§1. A universal property of the complex cobordism.

If $X$ is a space, let $\Omega^s(X) = \oplus_{s \in \mathbb{Z}} \Omega^s(X)$ be the complex cobordism ring of $X$. It is the cohomology theory associated to the Thom spectrum of the infinite unitary group. When $X$ is a manifold $\Omega^*(X)$ may be described as follows:

$$\Omega^*(X) = \text{equivalence classes of proper complex-oriented maps } f: Z \to X$$

A complex orientation on $f$ is a reduction of the stable normal bundle $\nu$ of $f$ from $O$ to $U$.

Two such maps $f: Z \to X$ and $f': Z' \to X$ are equivalent if they are cobordant, that is if there is a complex-oriented map $h: W \to X \times \mathbb{R}$ of and transversal cartesian squares:

$$\begin{array}{ccc}
Z & \xrightarrow{\alpha} & W \\
\downarrow f & & \downarrow h \\
X & \xrightarrow{\alpha} & X \times \mathbb{R}
\end{array} \quad \begin{array}{ccc}
Z' & \xrightarrow{\alpha'} & W' \\
\downarrow f' & & \downarrow h' \\
X & \xrightarrow{\alpha'} & X' \times \mathbb{R}
\end{array}$$

such that the induced isomorphisms of $\alpha f$ (resp. $\alpha' f'$) with $\alpha^* \nu$ (resp. $\alpha'^* \nu$) are compatible with the orientation.
Structure of $\Omega^*(X)$:

Sum:

\[ [Z \xrightarrow{f} X] + [Z' \xrightarrow{f'} X] = [Z \cup Z' \xrightarrow{f+f'} X]. \]

Then

\[ \Omega^*(X) = \bigoplus_{\varphi \in \mathbb{Z}} \Omega^\varphi(X) \]

\[ \Omega^\varphi(X) \text{ subset of elements represented by maps } \]
\[ f: Z \to X \text{ of codim } \varphi \text{ where } \]
\[ \text{codim}_Z f = \dim \nu_f = \dim k(Z \times_f X) - \dim Z \]
\[ = -\dim_Z f. \]

Moreover

zero: \[ [f: \emptyset \to X]. \]

Inverse of $f: Z \to X$ is the same map but when
the new orientation is defined as follows: Given an isomorphism
of $\nu_f$ with stable complex bundle $E$, then new isomorphism
is $\nu_f + 1 \cong E + C \cong E + C$.

Inverse image: Given $u: X \to Y$ and $f: Z \to Y$
proper oriented, without changing the element of $\Omega(X)$ represented
by $f$ we may move it to be transversal to $E_f$ and then form
the fibre product $\square_{X \times Y} Z \setminus E_f$. 

Theorem
Using the canonical isomorphism $u^* f = v_f$, one obtains an orientation of $f'$ so we define

$$u^* [\begin{array}{ccc} Z & f & Y \\ X & x & X' \end{array}] = [\begin{array}{ccc} X & x & Z' \\ Y & f' & X' \end{array}]$$

Then

$$u^*: \Omega(Y) \rightarrow \Omega(X)$$

is well-defined and compatible with addition and makes $\Omega$ into a contravariant functor from manifolds to abelian groups. Let

$$u: \Omega^n(Y) \rightarrow \Omega^n(X).$$

Given $\phi: X \rightarrow Y$ proper and oriented, get

$$u_*: \Omega(X) \rightarrow \Omega(Y)$$

$$u_* [\begin{array}{ccc} Z & f & X \\ Y & \end{array}] = [\begin{array}{ccc} Z & u_* f & Y \\ X & \end{array}]$$

If $u$ is of dimension $q$, then

$$u: \Omega^k(X) \rightarrow \Omega^{k+q}(Y).$$

Products:

$$[\begin{array}{ccc} Z & f & X \\ \end{array}] \cdot [\begin{array}{ccc} Z' & f' & X' \\ \end{array}] = [\begin{array}{ccc} Z \times Z' & f \times f' & X \times X' \end{array}]$$

$$\Omega^q(X) \otimes \Omega^q(X') \rightarrow \Omega^{q+q}(X \times X')$$

It makes $\Omega^*(X)$ into a commutative ring with

$$\Delta: \Omega^*(X) \otimes \Omega^*(X) \rightarrow \Omega^*(X \times X)$$

where $\Delta: X \rightarrow X \times X$ is the diagonal map.
Properties of this structure:

\[ X \mapsto \Omega(X) \text{ with } f \mapsto f^* \text{ is a contravariant functor on the} \]
\[ \text{category of } C^\infty \text{ manifolds and } C^\infty \text{ maps}. \]

1) \[ X \mapsto \Omega(X) \text{ with } f \mapsto f_* \text{ is a covariant functor on the} \]
\[ \text{category with objects } C^\infty \text{ manifolds and having proper-oriented maps for morphisms}. \]

2) \[
\begin{array}{c}
X' \xrightarrow{g'} X \\
\downarrow f' \downarrow f \\
Y' \xrightarrow{g} Y
\end{array}
\]
is a cartesian square of manifolds, and if \( f, g \) are transversal
\( f \circ g : X \times Y' \to Y \times Y \) is transversal to \( \Delta_Y \) \( \) if \( f \) is proper
\( g' \) is oriented and if \( f'_* \) is oriented by means of the canonical
transformation \( g^* \nu_f = \nu_{f'_*} \) then
\[ g^* f^* = f'_* g'^* \]

3) \( f, g : X \to Y \) are maps and \( f \circ g \), then \( f^* = g^* \).
\( \text{(equivalently if } f : X \to Y \text{ is a homotopy equivalence, then} \)
\( f^* : \Omega(Y) \to \Omega(X) \text{ is bijective).} \)

4) (Addition). If \( X = X_1 \sqcup X_2 \), then
\[
\left( w_1^*, w_2^* \right) : \Omega(X) \xrightarrow{\sim} \Omega(X_1) \times \Omega(X_2)
\]
\[ \text{and } f^* \text{ is compatible with addition}. \]
5) (Multiplication). Given \( f: X \to Y, f': X' \to Y' \), properly oriented,
then
\[
(f \circ f')(x \otimes x') = \sqrt{f_* x \otimes f'_* x'}
\]

Proposition (i): Let \( \Omega \) be a contravariant functor on the category of manifolds with values in \( \text{Set} \) endowed with a Gysin homomorphism satisfying axioms 1), 2), 3). If \( x \in \Omega(\text{pt}) \), then there is a unique natural transformation \( \Theta: \Omega \to \Omega \) of functors compatible with Gysin such that \( \Theta(1) = x \), where \( 1 \in \Omega(\text{pt}) \) denotes the element \([id: \text{pt} \to \text{pt}]\).

(ii) Let \( \Omega \) be a functor as in (i) but with values in \( \text{Ab} \), \( \text{Gr} \) and suppose \( \Omega \) also satisfies (4). Then \( \Theta: \Omega \to \Omega \) is compatible with addition.

(iii) Let \( \Omega \) be a contravariant functor on \( \text{Man} \) with values in \( \text{Rings} \) endowed with a Gysin homomorphism for properly oriented maps. If \( \Theta \) satisfies 1)-5), then there is a unique natural transformation \( \Theta: \Omega \to \Omega \) compatible with ring structure and Gysin homomorphism.

Proof: (i): Given \([Z \to X]\) an element of \( \Omega(X) \), we of course have to define
\[
\Theta[Z \to X] = f_* \pi_Z^* x
\]
where \( \pi_Z: Z \to X \) is the unique map. To
Note that the $\Sigma \Omega$ homomorphism is not a ring homomorphism in $\Omega(Y)$-homomorphism in virtue of the projection formula

$$f_x \cdot x \cdot f^* y = f_x \cdot y$$

which may be deduced from the following diagram:

$$\begin{array}{c}
X \xrightarrow{(id, f)} X \times Y \\
\downarrow f \downarrow f \times \text{id} \\
Y \xrightarrow{\Delta} Y \times Y
\end{array}$$

$$f_x \cdot (id, f)^* (x \boxtimes y) = f_x \cdot (x \cdot f^* y)$$

$$\Delta^*(f \times id)^* (x \boxtimes y) = f_x \cdot x \cdot y$$

We have the formula

$$\begin{cases} 
\alpha \cdot \beta = \Delta^* (\alpha \boxtimes \beta) \\
\alpha \boxtimes \beta = pr_1^* \alpha \cdot pr_2^* \beta
\end{cases}$$

Showing that the ring structure on $\Omega$ is equivalent to the external product $\boxtimes$.

Of course, the existence of identity together with an element $1 \in \Omega(pt)$, such that $1 \boxtimes \alpha = \alpha, \alpha \boxtimes 1 = \alpha$. 
show well-defined suppose \([Z \xrightarrow{f} X] = [Z' \xrightarrow{f'} X']\). Then have

\[
\begin{array}{c}
Z \xrightarrow{f_0} W \leftarrow d_1 \ x' \\
\downarrow f^* \quad \downarrow h \quad \downarrow f' \\
X \leftarrow X \times R \leftarrow \hat{R}
\end{array}
\]

\[
\begin{align*}
f_*^* \pi_Z^* \alpha &= f_*^* f_0^* \pi_W^* \alpha \quad = i_*^* h_*^* \pi_W^* \alpha \\
&= i'_* h_*^* \pi_W^* \alpha \quad \text{(homotopy axiom 3)}.
\end{align*}
\]

Thus we have defined \(\Theta: \Omega(X) \rightarrow Q(X)\) for every \(X\).

Given \(u: X' \rightarrow X\) represented as \([Z \xrightarrow{f} X]\) where \(f\) is transversal to \(u\) and form

\[
\begin{array}{c}
Z' \xrightarrow{u'} Z \\
\downarrow f' \quad \downarrow f \\
X' \xrightarrow{u} X
\end{array}
\]

Then

\[
\Theta(u^* \xi) = \Theta [Z' \xrightarrow{f'} X'] = f'_* \xi_{Z'} = f'_* u'^* \xi_Z = u^* f_* \xi_Z = u^* \Theta [Z \xrightarrow{f} X] = u^* \Theta \xi.
\]
Similarly by axiom 1), \( \Theta \) is compatible with \( f_* \), proving (i).

(ii). Recall \([Z \xrightarrow{f} X] + [Z' \xrightarrow{f'} X] = [Z \cup Z', h_f' \xrightarrow{} X] \).

\[
\Theta(y + y') = (f + f')_* (\frac{\Pi^*}{Z \cup Z'} x)
\]

Now:
\[
(m_1^*, m_2^*): Q(Z \cup Z') \rightarrow Q(Z) \times Q(Z')
\]
and
\[
(m_1^*, m_2^*) (\frac{\Pi^*}{Z \cup Z'} x) = (\frac{\Pi^*}{Z} x, \frac{\Pi^*}{Z'} x)
\]

Let \( \beta = (m_1^* \frac{\Pi^*}{Z} x + m_2^* \frac{\Pi^*}{Z'} x) \in Q(Z \cup Z') \)

Then \( m_1^* \beta = m_1^* m_1^* \frac{\Pi^*}{Z} x + m_2^* m_2^* \frac{\Pi^*}{Z'} x \)

\[
\begin{array}{c}
Z \xrightarrow{id} Z \\
\downarrow \text{id} \quad \downarrow m_1 \\
Z \cup Z' \xrightarrow{\text{cartesian}} \quad m_1^* m_1^* x = 0
\end{array}
\]

\[
(\text{because } Q(\phi) \xrightarrow{} Q(\phi) \times Q(\phi) \text{ for an abelian } \phi \rightarrow Q(\phi) = 0).
\]

Thus \( m_1^* \beta = \frac{\Pi^*}{Z} x \) and similarly

\[
\begin{array}{c}
Z \xrightarrow{id} Z \\
\downarrow \text{id} \quad \downarrow m_2 \\
Z \cup Z' \xrightarrow{\text{cartesian}} \quad m_2^* m_2^* x = 0
\end{array}
\]

\[
\text{(because } Q(\phi) \xrightarrow{} Q(\phi) \times Q(\phi) \text{ for an abelian } \phi \rightarrow Q(\phi) = 0).
\]

Thus \( m_1^* \beta = \frac{\Pi^*}{Z} x \) and similarly

\[
\begin{array}{c}
\text{So}
\end{array}
\]

\[
\text{So}
\]
Here assumed \( f_* \) is a homomorphism. Better method is as follows: Take

\[ \beta = m_1 f_* \pi_2^* \alpha + m_2 f'_* \pi_{2'}^* \alpha \in Q(X + X) \]

Then note that

\[ \operatorname{in}^* \beta = f_* \pi_2^* \alpha = \operatorname{in}^* f_* \operatorname{in}^* \pi_2^* \alpha = \operatorname{in}^* (f_* f'_*) \pi_{2+2'}^* \alpha \]

Similarly

\[ \operatorname{in}^* \beta = \operatorname{in}^* (f + f')_* \pi_{2+2'}^* \alpha \]

Now apply the map

\[ Q(X + X) \xrightarrow{\operatorname{V}_*} Q(X) \]
(iii). As 1 ∈ \( Q(\mathfrak{p}t) \) is the identity element of this ring, \( \Theta \) must send 1 to the identity element 1 of \( Q(\mathfrak{p}t) \). By the preceding we therefore has a unique natural trans of \( \Theta : \mathcal{L} \rightarrow \mathcal{Q} \) compatible with \( \text{Lyshin} \) and additions.

