dll] \arrow[d] \\
K(X,\mathbb{Z}/p)\otimes_{K(pt,\mathbb{Z}/p)} B_* & & \Omega(X,\mathbb{Z}/p) \arrow[ll, dashed, "?"']
\end{tikzcd}$$

$pE \sim 0$

$c_T(E)^p = 0$

not clear!

$$K(X)\otimes_{\mathbb{Z}} B_n \xleftarrow{\cong} \Omega(X)\otimes_{\Omega(pt)} B_n$$

$$\downarrow p$$

$$K(X) \longrightarrow \Omega(X)\otimes_{\Omega(pt)} B_n$$

$\Omega(X, \mathbb{Z}/p\mathbb{Z})$

operations

$$\mathbb{Z}[A]\otimes TX \longrightarrow T(A\otimes X)$$

$$\pi_p(\mathbb{Z}[A])\otimes \pi_q(TX) \longrightarrow \pi_{p+q}(T(A\otimes X))$$

in fact have

$$\Gamma A\otimes TX \longrightarrow T(A\otimes X)$$

if $T$ polynomial

~~if T of degree~~ $n$

$$A = kS^2$$

$\Gamma$

$$\boxed{\cancel{R}[kS^2]}$$

$\Gamma_p V$ 0, 1 $p-1, p$

✓✓ ✓✓

$S_p V$ $(0\ 1)(p-1, p)(2(p-1), 2p-1)$

✓ ✓

Conclusion:

$$\varprojlim_n H^{*+n}(\Gamma_p k[n])$$

$$\uparrow s$$

$$\varprojlim H^{*+n}(S_p k[n])$$

$* \geq 2$

~~otherwise~~ and it kills $P^0$ and $\beta$

cosimplicially $\varprojlim_n H^{*+n}(\Gamma k[n])$

dual to $\varinjlim_n H_{*+n}(S\, k[n])$

take $\boxed{S_p V}$ $\boxed{\text{stably}}$. $0 \to \Lambda^p V \longrightarrow S_{p-1}V\otimes V \quad S_p V \to 0$

$$H_q(S_p V) \xrightarrow{\sim} H_{q-(p-1)}(\Lambda_p V) \xrightarrow{\sim} H_{q-2(p-1)}(\Gamma_p V)$$

$H_0(V)$ $\quad 0 \to V \longrightarrow S_p V \longrightarrow \Gamma_p V \longrightarrow V \longrightarrow 0$

$\cong$

$0 \to H_1(S_p V) \xrightarrow{\sim} H_1(\Gamma_p V) \to 0 \qquad H_q(\Gamma_p V) = H_q(S_p V) \qquad q \geq 2$

1. Coh. autos.
2. Coh. autos + char classes
3. Gen. Wu fmla
4. Adams thm on ch
5. Completeness of Wu relations

---

<u>Theorem of Adams</u> (in torsion-free case): $X$ torsion-free

$a \in H^{2n}(X,\mathbb{Z})$ $\quad \xi \in K(X)$ $\quad \mathrm{ch}\,\xi = \sum_{q\geq 0} a_{q+n}$ $\qquad a_n = a$, $\; a_{q+n} \in H^{2(q+n)}(X,\mathbb{Z})$

Then $p^{[\frac{n}{p-1}]} a_{q+n}$ is $p$-integral and

$$\mathscr{P}^{-1}(\rho_p a) = \sum_{i\geq 0} \rho_p \left\{ p^{i} a_{n+i(p-1)} \right\}$$

(also for $2n+1$ by suspension).

---

<u>Lemma</u>: Let $f$ be a root of $X^{p-1} = p$. Then

$\dfrac{fX}{1-e^{-fX}}$ is $p$-integral and

$$\equiv 1 + (X - X^p + X^{p^2} - \cdots)^{p-1} \quad \mathrm{mod}\ p.$$

---

better to have

$$\boxed{\frac{1-e^{-fX}}{fX} \equiv \sum_{j\geq 0} (-1)^j X^{p^j - 1} \quad \mathrm{mod}\, p}$$

Proof:
$$e^{aX} = \sum \frac{a^n}{n!} X^n$$

$$v\left( \frac{\left(p^{\frac{1}{p-1}}\right)^{n-1}}{n!} \right) = \frac{n-1}{p-1} - v(n!) > 0$$

unless where $n = p^j$ and then

$$v(n!) = \frac{n-1}{p-1}.$$

$\therefore$ p-integral. Now

$$\frac{(p^j)!}{p^r} \equiv ? \quad (p) \qquad r = \frac{p^j - 1}{p-1}$$

Split up nos. $\leq p^j$ into those prime to and those divisible by $p$.

$$\boxed{(-1)^{p^{j-1}}} \qquad \frac{(p^{j-1})!}{p^{r - p^{j-1}}} \qquad \frac{p^j-1}{p-1} - p^{j-1} = \frac{p^{j-1}-1}{p-1}$$

Wilson

$\therefore$ you find
$$\frac{(p^j)!}{p^{1+\cdots+p^{j-1}}} \equiv (-1)^j \bmod p.$$

QED.

