§1. A universal property of complex cobordism.

If X is a space, let $\Omega^*(X) = \bigoplus_{q\in\mathbb{Z}} \Omega^q(X)$ be the complex cobordism ring of X. It is the generalized cohomology theory associated to the Thom spectrum of the infinite unitary group. When X is a manifold $\Omega^*(X)$ it may be described as follows:

$\Omega^*(X)$ = equivalence classes of proper complex-oriented maps $f: Z \to X$. A complex orientation on f is a reduction of the structural group of the stable normal bundle of f from O to U.

~~$\operatorname{codim} f = -\dim f = \dim X - \dim Z$~~

Two such maps $f: Z \to X$ and $f': Z' \to X$ are equivalent if they are cobordant, that is, if there is a proper complex-oriented map $h: W \to X\times\mathbb{R}$ of and transversal cartesian squares

$$\begin{tikzcd}
Z \arrow[r, "j_0"] \arrow[d, "f"] & W \arrow[d, "h"] & Z' \arrow[l, "j_1"'] \arrow[d, "f'"] \\
X \arrow[r, "i_0"] & X\times\mathbb{R} & X \arrow[l, "i_1"']
\end{tikzcd}$$

such that the induced isomorphisms of ν_f (resp $\nu_{f'}$) with $j_0^*\nu_h$ (resp $j_1^*\nu_h$) are compatible with the orientation

Structure of $\Omega^*(X)$:

Sum: ~~abelian group~~: $[Z \xrightarrow{f} X] + [Z' \xrightarrow{f'} X] = [Z \amalg Z' \xrightarrow{f+f'} X]$.

Then

$$\Omega^*(X) = \bigoplus_{g \in \mathbb{Z}} \Omega^g(X)$$

$\Omega^g(X)$ subset of elements represented by maps $f: Z \to X$ which are of codim g at all points of Z where

$$\operatorname{codim}_z f = \dim_z \nu_f$$

$$= \dim_{f(z)} X - \dim_z Z$$

$$\operatorname{codim}_z f = -\dim_z f.$$

Moreover

~~$\Omega^g(X)$~~

zero: $[f: \phi \longrightarrow X]$.

inverse ~~of~~ $f: Z \to X$ is the same map but when the new orientation is defined as follows: Given an isom. of ν_f with stable complex bundle E, then new isomorphism is $\nu_f + 1 \simeq E + \mathbb{C} \overset{id_E + -}{\simeq} E + \mathbb{C}$.

~~[illegible]~~

Inverse image: Given $u: X \to Y$ and $f: Z \to Y$ proper + oriented, without changing the element of $\Omega(X)$ represented by f we may move it to be transversal to ~~f~~ u and then form the fibre product ~~[illegible] $\to X$, [illegible] the normal~~

$$
\begin{tikzcd}
X \times_Y Z \arrow[r, "u'"] \arrow[d, "f'"'] & Z \arrow[d, "f"] \\
X \arrow[r, "u"] & Y
\end{tikzcd}
$$

Using the canonical isomorphism $u'^{*}\nu_f \cong \nu_{f'}$ one obtains an orientation of f' so one defines

$$u^*[Z \xrightarrow{f} Y] = [X \times_Y Z \xrightarrow{f'} X]$$

Then

$$u^* : \Omega(Y) \longrightarrow \Omega(X)$$

is well-defined and compatible with addition and makes Ω into a contravariant functor from manifolds to abelian groups.
$u : \Omega^n(Y) \longrightarrow \Omega^n(X)$.

<u>Gysin homomorphism or direct image:</u> Given $u : X \to Y$ proper + oriented get

$$u_* : \Omega(X) \longrightarrow \Omega(Y)$$

$$u_*[Z \xrightarrow{f} X] = [Z \xrightarrow{uf} Y]$$

If u is of dimension q, then $u : \Omega^k(X) \longrightarrow \Omega^{k+q}(Y)$.

<u>Products</u>: $\underset{\alpha}{[Z \xrightarrow{f} X]} \cdot \underset{\beta}{[Z' \xrightarrow{f'} X']} = \underset{\alpha \boxtimes \beta}{[Z \times Z' \xrightarrow{f \times f'} X \times X']}$

$$\Omega^q(X) \otimes \Omega^{q'}(X') \longrightarrow \Omega^{q+q'}(X \times X')$$

It makes $\Omega^*(X)$ into a (anti-) commutative ring with

$$\alpha \cdot \beta = \Delta^*(\alpha \boxtimes \beta)$$

where $\Delta : X \longrightarrow X \times X$ is the diagonal map

Properties of this structure:

$\left\{\begin{array}{l} X \mapsto \Omega(X) \text{ with } f \longmapsto f^* \text{ is a contravariant functor on the} \\ \text{category of } C^\infty \text{ manifolds and } C^\infty \text{ maps. ~~with~~} \end{array}\right.$

1) $\left\{\begin{array}{l} X \mapsto \Omega(X) \text{ with } f \longmapsto f_* \text{ is a covariant functor on the} \\ \text{category with objects } C^\infty \text{ manifolds and having proper-oriented} \\ \text{maps for morphisms.} \end{array}\right.$

2) If

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

is a cartesian square of manifolds, and if f, g are transversal (that is, $f \times g : X \times Y' \longrightarrow Y \times Y$ is transversal to the diagonal) if f is proper & oriented and if f' is oriented by means of the canonical isomorphism $g'^* \nu_f \simeq \nu_{f'}$, then

$$g^* f_* = f'_* g'^*$$

3) If $f, g : X \longrightarrow Y$ are maps and $f \sim g$, then $f^* = g^*$. (equivalently if $f : X \longrightarrow Y$ is a homotopy equivalence, then $f^* : \Omega(Y) \longrightarrow \Omega(X)$ is bijective).

4) (Addition). If $X = X_1 \amalg X_2$, then

$$(in_1^*, in_2^*) : \Omega(X) \xrightarrow{\sim} \Omega(X_1) \times \Omega(X_2)$$

and f_* is compatible with additions

5) (Multiplication). Given $f: X \to Y$, $f': X' \to Y'$, proper + oriented, then

$$(f\times f')_*(x \boxtimes x') = f_*x \boxtimes f'_*x'$$

$(-1)^{(\deg f')(\deg x)}$. You must orient $f\times f'$ and you choose the composite of the orientations of $(f\times id)\circ(id\times f')$ so $(f\times f')_* \overset{\text{defn}}{=} (f\times id)_*(id\times f')_*$. Check that (orient of $f\times f'$) $= (-1)^{\deg f\cdot\deg f'}$ (orient. of $f'\times f$)

(Insert page 5')

<u>Proposition (i)</u>: Let Q be a contravariant functor on the category of manifolds with values in (Sets) endowed with a Gysin homomorphism (for proper-oriented maps) satisfying axioms 1), 2), 3). If $\alpha \in Q(pt)$, then ~~there~~ is a unique natural transformation $\theta: \Omega \to Q$ of functors compatible with Gysin such that $\theta(1) = \alpha$, where $1 \in \Omega(pt)$ denotes the element $[id: pt \to pt]$.

(ii) Let Q be a functor as in (i) but with values in (Ab), and suppose Q also that satisfies (4). Then $\theta: \Omega \to Q$ is compatible with addition.

(iii) Let Q be a contravariant functor on Man with values in (Rings) endowed with a Gysin homomorphism for proper-oriented maps. If θ satisfies 1)-5), then there is a unique natural transf. $\theta: \Omega \to Q$ compatible with ~~additi~~ ring structure and Gysin homomorphism.

<u>Proof</u>: (i): Given an element $[Z \xrightarrow{f} X]$ of $\Omega(X)$ we of course have to define

$$\theta[Z \xrightarrow{f} X] = f_* \pi_Z^* \alpha$$

where $\pi_Z: Z \to pt$ is the unique map. To

Note that the Gysin homomorphism $(f_*: \Omega(X) \longrightarrow \Omega(Y))$ is not ~~compat~~ a ring homomorphism ~~instead~~ and is ~~an~~ just a $\Omega(Y)$-homomorphism in virtue of the projection formula

$$f_*(x \cdot f^* y) = f_* x \cdot y$$

which may be deduced from ~~We note that this follows from~~

$$\begin{array}{ccc} X & \xrightarrow{(id,f)} & X \times Y \\ \downarrow f & & \downarrow f\times id \\ Y & \xrightarrow{\Delta} & Y \times Y \end{array}$$

$$f_*(id,f)^*(x \boxtimes y) = f_*(x \cdot f^* y)$$

$$\parallel$$

$$\Delta^*(f \times id)_*(x \boxtimes y) = f_* x \cdot y$$

~~[illegible]~~ We have the formula

$$\begin{cases} \alpha \cdot \beta = \Delta^*(\alpha \boxtimes \beta) \\ \alpha \boxtimes \beta = pr_1^* \alpha \cdot pr_2^* \beta \end{cases}$$

Showing that ~~the algebra~~ internal ring structure on Ω is ~~that~~ equivalent to the external product $\boxtimes$, ~~and that one is~~ ~~[illegible] Of course the existence of identity~~ together with an element $1 \in \Omega(pt)$. such that $1 \boxtimes \alpha \simeq \alpha$, $\alpha \boxtimes 1 \simeq \alpha$.

show well-defined suppose $[Z \xrightarrow{f} X] = [Z' \xrightarrow{f'} X]$. Then $\overbrace{\text{we}}$ have

$$\begin{array}{ccccc} Z & \xrightarrow{j_0} & W & \xleftarrow{j_1} & Z' \\ \downarrow f & & \downarrow h & & \downarrow f' \\ X & \xrightarrow[i_0]{} & X \times \mathbb{R} & \xleftarrow[i_1]{} & X \end{array}$$

So

$$f_* \pi_Z^* \alpha = f_* j_0^* \pi_W^* \alpha = i_0^* h_* \pi_W^* \alpha$$

$$= i_1^* h_* \pi_W^* \alpha \qquad \text{(homotopy axiom 3).}$$

$$= f'_* j_1^* \pi_W^* \alpha = f'_* \pi_{Z'} \alpha$$

Thus we have defined $\theta : \Omega(X) \longrightarrow Q(X)$ for every X.
Given $u : X' \longrightarrow X$ ~~and $f: Z \to X$ proper oriented~~
an element γ of $\Omega(X)$ we may suppose γ represented as $[Z \xrightarrow{f} X]$ where f is transversal to u and form

$$\begin{array}{ccc} Z' & \xrightarrow{u'} & Z \\ \downarrow f' & & \downarrow f \\ X' & \xrightarrow{u} & X \end{array}$$

Then

$$\theta(u^*\gamma) = \theta\, [Z' \xrightarrow{f'} X']$$

$$= f'_* \alpha_{Z'} = f'_* u'^* \alpha_Z$$

$$= u^* f_* \alpha_Z = u^* \theta\, [Z \xrightarrow{f} X]$$

$$= u^* \theta \gamma.$$

Similarly by axiom 1), θ is compatible with f_*, proving (i)

(ii). Recall $\left[Z \xrightarrow[\gamma]{f} X\right] + \left[Z' \xrightarrow[\gamma']{f'} X\right] = \left[Z \amalg Z' \xrightarrow{f+f'} X\right]$.

$$\therefore \quad \theta(\gamma+\gamma') = (f+f')_*\left(\pi^*_{Z\amalg Z'}\alpha\right)$$

Now
$$(in_1^*, in_2^*) : Q(Z \amalg Z') \longrightarrow Q(Z) \times Q(Z')$$
and
$$(in_1^*, in_2^*)\left(\pi^*_{Z \amalg Z'}\alpha\right) = \left(\pi_Z^*\alpha, \pi_{Z'}^*\alpha\right)$$

Let
$$\beta = \left(in_{1*}\,\pi_Z^*\alpha + in_{2*}\,\pi_{Z'}^*\alpha\right) \in Q(Z \amalg Z')$$
Then
$$in_1^*\beta = in_1^*\,in_{1*}\,\pi_Z^*\alpha + in_1^*\,in_{2*}\,\pi_{Z'}^*\alpha$$

$$\begin{tikzcd}
Z \arrow[r, "id"] \arrow[d, "id"'] & Z \arrow[d, "in_1"] \\
Z \arrow[r, "in_1"] & Z \amalg Z'
\end{tikzcd} \qquad \text{cartesian} \Longrightarrow in_1^*\,in_{1*} = id$$

$$\begin{tikzcd}
\phi \arrow[r] \arrow[d] & Z' \arrow[d, "in_2"] \\
Z \arrow[r, "in_1"'] & Z \amalg Z'
\end{tikzcd} \qquad \text{cartesian} \Longrightarrow in_1^*\,in_{2*} = 0$$

(because $Q(\phi) \xrightarrow{\sim} Q(\phi) \times Q(\phi)$ for an abelian gp $\Rightarrow Q(\phi) = 0$).

Thus
$$in_1^*\beta = \pi_Z^*\alpha \qquad \text{and similarly}$$
$$in_2^*\beta = \pi_{Z'}^*\alpha \qquad \text{so}$$

$$\pi_{Z\sqcup Z'}^{*}\alpha = in_{1*}\,\pi_Z^{*}\alpha + in_{2*}\,\pi_{Z'}^{*}\alpha$$

$$\therefore\quad \theta(\gamma+\gamma') = \underbrace{(f+f')_*}_{\text{hom}}\,[\quad \text{"} \quad]$$

$$= f_*\,\pi_Z^{*}\alpha + f'_*\,\pi_{Z'}^{*}\alpha = \theta\gamma + \theta\gamma'. \quad ✓$$

~~Here assumed f_* is a homomorphism.~~ Better method is as follows: Take

$$\beta = in_{1*}f_*\,\pi_Z^{*}\alpha + in_{2*}f'_*\,\pi_{Z'}^{*}\alpha \;\in\; Q(X\sqcup X)$$

Then note that

$$in_1^{*}\beta = f_*\,\pi_Z^{*}\alpha = f_*\,in_1^{*}\,\pi_{Z\sqcup Z'}^{*}\alpha$$
$$= in_1^{*}\,(f\sqcup f')_*\,\pi_{Z\sqcup Z'}^{*}\alpha$$

$$\begin{tikzcd} Z \arrow[r, "in_1"] \arrow[d, "f"] & Z\sqcup Z' \arrow[d, "f\sqcup f'"] \\ X \arrow[r, "in_1"'] & X\sqcup X \end{tikzcd}$$

Similarly $in_2^{*}\beta = in_2^{*}\,(f\sqcup f')_*\,\pi_{Z\sqcup Z'}^{*}\alpha$

$$\Longrightarrow\quad in_{1*}f_*\,\pi_Z^{*}\alpha + in_{2*}f'_*\,\pi_{Z'}^{*}\alpha$$

Now apply the map $\quad Q(X\sqcup X) \xrightarrow{(\nabla)_*} Q(X)$

use 1) an

(iii). As $1 \in \Omega(pt)$ is the identity element of this ring θ must send 1 to the identity ellment 1 of $Q(pt)$. By the preceding we therefore has a unique natural transf. $\theta : \Omega \longrightarrow Q$ compatible with Gysin and additions. ~~Recall that~~ To show θ is a ring hom. it is enough to prove that

$$\theta\left([Z \xrightarrow{f} X] \boxtimes [Z' \xrightarrow{f'} X']\right)$$

$$\overset{?}{=}$$

$$\left(f_* 1_Z\right) \boxtimes \left(f'_* 1_{Z'}\right)$$

but

$$\theta\left([Z \xrightarrow{f} X] \boxtimes [Z' \xrightarrow{f'} X']\right)$$
$$= \theta\left([Z \times Z' \xrightarrow{f \times f'} X \times X']\right)$$
$$= (f \times f')_* \left(1_Z \boxtimes 1_{Z'}\right) \overset{5)}{=\!=} \left(f_* 1_Z\right) \boxtimes \left(f'_* 1_{Z'}\right)$$

---

§2. Cobordism of a projective bundle, the formal group law

<u>Add</u> to preceding section

Examples of (complex-) oriented maps.