To show \( \Theta \) is a ring hom., it is enough to prove that

\[
\Theta([Z \xrightarrow{f} X] \boxtimes [Z' \xrightarrow{f'} X'])
\]

It is 

\[
\Theta([f \ast 1_Z] \boxtimes [f' \ast 1_{Z'}])
\]

but

\[
\Theta([Z \xrightarrow{f} X] \boxtimes [Z' \xrightarrow{f'} X'])
\]

\[
= \Theta([Z \times Z' \xrightarrow{f \times f'} X \times X'])
\]

\[
= (f \times f')_\ast (1_Z \boxtimes 1_{Z'}) \overset{5)}{=} (f \ast 1_Z) \boxtimes (f' \ast 1_{Z'})
\]

Additional to preceding section

Examples of (complex-) oriented maps.

(i) An embedding whose normal bundle is endowed with a complex structure.

(ii) A submersion \( f: X \to Y \) such that the tangent bundle along the fibers is endowed with a complex structure (hence a family of almost complex manifolds parameterized by \( Y \)).

(iii) Any map of complex manifolds has a canonical complex orientation.

Proposition: Let \( i: Y \to X \) be an embedding endowed with orientation (thus the normal bundle of \( i \) is endowed with an isomorphism \( E + N \cong \mathbf{C}^n \)). Then let \( j: U \to X \) be the complement of \( Y \). Then

\[
\Omega(Y) \xrightarrow{j^*} \Omega(X) \xrightarrow{i^*} \Omega(X_u)
\]

is exact.

Proof: \( j^*i^* = 0 \) is clear.

Conversely if \( \alpha \in \text{Ker} j^* \) represent \( \alpha \) by \( f_\ast 1 \), \( f: Z \to X \). Then \( j^*\alpha \) is represented by \( f^{-1}U \).

As \( j^*\alpha = 0 \), \( f^{-1}U \to U \) is cobordant to zero, e.g. \( \exists W \xrightarrow{h} U \to \mathbb{R} \) proper oriented with \( h^{-1}(\mathbb{R} \times 0) \to U \)

such that \( f^{-1}U \to U \) and with \( h^{-1}(\mathbb{R} \times 1) = \emptyset \). Let \( N \) be a tubular
neighborhood of $Y$ in $X$ and let $\varphi$ be a nice distance-squared function from $Y$. Using $W \rightarrow (1, \infty) \times (0, \infty)$, one gets a cobordism from $Z \rightarrow X$ to something whose image lies in $N$. Thus may assume $f: Z \rightarrow N$, but then we may homotope $Z$ down into $Y$, so may assume $f: Z \rightarrow Y$, whence $f$ is oriented since $i$ is oriented.

\[
\begin{array}{c}
Z \\
\downarrow i \\
Y \\
\downarrow \\
X
\end{array}
\]

and so $f_*(1) = i_*(g_*(1))$.

(Short def of $c_1$)

Then: Proposition: Let $E$ be a complex vector bundle over $X$.
Let $f: P(E) \rightarrow X$ be the projective bundle of lines in $E$.
Let $O(1)$ be the canonical line bundle.

\[
f_* \{ \text{holomorphic sections of } O(n) \} = S^n \hat{E}.
\]

Let $\mathfrak{g} = c_1(O(1))$. Then $\Omega(P(E))$ is a free $\Omega(X)$-module with basis $1, \mathfrak{g}, \cdots, \mathfrak{g}^{n-1}$.

Proof: $P(E) \xrightarrow{\pi} E \xrightarrow{\pi} X \xrightarrow{i} E$

$E = \{ (l, v) \mid l \text{ line in } E, v \in l \}$.
Let \( E \) be a complex vector bundle over \( X \) of dimension \( n \) and let \( i : X \to E \) be the zero section. Let \( s : X \to E \) be a section transversal to \( i \) and form the fiber product

\[
\begin{array}{ccc}
Y & \xrightarrow{y} & X \\
\downarrow y' & & \downarrow i \\
X & \xrightarrow{s} & E
\end{array}
\]

Then \( Y = s^{-1}(0) \) is an oriented submanifold of \( X \), in fact the normal bundle of \( Y \) is canonically isomorphic to \( i_\ast T^*_X E \). Moreover

\[
i^\ast_1 = i^\ast_1 = i^\ast_1 = i^\ast_1 = i^\ast_1.
\]

Call \( i^\ast_1 \) the "tuner class" of \( E \); we will see later that it is the \( n \)th Chern class of \( E \), denoted \( c_n(E) \). Its formation is compatible with base change.

**Proof of prop.** On \( PE \) we have exact sequence

\[
0 \to O(0) \to f^\ast E \xrightarrow{\pi} F \to 0
\]

**Lemma 1:** \( f_\ast c_{n-1}(F) \) is a unit in \( \Omega(X) \).

**Proof:** Let \( s \) be a generic section of \( E \) with zero section \( \overline{i}' : \overline{Y} \to X \). Let \( \overline{g} : \overline{Y} \to X \) be the complement of \( Y \), let \( \overline{i} : \overline{U} \to PE|\overline{U} \) be the section given by the line generated by the section \( s \) over \( U \). Then \( \overline{i}_\ast 1 = \overline{g}^\ast c_{n-1}(F) \...
In effect the image of $t$ is where the section of $j^*F^*E$ given by $s$ is contained in $O(-1)$, i.e., where the section $\pi s$ of $j^*F$ vanishes. To make a local calculation to convince yourself that this means that

$$
\begin{array}{c}
U \\ \downarrow t \\
\pi s \\
\end{array}
\xrightarrow{t}
\begin{array}{c}
P(j^*E) \\
\downarrow s \\
l_{section} \\
\end{array}
\xrightarrow{t}
\begin{array}{c}
P(g^*F) \\
\downarrow s \\
\end{array}
\xrightarrow{t}
\begin{array}{c}
\pi s \\
\end{array}
$$

is the correct diagram. So $t_\ast 1 = c_{n-1}(g^*F) = j^*c_{n-1}(F)$.

Thus

$$
j^\ast \left( f_\ast c_{n-1}(F) - 1 \right) = f_\ast' \left( j^\ast c_{n-1}(F) - 1 \right)
= f_\ast' t_\ast 1 - 1 = 0.
$$

So by previous prep

$$
f_\ast c_{n-1}(F) - 1 = \frac{c_{n-1}(F)}{f_\ast c_{n-1}(F) - 1}
$$

is nilpotent.

Thus $f_\ast c_{n-1}(F)$ is a unit.
Thus we have found a canonical element \( u(E) = \frac{e_{n-1}}{f'} \) such that \( f'_* u(E) = 1 \). It's compatible with base change.

So now let \( X'(E) \) denote scheme of \( E \), \( X' \subset X \).

Then it is an iterated succession of projective fibres so one gets a canonical element \( v(E) \in \Omega(D(E)) \) with \( f'_* v(E) = 1 \).

Claim: it suffices to prove this. Form \( \mathcal{P}(g^* E) \xrightarrow{f'} D(E) \).

\[
\Omega(P(E)) \xrightarrow{g^*} \Omega(P(E)) \\
\Omega(D(E)) \xrightarrow{g^*} \Omega(X)
\]

In effect, we know that \( g'_* = g^* \). Given \( z \in \Omega(PE) \), write \( g'^* z = \sum_{i=0}^{n-1} f'^* a_i (i')^i \).

Multiply by \( v(E) = f'^* v(E) \) and apply \( g'_* \), using that \( g'^* v(E') = 1 \), we get

\[
z = \sum_{i=0}^{n-1} g'_* \left[ (f'^* a_i) v(E') \right] i^i
\]

This shows that \( 1, i, \ldots, i^{n-1} \) span \( \Omega(P(E)) \) in \( \Omega(X) \) modulo similarly as \( g^* \) is injective, they are independent!
so now we can assume that \( E = L_1 + \ldots + L_n \) where the \( L_i \) are line bundles. Let \( E_\emptyset = L_1 + \ldots + L_n \) and use induction on \( \emptyset. \) Then

\[
\begin{array}{ccc}
\mathbb{P}L_n & \to & \mathbb{P}E \\
\times & \downarrow \scriptstyle{s} & \searrow \scriptstyle{j} \\
\mathbb{P}(E - L_n) & \to & \mathbb{P}(E_{n-1})
\end{array}
\]

By prop earlier

\[
\Omega(X) \xrightarrow{s*} \Omega(\mathbb{P}E) \xrightarrow{j*} \Omega(\mathbb{P}E_{n-1})
\]

is exact. By induction \( j* (\mathcal{O}_E) = \mathcal{O}_{E_{n-1}} \) is exact. As induction hypothesis \( j* \) surjective, so sequence is split exact and \( \Omega(\mathbb{P}E) \) has the basis \( s*1, 1, \ldots, 1, \ldots, 1. \)

\[\text{Let } H_i = \text{ the hyperplane } \mathbb{P}(E_1 + \ldots + E_i + \ldots + E_n) \subset \mathbb{P}E \]
or equivalently where the section \( \mathcal{O} \to f*E \otimes \mathcal{O}(1) \to f*L_i \otimes \mathcal{O}(1) \)

vanishes. Thus

\[ [H_i \to \mathbb{P}E] = c_1(f*L_i \otimes \mathcal{O}(1)) \]

and as

\[ s(x) = \bigcap_{i<n} H_i \implies s*1 = \prod_{i=1}^{n-1} c_1(f*L_i \otimes \mathcal{O}(1)) \]
Case 1: Assume all $L_i$ are trivial

Then $s \cdot 1 = c_1(O(1))^{n-1} = q^{n-1}$ and we are finished. Conclude also that $q^n = 0$.

$$\Omega^*(X \times \mathbb{P}^n) = \Omega^*(\mathbb{X}) \left[ \frac{i}{(q^{n+1})} \right]$$

Recall that any line bundle is induced from the bundle $O(1)$ on $\mathbb{P}^n$ for some $n$. Define elements $a_{kL} \in \Omega^*(\mathbb{P}^n)$ for $k \in \mathbb{N}$ by

$$c_1(O(1) \otimes O(1)) = \sum_{k \in \mathbb{N}} a_{kL} c_1(O(1))^{k-1}$$

One sees that $a_{kL}$ is independent of $n$, hence there is a power series $F(X, Y) \in \Omega^*(\mathbb{P}^n)[[X, Y]]$ uniquely determined by

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2))$$

for all line bundles $L_1, L_2$ over any manifold $X$. By checking the bundle $O(1) \otimes O(1) \otimes O(1)$ on $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, one sees that $F$ is associative over $\Omega^*(\mathbb{P}^n)$ i.e.