# Locally isomorphic group laws.

$$(R_1, F_1) \qquad (R_2, F_2)$$

these are ~~locally~~ isomorphic if one can be obtained from the other by base extension

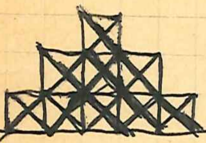
---

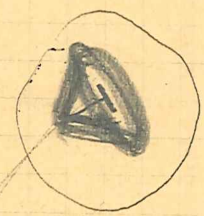
$$\cancel{R_1 \xrightarrow{f} R_2}$$

$$G_1 \qquad G_2$$

$$X_1 \underset{f}{\overset{g}{\rightleftarrows}} X_2$$

$$\begin{cases} G_1 \simeq f^* G_2 \\ G_2 \simeq g^* G_1 \end{cases}$$





$$\begin{tikzcd} X_1 \; (G_1 + \gamma_1) \arrow[dr] & & X_2 \; (G_2 + \gamma_2) \arrow[dl] \\ & \operatorname{Spec} L \; (G_{univ}, \gamma_{univ}) & \end{tikzcd}$$

---

The situation is as follows: ~~Given $X_1$ $X_2$~~

$X, Y$ with formal groups $G, H$ resp.

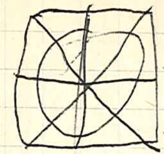
say locally equivalent iff $\exists$ faithfully flat q.c. map

$$X' \xrightarrow{f} X$$

$\ni$ $f^*G$ is equivalent to $H$ over $Y$

geometric fact: Conner-Floyd theorem

$$K(pt) \otimes_A A\Omega(X) \xrightarrow{\sim} K(X)$$



**Proof of the theorem:** After base extension from ~~A~~ A to B the universal law becomes canonically isomorphic to that of K. hence the isomorphism

$$(A) \qquad B \otimes_{\Omega(pt)} \Omega \cong B \otimes_k K$$

This is in reality the composition

$$B \otimes_{u\text{-}\Omega(pt)} \Omega \underset{\substack{\uparrow \\ \text{change} \\ \text{of law}}}{\cong} B \otimes_{v\text{-}\Omega(pt)} \Omega \cong B \otimes_{\mathbb{Z}} (\mathbb{Z} \otimes_{\Omega(pt)} \Omega) \underset{\substack{\uparrow \\ CF}}{\cong} B \otimes_{\mathbb{Z}} K$$

---

So we have the sequence

$$\begin{tikzcd}
A \otimes_{\Omega(pt)} \Omega \arrow[r] & B \otimes_{\mathbb{Z}} K \arrow[r, shift left] \arrow[r, shift right] & B \otimes_{\mathbb{Z}} \Gamma \otimes_{\mathbb{Z}} K \\
A \otimes_{\Omega(pt)} \Omega \arrow[r] & B \otimes_{\Omega(pt)} \Omega \arrow[u, "\wr"] \arrow[r, shift left] \arrow[r, shift right] & B \otimes_A B \otimes_{\Omega(pt)} \Omega \arrow[u, "\wr"]
\end{tikzcd}$$

exactness here since $A \to B$ f.f.

have to understand commutativity then ~~[illegible]~~ it follows that the upper sequence is exact.

K-theory char. nos.

<u>outline</u>

~~Results~~:

<u>Proof of main result.</u>

$$A = \Omega(pt)/p^n\,[P_{p-1}^{-1}]$$
$$B = K_0(MU)/p^n$$
$$\Gamma = \mathrm{Map\,cont}\,(\mathbb{Z}_p^*, \mathbb{Z}/p^n\mathbb{Z})$$

The ~~basic~~ geometric maps are

$$A\Omega(X) \xrightarrow{\mu} B \otimes K_0(X) \overset{\Delta_B \otimes id}{\underset{id \otimes \Delta_K}{\rightrightarrows}} B \otimes \Gamma \otimes K_0(X)$$

so you must know how $\Gamma$ acts on $B$ and on $K(X)$.

~~The basic alg. fact is that the action map $B \to B \otimes_k \Gamma$ induces an isomorphism~~

$$\cancel{B \otimes_A B \xrightarrow{\sim} B \otimes_k \Gamma}$$

~~i.e. $\Gamma$ acts freely on $B$ and the quotient is $A$.~~

<u>Alg. facts:</u>

(i) $A \longrightarrow B \overset{id \otimes 1}{\underset{\Delta}{\rightrightarrows}} B \otimes_k \Gamma$ exact $\qquad k = \mathbb{Z}/p^n\mathbb{Z}$

(ii) $A \longrightarrow B$ faithfully flat ~~[illegible]~~

(iii) $B \otimes_A B \xrightarrow{\cong} B \otimes_k \Gamma$

$$b_1 \otimes b_2 \longmapsto b_1 \cdot \Delta b_2$$

Observe that (ii) and (iii) imply (i).

Situation: To show

$X \longmapsto \Omega(\cancel{X})_{(p)}[P_{p-1}^{-1}]$ is equivalent in some sense to K theory. Idea is that

$$\Omega(X)_{(p)}[P_{p-1}^{-1}] = Q(X)$$

has a group law of height 1 over $\Omega(pt)_{(p)}[P_{p-1}^{-1}] = Q(pt)$ hence after flat base extension

$$\boxed{Q'(X) = Q' \otimes_Q Q(X)}$$

~~should~~ theref should exist a map

$$K(X) \longrightarrow Q'(X) \quad \cancel{\text{should}}$$

inducing an isomorphism with K theory.