(i) An embedding whose normal bundle is endowed with a complex structure.

(ii) A submersion $f: X \to Y$ such that the tangent bundle along the fibers is endowed with a complex structure (hence X is a family of almost complex manifolds parameterized by Y)

(iii) Any map of complex manifolds has a canonical complex orientation.
(one needs this for $s_* 1 = c_{n-1}(F)$ below)

<u>Proposition:</u> Let $i: Y \to X$ be an embedding endowed with orientation (thus the normal bundle E of i is endowed with an isomorphism $E + N \simeq$ a complex vector bundle over Y for some N). Let $j: U \to X$ be the complement of Y. Then

$$\Omega(Y) \xrightarrow{i_*} \Omega(X) \xrightarrow{j^*} \Omega(U)$$

is exact.

<u>Proof</u>: $j^* i_* = 0$ clear.

Conversely if $\alpha \in \operatorname{Ker} j^*$ represent α by $f_* 1$, $f: Z \to X$. Then $j^*\alpha$ is represented by $f^{-1}U$. As $j^*\alpha = 0$, $f^{-1}U \to U$ is cobordant to zero e.g. $\exists\, W \xrightarrow{h} U \times \mathbb{R}$ proper oriented with $h^{-1}(U \times 0) \to U$ isom. to $f_U: f^{-1}U \to U$ and with $h^{-1}(U \times 1) = \emptyset$. Let N be a tubular neighborhood of Y in X and let φ be a nice distance-squared function from Y. Using $W \xrightarrow{(h,\varphi)} U \times (0,\infty)$, one gets a cobordism from $Z \to X$ to something whose image lies in N. Thus may assume $f: Z \to N$; but then we may homotop Z down into Y, so may assume $f: Z \to Y$, whence f is oriented since i is

$$\begin{tikzcd} Z \arrow[d, "g"'] \arrow[dr, "f \text{ oriented}"] & \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

and so $f_*1 = i_*(g_*1)$.

(Insert defn of c_1)

Thm: ~~Proposition:~~ Let E be a complex vector bundle of dim. n over X. Let $f: \mathbb{P}\check{E} \to X$ be the projective bundle of lines in $\check{E}$, let $\mathcal{O}(1)$ be the canonical line bundle

$$f_*\{\text{sheaf of holomorphic sections of } \mathcal{O}(n)\} = S_n \check{E}.$$

Let $\xi = c_1(\mathcal{O}(1)) \in \Omega^2(\mathbb{P}\check{E})$. Then $\Omega(\mathbb{P}\check{E})$ is a free $\Omega(X)$-module with basis $1, \xi, \dots, \xi^{n-1}$.

~~Proof:~~

$$\begin{tikzcd} \mathbb{P}E \arrow[r, "j"] \arrow[d, "g"'] & \tilde{E} \arrow[d, "f"] \\ X \arrow[r, "i"'] & E \end{tikzcd}$$

~~i zero section~~

~~$\tilde{E} = \{(l, v) \mid l \text{ line in } E,\ v \in l\}$.~~

Let E be a complex vector bundle over X of dimension n and let $i : X \to E$ be the zero section. Let $s : X \to E$ be a section transversal to i and form the fiber product

$$\begin{tikzcd} Y \arrow[r, "i'"] \arrow[d, "i'"'] & X \arrow[d, "i"] \\ X \arrow[r, "s"'] & E \end{tikzcd}$$

Then $Y = s^{-1}(0)$ is an oriented submanifold of X, in fact the normal bundle of i' is canonically isomorphic to $i'^* E$. Moreover

$$* \qquad i'_* 1_Y = i'_* i'^* 1_X = s^* i_* 1_X = i^* i_* 1_X .$$

~~We will see~~ Call $i^* i_* 1$ the "Euler class" of E; we will see later that it is the nth Chern class of E, denoted $c_n(E)$. It's formation is compatible with base change.

<u>Proof of prop.</u> On $\mathbb{P}E$ we have exact sequence

$$0 \to \mathcal{O}(-1) \to f^* E \xrightarrow{\pi} F \to 0$$

~~Claim that $f_*(c_{n-1}(F))$ is a unit in Ω~~

<u>Lemma 1</u>: $f_* c_{n-1}(F)$ is a unit in $\Omega(X)$.

<u>Proof</u>: Let s be a generic section of E with zero submanifold $i' : Y \to X$. Let $j : U \to X$ be the complement of Y, let $t : U \to \mathbb{P}E|U$ be the section given by the line generated by the section s over U. Then $t_* 1 = j'^* c_{n-1}(F)$

$$\begin{tikzcd} \mathbb{P}(j^*E) \arrow[r, "j'"] \arrow[d, "f'"] & \mathbb{P}E \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

In ~~effect~~ the image of t is where the section $\hat{s}$ of $f^* j^*E$ given by s ~~___~~ is contained in $\mathcal{O}(-1)$, i.e. where the section $\pi\hat{s}$ of j^*F vanishes. ~~Thus~~ So make a local calculation to convince yourself that this means that

$$\begin{tikzcd} U \arrow[r, "t"] \arrow[d, "t"] & \mathbb{P}(j^*E) \arrow[d, "0\text{ section}"] \\ \mathbb{P}(j^*E) \arrow[r, "\pi\hat{s}"'] & j^*F \end{tikzcd}$$

is the correct ~~same~~ diagram + so $t_*1 = c_{n-1}(j^*F) = j^*c_{n-1}(F)$

Thus

$$j^*\left(f_* c_{n-1}(F) - 1\right) = f'_*\left(j'^* c_{n-1}(F)\right) - 1$$

$$= f'_* t_* 1 - 1 = 0.$$

so by previous prop

$$f_* c_{n-1}(F) - 1 = \underbrace{i_* z}_{\text{nilpotent.}}$$

Thus $f_* c_{n-1}(F)$ is a unit.

Thus we have found a canonical element $u(E) = \dfrac{c_{n-1}(F)}{f^*(f_* c_{n-1} F)}$ such that $f_* u(E) = 1$. It's compatible with base changes. So now let X' $D(E)$ = drapeaux scheme of E, X' $D(E) \xrightarrow{g} X$. Then it is an iterated succession of projective fibres so one gets a canonical element $v(E) \in \Omega(D(E)) \ni f_* v(E) = 1$. Claim it suffices to prove thm. form $\mathbb{P}(\underset{\overset{\|}{E'}}{g^* E}) \xrightarrow{f'} D(E)$.

$$
\begin{array}{ccc}
\Omega(\mathbb{P}E') & \xleftarrow{g'^*} & \Omega(\mathbb{P}E) \\
\uparrow f'^* & & \uparrow f^* \\
\Omega(X') & \xleftarrow{g^*} & \Omega(X)
\end{array}
$$

In effect we know that $\xi' = g'^* \xi$. Given $z \in \Omega(\mathbb{P}E)$ write

$$g'^* z = \sum_{i=0}^{n-1} f'^* a_i \, (\xi')^i$$

Multiply by $v(E') = f'^* v(E)$ and apply g'_* using that $g'_* v(E') = 1$ we get

$$z = \sum_{i=0}^{n-1} \underbrace{g'_*\left[(f'^* a_i)\, v(E')\right]}_{\substack{\| \\ g'_* f'^*(a_i v(E)) \\ \| \\ f^* g_*(a_i v(E))}} \xi^i$$

$$f^* g_*(a_i v(E)) \cdot \xi^i$$

$$
\begin{array}{ccc}
\mathbb{P}E' & \xrightarrow{g'} & \mathbb{P}E \\
\downarrow f' & & \downarrow f \\
X' & \xrightarrow[g]{} & X
\end{array}
$$

This shows that $1, \xi, \dots, \xi^{n-1}$ span $\Omega(\mathbb{P}E)$ ~~and~~ as an $\Omega(X)$ module. Similarly as g^* is injective, they are independent!

So now we can assume that $E = L_1 + \cdots + L_n$ where the L_i are line bundles. Let $E_q = L_1 + \cdots + L_q$ and use induction on q. Then

$$\cancel{\mathbb{P}L_{n-1} \to \mathbb{P}E_n \xleftarrow{s} \mathbb{P}L_n}$$

$$\begin{tikzcd}
\mathbb{P}L_n \arrow[r] & \mathbb{P}E & \mathbb{P}E - \mathbb{P}L_n \arrow[l, hook'] \arrow[d, "\text{line bundle}"] \\
X \arrow[u, phantom, "\cap"] \arrow[ur, "s"] & & \mathbb{P}(E_{n-1}) \arrow[ul, "j"]
\end{tikzcd}$$

By prop earlier

$$\Omega(X) \xrightarrow{s_*} \Omega(\mathbb{P}E) \xrightarrow{j^*} \Omega(\mathbb{P}E_{n-1})$$

is exact. ~~By induction~~ as $j^*(\mathcal{O}_{\mathbb{P}E}(1)) = \mathcal{O}_{\mathbb{P}E_{n-1}}(1)$ the induction hypothesis $\Longrightarrow$ j^* surjective. As $f_* s_* = id$ s_* is injective, so sequence is split exact and $\Omega(\mathbb{P}E)$ has the basis $s_*1, 1, \xi, \ldots, \xi^{n-2}$. ~~But $\mathbb{P}L_n$ is where the section $\mathcal{O} \to f^*E \otimes \mathcal{O}(1) \to f^*E_{n-1} \otimes \mathcal{O}(1)$ vanishes so~~

[illegible]

Let H_i = the hyperplane $\mathbb{P}(E_1 + \cdots + \hat{E}_i + \cdots + E_n) \subset \mathbb{P}E$ or equivalently where the section $\mathcal{O} \longrightarrow f^*E \otimes \mathcal{O}(1) \to f^*L_i \otimes \mathcal{O}(1)$ vanishes. Thus

$$[H_i \to \mathbb{P}E] = c_1(f^*L_i \otimes \mathcal{O}(1))$$

and as

$$s(X) = \bigcap_{i<n} H_i \quad \Longrightarrow \quad s_*1 = \prod_{i=1}^{n-1} c_1(f^*L_i \otimes \mathcal{O}(1))$$

~~But in general if ν is a generic section of a line~~

<u>Case 1</u>: Assume all L_i are trivial

Then $s_* 1 = c_1(\mathcal{O}(1))^{n-1} = \xi^{n-1}$ and we are finished. Conclude also that $\xi^n = 0$ so

$$\boxed{\Omega^*(X \times \mathbb{P}^n) = \Omega^*(X)[\xi] / (\xi^{n+1})}$$

Recall that any line bundle is induced from the bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ for some n. Define elements $a_{k\ell}^{(n)} \in \Omega^{2-k-\ell}(pt)$ for $k, \ell \le n$ by

$$c_1(\underset{\mathbb{P}^n}{\mathcal{O}(1)} \boxtimes \underset{\mathbb{P}^n}{\mathcal{O}(1)}) = \sum_{\substack{k \le n \\ \ell \le n}} a_{k\ell}^{(n)} c_1(\mathcal{O}(1))^k \boxtimes c_1(\mathcal{O}(1))^\ell$$

One sees that $a_{k\ell}^{(n)}$ is independent of n, hence there is a power series $F(X,Y) \in \Omega^{\omega}(pt)[[X,Y]]$ uniquely determined by

$$\boxed{c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2))}$$

for all line bundles L_1, L_2 over any manifold X. ~~By checking the bundle $\mathcal{O}(1) \boxtimes \mathcal{O}(1) \boxtimes \mathcal{O}(1)$ on $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ one sees that F is associative & satisfies~~ Claim F is a formal commutative group law over $\Omega^{\omega}(pt)$ i.e.

$$F(X,0) = F(0,X) = X$$

$$F(X, F(Y,Z)) = F(F(X,Y), Z)$$

$$F(X,Y) = F(Y,X)$$

The associativity results by calculate in two ways c_1 of the bundle $\mathcal{O}(1) \boxtimes \mathcal{O}(1) \boxtimes \mathcal{O}(1)$ on $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$. One sees now that

$$F(X,Y) = X+Y+XY\,G(X,Y)$$

hence in general

$$c_1(L_1 \otimes L_2) - c_1(L_2) = c_1(L_1)\big(1 + \underset{\substack{\uparrow \\ \text{nilpotent}}}{c_1(L_2)}\, G(\ ,\ )\big)$$

hence

$$c_1(L_1 \otimes L_2) - c_1(L_2) = c_1(L_1) \cdot \text{unit}.$$

Returning to our proof one has:

$$(*) \qquad (\text{unit}) \cdot c_1(f^* L_i \otimes \mathcal{O}(1)) = c_1(\mathcal{O}(1)) - c_1(f^* L_i^\vee)$$

$$\therefore \quad s_* 1 = (\text{unit}) \cdot \prod_{i<n} \left[\xi - f^* c_1(L_i^\vee)\right]$$

$\therefore$ one concludes that $1, \xi, \dots, \xi^{n-2}, \xi^{n-1}$ is a basis for $\Omega(\mathbb{P}E)$ as claimed, proving the theorem.

<u>Cor:</u> If $E = L_1 + \dots + L_n$ then in $\Omega(\mathbb{P}E)$ we have the relation

$$\prod_{i=1}^{n} \left(\xi - f^* c_1(L_i^\vee)\right) = 0.$$

Proof: We know that $\prod_{i=1}^{n} c_1(f^*L_i \otimes \mathcal{O}(1)) = 0$ so the result follows from (*).