Claim: $F$ is a form.

$$F(X, 0) = F(0, X) = X$$

$$F(X, F(Y, Z)) = F(F(X, Y), Z)$$

$$F(X, Y) = F(Y, X)$$
The associativity results by calculating in two ways the bundle $\mathcal{O}(1) \otimes \mathcal{O}(1) \otimes \mathcal{O}(1)$ on $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$. One sees now that

$$F(x, y) = x + y + xy C(x, y)$$

hence in general

$$c_1(L_1 \otimes L_2) - c_1(L_2) = c_1(L_1)(1 + c_1(L_2)G(\cdot, \cdot))$$

unitotent hence

$$c_1(L_1 \otimes L_2) - c_1(L_2) = c_1(L_1) \cdot \text{unit}.$$

Returning to our proof, one has:

$$(\star) \quad \text{unit} \cdot c_1(f^* L_i \otimes \mathcal{O}(1)) = c_1(\mathcal{O}(1)) - c_1(f^* L_i)$$

$$\therefore \quad \mathbb{P} \cdot 1 = \text{unit} \cdot \prod_{i<n} [\mathcal{O}(1) - f^* c_1(L_i)]$$

$$\therefore \quad \text{one concludes that } 1, \varrho, \cdots, \varrho^{n-2}, \varrho^{n-1} \text{ is a basis for } \mathcal{O}(PE) \text{ as claimed, proving the theorem.}$$

Corollary: If $E = L_1 + \cdots + L_m$, then in $\mathcal{O}(PE)$ we have

$$\prod_{i=1}^m (\mathcal{O}(1) - f^* c_1(L_i)) = 0.$$
Proof: We know that \[ \prod_{i=1}^{\hat{n}} c_i(f^*\mathcal{L}; \Theta(1)) = 0 \]
so the result follows from (\textasteriskcentered).

Defn: If \( E \) is a complex bundle over \( X \), let \( \gamma = c_1(\Theta(1)) \) on \( PE \). Then the Chern classes \( c_i(E) \) are defined by

\[ f^* c_i(E) \gamma^{n-1} + \ldots + (-1)^{n-i} f^* c_{n-i}(E) = 0. \]

Proposition:
\[ c_n(E+E') = \sum_{i=0}^{\hat{c}} c_i(E)c_{n-i}(E') \]
\[ e_0(E) = 1. \]

If \( n = \dim E \), then
\[ c_n(E) = i^*(c_x 1) \quad \text{where} \quad i: X \to E \text{ zero}. \]

Proof: By passage to the associated flag bundles which induces an injective map on \( E \) we may assume that \( E \) and \( E' \) are split. The corollary on p.14 shows that \( f^* \text{injective} \)

\[ c_g(E) = g \text{th elementary symmetric fn in } c_1(L) \]

whence the first statement of the proposition follows. For the second we have \( E = L_1 + \ldots + L_n \) and set \( E_g = L_1 + \ldots + L_{g+1} \)
and let \( i: X \to E_g \) be the zero section of \( E_g \). Then
We have

\[ X \xrightarrow{i_0} E_{g-1} \xrightarrow{j} E_g. \]

So

\[ i_0^* \cdot i_0^* \cdot \chi = i_0^* \cdot \text{retraction}. \]

But

\[ j^*(f^* u) = j^*(f_0^* \cdot g^* w) = j^*(f_0^* \cdot \pi^* u) = (g \cdot f_0^* 1) \cdot u. \]

Therefore,

\[ i_0^* \cdot i_0^* \cdot \chi = i_0^* (f_0^* \cdot 1) \cdot i_0^* i_{g-1} \cdot i_0^* 1. \]

Induction

\[ c_i(L'_0) \quad c_{g-1}(E_{g-1}). \]

where \( L'_0 \) is \( E_g \) viewed as a bundle over \( E_{g-1} \).

**Proposition.** If \( i : Y \rightarrow X \) is an embedding with a complex structure on the normal bundle \( E \) of \( i \), then

\[ c_n(E) \cdot y \quad \text{all} \; y \in Y, \quad n = \dim E. \]

**Proof.** Let \( U \) be a tubular nbhd. of \( Y \) in \( X \); we may identify \( E \) with \( U \). Then we have a square...
\[ Y \xrightarrow{id} Y \]
\[ Y \xrightarrow{i} \mathcal{U} \xrightarrow{j} X \]

\[ i^* \gamma = i^* j^* c_y = i^* i^* \gamma \]

But \( i^* \pi : \mathcal{U} \rightarrow Y \) a retraction so

\[ \pi^* (i^* \pi^* \gamma) = i^* (i^* \pi^* \gamma) \]
\[ = i^* (c_y + 1) \pi^* \gamma \]
\[ = (i^* (c_y + 1)). \gamma \]

But by definition of identifying \( \mathcal{U} \) and \( \mathcal{E} \), we have

\[ i^* i^* 1 = c_\mathcal{E}(E). \]

Comments. Add a general discussion on the Euler class after proof of first proposition proving the \( i^* \gamma = (i^* \pi^*) \gamma \) formula for an embedding. Then prove projective bundle theorem for \( E = L, \) the where \( L \)'s are trivial to get the formal group laws, then general \( L \)'s and then general case.
Cartier style proofs of Lazard theorem

Lemma: Given a formal Lie group $G$ in $d$-variables over $R$ in the sense of Cartier and

\[ f: D \rightarrow G \]

a curve in the formal group. Then $f$ is isomorphism if and only if

\[ F^q f = 0 \text{ for all } q \geq 2 \]

Proof: \[ \hat{W}_R \] be the formal Witt group over $R$ which associates to each $R$-algebra $A$ the group of power series

\[ 1 + c_1 t + \ldots, \quad c_i \in A, \quad c_i \text{ nilpotent} \]

and all but a finite number of $c_i$ are zero. Given $f: D \rightarrow G$ there is a unique homomorphism $h: \hat{W}_R \rightarrow G$ such that

\[ h(1 + at) = f(a) \]

for all $a \in D(A)$ and all $R$-algebras $A$. In fact one defines $h$ by the formula

\[ h(1 + c_1 t + \ldots + c_n t^n) = f(\lambda_1) + \cdots + f(\lambda_n) \]

where the $\lambda_i$ satisfy

\[ 1 + c_1 t + \ldots + c_n t^n = \prod_{i=1}^n (1 + \lambda_i t) \]

By hypothesis

\[ (F^q f)(a x) = f(z_1) + \cdots + f(z_n) = 0 \]

where

\[ \prod (t - z_i) = t^d - x \quad \text{or} \quad \prod (1 + t z_i) = 1 - (-1)^d t \]
Thus we are told that
\[ h(1 + at^b) = 0 \]
if \( a \in D(\cdot) \) and \( \beta > 1 \). But in fact any element
\[ 1 + c_1 t + \cdots + c_n t^n \]
\( \in W(A) \) may be factored
\[ 1 + c_1 t + \cdots + c_n t^n = (1 + a_1 t)(1 + a_2 t^2) \cdots (1 + a_N t^N) \]
(enough to do this universally if \( \beta \) when \( A = R[[c_1, \ldots, c_n]]/(c_1, \ldots, c_n)^\beta \)
whence in fact one sees that \( A_R \) has a certain filtration
and the product is finite). As \( h \) is a homomorphism we
see that
\[ h(1 + c_1 t + \cdots + c_n t^n) = h(1 + c_1 t). \]
Thus
\[ f(\lambda_1) + f(\lambda_2) = h(1 + (\lambda_1 + \lambda_2) t + \lambda_1 \lambda_2 t^2) \]
\[ = h(1 + (\lambda_1 + \lambda_2) t) = f(\lambda_1 + \lambda_2) \]
so \( f \) is a homomorphism. \( \text{qed.} \)

\[ \text{Corollary: Suppose } R \text{ has characteristic } p \text{ and that } F \text{ is a typical group law of height } \infty. \text{ Then } F(x, y) = x + y. \]

\[ \text{Proof: } F \text{ gives rise to a formal group } G \text{ together with } \text{an isomorphism } F : D \to G. \text{ If typical, } F_0 = 0 \text{ (if } (b, p) = 1 \text{)} \]
But in char. \( p \), \( V_p F_0 = p \) on curves. By height \( \infty \), \( p = 0 \) and
so \( F_0 = 0 \) since \( V_p \) is injective. Thus by lemma, \( Y \) is a
homomorphism, so
\[ F(x, y) = F^\prime(x + y, y) = x + y. \text{ \( \text{qed.} \) } \]
Proposition: Let \( R \) be a ring and let \( F \) be a group law such that \( p_F(x) \equiv 0 \mod pR \) for all primes \( p \). Then \( F \) is isomorphic to the additive law.

Proof (using Cartier theory). We let \( G \) be the formal group defined by \( F \). We shall construct a coordinate \( \gamma : D \rightarrow G \) such that \( F^\gamma = 0 \) for all primes \( p \). It follows from the preceding lemma that \( \gamma \) is a homomorphism, hence it gives an isomorphism of \( G \) with \( D \).

We suppose \( \gamma \) has been constructed so that \( F^\gamma = 0 \) for all primes \( p \); we will show how to modify \( \gamma \) in degrees \( \geq p \) so as to satisfy in addition \( F_p\gamma = 0 \). It's clear that the limit of these \( \gamma \)'s is what we need.

Suppose we can find a curve \( \mu : D \rightarrow G \) such that

\[
F_p\gamma = p\mu
\]

Then if \( q \) is a prime \( < p \)

\[
pq_q\mu = F^\mu = Fp\gamma = Fp\gamma = 0,
\]

hence the curve

\[
\mu' = \mu - \left( \frac{1}{q} \right) F^\mu
\]

is defined. Now

\[
\nu = \nu' = p\mu - \left( \frac{1}{q} \right) F^\nu = p\mu = F\gamma
\]
Moreover, if \( g' \) is a prime number \(< g \) and if we know that
\[ F_{g'} \mu = 0, \]
then
\[ F_{g'} \mu' = F_{g'} \mu - \left( \frac{1}{g'} \right) F_{g} F_{g'} \mu = 0. \]

Therefore by an evident induction we may modify \( \mu \) and so assume that
\[ F_{g} \mu = 0 \quad \text{all primes } g < p. \]

Then set
\[ g' = g - V_p \mu. \]

Then \( g' \) and \( g \) differ in degrees \( g' \geq g \) and
\[ F_{g} g' = F_{g} (g - V_p \mu) = 0 \quad \text{if } g < p, \]
\[ F_{p} g' = F_{p} (g - p \mu) = 0, \]
and we are finished.

It remains to show that we can solve (2). By hypothesis,
\[ p_F(X) \equiv 0 \quad \text{mod } p, \]
and one knows, in general that
\[ p_F(X) = pX \quad \text{mod deg } 2. \]

Thus there is a \( G(X) \in \mathbb{R}[X] \) such that
\[ p_F(X) = p G(X). \]
Next note that \( p \mid \Gamma_p F_p \equiv p \) hence if \( x \in D(A) \) where \( A \) is an \( R \)-algebra, then

\[
\left( \gamma^{-1} \circ \Gamma_p F_p \right)(x) \equiv \left( \gamma^{-1} \circ p \right)(x) = p F(x) \equiv 0 \mod p
\]

while

\[
\left( \gamma^{-1} \circ \Gamma_p F_p \gamma \right)(x) = \gamma^{-1}(p F \gamma(x)).
\]

Thus there is a power series \( \gamma(x) \in R[[X]]^+ \) with

\[
\left( \gamma^{-1} \circ F_p \gamma \right)(x) = p \cdot \gamma(x)
\]

\[
= p \cdot G(G^{-1}(\gamma x))
\]

\[
= p \cdot G(\gamma x)
\]

\[
= p F_G(x)
\]

\[
= (\gamma^{-1} \circ p \gamma)(G^{-1}(\gamma x))
\]

\[
= \gamma^{-1}(p \mu(x))
\]

where \( \mu(x) = \gamma(G^{-1}(\gamma x)). \) Thus \( F_p \gamma = p \mu \) as desired, and so the proof of the proposition is complete. \( \text{q.e.d.} \)
§1. Cobordism as a universal cohomology theory.