---

$$Q' = \cancel{\Omega(pt)_{(p)}\otimes} \quad \Omega(pt)_{(p)}[P_{p-1}^{-1}]$$

to define a ring hom $K \to Q'$ we need

$$\begin{tikzcd} & K \arrow[r] & Q' \\ \Omega \arrow[ur] \arrow[urr, "\hat{\varphi}"'] & & \end{tikzcd}$$

a power series $\varphi \in Q'[[X]]$ $\cancel{\varphi}$ $\bar{\varphi} \in N(Q')$ with

$$\boxed{\bar{\varphi} * F^{Q'} = X+Y-XY}$$

in other words ~~Spec~~

$$\begin{tikzcd} & \mathrm{Spec}\, Q' \arrow[dl] \arrow[d, "\text{open}"] \\ N \arrow[r, "\hat{G}_m \text{ orbit}"'] & \mathrm{Spec}\, L_{(p)} \end{tikzcd}$$

*Paradox:*

Let $A$ be a complete ~~[illegible]~~ discrete valuation ring of unequal char with residue field $k$ of char $p$ and let $F_0$ be a formal group law over $k$ of height 1 not isomorphic to $\hat{\mathbb{G}}_m$. Let $F$ be a lifting of $F_0$ to $A$ (exists ~~[illegible]~~ by Lazard) and is unique up to isomorphism by Lubin-Tate. In fact as the endos of $F \simeq \mathbb{Z}_p$ ~~[illegible]~~ are ~~[illegible]~~ faithfully represented as endos. of $F_0$ $F$ is unique up to canonical isomorphism. In other words given $F, F'$ with ~~[illegible]~~ $\bar{F} = \bar{F'} = F_0$ ($-$ = reduction mod $m$) there is a unique power series $\varphi(X) \equiv X$ mod $m$ and ~~mod deg 2~~ ($\varphi(X) \equiv uX$ mod deg 2, $u \in A^*$) such that $\varphi * F = F'$.

Now by taking a ~~separable~~ Galois extension $k'$ of $k$ of group $(\mathbb{Z}_p)^*$, ~~[illegible]~~ $F_0$ becomes isomorphic to $\hat{\mathbb{G}}_m$ (over $k'$). Let $A'$ be the unramified extension of $A$ ~~[illegible]~~ with residue field $k'$. It follows by what we said above that $F$ is ~~[illegible]~~ isomorphic to $\hat{\mathbb{G}}_m$ over $A'$. **Hence** $\exists$ ~~unique~~ power series

$$\varphi(X) = uX + \cdots\cdots \in A'[[X]] \qquad u \in A'^*$$

~~[illegible]~~

which is unique up to an automorphism of $F$, i.e. $\mathbb{Z}_p^*$. with

$$\varphi F(\varphi^{-1}X, \varphi^{-1}Y) = X + Y - XY.$$

But over $K' = A' \otimes_{\mathbb{Z}} \mathbb{Q}$, ~~[illegible]~~ one can calculate $\varphi$ in terms of the logarithm. Thus let

$$l F(X,Y) = l(X) + l(Y)$$

$$\ell(X) = X + \cdots \quad \in K[[X]] \qquad K = A \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Then $\quad \Psi(X) = 1 - e^{-\ell(X)}$ satisfies

$$\Psi(F(X,Y)) = 1 - e^{-\ell(X)-\ell(Y)} = (1-e^{-\ell(X)}) + (1-e^{-\ell(Y)}) - (1-e^{-\ell(X)})(1-e^{-\ell(Y)})$$

$$= \Psi(X) + \Psi(Y) - \Psi(X)\Psi(Y).$$

or $\quad (\Psi * F)(X,Y) = X + Y - XY$

$$\Psi(X) = X + \cdots$$

now if $\quad \psi^a(X) = 1 - (1-X)^a$

$$\psi^a(X+Y-XY) = 1 - (1-X-Y+XY)^a$$

$$= 1 - (1-X)^a(1-Y)^a$$

$$= \psi^a(X) + \psi^a(Y) - \psi^a(X)\psi^a(Y).$$

$$\psi^a(X) = aX + \cdots$$

$$\boxed{\psi^u \circ \Psi = \varphi}$$

$$\varphi(X) = 1 - (1 - 1 + e^{-\ell(X)})^u = 1 - e^{-u\ell(X)}$$

$$\boxed{\varphi(X) = 1 - e^{-u\ell(X)}}$$

The paradox arises because $l(X)$ is defined over $A'$ and does not belong to $A$

$$\underbrace{1-\varphi(X)}_{\in A'[[X]]} = \underbrace{\left(e^{-l(X)}\right)^{u}}$$

Let $N^\times$ = power series ~~$l(X) = X + \cdots$~~

---

Conclude if $F$ is a group law of height 1 over $A$, ~~then~~ with logarithm series $l$, then there is a unit $u$ in the maximal unramified extension $A'$ of $A$ unique up to multiplication by elements of $\mathbb{Z}_p^*$ such that

$$\varphi(X) = 1-\left(e^{-l(X)}\right)^{u} \in A'[[X]]$$

Conversely given $\varphi(X)$ and a unit $u$ in $A'[[X]]$ one can solve above equation for $l(X)$ and ~~group~~ if $l(X) \in A[[X]]$ the corresponding group law should be of height 1 ?

$\therefore$ no paradox

height 1 Eisenstein polyps are $T + c$ where $v(c) = 1$.

---

Cartier claims that $\exists$ unique lifting from $\mathbb{F}_p$ to $\mathbb{Z}_p$, hence maps $\mathbb{Z}_p^* \longrightarrow$ units in $\mathbb{Z}_p[\mu_{p^\infty}]$ ?

March 10 - , 1969

formulation of R-R
Proof of Conner-Floyd thm. char K
universal prop of $\Omega T$
Adams spec seq same for $\Omega + \Omega T$

## Characterizing cohomology theories by their formal group laws:

$\boxed{\text{K theory}}$: Given a multiplicative cohomology functor $Q$ on manifolds endowed with Gysin homomorphism for complex oriented maps whose group law is $F(X,Y) = X+Y-XY$ there is a unique ~~[illegible]~~ natural transformation $K \longrightarrow Q$ compatible with products + Gysin.