Defn: If E is a complex bundle over X, let $\xi = c_1(\mathcal{O}(1))$ on $\mathbb{P}E$. Then the Chern classes $c_i(E) \in \Omega^{2i}(X)$ are defined by

$$\xi^n - f^*c_1(E)\,\xi^{n-1} + \cdots + (-1)^n f^*c_n(E) = 0.$$

Proposition:
$$c_n(E+E') = \sum_{i=0}^{n} c_i(E)\,c_{n-i}(E')$$
$$c_0(E) = 1.$$

~~$c(L) = c_1(L)$ old sense~~

If $n = \dim E$, then

$$c_n(E) = i^*(i_* 1) \quad \text{where } i: X \to E \text{ zero se}$$

Proof: By passage to the ~~…~~ associated flag bundles which induces an injective map on Ω we may assume that E and E' are split. The ~~first~~ corollary on p. 14 shows that (f^* injective)

$$c_q(E) = q\text{th elementary symmetric fn in } c_1(L_i)$$

whence the first statement of the proposition follows. For the second we have $E = L_1 + \cdots + L_n$ & set $E_q = L_1 + \cdots + L_q$. and let $i: X \to E_q$ be the zero section of E_q. Then one has

$$\begin{tikzcd} X \arrow[r, "\iota_{q-1}"] \arrow[rr, bend left, "\iota_q"] & E_{q-1} \arrow[r, "j"] & E_q \end{tikzcd}$$

so

$$\iota_q^* \iota_{q*} x = \cancel{}$$
$$= \iota_{q-1}^* j^* j_* \iota_{q-1*} x$$

But
$$j^*(j_* u) = j^*(j_* j^* \pi^* u) \qquad \pi : E_q \xrightarrow{\text{retraction}} E_{q-1}$$
$$= j^*(j_* 1 \cdot \pi^* u)$$
$$= (j^* j_* 1) \cdot u$$

$$\therefore \quad \iota_q^* \iota_{q*} 1 = \underbrace{\iota_{q-1}^*(j^* j_* 1)}_{\iota_{q-1}^*(c_1 L'_q) \,=\, c_1(L_q)} \cdot \underbrace{\iota_{q-1}^* \iota_{q-1*} 1}_{c_{q-1}(E_{q-1}) \quad \text{induction}}$$

where L'_q is E_q viewed as a bundle over E_{q-1}.

Proposition: If $i : Y \longrightarrow X$ is an embedding with a complex structure on the normal bundle E of i, then

$$i^*(i_* y) = c_n(E) \cdot y \qquad \text{all } y \in Y, \quad n = \dim E.$$

Proof: Let U be a tubular nbd. of Y in X; we may identify E with U. Then we have a square

$$\begin{tikzcd} & Y \arrow[r, "id"] \arrow[d, "\iota'"] & Y \arrow[d, "i"] \\ Y \arrow[r, "\iota"] & U \arrow[r, "j"] & X \end{tikzcd}$$

$$\therefore \quad \iota^* \iota_* y = \iota'^* j^* \iota_* y = \iota'^* \iota'_* y$$

But $\exists\ \pi : U \to Y$ a retraction so

$$\begin{aligned} &= \iota'^* \iota'_* (\iota'^* \cdot \pi^* y) \\ &= \iota'^* (\iota'_* 1 \cdot \pi^* y) \\ &= (\iota'^* \iota'_* 1) \cdot y \end{aligned}$$

but ~~[illegible]~~ identifying U and E we have $\quad \iota'^* \iota'_* 1 = c_n(E)$.

~~[illegible]~~

Comments. Add a general discussion on the Euler class after proof of first proposition (and transitivity formula) proving the $\iota^* \iota_* y = (\iota^* \iota_* 1) \cdot y$ formula for an embedding. Then prove projective bundle theorem for $E = L_1 + \cdots + L_n$ where L_i's are trivial to get the formal group law, then general L_i's and then general case.

Cartier style proofs of Lazard theorems

Lemma: Given a formal Lie group G in d-variables over R in the sense of Cartier and

$$f: D \longrightarrow G$$

a curve in the formal group. Then f is a homomorphism if and only if $F_q f = 0$ for all $q \geq 2$

($\Rightarrow$) obvious
($\Leftarrow$):

Proof: Let $\widehat{W}_R$ be the formal Witt group over R [redacted] which associates to each R-algebra A the group of power series

$$1 + c_1 t + \cdots + \cdots \qquad c_n \in A, \quad c_n \text{ nilpotent}$$

and all but a finite number of c_i are zero. Given $f: D \to G$ there is a unique homomorphism $h: \widehat{W}_R \longrightarrow G$ such that

$$h(1 + at) = f(a)$$

for all $a \in D(A)$ and all R-algebras a. In fact one defines h by the formula

$$h(1 + c_1 t + \cdots + c_n t^n) = f(\lambda_1) + \cdots + f(\lambda_n)$$

where the λ_i ~~[illegible]~~ satisfy

$$1 + c_1 t + \cdots + c_n t^n = \prod_{i=1}^{n} (1 + \lambda_i t).$$

One must know some descent theory for G to define this properly. Thus it is

$$\mathrm{tr}^{G}_{A[Z]/(Z^n - c_1 Z^{n-1} + \cdots)} f(Z)$$

Note this descent needed to define F_q

By hypothesis

$$(F_q f)(x) = f(z_1) + \cdots + f(z_q) = 0$$

for all $x \in D(\cdot$

where

$$\prod (t - z_i) = t^q - x \quad \text{or} \quad \prod_{i=1}^{n} (1 + t z_i) = 1 - (-1)^q x t$$

Thus we are told that

$$h(1+at^q)=0$$

if $a\in D(\cdot)$ and $q>1$. But in fact any element $1+c_1t+\cdots+c_nt^n$ of $W(A)$ may be factored

$$1+c_1t+\cdots+c_nt^n=(1+a_1t)(1+a_2t^2)\cdots\cdots(1+a_Nt^N)$$

(Enough to do this universally ie when $A=R[c_1,\ldots,c_n]/(c_1,\ldots,c_n)^\nu$ whence in fact one sees that a_k has a certain filtration and the product is finite). As h is a homomorphism we see that

$$h(1+c_1t+\cdots+c_nt^n)=h(1+c_1t).$$

Thus

$$\begin{aligned}f(\lambda_1)+f(\lambda_2)&=h\left(1+(\lambda_1+\lambda_2)t+\lambda_1\lambda_2t^2\right)\\&=h\left(1+(\lambda_1+\lambda_2)t\right)=f(\lambda_1+\lambda_2)\end{aligned}$$

so f is a homomorphism. qed.

<u>Corollary</u>: Suppose R has characteristic p and that F is a typical group law of height ∞. Then $F(X,Y)=X+Y$.

<u>Proof</u>: F gives rise to a formal group G together with an isomorphism $\gamma: D\longrightarrow G$ such that $F(x,y)=\gamma^{-1}(\gamma x+\gamma y)$. γ is typical $\Rightarrow F_q\gamma=0$ $(q,p)=1$. But in char. p, $V_pF_p=p$ on curves. By height ∞, $p=0$ and so $F_p=0$ since V_p is injective. Thus by the lemma, γ is a homomorphism, so $F(x,y)=\gamma^{-1}(\gamma x+\gamma y)=x+y$. qed.

(Lazard)

Proposition: Let R be a ring and let F be a group law such that $p_F(X) \equiv 0 \mod pR$ for all primes p. Then F is isomorphic to the additive law.

Proof (using Cartier theory). We let G be the formal group defined by F. We shall construct a coordinate $\gamma : D \longrightarrow G$ such that $F_q \gamma = 0$ for all primes q. It follows from the preceding lemma that γ is a homomorphism, hence it gives an isomorphism of G with D.

We suppose γ has been constructed so that $F_q \gamma = 0$ for all primes $q < p$; we will show how to modify γ in degrees $\geq p$ so as to satisfy in addition $F_p \gamma = 0$. It's clear that the limit of these γ's is what we need.

Suppose we can find a curve $\mu : D \longrightarrow G$ such that

$$(*) \qquad F_p \gamma = p\mu$$

Then if q is a prime $< p$

$$pF_q\mu = F_q F_p \gamma = F_p F_q \gamma = 0,$$

hence the curve

$$\mu' = \mu - \left(\tfrac{1}{q}\right) V_q F_q \mu$$

(here $\left(\frac{1}{q}\right)$ is an integer $r \ni qr \equiv 1 \pmod p$).

is defined. Now

$$p\mu' = p\mu - \left(\tfrac{1}{q}\right) V_q\, pF_q \mu = p\mu = F_p \gamma$$

$$F_q \mu' = F_q \mu - \left(\tfrac{1}{q}\right) q F_q \mu = 0 \qquad \left\{ \begin{array}{l} \text{since } \left(\tfrac{1}{q}\right) q \equiv \\ \text{mod } p \; + \\ p F_q \mu = 0. \end{array} \right.$$

Moreover if q' is a prime number $< q$ and if we know that $F_{q'} \mu = 0$, then

$$F_{q'} \mu' = F_{q'} \mu - \left(\tfrac{1}{q}\right) V_q F_q F_{q'} \mu = 0.$$

Therefore by an evident induction we may modify μ and so assume that

$$F_q \mu = 0 \qquad \text{all prime } q < p.$$

Then set

$$\gamma' = \gamma - V_p \mu$$

Then γ' and γ differ only in degrees $\geq p$ and

$$F_q \gamma' = F_q \gamma - V_p F_q \mu = 0 \qquad \text{if } q < p$$

$$F_p \gamma' = F_p \gamma - p \mu = 0,$$

and we are finished.

It remains to show that we can solve (*). By hypothesis

$$p_F(X) \equiv 0 \mod p$$

and one knows in general that

$$p_F(X) \equiv pX \mod \deg 2.$$

Thus there is a $G(X) \in R[[X]]$ such that

$$p_F(X) = p\, G(X)$$

$$G(X) \equiv X \quad \text{mod deg } 2.$$

Next note that modulo p, $V_pF_p = p$ hence if $x \in D(A)$ where A is an R-algebra, then

$$(\gamma^{-1} \circ V_pF_p\gamma)(x) \equiv (\gamma^{-1} \circ p\gamma)(x) = p_F(x) \equiv 0 \mod p$$

while $(\gamma^{-1} \circ V_pF_p\gamma)(x) = \gamma^{-1}(F_p\gamma(x^p))$. Thus there is a power series $\psi(X) \in R[[X]]^+$ with

$$\begin{aligned}(\gamma^{-1} \circ F_p\gamma)(x) &= p \cdot \psi(X) \\ &= p \cdot G(G^{-1}(\psi x)) \\ &= p_F\, G^{-1}(\psi x) \\ &= (\gamma^{-1} \circ p\gamma)(G^{-1}(\psi x)) \\ &= \gamma^{-1}(p\,\mu(x))\end{aligned}$$

where $\mu(x) = \gamma(G^{-1}(\psi x))$. Thus $F_p\gamma = p\,\mu$ as desired, and so the proof of the proposition is complete. qed.

Cobordism theory and formal groups

§1. Cobordism as a universal cohomology theory.

Throughout this paper manifold means a C^∞ manifold with a Hausdorff topology having a countable basis for the open sets but not necessarily connected and the components may have different dimensions. However we shall insist that the dimension of the components be bounded. A morphism or map of manifolds will always be C^∞. If X is a manifold let τ_X be its tangent bundle. If $f: X \to Y$ is a map of manifolds, then its stable normal bundle denoted ν_f is the difference $\tau_Y - f^*\tau_X$ in the Grothendieck category $\underline{KO(X)}$ of real vector bundles over X. ν_f may be identified with an homotopy class of maps $\nu_f : X \longrightarrow \mathbb{Z} \times BO$, where $\mathbb{Z} \times BO$ is the Hopf space representing the functor $\underline{KO}$ i.e. $KO(X)$ equivalent to $\underline{\pi}(\mathrm{Hom}(X, BO))$.

Let H be a commutative associative Hopf space endowed with a map $H \longrightarrow \mathbb{Z} \times BO$ of Hopf spaces. By an H-orientation on f, we mean a lifting of $\nu_f : X \longrightarrow \mathbb{Z} \times BO$ to a map $X \longrightarrow H$. An H-oriented map is a map f provided with an H-orientation; we shall always denote the map $X \longrightarrow H$ by ν_f. Clearly

replace by next page

(i) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are H-oriented, then gf is H-oriented with

$$\nu_{gf} = \nu_f + f^*\nu_g$$

Let H be a commutative associative Hopf space endowed with a map $\xi : H \longrightarrow \mathbb{Z} \times BO$ of Hopf spaces. An H-orientation of f is by definition an object of the fiber over ν_f category of the map $\underline{H}(X) \longrightarrow \underline{KO}(X)$, that is, a map $u : X \longrightarrow H$ together with a homotopy class of homotopies from ξu to ν_f. ~~[illegible]~~ An H-oriented map is a map f endowed with an H-orientation; from now on we denote by ν_f the H-orientation of f. Then

(ii) If

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

is a cartesian square of manifolds where g and f are transversal (i.e. the map $f \times g : X \times Y' \longrightarrow Y \times Y$ is transversal to the diagonal submanifold Δ_Y), and if f is H-oriented, then f' is H-oriented with

$$\nu_{f'} = (g')^* \nu_f .$$

~~If X is a manifold and q is an integer we define $\Omega^q_H(X)$ to be the equivalence classes of proper H-oriented maps $f : Z \longrightarrow X$~~

Let $\Omega_H(X)$ be the set of equivalence classes of proper H-oriented maps $f : Z \to X$ where such a map f is equivalent to $f' : Z' \longrightarrow X$ if there exists a transversal cartesian diagram

$$\begin{tikzcd} Z \amalg Z' \arrow[r] \arrow[d, "f \amalg f'"] & W \arrow[d, "h"] \\ (X \times 0) \amalg (X \times 1) \arrow[r] & X \times \mathbb{R} \end{tikzcd}$$

where h is proper and H oriented, and where the orientations of f, f'

come from the orientation of h. It is easy to see that this relation (?) is an equivalence relation and that $\Omega_H(X)$ is an abelian monoid where the sum of ~~[illegible]~~ the equivalence classes containing $f: Z \to X$ and $f': Z' \to X$ is the equivalence class of $f+f': Z \sqcup Z' \to X$. ~~[illegible]~~ Let $\Omega^q_H(X)$ be the submonoid of $\Omega_H(X)$ consisting of elements represented by maps $f: Z \to X$ of dimension $-q$, i.e. $\dim \nu_f = q$. Then

$$\bigoplus_q \Omega^q_H(X) \xrightarrow{\sim} \Omega_H(X)$$

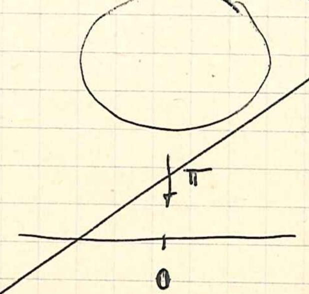
~~(Hard problem with cohomology.)~~

~~To remedy this defect we can either~~

~~(a) require dimension to be bounded~~ ~~(so that the embedding thms. hold)~~

~~(b) change definition of proper maps.~~

$\Omega_H(X)$ is not necessarily a group, however if H has an ~~[illegible]~~ inverse then it is: (Given $f: Z \to X$ with H-reduction consider the map $f \times \pi: Z \times S^1 \to X \times \mathbb{R}^1$, where π is as follows:



then $\tau_{Z\times S^1} = pr_1^* \tau_Z + 1$ in $KO(Z \times S^1)$

$\tau_{X\times\mathbb{R}^1} = pr_1^* \tau_X + 1$ in $KO(X \times \mathbb{R}^1)$

Cobordism as a universal cohomology theory

In this section we show how cobordism generalized cohomology theories can be characterized as a universal functor on the category of C^∞ manifolds endowed with Gysin morphisms for proper maps possessing a particular kind of orientation. This idea was suggested by Grothendieck's theory of motives in algebraic geometry. It permits one to define cobordism groups without using manifolds with boundary and as we plan to show in later papers leads to various generalizations such as cobordism for manifolds over a base manifolds and equivariant cobordism theory. Although the universal approach is not indispensable for the present paper, it furnished the motivation for many of the results.