Throughout this paper, manifold means a $C^\infty$ manifold with a Hausdorff topology having a countable basis for the open sets but not necessarily connected, and the components may have different dimensions. A morphism or map of manifolds will always be $C^\infty$. If $X$ is a manifold let $T_X$ be its tangent bundle. If $f : X \to Y$ is a map of manifolds, then its stable normal bundle $\nu_f$ is denoted $Ty$ is the difference $\nu_f \subset Ty$ in the category $KO(X)$ of real vector bundles over $X$.

$\nu_f$ may be identified with an homotopy class of maps $\nu_f : X \to \mathbb{Z} \times BO$, where $\mathbb{Z} \times BO$ is the Hopf space representing the functor $KO$ equivalent to $\pi_1(Hom(X, BO))$. Let $H$ be a Hopf space endowed with a map $H \to \mathbb{Z} \times BO$ of Hopf spaces. By an $H$-orientation of $f$, we mean a lifting of $\nu_f : X \to \mathbb{Z} \times BO$ to a map $X \to H$. An $H$-oriented map is a map $f$ provided with an $H$-orientation; we shall always denote the map $X \to H$ by $\nu_f$. Clearly:

(i) If $X \to Y \to Z$ are $H$-oriented, then $gf$ is $H$-oriented with $\nu_{gf} = \nu_f + f^* \nu_g$. 


Let $H$ be a commutative associative Hopf space endowed with a map $\eta : H \to \mathbb{Z} \times \mathcal{B}$ of Hopf spaces.

An $H$-orientation of $f$ is by definition an object of the fiber category of the map $H(X) \to KO(X)$, that is, a map $u : X \to H$ together with a homotopy class of homotopies from $\eta u$ to $\eta f$. An $H$-oriented map is a map $f$ endowed with an $H$-orientation; from now on we denote by $\mathcal{H}^f$ the $H$-orientation of $f$. Then
(ii) If

\[ \begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y 
\end{array} \]

is a cartesian square of manifolds where \( g \) and \( f \) are transversal (i.e. \( \text{transversal} \) the map \( f \times g : X \times Y \to Y \times Y \) is transversal to the diagonal submanifold \( \Delta Y \)), and if \( f \) is H-oriented, then \( f' \) is H-oriented with

\[ \nu_{f'} = (g')^* \nu_f. \]

If \( X \) is a manifold and \( g \) is an integer we define \( \Omega^g_H(X) \) to be the equivalence classes of proper H-oriented maps \( f : Z \to X \).

Let \( \Omega_H(X) \) be the set of equivalence classes of proper H-oriented maps \( f : Z \to X \) where such a map \( f \) is equivalent to \( f' : Z' \to X \) if there exists a diagram

\[ \begin{array}{ccc}
Z & \xrightarrow{f_\ast f'} & W \\
\downarrow f & & \downarrow h \\
(X \times 0) \sqcup (X \times 1) & \xrightarrow{h} & X \times \mathbb{R}
\end{array} \]

where \( h \) is proper and H-oriented, and where the orientations of \( f, f' \).
Then
\[ \mathbb{2} \times \mathbb{S}^2 = \mathbb{P}(\mathbb{T}^2 + 1) \text{ in } \mathbb{K}^0(\mathbb{Z} \times \mathbb{S}^2) \]

Consider the map
\[ f \times \mathbb{R} : \mathbb{2} \times \mathbb{S}^2 \to \mathbb{X} \times \mathbb{R} \]

where it is easy to see that the relation
\[ f ; Z \to X \text{ and } f' ; Z' \to X \]

is an equivalence relation and that \( \mathcal{H}(X) \) is commutative.

Let \( \mathcal{H}(X) \) be the equivalence class of \( f ; Z \to X \).

where the image of elements represented by \( f ; Z \) is a commutative diagram containing the equivalence class of \( f ; Z' \to X \).

\[ \mathcal{H}(X) \]

\[ \text{dim } Y = 8. \]

Then
Cobordism as a universal cohomology theory

In this section we show how cobordism generalized cohomology theories can be characterized as a universal functor on the category of $C^\infty$ manifolds endowed with Gysin morphisms for proper maps possessing a particular kind of orientation. This idea was suggested by Grothendieck's theory of motives in algebraic geometry. It permits one to define cobordism groups without using manifolds with boundary and as we plan to show in later papers, it leads to various generalizations such as cobordism for manifolds over a base manifolds and equivariant cobordism theory. Although the universal approach is not indispensable for the present paper, it furnished the motivation for many of the results.

We shall consider only $C^\infty$ manifolds which are Hausdorff, countably compact, and of bounded, not necessarily constant, dimension. Such a manifold possesses a closed embedding in Euclidean space $\mathbb{R}^N$ which is unique up to isotopy for $N$ sufficiently large. Let $\text{Man}$ be the category of $C^\infty$ manifolds and $C^\infty$ maps.
Enclosed is an announcement of some work of mine in cobordism theory which uses your theory of typical curves. Because I had to keep the thing under 8 pages, it was impossible to include any of the categorical considerations which motivated my results. Without these, the material in the last two sections is I think somewhat incomprehensible, so I'm going to try in this letter to explain my point of view.

I have gained a great deal of insight into cobordism theory by comparing it with Grothendieck's theory of motives, which I now like to think of as the analogue of cobordism theory in algebraic geometry. Indeed you have probably heard the theory of motives described as the universal cohomology theory for schemes, through which many other, such as $l$--adic, Hodge, de Rham, or crystalline, must factor. With this in mind I shall review the definition of the cobordism generalized cohomology theories. Let $\text{Man}$ be the category of $C^\infty$ manifolds and $C^\infty$ maps. I do not require that manifolds be connected or that their components have the same dimension, but I want the dimension to be bounded so that any manifold possesses an embedding in Euclidean space $\mathbb{R}^N$, which is unique up to isotopy if $N$ is sufficiently large. Given a morphism $f: X \rightarrow Y$ in $\text{Man}$ it can be factored

\[ X \xrightarrow{(f, s)} Y \times \mathbb{R}^N \xrightarrow{p_1} Y \]
where $i$ is an embedding. The normal bundle of $(f,i)$ has a well-defined stable isomorphism class independent of the choice of the embedding. If $G$ is one of the infinite classical groups such as $O, U, Sp, \text{etc.}$, then by a $G$-oriented map $f$, I mean a map whose stable normal bundle is endowed with a $G$-orientation, that is, a reduction of the structural group to $G$; the resulting $G$-oriented stable bundle will be denoted $\nu_f$. There are two sotries:

1) If $f : X \to Y$ and $g : Y \to Z$ are $G$-oriented, then $gf : X \to Z$ is $G$-oriented with

$$\nu_{gf} \cong \nu_f + f^* \nu_g.$$

2) Let

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y \\
\end{array}
$$

be a cartesian square in $\text{Man}_G$, where $f$ and $g$ are transversal (i.e., the map $f \times g : X \times Y' \to Y \times Y$ is transversal to the diagonal submanifold). If $f$ is $G$-oriented, then $f'$ is $G$-oriented with

$$\nu_{f'} \cong (g')^* \nu_f.$$

Given a manifold $Y$, let $\Omega G(Y)$ be the equivalence classes...
of maps $f$ in $\text{Man}$ which are proper, $G$-oriented and have
target $Y$; two such maps $f_i : X_i \to Y$, $i = 0, 1$ are
equivalent if they are cobordant, which means that there exists
a $G$-oriented proper map $h : Z \to Y \times \mathbb{R}$ and a diagram

$$
\begin{array}{c}
X_0 \xrightarrow{d_0} Z \leftarrow f_1 \quad X_1 \\
\downarrow f_0 \quad \downarrow h \quad \downarrow f_1 \\
Y \rightarrow Y \times \mathbb{R} \leftarrow Y \\
y \mapsto (y, 0) \quad y \mapsto (y, 1)
\end{array}
$$

where the two squares are transversal cartesian and the
orientation of $h$ induces that of the $f_i$ as described above in 2.)

Denoting the equivalence class of $f : X \to Y$ by $[f : X \to Y]$, $\Omega G(Y)$
is an abelian group with

$$
[f_1 : X_1 \to Y] + [f_2 : X_2 \to Y] = [(f_1, f_2) : X_1 \sqcup X_2 \to Y],
$$
in fact it is a graded abelian group

$$
\Omega G(Y) = \bigoplus_{g \in \mathbb{Z}} \Omega G^g(Y)
$$

where elements in $\Omega G^g(Y)$ are represented by maps $f : X \to Y$ which
are of relative dimension $-g$ at all points of $X$. (If you happen to
know some cobordism theory.) $\Omega G(Y)$ is the "generalized" cohomology
of the space $Y$ with values in the Thom spectrum $\text{MG}$, i.e.

$$
\Omega G^g(Y) \cong \varinjlim_k \left( \sum^{g + k} Y \text{, } \text{MG}(k) \right)
$$
Call $\Omega G(Y)$ the $G$-cobordism group of $Y$. $G$-cobordism possesses two variances with respect to $Y$. First of all, if $g: Y' \to Y$ is a proper $G$-oriented map one has the so-called Borsuk homomorphism

$$g_*: \Omega G(Y') \to \Omega G(Y)$$

$$g_* [f: X \to Y] = [gf: X \to Y]$$

where $gf$ is oriented as in 1) above. Secondly if $g: Y' \to Y$ is an arbitrary morphism in $\text{Man}$ and if $f: X \to Y$ is proper and $G$-oriented, then by the Thom transversality theorem $g$ may be moved transversally to $f$ and we can form the fiber product $\text{pr}_1: X \times_Y Y' \to Y'$, which is proper and $G$-oriented (using 2). The resulting element of $\Omega G(Y')$ depends only on $[f: X \to Y]$, and so a map

$$g^*: \Omega G(Y) \to \Omega G(Y')$$

is defined. Here are some basic properties of this structure:

I. $Y \mapsto \Omega G(Y), f \mapsto f^*$ is a contravariant functor from $\text{Man}$ to $\text{Ab}$.

II. $Y \mapsto \Omega G(Y), f \mapsto f_*$ is a covariant functor from the category of manifolds and proper $G$-oriented maps to $\text{Ab}$.

III. (Borsuk commutes with transversal base change). Given a transversal cartesian square in $\text{Man}$.
where $f$ is proper and $G$-oriented and where $f'$ is endowed with the induced orientation (2), then we have

$$g^*f^* = f'(g')^*.$$  

IV. (homotopy) If $f$ and $g : X \to Y$ are maps in $\text{Man}$ which are homotopic (by a homotopy $h : X \times I \to Y$ which is $C^\infty$), then $f^* = g^*$. 

V. If $\text{in}_i : X_i \to X$ $i = 1, 2$ are the inclusions, then

$$(\text{in}_1^*, \text{in}_2^*) : \Omega G(X_1 \sqcup X_2) \to \Omega G(X_1) \times \Omega G(X_2).$$

After all these preliminaries I come to the main point. If $X$ is a manifold, let $1_X \in \Omega G(X)$ be the class of $\text{id}_X$. Let $e$ be the final object of $\text{Man}$. 

**Proposition:** (universal property of $\Omega G$). Let $Y \to Q(Y)$ $f \mapsto f^*$ be a contravariant functor from $\text{Man}$ to $\text{Ab}$-enriched with a bijective homomorphism $f \mapsto f_*$ for proper $G$-oriented maps. Assume that condition $I - V$ hold with $\Omega G$ replaced by $Q$. Then given an element $\lambda \in Q(e)$, there is a unique natural transformation $\theta : \Omega G \to Q$.
compatible with Gysin homomorphism such that \( \Theta(\alpha_0) = \lambda \).

This is essentially trivial. It is clear that \( \Theta: \Omega G(Y) \rightarrow \Omega(Y) \) is given by

4) \[ \Theta(\alpha: X \rightarrow Y) = f_\# \pi_X^* \lambda \]

where \( \pi_X: X \rightarrow \mathcal{E} \) is the canonical map. To see this is well-defined suppose given two representatives \( f_i: X_i \rightarrow Y \) for an element of \( \Omega G(Y) \); then there is a diagram 3) above and so

\[
\begin{align*}
(f_0)_\pi X_0^* \lambda &= (f_0)_\pi f_0^* (\pi_X^* \lambda) \\
&= i_0^* h_\pi (\pi_X^* \lambda) \\
&= i_1^* h_\pi (\pi_Y^* \lambda) \\
&= (f_1)_\pi f_1^* (\pi_Y^* \lambda) = (f_1)_\pi \pi_X^* \lambda.
\end{align*}
\]

The rest is checking that \( \Theta \), as defined by 4), is compatible with everything.