*Proof:* The universal theory is clearly ~~[illegible]~~ $\Omega \otimes_L K(pt)$ where $L$ is the Lazard ring. As $L \simeq \Omega(pt)$ (my thm.) and ~~[illegible]~~ $\Omega \otimes_{\Omega(pt)} K(pt) \simeq K$ (Conner + Floyd) the result follows.

But the result is more elementary. In effect ~~[illegible]~~ ~~a ring homomorphism [illegible] by the splitting principle [illegible] to extend natural transformation Pic $\to \Omega \otimes_L K(pt)$ to each~~ given a multiplicative cohomology theory $Q$ with Gysin satisfying the splitting principle ~~[illegible]~~ and a natural transformation $\mathrm{Pic} \longrightarrow Q$ (given by a power series $a(X) \in Q(pt)[[X]]$), there is a unique additive extension $\varphi : K \longrightarrow Q$ given by

$$\varphi(E) = \underset{Q(P\check{E}) \to Q(X)}{\mathrm{Trace}}\ \varphi(\mathcal{O}(1)).$$

$\varphi$ is a ring homomorphism iff $\varphi : \mathrm{Pic} \longrightarrow Q^*$ is a group homomorphism iff the power series $a(X)$ satisfies

$$a(F^Q(X,Y)) = a(X)\,a(Y)$$

iff $a$ is a character of the formal group of $Q$.

Therefore if $Q$ has the group law of $K_*$ theory ~~[illegible]~~

then $a(X) = \frac{1}{1-X}$ satisfies this equation and so we obtain a ring homomorphism $\varphi : K \to Q$ such that

$$\varphi(c_1^K L) = \varphi(1 - L^\vee) = 1 - \varphi(L)^{-1} = 1 - a(c_1^Q L)^{-1}$$
$$= 1 - (1 - c_1^Q L)$$
$$= c_1^Q L.$$

Let $\rho : \Omega \to K$ be the canonical transformation. One knows that $\rho$ is onto in fact has an additive section $u$ given by

$$u(L) = 1 - c_1^\Omega(L^\vee) = 1 - I\{c_1^\Omega(L)\}$$

Then $\rho\varphi : \Omega \to Q$ is multiplicative and carries $c_1^\Omega(L)$ to $c_1^Q(L)$. Thus by RR-Thom theorem

$$\rho\varphi = \hat{b} \quad \text{where} \quad b(X) = 1$$

$$\text{(recall } \bar{b}(X) = Xb(X) \ \ \& \ \ \hat{b}\, c_1^Q(L) = \bar{b}(c_1^Q(L))\text{)}.$$

~~so~~ we have that $\rho\varphi$ is

$$\rho\varphi(f_* x) = f_*\left(\bar{b}(\nu_f)\ \rho\varphi x\right) = f_* \rho\varphi x.$$

Thus $\rho\varphi$ is compatible with Gysin and hence as $\rho$ is surjective, $\varphi$ is compatible with Gysin. This shows that $K$ has the universal property ~~as well as the fact~~ desired as well as the Conner-Floyd result

$$\boxed{\Omega \otimes_L K(pt) \xrightarrow{\sim} K}$$

as both sides have the same universal property.

Remarks 1:

Actually one doesn't need to know that $\rho$ is surjective since one has the following result proved exactly by R-R argument.

Lemma: Let $Q_1, Q_2$ be mult. coh. theories with Gysin + splitting principles. Assume $\Gamma : Q_1 \to Q_2$, natural is a ring homomorphism $\ni$ $\Gamma\{c_1^{Q_1}(L)\} = c_1^{Q_2}(L)$. Then $\Gamma$ commutes with Gysin homomorphism.

(Observe this result implies old R-R argument, e.g. given $\beta : \Omega \to Q$ a natural ring hom $\ni$ $\bar{\beta}(1) \in Q(pt)^*$ then $\beta(f_* x) = f_*(\bar{\beta}(\nu_f)\beta x)$. In effect endow Q with new Gysin $f_! x = f_*(\bar{\beta}(\nu_f)x)$, whence one wants to show that $\beta : \Omega \to Q$ with new Gysin is compatible with Gysin and this follows from the lemma).

Remark 2: Observe we get quite easily that

$$\Omega(pt) \otimes_L K(pt) \simeq K(pt)$$

Now by Lazard we know that $L \to \Omega(pt)$ is injective so one might hope to prove that $L \cong \Omega(pt)$ using this isomorphism. However unfortunately $L \to K(pt)$ isn't graded so one can't even get $\Omega(pt)_{odd} = 0$. Perhaps one should also use connected K-theory? Unfortunately $\Omega^* \to k^*$ is not onto so this method fails. Use of $K^*$ might at best tell us that ~~the~~ $\Omega(pt) \otimes_L \mathbb{Z}[T,T^{-1}] \xrightarrow{\sim} \mathbb{Z}[T,T^{-1}]$, which then implies the old result by setting $T=1$. But still we don't get that $\Omega(pt)_{odd} = 0$.

---

Remark 3: Additive extensions of a natural transformation $\text{Pic} \longrightarrow Q$ may be generalized as follows. Let $G$ be a comm. formal group over $Q(pt)$ and suppose given a natural transformation $\varphi : \text{Pic}X \longrightarrow G(Q(X))$. Then extend $\varphi$ to $K$ by

$$\varphi(E) = \text{Norm}_{Q(\mathbb{P}\check{E}) \to Q(X)} \varphi(\mathcal{O}(1))$$

where Norm is to be understood in the sense of algebraic groups (Norm can be defined for $G(B) \to G(A)$ where $G$ is a commutative algebraic group and $A \to B$ is finite locally free (Deligne). Should also work for formal groups).