We shall consider only C^∞ manifolds which are Hausdorff, countably compact, and of bounded, not necessarily constant, dimension. Such a manifold possesses ~~an~~ a closed embedding in Euclidean space $\mathbb{R}^N$ which is unique up to isotopy for N sufficiently large. Let <u>Man</u> be the category of manifolds and C^∞ maps.

~~Bures-sur-Yvette~~
~~July 1[illegible], 1969~~

~~Dear Mather~~,

Enclosed is an announcement of some work of mine in cobordism theory which uses your theory of typical curves. Because I had to keep the thing under 8 pages, it was impossible to include any of the categorical considerations which motivated my results. Without these, the material in the last two sections is I think somewhat incomprehensible, so I'm going to try in this letter to explain my point of view.

I have gained a great deal of insight into cobordism theory by comparing it with Grothendieck's theory of motives which I now like to think of as the analogue of cobordism theory in algebraic geometry. Indeed you have probably heard the ~~theory~~ theories of motives described as the universal cohomology theory for schemes, through which any other, such as l-adic, Hodge, de Rham, or crystalline, must factor. With this in mind I shall review the definition of the cobordism generalized cohomology theories. Let Man be the category of C^∞ manifolds and C^∞ maps. I do not require that manifolds be connected or that their components have the same dimension, but I want the dimension to be bounded so that any manifold possesses an embedding in Euclidean space $\mathbb{R}^N$, which is unique up to isotopy if N is sufficiently large. Given a morphism $f: X \longrightarrow Y$ in Man it can be factored

$$X \xrightarrow{(f,i)} Y \times \mathbb{R}^N \xrightarrow{pr_1} Y$$

where i is an embedding. The normal bundle of (f,i) has a well-defined stable isomorphism class independent of the choice of the embedding. If G is one of the infinite classical groups such as $\{1\}$, U, Spin, etc, then by a G-oriented map f I mean a map whose stable normal bundle is endowed with a G-orientation, that is, a reduction of the structural group to G; the resulting G-oriented stable bundle will be denoted ν_f. There are two sorites:

1.) If $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are G-oriented, then $gf: X \longrightarrow Z$ is G-oriented with

$$\nu_{gf} \cong \nu_f + f^*\nu_g .$$

2.) Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

be a cartesian square in $\underline{\text{Man}}$, where f and g are transversal (i.e. the map $f \times g: X \times Y' \longrightarrow Y \times Y$ is transversal to the diagonal submanifold). If f is G-oriented, then f' is G-oriented with

$$\nu_{f'} \cong (g')^* \nu_f .$$

Given a manifold Y, let $\Omega G(Y)$ be the equivalence classes of maps f in <u>Man</u> which are proper, G-oriented and have target Y; two such maps $f_i : X_i \longrightarrow Y$ $\quad i = 0, 1$ are equivalent if they are cobordant, which means that there exists a G-oriented proper map $h : Z \longrightarrow Y \times \mathbb{R}$ and a diagram

3)
$$\begin{tikzcd}
X_0 \arrow[r, "j_0"] \arrow[d, "f_0"'] & Z \arrow[d, "h"] & X_1 \arrow[l, "j_1"'] \arrow[d, "f_1"] \\
Y \arrow[r, "\iota_0"] & Y \times \mathbb{R} & Y \arrow[l, "\iota_1"']
\end{tikzcd}$$
$$y \longmapsto (y, 0) \qquad y \longmapsto (y, 1)$$

where the two squares are transversal cartesian and the orientation of h induces that of the f_i as described above in 2.)
Denoting the equivalence class of $f : X \to Y$ by $[f : X \to Y]$, $\Omega G(Y)$ is an abelian group with

$$[f_1 : X_1 \to Y] + [f_2 : X_2 \longrightarrow Y] = [(f_1, f_2) : X_1 \amalg X_2 \longrightarrow Y],$$

in fact it is a graded abelian group

$$\Omega G(Y) = \bigoplus_{q \in \mathbb{Z}} \Omega G^q(Y),$$

where elements in $\Omega G^q(Y)$ are represented by maps $f : X \longrightarrow Y$ which are of relative dimension $-q$ at all points of X. ((If you happen to know some cobordism theory.) $\Omega G(Y)$ is the "generalized" cohomology of the <u>space</u> Y with values in the Thom spectrum MG, i.e.

$$\Omega G^q(Y) \cong \varinjlim_k [\Sigma^{-q+k} Y, MG(k)] \Big)$$

Call $\Omega G(Y)$ the G-cobordism group of Y. G-cobordism possesses two variances with respect to Y. First of all if $g: Y' \longrightarrow Y$ is a proper G-oriented map one has the so-called Gysin homomorphism

$$g_* : \Omega G(Y') \longrightarrow \Omega G(Y)$$
$$g_* [f: X \to Y'] = [gf: X \to Y]$$

where gf is oriented as in 1) above. Secondly if $g: Y' \to Y$ is an arbitrary morphism in Man and if $f: X \to Y$ is proper and G-oriented, then by the Thom transversality theorem g may be moved transversally to f and we can form the **fiber product** $pr_2 : X \times_Y Y' \longrightarrow Y'$, which is proper and G-oriented using 2). The resulting element of $\Omega G(Y')$ depends only on $[f: X \to Y]$, and so a map

$$g^* : \Omega G(Y) \longrightarrow \Omega G(Y')$$

is defined. Here are some basic properties of this structure:

I. $Y \mapsto \Omega G(Y)$, $f \mapsto f^*$ is a contravariant functor from Man to Ab.

II. $Y \mapsto \Omega G(Y)$, $f \mapsto f_*$ is a covariant functor from the category of manifolds and proper G-oriented maps to Ab.

III. (Gysin commutes with transversal base change). Given a transversal cartesian square in Man

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

where f is proper and G-oriented and where f' is endowed with the induced orientation (2.), then we have

$$g^* f_* = f'_*(g')^* .$$

$\underline{\text{IV}}$. (homotopy) If f and $g: X \longrightarrow Y$ are maps in Man which are homotopic (by a ~~[illegible]~~ homotopy $h: X \times \mathbb{R} \longrightarrow Y$ which is C^∞), then $f^* = g^*$.

$\underline{\text{V}}$. If $in_i: X_i \longrightarrow X \quad i = 1,2$ are the inclusions, then

$$(in_1^*, in_2^*): \Omega G(X_1 \amalg X_2) \xrightarrow{\sim} \Omega G(X_1) \times \Omega G(X_2) .$$

After all these preliminaries I come to the main point. If X is a manifold, let $1_X \in \Omega G^0(X)$ be the class of id_X. Let e be the final object of Man.

Proposition: (universal property of ΩG). Let $Y \longmapsto$ ~~[illegible]~~ $Q(Y)$ $f \longmapsto f^*$ be a contravariant functor from Man to Ab endowed with a Gysin homomorphism $f \longmapsto f_*$ for proper G-oriented maps. Assume that conditions I–V hold with ΩG replaced by Q. ~~[illegible]~~ Then given an element $\lambda \in Q(e)$, there is a unique natural transformation

$$\theta: \Omega G \longrightarrow Q$$

<u>compatible with Gysin homomorphism such that</u> $\theta(1_e) = \lambda$.

This is essentially trivial. It is clear that $\theta : \Omega G(Y) \longrightarrow Q(Y)$ is given by

4) $$\theta[f : X \longrightarrow Y] = f_* \pi_X^* \lambda$$

where $\pi_X : X \longrightarrow e$ is the canonical map. To see this is well-defined suppose given two representatives $f_i : X_i \longrightarrow Y$ for an element of $\Omega G(Y)$; then there is a diagram 3) above and so

$$\begin{aligned}
(f_0)_* \pi_{X_0}^* \lambda &= (f_0)_* j_0^* (\pi_2^* \lambda) \\
& \qquad\qquad\qquad (\mathrm{III}) \\
&= \iota_0^* h_* (\pi_2^* \lambda) \\
& \qquad\qquad\qquad (\mathrm{IV}) \\
&= \iota_1^* h_* (\pi_2^* \lambda) \\
& \qquad\qquad\qquad\qquad (\mathrm{III}) \\
&= (f_1)_* j_1^* (\pi_2^* \lambda) = (f_1)_* \pi_{X_1}^* \lambda .
\end{aligned}$$

The rest is checking that θ, as defined by 4), is compatible with everything.

There is an external product operation

$$\Omega G(Y) \otimes_{\mathbb{Z}} \Omega G(Y') \longrightarrow \Omega G(Y \times Y')$$
$$\alpha \otimes \beta \longmapsto \alpha \boxtimes \beta$$

with unit $1_e \in \Omega G(e)$ given by

$$[f : X \longrightarrow Y] \boxtimes [f' : X' \longrightarrow Y'] = [f \times f' : X \times X' \longrightarrow Y \times Y']$$

where $f \times f'$ is G-oriented as follows: The orientations of f and f'

yield by 2) orientations of $f \times id_{Y'}$ and $id_X \times f'$, which in turn yields an orientation of $f \times f' = (f \times id_{Y'})(id_X \times f')$. (The orientation of $f' \times f$ is $(-1)^{(\deg f)(\deg f')}$ times that of $f \times f'$). $\Omega G(Y)$ becomes a Koszul commutative ring with the internal product

$$\alpha \cdot \beta = \Delta_Y^*(\alpha \boxtimes \beta)$$

where $\Delta_Y : Y \longrightarrow Y \times Y$ is the diagonal. There is a basic formula:

$\underline{\text{VI}}$. If $f : X \longrightarrow Y$, $f' : X' \longrightarrow Y'$ are proper and G-oriented, then

$$(f \times f')_*(x \boxtimes x') = (-1)^{(\deg f')(\deg x)} f_* x \boxtimes f'_* x'$$

for $x \in \Omega G(X)$ and $x' \in \Omega G(X')$.

The proposition may be augmented by the assertion that if Q has in addition products

$$\begin{array}{ccc} Q(Y) \otimes Q(Y') & \longrightarrow & Q(Y \times Y') \\ \alpha \otimes \beta & \longmapsto & \alpha \boxtimes \beta \end{array} \qquad 1 \in Q(e)$$

satisfying the formula of $\underline{\text{VI}}$ with ΩG replaced by Q, then the natural transformation θ given by the unit $1 \in Q(e)$ is compatible with products.

Therefore ΩG may be characterized as the initial object of the category of functors $Q : \underline{(Man)}^\circ \longrightarrow \underline{Ab}$ endowed with Gysin homomorphism for proper G-oriented maps and endowed with products such that the conditions $\underline{\text{I-VI}}$ hold. Note that $\Omega G(e)$ is just the

~~the~~ cobordism ring of compact G-oriented manifolds; and hence we have given a 'definition' of the latter not using manifolds with boundary!

I shall now turn to Chern classes and the formal group law. For this suppose that G is the infinite unitary group U; the cobordism theory ΩU will be called complex cobordism theory and denoted simply Ω. If E is a complex vector bundle of dimension n over a manifold X, then the zero section $i : X \to E$ is proper and U-oriented hence we can define the ~~[illegible]~~ nth Chern class of E to be

$$c_n^{\Omega}(E) = \iota^* \iota_* 1_X \in \Omega^{2n}(X).$$

Recalling how ι^* is defined, one sees that $c_n^{\Omega}(E)$ is the cobordism class of the zero submanifold of a generic section of E.

The following theorem permits one to define Chern classes of all dimensions following the old method of Grothendieck.

<u>VII</u>. If E is a complex vector bundle of dimension n over a manifold X, ~~then~~ let $\mathbb{P}(E')$ be the projective bundle of hyperplanes in E and let $\mathcal{O}(1)$ be the canonical quotient line bundle in $\mathbb{P}(E')$. Then $\Omega(\mathbb{P}E')$ is a free $\Omega(X)$-module with basis $1, \xi, \ldots, \xi^{n-1}$ where $\xi = c_1^{\Omega}(\mathcal{O}(1))$.

The Chern classes $c_i^{\Omega}(E) \in \Omega^{2i}(X)$ are defined as the coefficients of the relation giving the ring structure of $\Omega(\mathbb{P}E')$:

$$\xi^n - c_1^\Omega(E)\cdot\xi^{n-1} + \cdots\cdots + (-1)^n c_n^\Omega(E) = 0.$$

They satisfy the Whitney sum formula

$$c_n^\Omega(E'+E'') = \sum_{i+j=n} c_i^\Omega(E')\, c_j^\Omega(E'') \qquad \text{with } c_0^\Omega(E) = 1$$

but unlike Chern classes in integral cohomology the first Chern class of a tensor product of two line bundles $L_1 \otimes L_2$ <u>not</u> the sum of $c_1^\Omega L_1$ and $c_1^\Omega L_2$. Instead there is a formal commutative group law $F^\Omega(X,Y) = \sum a_{kl} X^k Y^l$ with $a_{kl} \in \Omega^{2-2k-2l}(e)$ such that

$$c_1^\Omega(L_1 \otimes L_2) = F^\Omega(c_1^\Omega L_1, c_1^\Omega L_2)$$

for all line bundles L_1 and L_2 over the same base. (The existence of this formal group law follows from the fact that $L_1\ L_2$ may be induced from $pr_1^* \mathcal{O}(1)$ and $pr_2^* \mathcal{O}(1)$ on $\mathbb{P}^n \times \mathbb{P}^n$ for n large and that

$$\Omega(\mathbb{P}^n \times \mathbb{P}^n) \cong \Omega(e)[x,y]/(x^{n+1}, y^{n+1})$$

where $x = c_1^\Omega(pr_1^* \mathcal{O}(1))$ and $y = c_1^\Omega(pr_1^* \mathcal{O}(1))$.)