There is an external product operation

\[
\Omega G(Y) \otimes \Omega G(Y') \rightarrow \Omega G(Y \times Y')
\]

with unit \( 1_\mathcal{E} \in \Omega G(\mathcal{E}) \) given by

\[
[f: X \rightarrow Y] \otimes [f': X' \rightarrow Y'] = [f \times f': X \times X' \rightarrow Y \times Y']
\]

where \( f \times f' \) is oriented as follows: The orientations of \( f \) and \( f' \)
yield by 2) orientations of \( f \times \text{id}_Y \) and \( \text{id}_X \times f' \), which in turn yields an orientation of \( f \times f' = (f \times \text{id}_Y)(\text{id}_X \times f') \). The orientation of \( f' \times f \) is \((-1)^{(\deg f')(\deg f)}\) times that of \( f \times f' \). \( QG(Y) \) becomes a Koszul commutative ring with the internal product

\[
\Delta^* (x \boxtimes y) = \Delta^* (x \boxtimes y)
\]

where \( \Delta_y : Y \to Y \times Y \) is the diagonal. There is a basic formula:

\[\text{VI. If } f : X \to Y, f' : X' \to Y' \text{ are proper and } G\text{-oriented, then}
\]

\[
(f \times f')_* (x \boxtimes y) = (-1)^{(\deg f')(\deg x)} f'_* x \boxtimes f'_* y
\]

for \( x \in QG(X) \) and \( x' \in QG(X') \).

The proposition may be augmented by the assertion that if \( Q \) has in addition products

\[
Q(Y) \otimes Q(Y') \longrightarrow Q(Y \times Y')
\]

\[
x \otimes y \mapsto x \boxtimes y
\]

satisfying the formula of VI with \( QG \) replaced by \( Q \), then the natural transformation \( \Theta \) given by the unit \( 1 \in Q(e) \) is compatible with products.

Therefore \( QG \) may be characterized as the initial object of the category of functors \( \text{Functor}_{\text{man}} \) endowed with \( G \)-equivariant isomorphism for proper \( G \)-oriented maps and endowed with products such that the conditions I-VI hold. Note that \( QG(e) \) is }
the cobordism ring of compact $G$-oriented manifolds, and hence we have given a definition of the latter not using manifolds with boundary!

I shall now turn to Chern classes and the formal group law. For this suppose that $G$ is the infinite unitary group $U$; the cobordism theory $\Omega U$ will be called complex cobordism theory and denoted simply $\Omega$. If $E$ is a complex vector bundle of dimension $n$ over a manifold $X$, then the zero section $i : X \to E$ is proper and $G$-oriented; hence we can define the $n$th Chern class of $E$ to be

$$c_\sigma^n(E) = i^* \cdot 1_X \in \Omega^{2n}(X).$$

Recalling how $i^*$ is defined, one sees that $c_\sigma^n(E)$ is the cobordism class of the zero submanifold of a generic section of $E$.

The following theorem permits one to define Chern classes of all dimensions following the old method of Grothendieck.

VII. If $E$ is a complex vector bundle of dimension $n$ over a manifold $X$, let $\mathbb{P}(E)$ be the projective bundle of hyperplanes in $E$ and let $\mathcal{O}(1)$ be the canonical quotient line bundle in $\mathbb{P}(E)$. Then $\Omega(\mathbb{P}(E))$ is a free $\Omega(X)$-module with basis $1, \frac{1}{2}, \ldots, \frac{1}{n}$, where $i = c_\sigma^1(\mathcal{O}(1))$.

The Chern classes $c_\sigma^i(E) \in \Omega^{2i}(X)$ are defined as the
coefficients of the relation giving the ring structure of $\Omega(\mathcal{E})$:

$$\mathcal{C}^n - \mathcal{C}^{n-1}(\mathcal{E}) \cdot 1 + \ldots + (-1)^n \mathcal{C}^0(\mathcal{E}) = 0.$$ 

They satisfy the Whitney sum formula

$$c^\alpha(E+E") = \sum_{\alpha'} c^\alpha(E') c^\alpha(E")$$

with $c^\alpha(\mathcal{E}) = 1$.

but unlike Chern classes in integral cohomology, the first Chern class of a tensor product of two line bundles is $L_1 \otimes L_2$ not the sum of $c_1^{\mathcal{L}_1}$ and $c_1^{\mathcal{L}_2}$. Instead there is a formal group law $F^\mathcal{E}(x,y) = \sum a_k x^k y^k$ with $a_k \in \mathcal{E}^{2k-2\alpha}(\mathcal{E})$

such that

$$c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = F^\mathcal{E}(c_1^{\mathcal{L}_1}, c_1^{\mathcal{L}_2})$$

for all line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on the same base. (The existence of this formal group law follows from the fact that $\mathcal{L}_1 \otimes \mathcal{L}_2$ may be induced from $\mathcal{P}_1^* \mathcal{O}(1)$ and $\mathcal{P}_2^* \mathcal{O}(1)$ on $\mathbb{P}^n \times \mathbb{P}^n$ for $n$ large and that

$$\Omega(\mathbb{P}^n \times \mathbb{P}^n) \cong \Omega(\mathcal{E})[x,y]/(x^{n+1}, y^{n+1})$$

where $x = c_1^{\mathcal{E}}(\mathcal{P}_1^* \mathcal{O}(1))$ and $y = c_1^{\mathcal{E}}(\mathcal{P}_2^* \mathcal{O}(1))$.)

Now if $Q$ is a functor $(\text{Man})^\circ \rightarrow \text{Ab}$ with Steen model for proper U-oriented maps and products satisfying I-VI, then by means of the canonical homomorphism $\Theta: \Omega \rightarrow Q$ we obtain Chern classes for complex bundles with values in $Q$ and a formal group law $F^Q$. Here are some examples:
(i) \( Q(X) = H^*(X, \mathbb{Z}) \). Then \( F^Q(x, y) = x + y \)

(ii) \( Q^1 = \mathbb{Z}/2\mathbb{Z} \)-graded complex \( K \)-theory. Thus \( Q(X) = K^+(X) \oplus K^-(X) \), where \( K^+(X) \) is the Grothendieck group on complex bundles on \( X \) and \( K^-(X) = \ker \left[ K^+(S^1 \times X) \to K(S^1 \times X) \right] \); the Gysin homomorphism is defined using the periodicity theorem of Bott. I've checked that for a vector bundle of rank \( n \)

\[
C_n^k(E) = \lambda_{-1} E^\vee \quad E^\vee = \text{dual bundle to } E
\]

hence that \( c^k_1(L) = 1 - L^{-1} \) and that

\[
F^k(X, Y) = x + y - xy
\]

(iii) \( Q = \mathbb{Z} \)-graded complex \( K \)-theory. Here \( Q(X) = \bigoplus_{n \in \mathbb{Z}} K^n(X) \)

where \( K^n = K^+ \) as above if \( n \) is even and \( K^- \) if \( n \) is odd. If \( \beta \in K^{-2}(pt) \) is the periodicity class, then the formal group law is \( x + y - \beta xy \).

Since \( Q \) is a universal such theory \( Q \), one conjectures that \( F^Q \) should be a universal group law. This is true and a proof is given in my announcement. The proof uses everything - Hazards' theorem on the structure of the universal group law - and Milnor's on the structure of \( Q(e) \). It would be nice to have a proof avoiding Milnor's results and using the universal nature of \( Q \), but I have not succeeded in finding one.
February 17, 1969

Equiariant Cobordism Theory

Outline:

1. Definition of $\Sigma G$ and basic properties.
2. Algebraic analysis of the group law coming from tensor product of line bundles.

[real cobordism eventually]

Let $G$ be a compact Lie group, $\mathcal{M}_G$ the category of $C^\infty$-manifolds on which $G$ acts and equivariant maps. Further we shall only consider $G$-manifolds with finitely many orbit types, since these are the ones which can be embedded in representations. If $f: X \to Y$ is a $G$-map, i.e. map in $\mathcal{M}_G$, then $xf = f^x - rx$ is a stable $G$-bundle over $X$ (the category of stable bundles is obtained by a Grothendieck construction from bundles and homotopy classes of isomorphisms by adjoining inverses). By a complex orientation of $f$ we mean an isomorphism of $f^*G$ with a stable complex $G$-bundle. From now on all bundles are complex.

By a (resp. additive, resp. multiplicative) ring cobordism theory $\mathcal{M}_G$ we shall mean the following data subject to the axioms below.
A rule associated to each $G$-manifold $X$ a set $Q(X)$ and to each $G$-map $f: X \to Y$ a set map $f^*: Q(Y) \to Q(X)$ and to each proper oriented $G$-map a set map $f_*: Q(X) \to Q(Y)$ such that

$$(gf)^* = f^* g^* \quad (id)^* = id$$

$$(gf)_* = g_* f_* \quad (id)_* = id$$

B. (Homotopy). If $R$ has trivial $G$-action, then

$$pr_*^*: Q(X) \to Q(X \times R)$$

C. (Transversality). If

$$X' \xrightarrow{g'} X$$
$$Y' \xrightarrow{g} Y$$

is cartesian in $M_{G\text{-}X}$ with $g'$ transversal to $g$, then

$$f^* f_* = f'_* g'^*$$

In the case of an additive cohomology theory we want $Q(X)$ to be an abelian group, $f^*$ and $f_*$ to be homomorphisms and

D. (Union). $Q(\phi) = 0$. 
\[ \mathrm{in}_1^* + \mathrm{in}_2^* : Q(X) \oplus Q(X') \xrightarrow{\cong} Q(X \sqcup X') \]

(\[ \Rightarrow (\mathrm{in}_1^*, \mathrm{in}_2^*) : Q(X \sqcup X') \xrightarrow{\cong} Q(X) \times Q(X') \])

(Problem: Do you have to insist that changing \( \overline{\eta} \) to \( \overline{\eta}^\star \) changes the case of a multiplicative cohomology theory?)

I want there to be given an element \( 1 \in Q(pt) \) and maps

\[ Q(X) \otimes Q(X') \longrightarrow Q(X \times X') \]

\[ a \otimes b \longmapsto a \otimes b \]

such that

\[ \begin{align*}
\text{associativity:} & \quad (a \otimes b) \otimes c = a \otimes (b \otimes c) \\
\text{unit:} & \quad \begin{cases}
1 \otimes a = a \\
a \otimes 1 = a.
\end{cases}
\end{align*} \]

(Problem: How to handle the anti-commutativity)

\[ \begin{align*}
(f \times g)^* (a \otimes b) &= f^* a \otimes f^* b \\
(f \circ g)^* (a \otimes b) &= f^* a \otimes g^* b.
\end{align*} \]
Mi now wish to determine the universal cohomology theory.

**Lemma 1.** Let $Q$ be a cohomology theory. Let $Q'(X) \subseteq Q(X)$ consisting of elements which can be represented in the form $\mathbb{Z} \to \text{pt}$.

Let $Q'(X) = \mathbb{Z}$, subset of $Q(X)$ consisting of elements which can be represented in the form $\tau^{*}g_{*} \pi^{*}1$.

$$\xymatrix{ X \ar[r]^i & X \times V \ar[d]^{\pi} \ar[r]^f & Y \ar[r]^{i'} \ar[d]^{f'} & X \times V \ar[d]^{\pi} \ar[r]^f & \text{pt} \ar@{^(->}[ld]_{\mathbb{Z}} \ar@{^(->}[ld]_{\tau^{*}g_{*} \pi^{*}1} }$$

where $f$ is proper oriented, and $V$ is a representation of $G$ and $i(x) = (x,0)$. Then $Q'$ is a sub-cohomology theory of $Q$.