---

Brown-Peterson theory: Given as standard (products, Gysin, splitting principle) theory $Q$ with values in $\mathbb{Z}_{(p)}$ algebras such that the group law $F^Q$ is typical, there is a unique hommorphism $BP \longrightarrow Q$ of theories.

Proof: (Follows from $\Omega \otimes_L LT \xrightarrow{\sim} BP$.

---

$H^*(?, \mathbb{F}_2)$: Universal standard unoriented theory with values in $\mathbb{F}_2$-algebras such that the formal group law is $X+Y$.

Proof: $\mathfrak{N}^*(X) \otimes_{\mathfrak{N}^*(pt)} \mathbb{F}_2 \xrightarrow{\sim} H^*(X, \mathbb{F}_2)$.

---

? Example: Consider $H^*(X, \mathbb{F}_p)$ ($p$ odd) as a standard cohomology theory with complex orientation. Then $x \longmapsto x + \beta x$, $\beta =$ Bockstein is an automorphism* of this theory whose effect on $c_1(L)$ is the identity since $\beta$ on $H^*(\mathbb{P}^n, \mathbb{F}_p)$ is zero. Thus by the lemma $(id + \beta)$ is compatible with Gysin homomorphisms. It follows that $H^*(?, \mathbb{F}_p)$ can't be characterized by its formal group law.

*False since $\beta x \cdot \beta y \neq 0$ ?

~~Cobordism theory~~

~~$\Omega$ as a universal cohomology theory~~

**Riemann–Roch lemma:** $Q_1, Q_2$ standard theories (splitting principle). Then if $\theta : Q_1 \to Q_2$ is a natural ring hom. ~~commuting~~ with $\theta(c_1^{Q_1} L) = c_1^{Q_2} L$, then $\theta f_*^{Q_1} = f_*^{Q_2}\theta$.

Proof: (1). True for a map $f : \mathbb{P}\check{E} \longrightarrow X$. In effect enough

$$c_1^{Q_i}(\mathcal{O}(1)) = \xi_i \qquad \theta\xi_1 = \xi_2$$

$$\theta \cdot f_*^1 (f^{1*}a \cdot \xi_1^q) = \theta a \ \theta f_*^1 \xi_1^q$$

$$f_*^2 \ \theta(f^{1*}a \cdot \xi_1^q) = f_*^2 f^{2*} \theta a \cdot \theta \xi_1^q = \theta a \cdot f_*^2 \theta \xi_1^q .$$

$$\theta \ f_*^1 \xi_1^q = \theta \ \mathrm{res} \begin{bmatrix} Z^q \omega^1 \\ \cancel{F(Z)} \\ \cancel{\Pi F_1(Z, I x_j)} \end{bmatrix} = \mathrm{res} \begin{bmatrix} Z^q \omega^2 \\ \Pi F_2(Z, I x_j^2) \end{bmatrix}$$

~~Since~~

$$\theta \ \mathrm{Norm}_{Q_1(\mathbb{P}\check{E})[[Z]] \to Q_1(X)[[Z]]} (F_1(Z, I_1 \xi_1))$$

$$\parallel$$

$$\mathrm{Norm}_{Q_2(\mathbb{P}\check{E})[[Z]] \to Q_2(X)[[Z]]} (F_2(Z, I_2 \xi_2))$$

since $\theta\xi_1 = \xi_2$ ✓

(2) $X \xrightarrow{f} Y$ proper with cx. orientation

$X \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^n$ embedding

get $X \longrightarrow X \times \mathbb{P}^2 \longrightarrow \mathbb{P}^n$ embed.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & Y \times \mathbb{P}^n \arrow[d, "\text{true here}"] \\ & Y \end{tikzcd}$$

thus can assume that $f$ is an embedding.

(3) $i : Y \longrightarrow X$ embedding $\ni$ normal bundle
$\cancel{N} = 2 \oplus E''$.

Now
$$\begin{tikzcd} \tilde{Y} \arrow[r, "j"] \arrow[d, shift left, "g"] & \tilde{X} \arrow[d, "f"] \\ Y \arrow[u, shift left, "s"] \arrow[r, "i"'] & X \end{tikzcd}$$

(i) $f_* 1$ is a unit in $Q_2$ (comes from $\Omega$)

(ii) $f^*(i_* y) = j_*(s_* 1 \cdot g^* y)$ (geometric argument.)

does this work equivariantly?
Yes see below

(iii) $s^1_* 1$ divisible by $j^* j_* 1$ (comes from $\Omega$)

(iv)

$\Theta\, s^1_* 1 = s^2_* 1$. ($s_* 1$ has a formula in terms of $\xi$)

(v) $\Theta(j^1_* 1) = j^2_* 1$ again by Chern classes
$j_* 1 = c_1(\mathcal{O}(-1)_{\tilde{X}})$.