Now if Q is a functor $(\underset{\sim}{\mathrm{Man}})^\circ \longrightarrow \underset{\sim}{\mathrm{Ab}}$ with Gysin homomorphism for proper U-oriented maps and products satisfying I–VI, then by means of the canonical homomorphism $\theta: \Omega \longrightarrow Q$ we obtain Chern classes for complex bundles with values in Q and a formal group law F^Q. Here are some examples:

(i) $Q(X) = H^*(X, \mathbb{Z})$. Then $F^Q(X,Y) = X+Y$

(ii) Q = $\mathbb{Z}/2\mathbb{Z}$ - graded complex K-theory. Thus $Q(X) = K^+(X) \oplus K^-(X)$, where $K^+(X)$ is the Grothendieck group on complex bundles on X and $K^-(X) = \operatorname{Ker}\{K^+(S^1 \times X) \longrightarrow K(\{*\}\times X)\}$; the Gysin homomorphism is defined using the periodicity theorem of Bott. I've ~~computed~~ checked that for a vector bundle of dim n

$$c_n^K(E) = \lambda_{-1} E^\vee \qquad E^\vee = \text{dual bundle to } E$$

hence that $c_1^K(L) = 1 - L^{-1}$ and that

$$F^K(X,Y) = X + Y - XY$$

(iii) $Q = \mathbb{Z}$-graded complex K-theory. Here $Q(X) = \bigoplus_{n\in\mathbb{Z}} K^n(X)$ where $K^n = K^+$ as above if n even and K^- if n is odd. If $\beta \in K^{-2}(pt)$ is the periodicity class, then the formal group law is $X+Y-\beta XY$.

Since Ω is a universal such theory Q, one conjectures that F^Ω should be a universal group law. This is true and a proof is given in my announcement. The proof uses everything - ~~[illegible]~~ Lazard's theorem on the structure of the universal group law and Milnor's on the structure of $\Omega(e)$. It would be nice to have a proof ~~[illegible]~~ avoiding Milnor's results and using instead the universal nature of Ω, but I have not succeeded in finding one.

February 19, 1969. { a first attempt to ~~[illegible]~~ write details for Ω_G; unfinished

Equivariant Cobordism Theory

Outline:

1. Definition of Ω_G and basic properties.
2. Algebraic analysis of the group law coming from tensor product of line bundles.

[real cobordism eventually]

Let G be a compact Lie group, Man_G the category of C^∞-manifolds on which G acts and equivariant maps; ~~[illegible]~~ further we shall only consider G-manifolds with finitely many orbit types, ~~If $f: X \to Y$ is a map~~ since ~~for~~ these ~~[illegible]~~ are the ones which can be embedded in representations. If $f: X \longrightarrow Y$ is a G-map, ie. map in Man_G, then $\nu_f = f^*\tau_Y - \tau_X$ is a stable ~~[illegible]~~ G-bundle over X (the category of stable bundles is obtained by a Grothendieck construction from bundles and homotopy classes of isomorphisms by adjoining inverses). By a complex orientation of f we mean an isomorphism of ν_f with a stable ~~[illegible]~~ complex G-bundle. From now on all bundles are complex.

By a ~~cohomology theory~~ (resp. additive, resp multiplicative) cohomology theory Q on Man_G (with Gysin homomorphism for complex oriented maps) we shall mean the ~~following~~ data subject to the axioms below.

a rule associating to each G-manifold X a ~~set (resp abelian group, resp a ring) $Q(X)$ and to~~ set $Q(X)$ and to each G-map $f: X \longrightarrow Y$ a set map $f^*: Q(Y) \longrightarrow Q(X)$ and to each proper oriented G map a set map $f_*: Q(X) \longrightarrow Q(Y)$ such that

A.
$$(gf)^* = f^* g^* \qquad (id)^* = id$$
$$(gf)_* = g_* f_* \qquad (id)_* = id$$

B. (Homotopy). If $\mathbb{R}$ has trivial G-action, then

$$pr_1^* : Q(X) \xrightarrow{\sim} Q(X \times \mathbb{R})$$

C. (transversality). If

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

is cartesian in Man_G with g transversal to f, then

$$g^* f_* = f'_* g'^*$$

In the case of an additive cohomology theory we want $Q(X)$ to be an abelian gp, f^* and f_* to be homomorphisms and

D. (union).

$$Q(\phi) = 0.$$

$$in_{1*} + in_{2*} : Q(X) \oplus Q(X') \xrightarrow{\cong} Q(X \amalg X')$$

$$\left(\Longrightarrow \ (in_1^*, in_2^*) : Q(X \amalg X') \xrightarrow{\cong} Q(X) \times Q(X')\right).$$

(Problem: Do you have to insist that changing ν_f to $\overline{\nu_f}$ changes f_* into $-f_*$?)

In the case of a multiplicative cohomology theory

~~I want Q(X) to there to be given maps~~

$$\cancel{Q(X) \otimes Q(X') \longrightarrow Q(X \times X')}$$

$$\cancel{a \otimes b \longmapsto a \boxtimes b}$$

~~I want Q(X) to be~~ I want ~~the~~ there to be given an element $1 \in Q(pt)$ and maps

$$Q(X) \otimes Q(X') \longrightarrow Q(X \times X')$$
$$a \otimes b \longmapsto a \boxtimes b$$

such that

E. **associativity**: $(a \boxtimes b) \boxtimes c = a \boxtimes (b \boxtimes c)$

unit: $\begin{cases} 1 \boxtimes a \text{ “}=\text{” } a \\ a \boxtimes 1 = a. \end{cases}$

(Problem: How to handle the anti-commutativity)

$$\begin{cases} (f \times g)_* (a \boxtimes b) = f_* a \boxtimes f_* b \\ (f \times g)^* (a \boxtimes b) = f^* a \boxtimes g^* b. \end{cases}$$

We now wish to determine the universal coh. theory.

<u>Lemma 1</u>: Let Q be a cohomology theory and let $1 \in Q(pt)$. ~~Then the~~
Let $Q'(X) =$ subset of $Q(X)$ consisting of elements which can be represented in the form $\iota^* g_* \pi^* 1$

$$\begin{tikzcd} & Z \arrow[r, "\pi"] \arrow[d, "g"] & pt \\ X \arrow[r, "i"] & X \times V & \end{tikzcd}$$

where g is proper + oriented, ~~and~~ where V is a representation of G and $i(x) = (x, 0)$. Then Q' is a subcohomology theory of Q.

<u>Proof</u>: Let $f: X \longrightarrow Y$ be proper and oriented. Then $f_* \iota^* g_* \pi^* 1 = \iota'^*(f'_* g_* \pi^* 1$, where we use the cartesian square

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & X \times V \arrow[d, "f'"] \\ Y \arrow[r, "i'"] & X \times V \end{tikzcd} \qquad \begin{cases} f' = f \times id \\ \iota'(y) = (y, 0). \end{cases}$$

Thus $f_* Q'(X) \subset Q'(Y)$.

~~Then~~ Let $f: Y \longrightarrow X$ be arbitrary. Then if we factor

$$\begin{tikzcd} Y \arrow[r, "i'"] \arrow[dr, "f"'] & Y \times W \arrow[dr, "f'"] & \\ & X \arrow[r, "i"] & X \times V \end{tikzcd}$$

, where W is a representation and f' is smooth; we have

$$\begin{tikzcd}
 & Z' \arrow[r, "f''"] \arrow[d, "g'"'] \arrow[dr, phantom, "cart"] & Z \arrow[r, "\pi"] \arrow[dd, "g"] & pt \\
Y \arrow[r, "i''"] \arrow[dr, "f"'] & Y\times W \arrow[dr, "sm\ f'"] & & \\
 & X \arrow[r, "i"'] & X\times V &
\end{tikzcd}$$

and we have $f^*i^*g_*\pi^*1 = i''^*f'^*g_*\pi^*1 = i''^*g'_*(\pi f'')^*1$. To construct this factorization ~~choose an embedding~~ first factor f into an embedding followed by a smooth map, then by means of the exponential one gets an embedding into a vector bundle following by a smooth map; finally the bundle is the quotient of a trivial bundle $Y\times W$. Thus have

$$\begin{tikzcd}
Y \arrow[r] \arrow[dr] & Y\times W \arrow[d, "smooth"] \\
 & E \arrow[r, "smooth"] & X\times V
\end{tikzcd}$$

~~Construction of Ω_G: Consider triples (V, Z, g) where V is a representation of G and $g: Z \to X\times V$ is proper oriented.~~

If X is a G-manifold, let $\mathcal{C}_G^q(X)$ be the bordism classes of proper oriented G-maps $f: Z \to X$ of codimension q. Set

$$\Omega_G^q(X) = \varinjlim \mathcal{C}_G^q(X\times V)$$

where V runs over the category ~~of representations and maps of G~~

consisting of ~~sets~~ representations of G and surjective maps. The first thing to show is that this limit may be calculated as a filtered **inductive** limit. However the category is filtering up to homotopy, e.g.

$$V, V' \quad \text{have} \quad V \oplus V' \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} V, V' \quad \text{surj.}$$

$$V \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} V' \quad \text{two surjections}$$

Then consider

$$V \oplus V' \xrightarrow{pr_1} V \quad \text{surjective}$$

$$(V \oplus V') \overset{\pi_1 pr_1}{\underset{pr_2}{\rightrightarrows}} V' \quad \text{homotopic trivially}$$

$$t\,\pi_1 pr_1 + (1-t)\,\mathrm{id}$$

similarly
$$V \oplus V' \overset{\pi_2 pr_1}{\underset{pr_2}{\rightrightarrows}} V' \quad \text{homotopic}$$

You had a better method **before**: To show that the category of thickenings of X

objects = embeddings $X \overset{i}{\hookrightarrow} Y$

morphisms =
$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, hook] & Y \arrow[d, "g"] \\ & Y' \end{tikzcd} \quad \text{with } g \text{ smooth}$$

This is filtering up to homotopy: Given

$$\begin{tikzcd}
 & & & Y' \arrow[d, "\text{homotopy equivalence}"] \\
 & Z \arrow[rr] \arrow[d, "\text{open}"] & & N \arrow[d, "\text{open}"] \\
X \arrow[r, hook] \arrow[ur, hook] \arrow[uurrr, bend left] & Y \arrow[rr, "{(g_1, g_2)}"] \arrow[drr, "g_i"', "\text{sm.}"] & & Y' \times Y' \arrow[d, "pr_i"] \\
 & & & Y'
\end{tikzcd}$$

Now it's clear that given any theory Q we have a map

$$\Omega_G^{q}(X) \longrightarrow Q(X)$$

obviously universal. (Assumes Q just a functor to sets)

Additive structure: What do we need to guarantee that

$$\underline{\mathcal{C}_G^{q}(X)} \xrightarrow{\theta} Q(X)$$

is a homomorphism? Recall that given $f_1 : Z_1 \to X$, $f_2 : Z_2 \longrightarrow X$ proper + oriented, then sum in $\mathcal{C}_G(X)$ is defined by $(f_1)_* 1 + (f_2)_* 1 = (f_1 + f_2)_* 1$ where $f_1 + f_2$ is the canon. map $Z_1 \amalg Z_2 \longrightarrow X$. Now

$$\theta(f_* 1) = f_*^Q 1$$

so θ is a homomorphism if

$$\boxed{f_{1*}^Q 1 + f_{2*}^Q 1 = (f_1^Q + f_2^Q)_* 1}$$

~~What we need therefore seems to be that given $f_i : Z_i \to$~~

Correct condition is that given $f_i : Z_i \longrightarrow X$, $g_i : Z_i \longrightarrow Y$, then

$$\boxed{(f_1+f_2)_*(g_1+g_2)^* y = f_{1*}g_1^* y + f_{2*}g_2^* y}$$

Relation with preceding axiom:

$Z_i = \phi = X = Y$. Then you get for $y \in Q(\phi)$ that

$$y = y + y \implies y = 0.$$

Also we find that taking $X = Z_1 \amalg Z_2 = Y$ $\quad f_i = g_i = in_i$ that $f_1 + f_2 = g_1 + g_2 = id_{Z_1 \amalg Z_2}$

$$x = (in_1)_*(in_1^* x) + in_{2*}\, in_2^* x.$$

implying that

$$Q(Z_1) \oplus Q(Z_2) \simeq Q(Z_1 \amalg Z_2)$$

Conversely the latter implies with $x = (g_1+g_2)^* y$ $\quad y \in Q(Y)$ that

$$(f_1+f_2)_*(g_1+g_2)^* y = f_{1*}g_1^* y + f_{2*}g_2^* y.$$

$\therefore$ Same axiom.

$\boxed{\textbf{Multiplicative Structure}}$: ~~What do we need to guarantee that~~

~~Ω_G~~

product structure on Ω_G:

$$\begin{array}{cccc} & Z_1 & Z_2 & Z_1 \times Z_2 \\ & \downarrow & \downarrow & \downarrow \\ X & X \times V_1 & X \times V_2 & \cancel{X} \times X \times (V_1 \times V_2) \end{array}$$

we have a natural map

$$C^p_G(X) \otimes C^q_G(Y) \longrightarrow C^{p+q}_G(X \times Y)$$

$$\begin{array}{ccccc} Z_1 & & Z_2 & & Z_1 \times Z_2 \\ \downarrow & \times & \downarrow & \longmapsto & \downarrow \\ X & & Y & & X \times Y \end{array}$$

This we claim defines a map

$$\Omega^p_G(X) \otimes \Omega^q_G(Y) \longrightarrow \Omega^{p+q}_G(X \times Y).$$

Method is to ~~[illegible]~~ consider the functor

$$X \longmapsto \Omega_G(X \times Y)$$

observe it satisfies the axioms, hence given an element ~~of~~ $\alpha \in \Omega_G(Y)$ one gets a map

$$\Omega_G(X) \xrightarrow{\ \cup\alpha\ } \Omega_G(X \times Y)$$

necessary to check that isomorphism

$$Hom(\Omega_G, Q) = Q(pt)$$

is compatible with addition

$Z_i \times Y \xrightarrow{pr_2} Y$

Recall Q is a coh. theory which is additive! Claim then that $X \longmapsto Q(X \times Y)$. ~~with~~ $Q_Y(X)$ with $(f_Y)_* = (f_* \times id_Y)_*$ $(f_Y)^* = (f \times id_Y)^*$ is a coh. theory additive. For this one wants to know ~~that~~, given $y \in Q(Y)$ and $f_i : Z_i \to X$ ~~[illegible]~~ that

$$(f_1 Y)_* \, pr_2^* y + (f_2)_* \, pr_2^* y = (f_1 + f_2)_{Y*} \, pr_2^* y \checkmark$$

February 22, 1969

Checked steps of Groth. proof of R-R to see if they go in alg. geom. Problem with $\iota^* \iota_* x = c_d(\nu_i) \cdot x$.

Some remarks toward a theory of cobordism in algebraic geometry.