**Proof:** Let $f: X \to Y$ be proper oriented. Then $f'_{*} i^{*} g_{*} \pi^{*}1 = f'^{*} (f_{*} g_{*} \pi^{*}1)$, where we use the cartesian square

$$\xymatrix{ X \ar[r]^i \ar[d]^f & X \times V \ar[d]^{f'} \ar[r]^i' \ar[d]^{f'} & Y \ar[r]^{i'} \ar[d]^{f'} & X \times V \ar[d]_{\pi} \ar[r]^f & \text{pt} \ar@{^(->}[ld]_{\mathbb{Z}} \ar@{^(->}[ld]_{\tau^{*}g_{*} \pi^{*}1} }$$

Thus $f_{*} Q'(X) \subseteq Q'(Y)$.

Let $f: Y \to X$ be arbitrary. Then $f'$ we factor

$$\xymatrix{ Y \ar[r]^{i'} \ar[d]^f & Y \times W \ar[d]^{f'} \ar[r]^{i'} & X \times V \ar[d]_{\pi} \ar[r]^f & \text{pt} \ar@{^(->}[ld]_{\mathbb{Z}} \ar@{^(->}[ld]_{\tau^{*}g_{*} \pi^{*}1} }$$

Thus $f_{*} Q'(X) \subseteq Q'(Y)$.
where \( W \) is a representation and \( f' \) is smooth; we have

\[
\begin{array}{c}
\xymatrix{
Z' \\
Y' \\
\ar[u]^{g'} \ar[ur]^{f'} \ar[rr]^{\pi} & & \ar[u]^\pi pt \\
Y \\
\ar[u]_i Y \times W \\
\ar[u]^{f} \ar[rr]_{g} & & \ar[u]_{g} X \times V \\
X \\
\ar[u]_i \ar[ur]_{f} & & & & & \\
\end{array}
\]

and we have \( f'^*g_*\pi^*1 = c*f'^*g_*\pi^*1 = c_g (nf')^*1 \). To construct this factorization choose an embedding of the first factor \( f \) into an embedding followed by a smooth map, then by means of the exponential one gets an embedding into a vector bundle following by a smooth map; finally the bundle is the quotient of a trivial bundle \( Y \times W \).

Thus have

\[
\begin{array}{ccc}
Y & \longrightarrow & Y \times W \\
\downarrow & & \downarrow \text{smooth} \\
E & \longrightarrow & X \times V \\
\downarrow & \downarrow \text{smooth} & \downarrow \\
X & & V \\
\end{array}
\]

Consider triples \((\beta, Y, \mathcal{E})\) where \( X \) is a \( G \)-representation of \( G \).

If \( X \) is a \( G \)-manifold, let \( C^G_\mathcal{E}(X) \) be the bordism classes of proper oriented \( G \)-maps \( f: Z \rightarrow X \) of codimension \( g \).

let

\[
\Omega^G_g(X) = \lim_\rightarrow C^G_\mathcal{E}(X \times V)
\]

where \( V \) runs over the category of representations of \( G \).
consisting of representations of $G$ and surjective maps. The first thing to show is that this limit may be calculated as a filtered inductive limit. However the category is filtering up to homotopy, e.g.

\[ V, V' \text{ have } V \oplus V' \xrightarrow{\text{surj.}} V' \]

\[ V \xrightarrow{\pi_1} V' \quad \text{two surjections} \]

Then consider

\[ V \oplus V' \xrightarrow{\text{pr}_1} V \quad \text{surjective} \]

\[ (V \oplus V') \xrightarrow{\pi_1 \text{pr}_1} V' \quad \text{homotopic trivially} \]

\[ t \pi_1 \text{pr}_1 + (1-t) \text{id} \]

\[ \text{homotopic} \]

Similarly

\[ V \oplus V' \xrightarrow{\text{pr}_2} V' \quad \text{homotopic} \]

You had a better method before. To show that the category of thickenings of $X$ is

objects = embeddings $X \xrightarrow{i} Y$

morphisms = $X \xrightarrow{\gamma} Y$ with $\gamma$ smooth

This is filtering up to homotopy: Given
Now it's clear that given any theory $Q$, we have a map

$$\Omega^b_0(X) \rightarrow Q(X)$$

obviously universal. (Assumes $Q$ just a function to sets)

Additive structure: What do we need to guarantee that

$$C^b_G(X) \rightarrow Q(X)$$

is a homomorphism? Recall that given $f_1: Z_1 \rightarrow X$, $f_2: Z_2 \rightarrow X$ proper + oriented, then sum in $C^b_G(X)$ is defined by

$$(f_1)^*1 + (f_2)^*1 = (f_1 + f_2)^*1$$

where $f_1 + f_2$ is the canonical map $Z_1 \sqcup Z_2 \rightarrow X$. Now,

$$\Theta(f_1)^*1 = f_1^Q1$$

so $\Theta$ is a homomorphism if

$$f_2^Q1 + f_2^Q1 = (f_1 + f_2)^Q1$$

What we need therefore seems to be that given $f_2: Z_2 \rightarrow X$...
Correct condition is that given \( f_i: Z_i \to X \), \( g_i: Z_i \to Y \) then

\[
(f_1 + f_2) \times (g_1 + g_2)^* y = f_1 \times g_1^* y + f_2 \times g_2^* y
\]

Relation with preceding axiom:

\( Z_i = \phi \Rightarrow X = Y \). Then you get for \( y \in Q(\phi) \) that

\[
y = y + y \Rightarrow y = 0.
\]

Also we find that taking \( X = Z_1 \sqcup Z_2 = Y \), \( f_i = g_i = \text{id} \),

that \( f_1 + f_2 = g_1 + g_2 = \text{id}_{Z_1 \sqcup Z_2} \)

\[
x = (\text{id}_1)^* x + \text{id}_2^* x.
\]

implying that

\[ Q(Z_1) \oplus Q(Z_2) = Q(Z_1 \sqcup Z_2) \]

Conversely the latter implies with \( x = (g_1 + g_2)^* y, y \in Q(Y) \) that

\[
(f_1^* + f_2^*) \times (g_1 + g_2)^* y = f_1^* g_1^* y + f_2^* g_2^* y.
\]

:\( \text{Same axiom.} \)

---

**Multiplicative structure**

What do we need to guarantee that

the product structure on \( \Omega G \):

\[
\begin{align*}
X & \xrightarrow{Z_1} Z_1 \times X \times V_1, \\
X & \xrightarrow{Z_2} Z_2 \times X \times V_2, \\
X & \xrightarrow{Z_1 \times Z_2} X \times X \times (V_1 \times V_2)
\end{align*}
\]
we have a natural map
\[ C^p_G(X) \otimes C^q_G(Y) \longrightarrow C^{p+q}_G(X \times Y) \]
\[
\begin{array}{ccc}
Z_1 & \otimes & Z_2 \\
\downarrow & & \downarrow \\
X & \otimes & Y \\
\end{array}
\longrightarrow
\begin{array}{ccc}
Z_1 \times Z_2 \\
\downarrow \\
X \times Y \\
\end{array}
\]
This we claim defines a map
\[ \Omega^p_G(X) \otimes \Omega^q_G(Y) \longrightarrow \Omega^{p+q}_G(X \times Y). \]

Method is to consider the functor
\[ X \mapsto \Omega_G(X \times Y) \]
observe it satisfies the axioms, hence given an element \( \alpha \in \Omega_G(Y) \) one gets a map
\[ \Omega_G(X) \xrightarrow{U(\alpha)} \Omega_G(X \times Y) \]

necessary to check that isomorphism
\[ \text{Hom}(\Omega_G, \mathbb{Q}) = \mathbb{Q}(\text{pt}) \]
is compatible with addition

Recall \( \mathbb{Q} \) is a coh. theory which is additive! Claim then that \( X \mapsto \mathbb{Q}(X \times Y) \) with \( (f)_* = (f^* \circ \text{id}_Y)_* \)/(\( (f)_* = (f \times \text{id}_Y)^* \) is a coh. theory additive. For this one wants to know that, given \( y \in \mathbb{Q}(Y) \) and \( f_i : Z_i \to X \)
\[ f_1 \circ y \circ \text{pr}_2^* y + (f_2)_* \circ \text{pr}_2^* y = (f_1 + f_2)_* \circ \text{pr}_2^* y \]
February 22, 1969

Some remarks toward a theory of cobordisms in algebraic geometry.

Consider the category of quasi-projective non-singular varieties over an algebraically closed field. By a cohomology theory on this category I shall mean a contravariant functor to the category of rings endowed with a Gysin homomorphism for proper maps having the properties of K-theory, in particular, homotopy, transversal base change, half exactness, and splitting principle. I assume that a universal such theory exists and denote it $\Omega$. I now wish to prove Riemann-Roch under the following form:

$$\text{Hom}^+(K, Q^*) \rightarrow \left\{ \begin{array}{l}
\text{natural ring hom } \beta : \Omega \rightarrow Q \\
\text{such that for } \text{pt} \xrightarrow{s} \mathbb{P}^2 \xrightarrow{\pi} \text{pt} \text{ we have that } \pi_* \beta s_* 1 \in Q(\text{pt})^* \end{array} \right\}$$

are denote the RHS by $\text{Hom}^0(\Omega, Q)^*$.

The map is given by sending a characteristic class $\alpha : K \rightarrow Q^*$ into $\hat{\alpha} : \Omega \rightarrow Q$ given by

$$\hat{\alpha}(f_* 1) = f_* \alpha(v_f)$$

$\hat{\alpha}$ is well-defined because using $\alpha$ one defines a new Gysin homomorphism.
\[ f_! \alpha = f_\ast (\alpha(\nu_\ast \alpha)) \quad \alpha \in \Omega(X) \]

and hence by the universal property of \( \Omega \) one gets a unique map \( \hat{\alpha} : \Omega \to Q \) with

\[
\begin{align*}
\hat{\alpha}(f_\ast \alpha) &= f_\ast (\hat{\alpha}(\alpha)) \\
2 \hat{\alpha} f^\ast &= f^\ast \hat{\alpha}
\end{align*}
\]

To prove the R-R this one defines a map in the opposite direction. Thus starting with \( \beta \) one defines \( \beta \) on vector bundles by the formula

\[ \beta(l_\ast 1) = l_\ast \beta(E) \]

or \[ \beta(E) = \pi_\ast \beta l_\ast 1 \]

where \( i : X \to P(E+1) \) is the zero section of \( E \), and where \( \pi : P(E+1) \to X \)

\[ \text{Hypothesis: } \beta \text{ extends to a transf } K \to Q^* \text{ provided } \beta(1) \in Q(\text{pt})^*. \]

In other words

\[ \beta(E) = \beta(E') \beta(E'') \text{ if } 0 \to E' \to E \to E'' \to 0 \text{ is exact} \]

\[ \beta(E) \in Q(X)^*. \]

Observe this is true if \( \beta = \hat{\alpha} \) since

\[ \hat{\alpha}(E) = \pi_\ast \alpha l_\ast 1 = \pi_\ast (\pi_\ast \alpha(E)) = \alpha(E). \]
Thus we see immediately that $\alpha \rightarrow \beta$ is injective.

We shall now show that the $\beta$ satisfying the hypothesis we have $\widehat{\beta} = \beta$, thus identifying the image of $\alpha$ as those

The thing to prove is

$$\beta(f_x \alpha) = f_x(\beta(y f) \beta \alpha).$$

because then both $\beta$ and $\widehat{\beta}$ are transforms from $\Omega$ to $\Omega$ compatible with the $f$ defined using $\beta$ and hence are equal. This is a R-R statement and is proved following Grothendieck:

1) Formula true for $f: \mathbb{P}^n \rightarrow k$. Both sides are linear over $\Omega(k)$ so only have to check for $x = j_x \cdot 1$

where $j: \mathbb{P}^0 \hookrightarrow \mathbb{P}^n$.