(vi) $\quad j^*(j_* z) = (j^* j_* 1)\cdot z$ (will hold if locally $\tilde{X}$ retracts to $\tilde{Y}$)

---

$$\theta\, \iota^1_* y \overset{?}{=} \iota^2_* \,\theta y$$

By (i) $f_2^*$ is injective so to show

$$\theta f_1^* \iota^1_* y \overset{?}{=} f^{2*} \iota^2_* \theta y$$

$$\begin{array}{cc} \| & \| \\ \theta\, j^1_*(s^1_* 1\, g^{1*} y) & j^2_*(s^2_* 1 \cdot g^{2*}\theta y) \\ & \|\ \text{(iv)} \\ & j^2_*(\theta(s^1_* 1 \cdot g^{1*} y)). \end{array}$$

By (iii)

$$s^1_* 1 = j^{1*} j^1_* 1 \cdot a$$

$$\begin{aligned} s^1_* 1 \cdot g^{1*} y &= j^{1*} j^1_* 1 (a \cdot g^{1*} y) \\ &= j^{1*} j^1_* (a \cdot g^{1*} y) \qquad \text{(vi)} \\ &= j^{1*} z. \end{aligned}$$

$$\theta(j^1_* j^{1*} z) \overset{?}{=} j^2_* \cancel{\theta} j^{2*} \theta z$$

$$\|$$

$$\theta\, j^1_* 1 \cdot \theta z \overset{?}{=} j^2_* 1 \cdot \theta z$$

But $\theta\, j^1_* 1 = j^2_* 1$ by (iv).

Proof of (iii): Recall on $\mathbb{P}E$ that we have an exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow f^*E \xrightarrow{\pi} F \longrightarrow 0,$$

hence given an everywhere non-zero section $u$ of $E$ it gives a section $f^*u$ of $f^*E$ hence a section $\pi f^*u$ of $F$ which is transversal to zero and whose zerosubmanifold are precisely $s: X \longrightarrow \mathbb{P}E$. Thus

$$s_* 1 = c_{n-1}(F).$$

But

$$c_t(\mathcal{O}(-1)) \cdot c_t(F) = f^* c_t(E),$$

so

$$c_{n-1}(F) + c_1(\mathcal{O}(-1))\, c_{n-2}(F) = f^* c_{n-1}(E).$$

Therefore since $E = 2 + E''$ we have $c_{n-1}(E) = c_{n-1}(E'') = 0$, so

$$s_*(1) = c_{n-1}(F) = \left(-c_{n-2}(F)\right) \underbrace{c_1(\mathcal{O}(-1))}_{\substack{\| \\ j^* j_* 1}}$$

because ~~recall~~ near $\tilde{Y}$ that $\tilde{X} = \mathcal{O}_{\mathbb{P}E}(-1)$ and $j: \tilde{Y} \longrightarrow \tilde{X}$ is the zero section.

March 10, 1967

Proof that $N/H_F \xrightarrow{\sim} X_h$ over $\mathbb{F}_p$ for any law $F$ of height $h$ on page 10.

The scheme of endos of $\hat{\mathbb{G}}_m$ over $\mathbb{F}_p$:

Let $F(X,Y) = X+Y+XY$. Consider the functor which associates to each $\mathbb{F}_p$ algebra $A$ the group of power series $\varphi(X) = \sum_{n\geq 1} a_n X^n$ with coefficients in $A$ such that $F(\varphi(X),\varphi(Y)) = \varphi(F(X,Y))$. The group law comes from $(\varphi+\psi)(X) = F(\varphi(X),\psi(X))$. It's perhap simpler to use the power series $\varphi'(X) = 1+\varphi(X) = \sum_{n\geq 0} a_n X^n$, $a_0 = 1$ which satisfies

$$\varphi'(X+Y+XY) = \varphi'(X)\varphi'(Y).$$

Now

$$1+[p]_F(X) = (1+X)^p = 1+X^p$$

$$[p]_F(X) = X^p$$

so as an endo $\varphi$ satisfies $\varphi\circ[p] = [p]\circ\varphi$ we have

$$\varphi(X^p) = \varphi(X)^p$$

or that

$$\sum_{n\geq 1} a_n X^{pn} = \sum_{n\geq 1} a_n^p X^{np}$$

Thus

$$a_n = a_n^p \quad \text{for all } n.$$

~~Next note that if $a^p = a$ in $A$, then~~

$$\cancel{\varphi^n(X) = aX^{p^n}}$$

~~is an endo. i.e.~~ $\cancel{a(X+Y+XY)^{p^n} = aX^{p^n} + aY^{p^n} + a}$

<u>Lemma</u>: If $a$ is an element of a $\mathbb{Z}_{(p)}$-algebra $R$ such that $a(a-1)\cdots(a-p+1)=0$, then

$$\varphi(X)=\sum_{i=1}^{p-1}\binom{a}{i}X^i \qquad\qquad \binom{a}{i}=\frac{a(a-1)\cdots(a-i+1)}{i!}$$

is an endomorphism of the formal group law $X+Y+XY$.

<u>Proof</u>: We may suppose $R=\mathbb{Z}_{(p)}[T]/(T(T-1)\cdots(T-p+1))$ and hence we may suppose $R$ is torsion-free. Thus working in $R\otimes\mathbb{Q}$ we have

$$\sum_{i=0}^{p-1}\binom{a}{i}X^i=\sum_{i=0}^{\infty}\binom{a}{i}X^i=e^{a\log(1+X)}$$

In particular

$$\sum_{i=0}^{p-1}\binom{a}{i}(X+Y+XY)^i=\sum_{i=0}^{p-1}\binom{a}{i}X^i\cdot\sum_{i=0}^{p-1}\binom{a}{i}Y^i$$

in $R\otimes\mathbb{Q}$, hence also in $R$. QED.

<u>Corollary</u>: If $a^p-a=0$ in an $\mathbb{F}_p$-algebra $A$, then

$$\sum_{i=1}^{p-1}\binom{a}{i}X^{ip^s} \qquad s \text{ integer } \geq 0.$$

in an endo of $X+Y+XY$.