Consider the category of quasi-projective non-singular varieties over an algebraically closed field. By a cohomology theory Q on this category I shall mean a contravariant functor to the category of rings endowed with a Gysin homomorphism for proper maps ~~[illegible]~~ having the properties of K-theory, in particular, homotopy, transversal base change, half exactness, and splitting principle. I assume that a universal such theory exists and denote it Ω. I now wish to prove Riemann-Roch under the following form:

$$\mathrm{Hom}^+(K, Q^*) \Rightarrow \left\{ \begin{array}{l} \text{natural ring homs } \beta : \Omega \longrightarrow Q \\ \text{such that for } pt \xrightarrow{s} \mathbb{P}^2 \xrightarrow{\pi} pt \\ \text{we have that } \pi_* \beta s_* 1 \in Q(pt)^* \end{array} \right.$$

we denote the ~~[illegible]~~ RHS by $\mathrm{Hom}^{\otimes\cdot}(\Omega, Q)^*$.
The map is given by sending a characteristic class $\alpha : K \longrightarrow Q^*$ into $\hat{\alpha} : \Omega \longrightarrow Q$ given by

$$\hat{\alpha}(f_* 1) = f_* \alpha(\nu_f)$$

~~[illegible]~~ $\hat{\alpha}$ is well-defined because using α one ~~[illegible]~~ defines a new Gysin homomorphism

$$f_! x = f_*(\alpha(\nu_f) x) \qquad x \in Q(X)$$

and hence by the universal property of Ω one gets a unique map $\hat{\alpha} : \Omega \longrightarrow Q$ with

$$\begin{cases} \hat{\alpha}(f_* x) = f_!(\hat{\alpha} x) \\ \hat{\alpha} f^* = f^* \hat{\alpha}. \end{cases}$$

To prove the R-R thm one defines a map in the opposite direction. Thus starting with β one defines $\bar{\beta}$ on vector bundles by the formula

$$\beta(\iota_* 1) = \iota_* \bar{\beta}(E)$$

$$\text{or} \quad \bar{\beta}(E) = \pi_* \beta\, \iota_* 1$$

These formulas are equivalent since one has exact sequence $0 \to Q(X) \xrightarrow{i_*} Q(\mathbb{P}(E+1)) \longrightarrow Q(\mathbb{P}E) \to 0$ by splitting principle, hence $\beta \iota_* 1 = \iota_* \bar{\beta}(E)$ some $\bar{\beta}(E) \in Q(X)$ and so $\bar{\beta}(E) = \pi_* \beta i_* 1$

where $i : X \longrightarrow \mathbb{P}(E+1)$ given by the zero section of E, and where $\pi : \mathbb{P}(E+1) \longrightarrow X$ is the structural map.

<u>Hypothesis:</u> $\bar{\beta}$ extends to a transf $K \longrightarrow Q^*$ provided $\bar{\beta}(1) \in Q(pt)^*$. In other words

ⓐ $\bar{\beta}(E) = \bar{\beta}(E')\beta(E'')$ if $0 \to E' \to E \to E'' \to 0$ is exact

ⓑ $\bar{\beta}(E) \in Q(X)^*$.

Observe this is true if $\beta = \hat{\alpha}$ since

$$\bar{\hat{\alpha}}(E) = \pi_* \hat{\alpha}\, \iota_* 1 = \pi_* \iota_* \alpha(E) = \alpha(E).$$

Thus we see immediately that $\alpha \longmapsto \hat{\alpha}$ is injective. We shall now show that for the β satisfying the hypothesis we have $\hat{\beta} = \beta$, thus identifying the image of ^ as those β satisfying the hypothesis. The thing to prove is

$$\boxed{\beta(f_* x) = f_*(\bar{\beta}(\nu_f)\,\beta x).}$$

because then both β and $\hat{\beta}$ are transf. from Ω to Q compatible with the $f_!$ defined using $\bar{\beta}$ and hence are equal. This is a R-R statement and is proved following Grothendieck:

1) Formula true for $f: \mathbb{P}^n \longrightarrow pt$. Both sides are linear over ~~$\Omega(pt)$~~ $\Omega(pt)$ so only have to check for $x = j_* 1$ where $j: \mathbb{P}^q \hookrightarrow \mathbb{P}^n$. Reduction to the case where $x = 1$: Note that there is a trans. cartesian diagram

$$\begin{array}{ccc} \mathbb{P}^q & \xrightarrow{\;j\;} & \mathbb{P}^n \\ \downarrow j & & \downarrow i \\ \mathbb{P}^n & \xrightarrow[\;s\;]{} & \mathbb{P}(\oplus_{n-q} O(1) + 1/\mathbb{P}^n) \end{array}$$

where i (resp s) comes from the zero section (resp generic section) of $\oplus_{n-q} O(1)$. Thus

$$j_* 1 = j_* j^* 1 = s^* i_* 1 = i^* i_* 1$$

and

$$\beta(j_* 1) = i^* \beta\, i_* 1 = i^* i_* \bar{\beta}(O(1))^{n-q} = j_* 1 \cdot \bar{\beta}(O(1))^{n-q}$$

Thus

$$\beta(f_* x) = \beta(f_* j_* 1) = \beta(g_* 1)$$

$$f_*(\bar\beta(\nu_f)\cdot x) = f_*\left(\bar\beta(O(1))^{-n-1}\cdot\bar\beta(O(1))^{n-q} j_* 1\right)$$

$$= g_*\ \bar\beta(\nu_g) \qquad \text{where } g: \mathbb{P}^q \to pt.$$

It therefore remains to prove for $f: \mathbb{P}^n \to pt$ and $x = 1$. Recall that as Q satisfies splitting principle, there is a universal formula

$$\bar\beta(L) = \varphi(c_1(L)) \qquad \beta c_1(L) = \bar\varphi(c_1(L))$$

for a uniquely determined $\varphi(X) \in Q(pt)[[X]]$. Then

$$\beta F^\Omega = \bar\varphi * F^Q$$

so

$$\beta\omega^\Omega = \frac{dZ}{(\beta F^\Omega)_X(0,Z)} = \frac{dZ}{\frac{\partial}{\partial X}\bar\varphi(F^Q(\bar\varphi^{-1}X, \bar\varphi^{-1}Z))\Big|_{X=0}}$$

$$= \frac{dZ}{\bar\varphi'(\bar\varphi^{-1}Z)\, F_X^Q(0,\bar\varphi^{-1}Z)} = \frac{dZ}{\bar\varphi'(\bar\varphi^{-1}Z)\, F_X^Q(0,\bar\varphi^{-1}Z)}$$

$$= \frac{\bar\varphi'(W)\,dW}{\bar\varphi'(W)\, F_X^Q(0,W)} = \frac{dW}{F_X^Q(0,W)} \qquad \text{if } Z = \bar\varphi(W)$$

$\therefore \quad \beta\omega^\Omega =$ transform of ω^Q under $\bar\varphi$ (or $\bar\varphi^{-1}$?)

So

$$\beta(f_* 1) = \beta\ \mathrm{res}\ \frac{\omega^\Omega}{Z^{n+1}} = \mathrm{res}\ \frac{\beta\omega^\Omega}{Z^{n+1}} = \mathrm{res}\ \frac{dW}{(\bar\varphi W)^{n+1} F_X^Q(0,W)}$$

$$f_*(\bar\beta(\nu_f)) = \mathrm{res}\ \frac{\omega^Q}{\varphi(Z)^{n+1} Z^{n+1}} = \mathrm{res}\ \frac{\omega^Q}{\bar\varphi(Z)^{n+1}}$$

QED.

Remarks: Above proof uses the formula for f_* as a residue for the theory Q. We still have to prove this from the splitting principle.

2) Now take $f : X \longrightarrow Y$ and choose an embedding

$$X \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^N$$

and thus construct a factorization of f into

$$X \xrightarrow{\ i\ } Y \times \mathbb{P}^N \xrightarrow{\ pr_1\ } Y$$

where i is a closed immersion whose normal bundle E is isomorphic to $2 \oplus E'$. We know R-R for pr_1, hence it remains to prove it for i.

Changing notation let $i : Y \longrightarrow X$ be a closed embedding factoring into $Y \hookrightarrow Y \times \mathbb{P}^2 \hookrightarrow X$ and blow X up along Y

$$\begin{tikzcd} \tilde{Y} \arrow[r, "j"] \arrow[d, "g"] & \tilde{X} \arrow[d, "f"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

$$E = \nu_i$$

$$\tilde{Y} = \mathbb{P}E \qquad\qquad 0 \longrightarrow \mathcal{O}_{\tilde{Y}}(-1) \longrightarrow g^*E \longrightarrow F \longrightarrow 0$$

Let $s : Y \longrightarrow \tilde{Y}$ be the section obtained from a splitting

$$E = 1 \oplus E'$$

Then we claim the following formulas hold

(i) $f_* 1 \in Q(X)^*$

(ii) $f^* i_* y = j_* (s_* 1 \cdot g^* y)$

(iii) $s_* 1$ is divisible by $j^* j_* 1$ in $\Omega(\tilde{Y})$

(iv) $\beta s_* 1 = \bar{\beta}(F) \cdot s_* 1$

(v) $\beta(j_* 1) = j_*(\bar{\beta}(\nu_j))$.

Assuming these (and [vi] below) we prove R-R. for i

$$\beta i_* y \overset{?}{=} i_*(\bar{\beta}(E) \cdot \beta y)$$

By (i) f^* is injective so applying f^* + using (ii) we have

$$f^* \beta i_* y = \beta j_*(s_* 1 \cdot g^* y)$$

$$f^* i_*(\bar{\beta}(E) \cdot \beta y) = j_*(s_* 1 \cdot \bar{\beta}(g^* E) \cdot \beta g^* y)$$

By hypothesis

$$\bar{\beta}(g^* E) = \bar{\beta}(F) \cdot \bar{\beta}(\mathcal{O}_{\tilde{Y}}(-1)) = \bar{\beta}(F) \cdot \bar{\beta}(\nu_j)$$

so that we have to show

$$(\#) \qquad \beta j_*(s_* 1 \cdot g^* y) \overset{?}{=} j_*(s_* 1 \cdot \bar{\beta}(F) \cdot \bar{\beta}(\nu_j) \beta g^* y)$$

$$= j_*(\bar{\beta}(\nu_j)\, \beta \cdot (s_* 1 \cdot g^* y)) \qquad \text{(iv)}$$

By (iii) we have

$$s_* 1 = u \cdot j^* j_* 1$$

so

$$s_* 1 \cdot g^* y = (u \cdot g^* y) \cdot j^* j_* 1$$

NEED. (vi) $j^*(j_* z) = (j^* j_* 1)\cdot z$ for Ω.

Assuming this we have

$$s_* 1 \cdot g^* y = (u\cdot g^* y)\; j^* j_* 1$$
$$= j^*[j_*(u\cdot g^* y)] = j^* a$$

for some $a \in \Omega(\tilde{X})$. Then the two sides of (#) become

$$\beta j_*(j^* a) = \beta j_* 1 \cdot \beta a$$

$$j_*\left(\bar{\beta}(\nu_j)\cdot \beta j^* a\right) = j_* \bar{\beta}(\nu_j)\cdot \beta a$$

so by (v) we are done.

(ii):

$$s_* y = s_*(s^* g^* y) = s_* 1 \cdot g^* y$$

Thus (ii) is equivalent to

$$f^* i_* y = j_* s_* y .$$

This we may prove as follows: Since we have arranged that $Y \times \mathbb{P}^2 \hookrightarrow X$ we have a family of embeddings $i_t : Y \longrightarrow X$ such that $i_0 = 1$, $i_1(Y)\cap Y = \phi$. Thus we have a cartesian diagram

$$\begin{tikzcd}
Y \arrow[r, "f^{-1}i_t"] \arrow[d, equal, "id"'] & \tilde{X} \arrow[d, "f"] \\
Y \arrow[r, "i_t"] & X
\end{tikzcd}$$

since f is an isomorphism over $X-Y$, and so

$$f^* \iota_* y = f^*(\iota_t)_* y = (f^{-1} i_t)_* y$$

? this uses homotopy axiom for f_*

but letting t go to zero one puts $f^{-1} i_t$ in a family of maps $Y \times \mathbb{P}^2 \longrightarrow \tilde{X}$ which is js at $t=0$. $\therefore$

$$f^* \iota_* y = j_* s_* y.$$

(i): Let $k : X-Y \longrightarrow X$ be the inclusion. Then $k^*(f_* 1 - 1) = 0$ so $f_* 1 - 1 = i_* y$. But

$$(i_* y)^2 = 0$$

since Y can be deformed off itself. Thus

$$(f_* 1)^{-1} = (1 + i_* y)^{-1} = 1 - i_* y$$

exists.

(iii): Recall that $E = 1 + E' = 1 + (1 + E'')$. Thus we have $\Omega(\mathbb{P}E) = \Omega(Y)[\xi] / \xi^2(\xi^{n-2} - c_1(E''^\vee)\xi^{n-3} + \cdots)$.

Moreover sY is where $\mathcal{O}(-1) \simeq f^* 1$ in $f^*E = f^*1 + f^*E'$. or where the section

$$\mathcal{O} \longrightarrow f^*E \otimes \mathcal{O}(1) \longrightarrow f^*E' \otimes \mathcal{O}(1)$$

is zero. Thus

$$s_* 1 = c_{n-1}(f^*E' \otimes \mathcal{O}(1))$$

$$= c_{n-1}(f^*E'' \otimes \mathcal{O}(1) \oplus \mathcal{O}(1)) = c_{n-2}(f^*E'' \otimes \mathcal{O}(1))\, c_1(\mathcal{O}(1))$$

So s_*1 is divisible by $\xi = c_1(\mathcal{O}(1))$. But

$$c_1(\mathcal{O}(-1)) = I(\xi) = \xi(1 + \underbrace{\text{higher terms}}_{\text{nilpotent}})$$

$\therefore$ s_*1 is divisible by $c_1(\mathcal{O}(-1)) = j^*j_*1$. (this follows by arg. given below)

(iv) As $E = 1 \oplus E'$ we have by definition of $\bar{\beta}$ that

$$\begin{aligned} \beta s_*1 &= s_*(\bar{\beta} E') \\ &= s_*(\bar{\beta}(s^*F)) \\ &= s_*1 \cdot \bar{\beta}(F) \end{aligned}$$

(v) As $\tilde{Y} \to \tilde{X}$ is a non-singular divisor there is a transversal cartesian square

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{j} & \tilde{X} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} \\ \tilde{X} & \xrightarrow{t} & \mathbb{P}(1 \oplus L) \end{array}$$

where L is the line bundle defined by $\tilde{Y}$ (e.g. $(\mathrm{Ker}\ \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{Y}})^{-1}$)

Thus

$$\begin{aligned} \beta(j_*1) &= \beta\, t^* k_* 1 \\ &= t^* \beta k_* 1 = t^* k_* \bar{\beta}(L) \qquad \text{defn of } \bar{\beta} \\ &= j_* j^* \bar{\beta}(L) \\ &= j_* \bar{\beta}(L_j). \end{aligned}$$

(vi). (I can't prove this without having "tubular neighborhoods".) If $Y \xrightarrow{i} X$ is a divisor we want to prove that

$$i^* i_* y = c_1(\nu_i) \cdot y$$

Can prove this in the following cases:

A. $y = i^* x$ because then have

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "j", shift right] \\ X \arrow[r, "s"'] & L \arrow[u, "\pi"', shift right] \end{tikzcd}$$

$$i_* 1 = j^* j_* 1 = c_1(L)$$

so $$i^* i_* y = i^* i_* i^* x = i^*(i_* 1 \cdot x)$$
$$= c_1(i^* L)\, i^* x = c_1(\nu_i) \cdot y.$$

B. If i can be moved to intersect itself transversally; more precisely if $\exists\, t \in \Gamma(X, L)$ with $t | Y$ transversal to zero-section.