Reduction to the case where $x = 1$: Note that there is a cartesian diagram

$$\begin{array}{ccc}
\mathbb{P}^0 & \rightarrow & \mathbb{P}^n \\
\downarrow i & & \downarrow i \\
\mathbb{P}^n & \rightarrow & \mathbb{P}(\mathcal{O}(1) + 1/\mathbb{P}^n)
\end{array}$$

where $i^*$ is the zero section (resp generic section) of $\mathcal{O}(1)$. Thus

$$j_x \cdot 1 = j_x j^* \cdot 1 = s \cdot \iota \cdot 1 = \iota \cdot \iota \cdot 1$$

and

$$\beta(j_x \cdot 1) = \iota \cdot \beta \iota \cdot 1 = \iota \cdot \iota \cdot \beta(\mathcal{O}(1)) = j_x \cdot \beta(\mathcal{O}(1)) \cdot 1 = j_x \cdot 1 \cdot \beta(\mathcal{O}(1)) \cdot 1$$
Thus
\[ \beta(f_* x) = \beta(f_* g^* 1) = \beta(g^* 1) \]
\[ f_* (\beta(dy) - x) = f_* (\beta(0y))^{-1} \cdot \beta(0y)^{g^* 1} \]
\[ = g^* \beta(dy) \quad \text{where } g: \mathbb{P}^n \to \mathbb{P}^l \]

It therefore remains to prove for \( f: \mathbb{P}^n \to \mathbb{P}^l \) and \( x = 1 \). Recall that as \( Q \) satisfies splitting principle, there is a universal formula
\[ \beta(L) = \varphi(c_1(L)) \quad \beta c_1(L) = \overline{\varphi}(c_1(L)) \]
for a uniquely determined \( \varphi(x) \in Q(\mathbb{P}^l)[[X]] \). Then

\[ \beta F^\Omega = \overline{\varphi} \ast F^\Omega \]

so
\[ \beta \omega^\Omega = \frac{dZ}{(\beta F_x(0, Z))} = \frac{dZ}{\overline{\varphi}(F_x(0, \overline{\varphi}^{-1} Z), \overline{\varphi}^{-1} Z) \bigg|_{X_0} \overline{\varphi}(F_x(0, \overline{\varphi}^{-1} Z), \overline{\varphi}^{-1} Z) \bigg|_{X_0}} \]

\[ = \frac{dZ}{\overline{\varphi}'(Z) F_x(0, \overline{\varphi}^{-1} Z)} = \frac{dZ}{\overline{\varphi}'(\overline{\varphi}^{-1} Z) F_x(0, \overline{\varphi}^{-1} Z)} \]

\[ = \frac{\overline{\varphi}(w) dw}{\overline{\varphi}'(w) F_x(0, w)} = \frac{dW}{F_x(0, w)} \quad \text{if } Z = \overline{\varphi}(w) \]

so
\[ \beta \omega^\Omega = \text{transform of } \omega^\Omega \text{ under } \varphi \text{ (or } \overline{\varphi}^{-1}?) \]

\[ \beta(f_* 1) = \beta \text{res } \frac{\omega^\Omega}{Z^{n+1}} = \text{res } \frac{\beta \omega^\Omega}{Z^{n+1}} = \text{res } \frac{dW}{(\varphi^* w)^{n+1} F_x(0, w)} \]

\[ f_*(\beta(dy)) = \text{res } \frac{\omega^\Omega}{\varphi(Z)^{n+1} Z^{n+1}} = \text{res } \frac{\omega^\Omega}{\varphi(Z)^{n+1}} \]

Q.E.D.
Remarks: Above proof uses the formula for $T_*$ as a residue for the theory $Q$. We still have to prove this from the splitting principle.

2) Now take $f: X \to Y$ and choose an embedding $X \times \mathbb{P}^2 \subset \mathbb{P}^N$ and thus construct a factorization of $f$ into

$$
\begin{array}{ccc}
X & \xrightarrow{i} & Y \times \mathbb{P}^N \\
\downarrow{i} & & \downarrow{p_1} \\
Y & \to & Y
\end{array}
$$

where $i$ is a closed immersion whose normal bundle $E$ is isomorphic to $2 \oplus E'$. We know $R-R$ for $p_1$, hence it remains to prove it for $i$.

Changing notation let $i: Y \to X$ be a closed embedding factoring into

$$
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{f} & \tilde{X} \\
\tilde{g} & & \tilde{f} \\
Y & \xrightarrow{i} & X
\end{array}
$$

and blow up $X$ up along $Y$

$$
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{f} & \tilde{X} \\
\downarrow{g} & & \downarrow{f} \\
Y & \xrightarrow{i} & X
\end{array}
$$

$$
E = \nu_i
$$

$$
\begin{array}{cccc}
\ast & \ast & \ast & \ast \\
Y = \mathbb{P}E & o & \mathcal{O}(\cdot)(-1) & g^*E & F & 0 \\
\end{array}
$$

Let $s: Y \to \tilde{Y}$ be the section obtained from a splitting $E$
Then we claim the following formulas hold:

(i) $f_1 \in Q(X)$
(ii) $f^*y = j_*(s_1 \cdot g^*y)$
(iii) $s_1$ is divisible by $j_*(y)$ in $R(Y)$
(iv) $\beta s_1 = \beta(F) \cdot s_1$
(v) $\beta j_*(s_1 \cdot g^*y) = j_*(\beta(y))$ (and (vi))

Assuming these, we prove B-R. for $i \geq i^*$ below:

\[ \beta i_*(\beta(E), y) = j_*(\beta(E) \cdot \beta y) \]

By (i) $f^*$ is injective, so applying $f^*$, using (ii) we have

\[ f^* \beta y = \beta j_*(s_1 \cdot g^*y) \]

\[ f^* i_*(\beta(E), y) = j_*(s_1 \cdot \beta(g^*E) \cdot \beta g^*y) \]

By hypothesis

\[ \beta(g^*E) = \beta(F) \cdot \beta(g^*E) = \beta(F) \cdot \beta(y) \]

so that we have to show

\[ \beta j_*(s_1 \cdot g^*y) = j_*(s_1 \cdot \beta(F) \cdot \beta(y) \cdot \beta g^*y) \]

\[ = j_*(\beta(y) \cdot \beta(s_1 \cdot g^*y)) \]

By (iii) we have

\[ s_1 = u \cdot j_f^*s_1 \]

so

\[ s_1 \cdot g^*y = (u \cdot g^*y) \cdot j_f^*s_1 \]
NEED (vi) \( \beta j^* (f^*a) = \beta j^* 1 \cdot \beta a \)

for some \( a \in \mathcal{P}(x) \). Then the two sides of (\#) become

\[
\beta j^* (f^*a) = \beta j^* 1 \cdot \beta a
\]

so by (v) we are done.

(ii):

\[
S_* y = S_* (s^*g^*y) = s^* 1 \cdot g^* y
\]

Thus (iii) is equivalent to

\[
f_* i_* y = i_* S_* y.
\]

This we may prove as follows: Since we have arranged that \( Y \times \mathbb{R}^2 \to X \) we have a family of embeddings \( i_k : Y \to X \) such that \( l_0 = 1, \ e_1 (y) \cap Y = \emptyset \). Thus we have a cartesian diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{i_k} & \tilde{X} \\
\| & \searrow & \downarrow f \\
Y & \xrightarrow{i_k} & X
\end{array}
\]
Since $f$ is an isomorphism over $X - Y$, and so
\[
(f^*t)_*y = f(t_*)(f^*y) = (f^{-1}t)_*y
\]
but letting $t$ go to zero one puts $f^{-1}t$ in a family of
maps $Y \times \mathbb{P}^2 \to \mathbb{P}^2$ which is $f_*$ at $t = 0$. 

\[
f^*t_y = f^*S_y
\]

(i): Let $k: X - Y \to X$ be the inclusion. Then
\[
k^*(f^*1 - 1) = 0 \quad \text{so} \quad f^*1 - 1 = i_*y.
\]

But
\[
(i_*y)^2 = 0
\]
since $Y$ can be deformed off itself. Thus
\[
(f_*1) - 1 = (1 + i_*y)^{-1} = 1 - i_*y
\]
exists.

(iii): Recall that $E = 1 + E' = 1 + (1 + E)$.
Thus we have
\[
\Omega(fE) = \Omega(Y)[\mathcal{O}] / \mathfrak{g}^2 \left( \mathfrak{g}_{n-2} - c(E^\vee, \mathcal{O}) \mathfrak{g}_{n-3} \ldots \right).
\]

Moreover $\mathfrak{g}^2$ s/ is where $\mathcal{O}(-1) \subset f^*1$ in $f^*E = f^*1 + f^*E'$
or $\mathfrak{g}^2$ where the section
\[
0 \to f^*E \otimes \mathcal{O}(1) \to f^*E' \otimes \mathcal{O}(1)
\]
is zero. Thus
\[
S_*1 = c_{n-1}(f^*E' \otimes \mathcal{O}(1))
\]
\[
= c_{n-1}(f^*E'' \otimes \mathcal{O}(1) + \mathcal{O}(1)) = E_{n-2}(f^*E' \otimes \mathcal{O}(1))c_{n-1}(\mathcal{O}(1))
\]
So \( s_*1 \) is divisible by \( j = c_1(\mathcal{O}(1)) \). But
\[
c_1(\mathcal{O}(1)) = I_{\mathcal{O}(1)}(s) = \delta(1 + \text{higher terms}) \quad \text{(illotted)}
\]

\[
\text{Hence } s_*1 \text{ is divisible by } c_1(\mathcal{O}(1-1)) = \delta^* j^*1. \quad \text{(This follows by arguments given below)}
\]

(iv) As \( E = 1 + E' \), we have by definition of \( \bar{s} \) that
\[
\beta s_*1 = s_*(\bar{s} E')
\]
\[
= \overline{s_{*}}(s(F))
\]
\[
= s_*1 \cdot \bar{s}(F)
\]

(v) As \( \tilde{Y} \rightarrow \tilde{X} \) is a divisor there is a cartesian square

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{f} & \tilde{X} \\
\downarrow{j} & & \downarrow{k} \\
\tilde{X} & \xrightarrow{\pi} & \mathbb{P}(1 + L)
\end{array}
\]

where \( L \) is the line bundle defined by \( \tilde{Y} \). (e.g. \( \ker O_{\tilde{X}} \rightarrow O_{\tilde{Y}} \))

Thus
\[
\beta(j_*1) = \beta \, t^*k_*1
\]
\[
= t^* \beta k_*1 = t^*k_* \bar{s}(L) \quad \text{def of } \beta
\]
\[
= j_*j^* \bar{s}(L)
\]
\[
= j_* \bar{s}(L)
\]
(vi). (I can't prove this without having "tubular neighborhood"). If \( Y \xrightarrow{i} X \) is a divisor, we want to prove that

\[ l^* l_* y = c_1(V_i) \cdot y. \]

Can prove this in the following cases:

A. \( y = \bar{d}^* x \) because then have

\[ Y \xrightarrow{i} X \xrightarrow{f} L \]

\[ X \xrightarrow{s} L \]

\[ l_\ast 1 = f^* f_* 1 = c_1(L) \]

\[ \Rightarrow \quad l^* (l_* y) = l^* (l_\ast 1) = l^* (c_1(L) \cdot x) = c_1 (l^* L) \cdot y = c_1 (V_i) \cdot y. \]

B. If \( i \) can be moved to intersect itself transversally, more precisely if \( \exists \epsilon \in \mathcal{P}(X) \) with \( \epsilon \mid Y \),

let \( Z_{\epsilon} = \) zero set of \( \epsilon + \epsilon^\ast \).

This is non-singular and we have

\[ (l_\epsilon)^* 1 = l^\ast 1. \]
B. If the embedding \( i \) belongs to a family of embeddings \( i_t : Y \to X \) such that \( i_1 \) and \( i_0 \) are transversal. Then

\[
\begin{array}{ccc}
Z & \xrightarrow{\delta} & Y \\
\downarrow i & & \downarrow i \\
Y & \xrightarrow{i} & X
\end{array}
\]

\[\iota_*^* y = j_* j^* y = j_* 1 \cdot y = \]

But \( j_* 1 = \iota_*^* \iota_1 = c_1 (\nu_i) \) by A.