<u>Proof</u>: $a^p-a=a(a-1)\cdots(a-p+1)$ in characteristic $p$, hence by the lemma

$$\sum_{i=0}^{p-1}\binom{a}{i}(X+Y+XY)^{ip^s}=\sum_{i=0}^{\infty}\binom{a}{i}(X^{p^s}+Y^{p^s}+X^{p^s}Y^{p^s})^i=\sum_{i=0}^{p-1}\binom{a}{i}X^{ip^s}\sum_{i=0}^{p-1}\binom{a}{i}Y^{ip^s}.$$

With the dot notation of page 1, let

$$\varphi^{\cdot}(X) = \sum_{n=0}^{\infty} a_n X^n \qquad a_0 = 1$$

be an endomorphism of $X+Y+XY$. Then

$$(1+X)^{-a_1} \cdot \varphi^{\cdot}(X) \quad \equiv \quad 1 \qquad \text{mod deg } 2.$$

is an endomorphism. ~~with~~ One knows that an endomorphism is always a power series in $X^{p^h}$ for some $h$. Thus one can repeat the above process getting an infinite product expansion

$$\varphi^{\cdot}(X) = (1+X)^{b_0}(1+X^p)^{b_1} \cdots\cdots$$

where the $b_i \in A$ satisfy $b_i^p = b_i$. Thus one concludes

$$a_n = \prod_{i\geq 0} \binom{b_i}{\varepsilon_i} \qquad \text{if} \quad n = \sum_{i\geq 0} \varepsilon_i p^i \quad 0 \leq \varepsilon_i < p$$

e.g. $\quad a_{p^k} = b_k \quad .$

Now I want to identify ~~the ring set~~ an endomorphism of $\hat{\mathbb{G}}_m$ over $A$ with a ~~~~ map of Spec $A$ into the profinite ring ~~group~~ $\mathbb{Z}_p$ regarded as a ring scheme in the canonical way. Now ~~~~ if ~~~~ $a^p = a$, then one obtains a partition of (Spec $A$)$=X$ into open and closed subscheme $X_i$, $i \in \mathbb{F}_p$ where $a = i$ on $X_i$. Therefore given an integer $n$ we get a partition $X_i \quad i \in \mathbb{Z}/p^n\mathbb{Z}$ where modulo degree $p^n + 1$ we have that

$$\varphi^i(X) \equiv (1+X)^i$$

~~[illegible]~~ or $X_i$. ~~[illegible]~~ One sees that sum and product of endos. corresponds to sum and product of ~~[illegible]~~ the ~~[illegible]~~ indices $i$, since

$$(1+X)^{i+j} = (1+X)^i(1+X)^j$$

$$\left(1+ (1+X)^j-1\right)^i = \left((1+X)^j\right)^i = (1+X)^{ji}$$

Thus the ring of endos. of $\hat{\mathbb{G}}_m$ mod degree $p^n+1$ over $A$ is the same as ~~[illegible]~~ the ring of points of the constant scheme $\underline{\mathbb{Z}/p^n\mathbb{Z}}$ with values in Spec $A$. ~~[illegible]~~ Letting $n \to \infty$ one finds the desired result:

<u>Proposition:</u> The ring scheme of endos. of $\hat{\mathbb{G}}_m$ over $\mathbb{F}_p$ is the profinite ring $\mathbb{Z}_p$ regarded as a "proconstant" ring scheme over $\mathbb{F}_p$, i.e. with affine algebra = all locally constant functions from $\mathbb{Z}_p$ to $\mathbb{F}_p$ with $\Delta_a$ and $\Delta_m$ induced by the addition + multiplication of $\mathbb{Z}_p$.

The group scheme of autos. of $\hat{\mathbb{G}}_m$ over $\mathbb{F}_p$ is the profinite group $\mathbb{Z}_p^*$ regarded as a affine group scheme over $\mathbb{F}_p$.

---

We work over $\mathbb{F}_p$. Let $N$ be the affine group scheme of power series $a_1 X + a_2 X^2 + \cdots$ where $a_1$ is a unit, let $N_1$ be the normal subgroup with $a_1 = 1$; then $N$ is a semi-direct product of $\mathbb{G}_m$ and $N_1$. Let $L$ be the Lazard ring over $\mathbb{F}_p$ and let $F$ be the canonical group law over $L$. Then $N$ acts on Spec $L$ by $(\varphi * F)(X,Y) = \varphi F(\varphi^{-1}X, \varphi^{-1}Y)$. If $\omega$ is the invariant differential of $F$, let

$$\omega = \sum_{n=0}^{\infty} P_n Z^n \, dZ \qquad P_0 = 1 \quad P_n \in L.$$

I claim that

$$(\underbrace{X * \cdots * X}_{p \text{ times}} =)\ [p]_F(X) = P_{p-1} X^p + \text{higher terms.}$$

To see this work in the integral Lazard ring

$$u(X) = \sum_{i=1}^{p-1} P_{i-1} \frac{X^i}{i}$$

$$v(X) = X + P_{p-1} \frac{X^p}{p}$$

so that

$$vu = \sum_{i=1}^{p} P_{i-1} \frac{X^i}{i} \equiv \ell(X) \quad \text{mod deg } p+1$$

hence

$$(u * F)(X,Y) \equiv X + Y + \lambda C_p(X,Y) \qquad \cancel{\text{mod deg } p+1}$$

$$((v \circ u) * F)(X,Y) \equiv X + Y \qquad ''$$

$$\therefore \quad v(X + Y + \lambda C_p(X,Y)) = v(X) + v(Y)$$

or

$$X + Y + \lambda C_p(X,Y) + P_{p-1} \frac{(X+Y)^p}{p} = X + P_{p-1}\frac{X^p}{p} + Y + P_{p-1}\frac{Y^p}{p}$$