Set Z_ε = zero set of $s + \varepsilon t$. This is non-singular $\underbrace{}_{}$ (for ε near 0) and we have that $(i_\varepsilon)_* 1 = i_* 1$. So

B. If the embedding i belongs to a family of embeddings $i_t : Y \to X$ such that i_1 and $i_0 = i$ are transversal. Then

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[d, "j"] & Y \arrow[d, "i"] \\ Y \arrow[r, "i_1"] & X \end{tikzcd}$$

$$i^* i_* y = j_* j^* y = j_* 1 \cdot y$$

But $j_* 1 = i^* i_* 1 = c_1(\nu_i)$ by A.

~~Unfortunately neither of these covers~~

C. When there is a neighborhood U of Y in X admitting a retraction back to Y.

$$\begin{tikzcd} & Y \arrow[r, "id"] \arrow[d, "i'"] & Y \arrow[d, "i"] \\ Y \arrow[r, "i'"] \arrow[rr, bend right, "i"'] & U \arrow[r, hook, "\alpha"] & X \end{tikzcd}$$

cartesian

Then

$$i^* i_* y = i'^* \alpha^* i_* y = i'^* i'_* y = c_1(\nu_{i'}) \cdot y = c_1(\nu_i) y$$

using A.

Unfortunately neither of these covers the situation in algebraic geometry. Observe that C. covers equivariant cobordism.

To prove R-R we needed the following:

properties of Ω (beyond the functorial ones, i.e. its universal property)

- splitting principle + Chern classes + formal group law
- half exactness
- f_* for $f: \mathbb{P}^n \longrightarrow pt$ as a residue
- $j^* j_* z = (j^* j_* 1) \cdot z$ for a codimension 1 embedding

properties of Q

- splitting principle
- f_* for $f: \mathbb{P}^n \longrightarrow pt$ as a residue (this follows from the splitting principle and the fact that for $\mathbb{P}^n$ we have
$$Q(pt) \otimes_{\Omega(pt)} \Omega(\mathbb{P}^n) \xrightarrow{\sim} Q(\mathbb{P}^n)\Big).$$

Existence of $\Omega \otimes_{\mathbb{Z}} Q$ in algebraic geometry is now clear. In effect if Q is a cohomology theory with Gysin homomorphism and products satisfying the splitting principle and moreover permitting one to construct (if this doesn't already follow) Chern classes, then one can construct a unique morphism of theories

$$K \otimes_{\mathbb{Z}} Q[P_1, P_2, \ldots] \longrightarrow Q$$

where Gysin on the left is rigged so as to be compatible with the formal group law of Q. This is essentially Grothendieck's result that over Q one can calculate Ω as the universal recipient for the Chern classes

<u>Proofs of @:</u> Suffices (by splitting principle) to show that

$$\bar\beta(E) = \prod_{i=1}^{n} \bar\beta(L_i)$$

if E has a flag with quotients L_i. So consider

$$s: X \underset{\xleftarrow{\ f\ }}{\longrightarrow} P(1+E)$$

Then

$$s_* 1 = \prod_{j=1}^{n} c_1(\mathcal{O}(1) \otimes f^* L_j)$$

(This formula which we have proved when $E = L_1 + \cdots + L_n$ probably also goes in algebraic geometry (?)), so

$$\beta s_* 1 = \prod_{j=1}^{n} \beta c_1(\mathcal{O}(1) \otimes f^* L_j)$$

For a vector bundle L with zero section $s: X \to P(1+E)$ we have $s^* s_* 1 = c_1(L)$ ~~and so $\bar\beta(E) = \beta s^* s_* 1 = s^* s_* \bar\beta(E)$~~ (see argument C, page 11) so

$$\beta\, c_1(L) = \beta s^* s_* 1 = s^* \beta s_* 1 = s^* s_* \bar\beta(L)$$
$$= c_1(L) \cdot \bar\beta(L) \qquad \text{(argument A, p. 11)}$$

Thus

$$\beta s_* 1 = \prod_{j=1}^{n} c_1(\mathcal{O}(1) \otimes f^* L_j) \cdot \bar\beta(\mathcal{O}(1) \otimes f^* L_j)$$

$$= s_* 1 \cdot \prod_{j=1}^{n} \bar\beta(\mathcal{O}(1) \otimes f^* L_j) = s_* s^* \left(\prod \cdots\right)$$

$$= s_* \prod_{j=1}^{n} \bar\beta(L_j) \qquad \therefore\ \bar\beta(E) = \prod_j \bar\beta(L_j).$$

Proof of (b): We have just shown that if E is a bundle over X and $g: Y \longrightarrow X$ is the flag bundle, then

$$\bar\beta(g^*E) = \prod_j \bar\beta(L_j)$$

$$= \prod_j \bar\beta(1)\left[1 + c_1(L_j)\, Q(c_1(L_j))\right]$$

$$g^*\left[\bar\beta(1)^n \Phi(c_1(E), \dots, c_n(E))\right]$$

where $\Phi(c_1(E), \dots, c_n(E)) \equiv 1$ mod nilpotent elements.

Thus

$$\bar\beta(E) = \bar\beta(1)^{rg\, E}\, \Phi(c_1(E), \dots, c_n(E)) \in Q(X)^*.$$

Conclusion: I am now sure that for ordinary complex cobordism I can prove the Thom, R-R isomorphism

$$\mathrm{Hom}^+(K, Q^*) \simeq \mathrm{Hom}^\bullet(\Omega, Q)'$$

using only the splitting principle for Q. The argument ought also to generalize without difficulty to equivariant cobordism (with G abelian for splitting principle) once we understand the formal group + residue picture better

February 25, 1969: | most of this is completely wrong due to fact that U_G doesn't satisfy proj. bundle theorem.

Equivariant cobordism and formal group laws.

In the following G will denote a compact abelian Lie group and $\hat{G}$ its character group. We let Q be a cohomology theory on the category of G-manifolds with all the properties we need. Ω_G denotes the universal such theory. We assume

$$\mathrm{Hom}^+(K_G, Q^*) \cong \mathrm{Hom}^{\otimes}(\Omega_G, Q)' = \{\beta : \beta \text{ natural ring hom. such that } \bar{\beta}(V) \in Q(pt)^* \text{ for all representations of } G\}.$$

and we want to determine the left side. By splitting principle

$$\mathrm{Hom}^+(K_G, Q^*) = \mathrm{Hom}(\mathrm{Pic}_G, Q^*),$$

in other words to give an additive transformation $\alpha : K_G \to Q^*$ is the same as giving its effect on line bundles.

In fact we have the formula

$$\boxed{\alpha(E) = \mathrm{Norm}_{\Omega(\mathbb{P}\check{E})/\Omega(X)} \alpha(\mathcal{O}(1)).}$$

Lemma: Let A be a ring and let $c_1, \dots, c_n \in A$. Let

$$A[x_1, \dots, x_n] = A[X_1, \dots, X_n] / (c_j = \sigma_j(X_1, \dots, X_n))_{j=1, \dots, n}$$

Then $$A[x_1] \simeq A[X_1]/(X_1^n - c_1 X_1^{n-1} + \cdots)$$ and

$$\mathrm{Norm}_{A[x_1] \to A}\, b(x_1) = \prod_{j=1}^{n} b(x_j) \quad \text{in } A[x_1, \dots, x_n].$$

Proof: The assertions are evidently compatible with base extension $A \to A'$ so that one can replace A by a polynomial ring $B[C_1, \dots, C_n]$. In this case it is well known that $A[x_1, \dots, x_n]$ is a polynomial ring and that $B[C_1, \dots, C_n]$ is the ring of invariants for the symmetric group. Moreover one may adjoin elements Z_i to A, the successive roots of $Z^n - C_1 Z^{n-1} + \cdots$, and obtain a ring of rank $n!$ over A mapping onto $A[x_1, \dots, x_n]$. It follows that they are equal so $A[x_1] \cong A[Z_1]/(Z_1^n - c_1 Z_1^{n-1} + \cdots)$ as claimed. To prove the norm formula one may embed A into a larger ring and so may assume that $c_i = \sigma_i(\lambda_1, \dots, \lambda_n)$ where $(\lambda_i - \lambda_j)^{-1} \in A$ for $i \neq j$. Then

$$A[x_1] \simeq \prod_{j=1}^{n} A$$

where x_1 acts as λ_j on the ith factor

so

$$N_{A[X_1] \to A}\, b(x_1) = \prod_{j=1}^{n} b(\lambda_j).$$

By symmetric functions thm, this equals $P(c_1, \dots, c_n)$ for some P hence in $A[x_1, \dots, x_n]$ is also equal to $\prod_{j=1} b(x_j)$. QED.

<u>Lemma</u>: Let $\varphi(X) \in A[X]\Big/\Big(\prod_{i=1}^{n}(X-\lambda_i)\Big)$. Then φ is a unit iff $\varphi(\lambda_i) \in A^*$ for all i.

<u>Proof</u>: $\Longrightarrow$ obvious

$\Longleftarrow$

$$N_{A[x]/A}\,\varphi = \prod_{i=1}^{n}\varphi(\lambda_i) \in A^*$$

As norm is determinant of multiplication it follows that $\varphi \in A[x]^*$.

Now every line bundle is induced from $\mathcal{O}(1)$ on $P\check{V}$ for some ~~representation~~ representation V of G. If $V = \sum_{\chi \in \hat{G}} n_\chi \chi$ in $R(G)$, where $\underline{n} = (n_\chi)$ is a ~~sequence~~ family with $n_\chi \geq 0$ and all but a finite number are zero, then

$$Q(P\check{V}) \cong Q(pt)[X] \Big/ \prod_\chi (X - c_1(\chi))^{n_\chi} \qquad X = c_1(\mathcal{O}(1))$$

Thus the operation $\alpha : K_G \longrightarrow Q^*$ is the same as an element

$$\varphi(X) \in \varprojlim_{\underline{n}} Q(pt)[X] \Big/ \prod_\chi (X - c_1(\chi))^{n_\chi}$$

such that $\quad \varphi(c_1(\chi)) \in Q(pt)^* \quad$ for all $\chi \in \hat{G}$.

Cartier's method of working with such a φ:
Consider the category of $Q(pt)$-algebras and the functor to sets given by

$$D(A) = \left\{ a \in A \;\middle|\; \exists\, \underline{n} \quad \prod_\chi (a - c_1(\chi))^{n_\chi} = 0 \right\}.$$

$$\cong \varinjlim_{\underline{n}} \mathrm{Hom}_{Q(pt)\text{-algs.}} \left(Q(pt)[X] \Big/ \left(\prod_\chi (X - c_1(\chi))^{n_\chi} \right), A \right).$$

Thus φ is the same thing as a natural transformation from D to $\mathbb{G}_m$, i.e. from $D(A)$ to A^*. Therefore we have

Proposition:

$$\mathrm{Hom}^{\otimes}(\Omega_G, Q)' \cong \mathrm{Hom}^{+}(K_G, Q^*)$$

$$\cong \mathrm{Hom}(\mathrm{Pic}_G, Q^*)$$

$$\cong \mathrm{Hom}(D, \mathbb{G}_m) \quad \text{as functors from } Q(pt)\text{-algs to sets}$$

$$\cong (Q^{(pt)}\{X\})^*$$

where

$$Q(pt)\{X\} = \varprojlim_{n} Q[X]\Big/\Big(\prod_{x}(X - c_1(x))^{n_x}\Big)$$

$$\cong \varprojlim_{V} Q(\mathbb{P}\check{V})$$

Tensor product of line bundles defines a map $\mathbb{P}\check{V} \times \mathbb{P}\check{W} \longrightarrow \mathbb{P}(\check{V}\otimes\check{W})$ and hence a map

$$\hat{Q}^{(pt)}[X]\Big/\Big(\prod_{x_1,x_2}(X - c_1(x_1 x_2))^{m_{x_1} n_{x_2}}\Big) \longrightarrow$$

$$\hat{Q}^{(pt)}[X]\Big/\Big(\prod(X - c_1(x))^{m_x}\Big) \otimes \hat{Q}^{(pt)}[X]\Big/\Big(\prod(X - c_1(x))^{n_x}\Big)$$

and so by passage to the limit a map

$$\Delta : Q(pt)\{X\} \longrightarrow Q(pt)\{X\otimes 1, 1\otimes X\}.$$

which is associative + commutative + hence is a generalized formal group law.

In terms of D we have a transformation

$$D_V(A) \times D_W(A) \longrightarrow D_{V\otimes W}(A)$$

and hence a group law on the functor D which we will denote by $*$.

Let

$$T = \varinjlim_{V} \operatorname{Hom}_{Q(pt)\text{-mod}}\left(Q(\mathbb{P}\check{V}),\ Q(pt)\right)$$

Then T is a commutative bigebra over $Q(pt)$ endowed with cogebra maps

$$\theta_\chi : Q(pt) \longrightarrow T$$

for each $\chi \in \hat{G}$. $\theta_\chi(\varphi(X)) = \varphi(c_1(\chi))$ for $\varphi(X) \in Q(\mathbb{P}\check{V})$. Thus one gets a map of bigebras

$$Q(pt)\left[\hat{G}\right] \longrightarrow T$$

and so a map of affine group schemes over $\operatorname{Spec} Q(pt)$

$$\operatorname{Spec} T \longrightarrow G \times \operatorname{Spec} Q(pt)$$

where we now denote by G the group scheme of multiplicative type over $\operatorname{Spec} \mathbb{Z}$ whose character group is $\hat{G}$. Thus

$$G = \operatorname{Spec} \mathbb{Z}[\hat{G}]$$

<u>Proposition:</u> If A is a $Q(pt)$-algebra, then an element of $(\operatorname{Spec} T')(A)$ is the same as a natural ring homomorphism $K_G \longrightarrow A \otimes_{Q(pt)} Q$.

<u>Proof:</u> An element ~~of~~ $(\operatorname{Spec} T')(A)$ is a homomorphism $\gamma : T' \longrightarrow A$ of $Q(pt)$-algebras and may be identified with an element γ' of $A\{X\}^*$ such that $\Delta\gamma' = \gamma' \otimes \gamma'$. This is the same as an operation $\mathrm{Pic}_G \longrightarrow (A \otimes_{Q(pt)} Q)^*$ ~~[illegible]~~ which is a homomorphism and hence the same as a ring homomorphism $K_G \longrightarrow A \otimes_{Q(pt)} Q$.

<u>Example 1):</u> $G = \{e\}$, $Q = H_*(\ ,\mathbb{Z})$. Then

$T' = H_*(\mathbb{P}^\infty; \mathbb{Z})$ = divided power algebra on 1-generator b.

If A is a $\mathbb{Z}$-algebra, then to give a homomorphism $T' \longrightarrow A$ is the same as giving elements $a_i \in A$ verifying

$$a_i a_j = \frac{(i+j)!}{i!\,j!} a_{i+j}$$

or setting $\quad \varphi(X) = \sum a_i X^i \qquad a_0 = 1$ we get an operation on line bundles

$$\tilde{\varphi}(L) = \sum a_i (c_1^H L)^i$$

such that $\quad \tilde{\varphi}(L_1 \otimes L_2) = \sum a_i \left(c_1^H L_1 + c_1^H L_2\right)^i$

$$= \sum_{j,k} a_{j+k} \frac{(j+k)!}{j!\,k!} (c_1^H L_1)^j (c_1^H L_2)^k = \tilde{\varphi}(L_1)\,\tilde{\varphi}(L_2).$$

Remark: If $F(X,Y)$ is a formal group law over a ring k, then the bigebra T of distributions on the formal group

$$T = \operatorname{Homcont}_{k\text{-modules}} (k[[X]], k)$$

is ~~the~~ coordinate ~~ring~~ of an affine group scheme Spec T over k. One checks that as predicted by Cartier duality a point of Spec T with values in a k-alg A is the same thing as a homomorphism of the formal group into $\hat{\mathbb{G}}_m$ over Spec A.

Note that this dual is quite different from the Tate p-divisible dual and the latter appears to be potentially more interesting.

2): $G = \{e\}$, $Q = K$. Then

$$\mathbf{T} = \mathrm{Hom}\,(K(P^\infty), \mathbb{Z}) = \bigoplus_n \mathbb{Z}\binom{D}{n}.$$ Thus a point of T with values in A is a binomial power series

$$\sum \binom{a}{n} X^n = (1+X)^a$$

with all its coefficients in A. Hence for $A = \mathbb{Z}$ any a works and we get the operator ψ^a.

3): Assume that $[c_1(\chi) - c_1(\chi')]^* \in Q(pt)^*$ for $\chi \neq \chi'$. Then by the Chinese remainder theorem for A a $Q(pt)$-alg.

$$A[X]\Big/\Big(\prod_\chi (X - c_1(\chi))^{n_\chi}\Big) \cong \prod_\chi A[X]\Big/\big((X - c_1(\chi))^{n_\chi}\big)$$

so

$$A\{X\} = \prod_\chi A[[X - c_1(\chi)]].$$

Equivalently

$$D(A) = \coprod_\chi D_\chi(A) \qquad \text{for the } * \text{ operation}$$

where $D_\chi(A) = \{a \mid \exists n \ (a - c_1(\chi))^n = 0\}$. Then in fact as abelian group functors we have

$$D(A) \cong \hat{G} \times D_0(A)$$

where $D_0(A)$ is endowed with a formal group law in the usual sense. Hence as topological Hopf algebras we have

$$Q(pt)\{X\} = Q(pt)^{\hat{G}} \hat{\otimes} Q(pt)[[X]]$$

To show that $Q(pt)\{X\} \cong Q(pt)^{\hat{G}} \otimes Q(pt)[[X]]$ as topological Hopf algs. 7

Example: Let $Q(pt) = \mathbb{F}_2$, let $G = \mathbb{Z}/2\mathbb{Z}$ and $c_1 : (\mathbb{Z}/2\mathbb{Z})^\wedge \longrightarrow \mathbb{F}_2$ be the only possible additive isomorphism. Then for any $\mathbb{F}_2$-algebra A we have

$$D(A) = \{ a \mid \exists n \ (a^2-a)^n = 0 \}.$$

Let the group law be given by

$$a * b = a + b.$$

Observe this is legitimate since

$$\left((a+b)^2 - (a+b)\right)^n = \left((a^2-a) + (b^2-b)\right)^n$$

and since $a - c_1(\chi_1)$ nilpotent order n, $b - c_1(\chi_2)$ nilpotent of order m $\Longrightarrow$ $a+b - c_1(\chi_1 \otimes \chi_2) = (a - c_1(\chi_1)) + (b - c_1(\chi_2))$ nilpotent of order $n+m-1$.

$$D(A) \simeq \hat{\mathbb{G}}_a(A) \times (\underline{\mathbb{Z}/2\mathbb{Z}})(A)$$
$$a \longmapsto \left((a^2-a), \lim_n a^{2^n}\right)$$

where $\underline{\mathbb{Z}/2\mathbb{Z}}$ is the constant group scheme

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{G}_a \longrightarrow \mathbb{G}_a$$
$$x \longmapsto x^2 - x$$

The inverse isomorphism is given by $(x, y) \longmapsto (x + x^2 + x^4 + \cdots + y)$

It follows that as topological Hopf algebras we have

$$\mathbb{F}_2\{X\} \cong \mathbb{F}_2[[Y]] \otimes \mathbb{F}_2[[T]]/(T^2-T) \qquad \Delta Y = Y \otimes 1 + 1 \otimes Y$$
$$\Delta T = T \otimes 1 + 1 \otimes T$$

Note that $\mathbb{F}_2^{\widehat{\mathbb{Z}/2\mathbb{Z}}} = \mathbb{F}_2^{\mathbb{Z}/2\mathbb{Z}} = \mathbb{F}_2 1 \oplus \mathbb{F}_2 \chi \qquad \chi^2 = \chi \qquad \Delta\chi = \chi \otimes 1 + 1 \otimes \chi$

~~[illegible]~~ and therefore

~~[illegible]~~

~~as Hopf algebras~~ $\mathbb{F}_2[[T]]/(T^2-T) \cong \mathbb{F}_2^{\widehat{\mathbb{Z}/2\mathbb{Z}}}$ just as we conjectured.

Let $k = \mathbb{Q}(pt)$ to simplify writing, and let

$$1 = \sum_{\chi \in \hat{G}} \delta_\chi$$

be the decomposition of 1 with respect to the isomorphism

$$k\{X\} \cong \prod_\chi k[[X - c_1(\chi)]]$$

As an idempotent in $k[[Y]]$ is determined by its reduction modulo the ideal (Y), it follows that δ_χ is the unique <u>idempotent</u> element of $k\{X\}$ such that

$$\delta_\chi(c_1(\chi')) = \begin{cases} 1 & \chi' = \chi \\ 0 & \chi' \neq \chi \end{cases}$$

Interpreting elements of $k\{X\}$ as natural transformations $D(A) \longrightarrow A$ we have that

$$\delta_\chi : D(A) \longrightarrow A$$

is the unique natural transformation such that

$$\delta_\chi(a)^2 = \delta_\chi(a)$$

$$\delta_\chi(c_1(\chi')) = \begin{cases} 1 & \chi = \chi' \\ 0 & \chi \neq \chi'. \end{cases}$$

This argument clearly also shows that for any k-algebra k' δ_χ in $k'\{X\}$ is characterized by the same properties. Thus we have

$$\delta_\chi(a * x) = \sum_{\chi_1 \otimes \chi_2 = \chi} \delta_{\chi_1^*}(a) \cdot \delta_{\chi_2}(x)$$

for $a \in D(k')$ since as natural transformations of $x \in D(A) \longrightarrow A$ for any k'-algebra A, they are both idempotent and have the same values for $x = c_1(\chi)$, by a similar argument which shows

$$\delta_\chi(a * c_1(\chi')) = \delta_{\chi \otimes (\chi')^{-1}}(a).$$

Therefore

$$\boxed{\Delta \delta_\chi = \sum_{\chi_1 \otimes \chi_2 = \chi} \delta_{\chi_1} \otimes \delta_{\chi_2}}$$

In other words we have a map of topological Hopf algebras

$$\begin{aligned} k^{\hat G} &\longrightarrow k\{X\} \\ f &\longmapsto \sum f(\chi)\,\delta_\chi \end{aligned}$$

which is a section of the map in the opposite direction given by $\varphi \longmapsto (\chi \mapsto \varphi(c_1(\chi)))$.

I Claim we have the following split exact sequence of abelian group functors

$$\begin{tikzcd} 0 \arrow[r] & D_0(A) \arrow[r, "i"'] & D(A) \arrow[r, "\pi"'] & \underline{\hat G}(A) \arrow[r] \arrow[l, bend right, "s"'] & 0 \end{tikzcd}$$

where $D_0(A)$ is the set of nilpotent elements of A with group law $*$, i in the inclusion, $\underline{\hat G}$ is the "discrete" group scheme associated to the group $\hat G$ i.e.

$$\underline{\hat{G}}(A) = \left\{ \text{partitions} \quad 1 = \sum_{x} a_x \;\middle|\; \begin{matrix} a_x \in A & a_x a_{x'} = 0 \quad x \neq x' \\ a_x = 0 & \text{almost all } x \end{matrix} \right\}$$

$$= \mathrm{Hom}(\operatorname{Spec} A, \hat{G}) = \text{locally constant fns on Spec}A \text{ with values in } \hat{G}$$

and where

$$\pi(a) = \left(1 = \sum{}' \delta_x(a)\right)$$

$$s\left(1 = \sum a_x\right) = \sum_{x}{}' a_x c_1(x).$$

s is well-defined because in fact if $z = \sum a_x c_1(x)$, we have

$$\prod_{x \ni a_x \neq 0} (z - c_1(x)) = 0$$

as one sees locally on Spec A. Given a partition $1 = \sum a_x$

$$A \xrightarrow{\sim} \prod_{x} A_{a_x}$$

$$\delta_x\left(\sum{}' a_x c_1(x)\right) = \delta_x(c_1(x')) = \left\{ \begin{matrix} 1 \\ 0 \end{matrix} = a_x \quad \text{in } A_{a_{x'}} \right.$$

for each $x' \Longrightarrow \pi s = id$. Finally we calculate $\operatorname{Ker} \pi = \{a \mid \delta_x(a) = 0 \text{ for all } x \neq 1\}$. But this means that the map $k\{X\} \longrightarrow A$ sending X to a has δ_x in its kernel for $x \neq 1$, hence it factors through $k[[X]] \longrightarrow A$, which is true iff a is nilpotent.

Thus from the exact sequence we have

$$D(A) \simeq D_0(A) \times \underline{\hat{G}}(A)$$

or an isomorphism of topological Hopf algebras

$$\boxed{k\{X\} \simeq k^{\hat{G}} \,\hat{\otimes}\, k[[X]]}$$

where $\Delta X = F(X\otimes 1, 1\otimes X)$ is an ordinary formal group law.

We can now read off the group-like elements in $k\{X\}$. First of all a group-like in $k^{\hat{G}}$ is the same as a homomorphism $\hat{G} \longrightarrow k^*$, or the same as a point $^{G(k)}$ of the algebraic group G with values in k. Thus

Proposition: 1) If $c_1(X) - c_1(X') \in Q(pt)^* = k^*$ for $X \neq X'$, then to each element $g \in G(k)$, there is a unique natural ring homomorphism $\theta_g : K_G \longrightarrow Q$ such that

$$\theta_g(X) = X(g) \qquad \text{~~over pt~~ in } Q(pt)$$

$$\theta_g(\mathcal{O}(1)) = 1$$ if $\mathcal{O}(1)$ is the canonical line bundle on $\mathbb{P}^n$ with trivial G-action.

2) If in addition $k \supseteq \mathbb{Q}$ and $\lambda \in k$, then there is a unique operation $ch_\lambda : K_G \to Q$ such that

$$ch_\lambda(X) = 1 \quad \text{in} \quad Q(pt)$$

$$ch_\lambda(\mathcal{O}(1)) = 1 + \lambda c_1(\mathcal{O}(1)) + \cdots\cdots$$

if $\mathcal{O}(1)$ is the line bundle on $\mathbb{P}^n$ with trivial G-action.

3) Under the hypotheses of 1)+2) any ring hom $K_G \to Q$ is is the extension of the homomorphism

$\Theta_g \cdot ch_\lambda : Pic_G \longrightarrow Q$ given by

$$(\Theta_g \cdot ch_\lambda)(L) = \Theta_g(L)\cdot ch_\lambda(L).$$

Problem: Can ~~[illegible]~~ you generalize your old result that $\Omega\otimes Q$ is calculable in terms of $K\otimes Q$ to obtain the localization $\Omega\otimes Q\left[\frac{1}{c_1(x)-c_1(x')}\right]_{x\neq x'}$ from K. ~~Note that the corresponding localization of K [illegible] namely~~

$$\cancel{K\otimes Q\left[\frac{1}{c_1^K(x)-c_1^K(x')}\right]_{x\neq x'} = K\left[\frac{1}{x-x'}\right]_{x\neq x'}\otimes Q}$$

~~might be zero, e.g.~~ $\cancel{G = \mathbb{Z}/6\mathbb{Z} \qquad R(G) = \mathbb{Z}[T]/(T^6-1)}$

$$\cancel{R(G)\otimes Q = Q\times Q\times Q[\omega]\times Q[\omega]}$$

$$\cancel{T\mapsto 1 \quad T\mapsto -1 \quad T\mapsto \omega \quad T\mapsto \omega^2}$$

~~In any case one sees that different [illegible] powers~~ $\cancel{T^i}$ ~~becomes the same in each factor hence~~

$$\cancel{R(G)\left[\frac{1}{T^i-T^j}\right]_{\substack{i\neq j\\ 0\le i<6}} = 0}$$

Note that $K\otimes Q\left[\frac{1}{c_1^K(x)-c_1^K(x')}\right]_{x\neq x'} = K\left[\frac{1}{x-x'}\right]_{x\neq x'}\otimes Q$ is ~~[illegible]~~ identically zero for the finite group $(\mathbb{Z}/2\mathbb{Z})^2$, since $R(G)$ = group ring $Q[(\mathbb{Z}/2\mathbb{Z})^2] = Q\times Q\times Q\times Q$ and in each factor two different elements of $(\mathbb{Z}/2\mathbb{Z})^2$ coincide

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Let Q be a theory as above such that $c_1(X) - c_1(X') \in Q(pt)^*$ and $\mathbb{Q} \subset Q(pt) = k$. Then the "formal group law" is known once one knows the character or equivalently the logarithm for the formal group law induced on D_0. One has necessarily that

$$\ell(X) = \sum_{n \geq 0}{}' P_{n-1} \frac{X^n}{n}$$

where $P_{n-1} = [\mathbb{P}_{n-1}]$ in k. In effect by the projective bundle formula one has

$$P_{n-1} = \operatorname{res}\left[\frac{d\ell(Z)}{Z^n}\right].$$

(Question: Does ω always $\exists$ for these formal groups?)

Thus to the theory Q we have the invariants

$$\begin{cases} c_1(X) \in Q(pt), & c_1(1) = 0 \\ P_n \in Q(pt), & P_n = 1 \end{cases}$$

which makes reasonable the following

<u>Conjecture</u>: $\Omega_G(pt)\left[\frac{1}{c_1(X) - c_1(X')}\right]_{X \neq X'} \otimes \mathbb{Q} \cong \mathbb{Q}\left[P_n, c_1(X), \frac{1}{c_1(X) - c_1(X')}\right]$

for $n > 0$, $X \neq 1$, $X \neq X'$.

To prove this conjecture we shall have to construct a theory Q with $Q(pt)$ the given candidate.