Thus

$$\lambda = - P_{p-1}$$

~~[illegible] as $u$ is defined over $\mathbb{Z}_{(p)}$ the formula makes sense [illegible]~~

$$(u * F)(X, Y) = X + Y - P_{p-1}\, C_p(X, Y) \qquad \left\{ \begin{array}{l} \cancel{\text{mod } p} \\ \text{mod deg } p+1 \end{array} \right.$$

Thus

$$[p]_{u*F}(X) \equiv pX - P_{p-1} \frac{p^p - p}{p} X^p \qquad ''$$

(Frölich, page 66). Now we can reduce this formula mod $p$ since $u$ is defined over $\mathbb{Z}_{(p)}$. This gives

$$[p]_{u*F}(X) = u \circ [p]_F \circ u^{-1} \equiv P_{p-1} X^p$$

$$\Longrightarrow \quad \boxed{[p]_F(X) \equiv P_{p-1} X^p \qquad \text{mod deg } p+1}$$

which was to be proved.

It follows that over $\mathbb{F}_p$ the element $P_{p-1}$ is invariant under the action of $N_1$. ~~and~~ Of course $\mathbb{G}_m$ acts by the degree: $a * P_{p-1} = a^{p-1} P_{p-1}$.

Set

$$X = \operatorname{Spec}(L[P_{p-1}^{-1}])$$

moduli scheme for formal group laws in characteristic $p$ of height 1. Then $N$ acts on $X$.

<u>Proposition:</u> Let $s$ be the section of $X$ over $\mathbb{F}_p$ given by the group law $X + Y + XY$. Then the stabilizer of $s$ in $N$ is the profinite group $\mathbb{Z}_p^*$ embedded as the power series

$$a \in \mathbb{Z}_p^* \longmapsto (1+X)^a = \sum_{i \geq 0} \binom{a}{i} X^i$$

Then the homogeneous space scheme $N/\mathbb{Z}_p^*$ exists and $s$ induces an isomorphism of schemes over $\mathbb{Z}_p^*$.

$$N/\mathbb{Z}_p^* \xrightarrow{\sim} X$$

<u>Proof:</u> Let $N_k$ be the normal subgroup of $N$ "consisting" of power series congruent to $X$ mod deg $k+1$. Then

$$N_{p^a-1} \cap \mathbb{Z}_p^* = 1 + p^a \mathbb{Z}_p \qquad a \geq 1$$

so that

$$(\mathbb{Z}/p^a\mathbb{Z})^* \hookrightarrow N/N_{p^a-1}$$

i.e. the former is a finite subgroup scheme of the latter, which of course if a linear algebraic group, the semi-direct product of $\mathbb{G}_m$ and a successive extension of $\mathbb{G}_a$'s. Now quotients by finite flat subgroup schemes exist (ref. ?). If $Y_a$ is the homogeneous scheme space $(N/N_{p^a-1})/(\mathbb{Z}/p^a\mathbb{Z})^*$, then $Y_a$ is affine (ref. ?), so we can form

$$Y = \varprojlim_a Y_a .$$

It pretty clear that $Y \simeq N/\mathbb{Z}_p^*$. In effect $N \longrightarrow Y$ is flat and the equivalence relation is that of the action of $\mathbb{Z}_p^*$. It seems likely that $N \to Y$ is faithfully flat; if so then $Y$ is the quotient on $N$ by $\mathbb{Z}_p^*$. (?)

(see remark 2 below). The map $Y \to X$ induced by $s$ is a monomorphism by the previous proposition. We shall now show that it is locally surjective for the ffqc topology and hence is an isomorphism. The idea is that the proof that a group law of height 1 is isomorphic to $\hat{G}_m$ over a separably closed field proceeds by extraction of separable equations which are always soluble by etale localization.

We follow the proof in Fröhlich. Let $F$ be a formal group of height 1 over a ring $K$ of characteristic $p$. Then $[p]_F(X) = f(X^p)$ where $f(X) = aX + \cdots$ with $a \in K^*$. Looking at the proof of lemma 3 page 77 one sees that after extracting roots of equations of the form

$$\left.\begin{array}{ll} X^{p-1} = a & a \in K^* \\ X^p - X - a = 0 & \end{array}\right\} \text{this gives rise to a faithfully flat ind-etale extension of } K.$$

one can find a power series $u(X) \in N(K)$ with $[p]_G(X) \doteq X^p$ where $G = u * F$. But now the group law $G$ is isomorphic to $X+Y+XY$. In effect as usual one does the obstruction theory and supposes

$$G(X,Y) = X + Y + XY + \lambda C_n(X,Y) \qquad \text{mod deg } n+1$$

where $\lambda \in K$, $\lambda \neq 0$. We may assume $n = p^e$ whence

$$X^p = [p]_G(X) = X^p + \lambda \underbrace{\frac{p^{p^a} - p}{p}}_{\neq 0} X^{p^a}$$

hence $\lambda = 0$. $\therefore$ No obstructions QED.

Remark: 1. It is desirable to understand Lazard's proof because it should actually construct $N$ as the ~~...~~ integral ind-etale extension of $X$ over which the group law becomes canonically isomorphic to $\hat{G}_m$.

2. ~~...~~ The difficulties in the preceding proof may be removed as follows: ~~...~~ Consider the ~~...~~ sheaves for the ffqc topology represented by $N$, $\mathbb{Z}_p^*$, and $X$. Then $\mathbb{Z}_p^*$ is the stabilizer of $\hat{G}_m$ so the quotient sheaf $N/\mathbb{Z}_p^*$ injects into $X$. But the map is onto, hence is an isomorphism. Hence the quotient of $N$ by $\mathbb{Z}_p^*$ as a scheme exists because it exists