C. When there is a neighborhood of \( Y \) in \( X \) admitting a retraction back to \( Y \),

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{id}} & Y \\
\downarrow \iota & \downarrow \iota \\
Y & \xrightarrow{\iota} & \cup c_x \to X
\end{array}
\]

Then

\[\iota_*^* y = \iota_*^* c_* \iota_* y = \iota_*^* \iota_*^* y = c_1 (\nu_i) \cdot y = c_1 (\nu_i) y\]

using A.

Unfortunately neither of these covers the situation in algebraic geometry. Observe that C covers equivariant cobordism.
To prove R-R we needed the following:

properties of $Q$ (beyond the functorial ones, i.e. its universal property)

- splitting principle
- Chern classes
- formal group law
- half exactness

$f^*$ for $f : \mathbb{P}^n \to \mathbb{P}^1$ as a residue

$g^* j^*_z = (g^* j^*_1) z$ for a codimension 1 embedding

properties of $Q$

- splitting principle
- $f^*$ for $f : \mathbb{P}^n \to \mathbb{P}^1$ as a residue
  (This follows from the splitting principle and the fact that for $\mathbb{P}^n$ we have
  \[ Q(\mathbb{P}^1) \otimes Q(\mathbb{P}^n) \longrightarrow Q(\mathbb{P}^n) \].

Existence of $Q \otimes \mathbb{Z}$ in algebraic geometry is now clear. In effect if $Q$ is a cohomology theory with Gysin homomorphism and products satisfying the splitting principle and (if this doesn't already follow) moreover permitting one to construct Chern classes, then one can construct a unique morphism of theories

$K_{\mathbb{Z}}Q[p_0, p_1, \ldots, I] \longrightarrow Q$

where Gysin on the left is rigged so as to be compatible with the formal group law of $Q$. This is essentially Grothendieck's result that over $\mathbb{Q}$ one can calculate $Q$ as the universal recipient for the Chern classes.
Proofs of \( \theta \): suffices to show that
\[
\bar{\beta}(E) = \prod_{i=1}^{n} \bar{\beta}(L_i)
\]

If \( E \) has a flag with quotients \( L_i \), to consider
\[
f: X \xrightarrow{\sim} \mathbb{P}(1+E)
\]
Then
\[
s_* 1 = \prod_{j=1}^{n} c_1(O(1) \otimes f^*L_j)
\]

(This formula which we have proved when \( E = L_1 + \ldots + L_n \) probably also goes in algebraic geometry, so
\[
\beta s_* 1 = \prod_{j=1}^{n} \beta c_1(O(1) \otimes f^*L_j)
\]

For a vector bundle \( E \) with zero section \( s: X \rightarrow \mathbb{P}(1+E) \) we have
\[
s^* s_* 1 = c_1(L)
\]

(see argument C3 page 11) so
\[
\beta c_1(L) = \beta s^* s_* 1 = s^* \beta s_* 1 = s^* s_* \bar{\beta}(L)
\]
\[
= \prod c_1(L) \cdot \bar{\beta}(L) \quad \text{(argument A, p. 11)}
\]
Thus
\[
\beta s_* 1 = \prod_{j=1}^{n} c_1(O(1) \otimes f^*L_j) \cdot \bar{\beta}(O(1) \otimes f^*L_j)
\]
\[
= s_* 1 \cdot \prod_{j=1}^{n} \bar{\beta}(O(1) \otimes f^*L_j) = s_* s^* (\prod_{j=1}^{n} \bar{\beta}(L_j))
\]
\[
= s_* \prod_{j=1}^{n} \bar{\beta}(L_j) \quad \therefore \bar{\beta}(E) = \prod_{j=1}^{n} \bar{\beta}(L_j).
\]
Proof of (a): We have just shown that if $E$ is a
bundle over $X$ and $g : Y \to X$ is the flag bundle then

\[
\bar{\beta}(g^*E) = \prod_j \bar{\beta}(L_j)
\]

\[
= \prod_j \left( \bar{\beta}(L_j) + e_i(L_j) \cdot Q(c_i(L_j)) \right)
\]

\[
g^* \left( \prod_i \sum c_i(E) \right)
\]

where $\prod_i (c_1(E), \ldots, c_n(E)) \equiv 1 \mod \text{nilpotent elements}$.

Thus

\[
\bar{\beta}(E) = \bar{\beta}(1)^{r^E} \prod (c_1(E), \ldots, c_n(E)) \in Q(X)^*.
\]

Conclusion: I am now sure that for ordinary
complex cobordism I can prove the Thom-R-R isomorphism

\[
\text{Hom}^+(K, \mathbb{Q}) \cong \text{Hom}^*(\Omega, \mathbb{Q})
\]

using only the splitting principle for $\mathbb{Q}$. The argument
ought also to generalize without difficulty to equivariant
cobordism $\Omega^G$ (with $G$ abelian for splitting principle).
Once we understand the formal group + residue picture better
Equivariant cobordism and formal group laws.

In the following $G$ will denote a compact abelian Lie group and $\hat{G}$ its character group. We let $Q$ be a cohomology theory on the category of $G$-manifolds with all the properties we need. $\Omega_G$ denotes the universal such theory. We assume

$$\text{Hom}^+(K,G^*) \cong \text{Hom}^0(\Omega_G,Q^*) = \{ \beta : \beta \text{ natural ring hom. such that } \beta(V) \in Q(pt)^* \text{ for all representations of } G \}$$

and we want to determine the left side. By splitting principle

$$\text{Hom}^+(K,G^*) = \text{Hom}(\text{Pic}_G,G^*)$$

in other words to give an additive transformation $\alpha : K \to G^*$, in effect on line bundles.

In fact we have the formula

$$\alpha(E) = \text{Norm}_{\Omega(\text{Pic}_G)/\Omega(G^*)} \circ \alpha(\mathcal{O}(E)).$$
Lemma: Let $A$ be a ring and let $c_0, \ldots, c_n \in A$. Let

$$A[x_0, \ldots, x_n] = A[x_1, \ldots, x_n] / (c_j = \sigma_j(x_0, \ldots, x_n))_{j=0, \ldots, n}$$

Then $A[x_1] \cong A[x_1] / (x_1^n - c_1 x_1^{n-1} + \cdots)$ and

$$\text{Norm}_{A[x_1] \to A} b(x_1) = \prod_{j=1}^n b(x_j) \quad \text{in } A[x_0, \ldots, x_n].$$

Proof: The assertions are evidently compatible with base extension $A \to A'$ so that one can replace $A$ by a polynomial ring $B[x_1, \ldots, x_n]$. In this case it is well known that $A[x_1, \ldots, x_n]$ is a polynomial ring and that $B[x_1, \ldots, x_n]$ is the ring of invariants for the symmetric group. Moreover one may adjoin elements $\mathbb{Z}/Z^n Z^{n-1} + \cdots$, and otherwise obtain a ring of rank $n!$ mapping onto $A[x_1, \ldots, x_n]$. It follows that they are equal so $A[x_1] \cong A[x_1] / (x_1^n - c_1 x_1^{n-1} + \cdots)$ as claimed. To prove the norm formula one may embed $A$ into a larger ring and so may assume that $c_i = \sigma_i(x_1, \ldots, x_n)$ where $(\lambda_i - \lambda_j)^{-1} \in A$ for $i \neq j$. Then

$$A[x_1] \cong \prod_{j=1}^n A,$$

where $x_1$ acts as $\lambda_j$ on the $i$th factor.

So

$$\text{Norm}_{A[x_1] \to A} b(x_1) = \prod_{j=1}^n b(x_j).$$

By symmetric functions theory, this equals $P(c_1, \ldots, c_n)$ hence in $A[x_0, \ldots, x_n]$ is also equal to $\prod_{j=1}^n b(x_j)$. Q.E.D.
Lemma: Let \( \varphi(x) \in A[x]/\left( \prod_{i=1}^{n} (x-x_i) \right) \). Then \( \varphi \) is a unit iff \( \varphi(x_i) \in A^* \) for all \( i \).

Proof: \( \Rightarrow \) obvious

\[
\prod_{i=1}^{n} \varphi(\lambda_i) \in A^*
\]

As \( \varphi \) is determinant of multiplication it follows that \( \varphi \in A[x]^* \).
Now every line bundle is induced from $\mathcal{O}(1)$ on $\mathbb{P}^n$ for some representation $V$ of $G$. If \( V = \sum_{x \in G} n_x x \) in $R(G)$, where \( n = (n_x) \) is a sequence family with $n_x \geq 0$ and all but a finite number are zero, then

\[
\mathbb{Q}(\mathbb{P}^n) = \mathbb{Q}(pt)[x] / \prod_{x \in G} (x - c_i(x))^{n_x},
\]

where $X = c_i(\mathcal{O}(1))$.

Thus the operation $d : K_G \to \mathbb{Q}^*$ is the same as an element

\[
\varphi(x) \overset{\text{def}}{=} \lim_n \mathbb{Q}(pt)[x] / \prod_{x \in G} (x - c_i(x))^{n_x},
\]

such that $\varphi(c_i(x)) \in \mathbb{Q}(pt)^*$ for all $x \in \hat{G}$.

Cartier's method of working with such a $\varphi$: Consider the category of $\mathbb{Q}(pt)$-algebras and the functor to sets given by

\[
D(A) = \left\{ a \in A \mid \exists n \quad \prod_{x \in \hat{G}} (a - c_i(x))^{n_x} = 0 \right\},
\]

\[
\cong \lim_n \text{Hom}_{\mathbb{Q}(pt)\text{-alg.}} (\mathbb{Q}(pt)[x] / \prod_{x \in \hat{G}} (x - c_i(x))^{n_x}, A).
\]

Thus $\varphi$ is the same thing as a natural transformation from $D$ to $\mathbb{G}_m$, i.e. from $D(A)$ to $A^*$. Therefore we have

\[
\varphi \in \text{Hom}_{\mathbb{Q}(pt)\text{-alg.}} (\mathbb{Q}(pt)[x] / \prod_{x \in \hat{G}} (x - c_i(x))^{n_x}, A).
\]
Proposition:

\[ \text{Hom}^\otimes(\Omega_G, Q) \cong \text{Hom}^+ (K_G, Q^*) \]

\[ \text{Hom}(\text{Pic}_G, Q^*) \]

\[ \text{Hom}(\text{D}, \text{Gm}) \text{ as functors from } Q(\text{pt})\text{-alg to sets} \]

\[ \text{Q}(\text{pt})\{x\} = \lim_{n \to \infty} \frac{\text{Q}[X]/(\prod (X-c_i(x))^{m_X_i})}{\text{Q}[X]/ (\prod (X-c_i(x))^{m_X_i})} \]

\[ \text{Q}(\text{pt})\{x\} = \lim_{n \to \infty} \text{Q}(\text{P}^n) \]

Tenser product of line bundles defines a map

\[ \text{P}^n \times \text{P}^m \to \text{P}(\text{V} \otimes \text{W}) \]

and hence a map

\[ \frac{\text{Q}[X]/(\prod (X-c_i(x))^{m_X_i})}{\text{Q}[X]/ (\prod (X-c_i(x))^{m_X_i})} \otimes \frac{\text{Q}[X]/(\prod (X-c_i(x))^{m_X_i})}{\text{Q}[X]/ (\prod (X-c_i(x))^{m_X_i})} \]

and so by passage to the limit a map

\[ \Delta : \text{Q}(\text{pt})\{x\} \to \text{Q}(\text{pt})\{x \otimes 1 \otimes X\} \]

which is associative + commutative + hence is a generalized formal group law.
In terms of $D$ we have a transformation

$$D_v(A) \times D_w(A) \rightarrow D_{v\circ w}(A)$$

and hence a group law on the functor $D$ which we will denote by $\ast$.

Let

$$T = \lim_{\rightarrow AV} \text{Hom}_{Q(pt)-\text{mod}}(Q(PV), Q(pt))$$

Then $T$ is a commutative bigebra over $Q(pt)$ endowed with coalgebra maps

$$\Theta_x : Q(pt) \rightarrow T$$

for each $x \in \hat{G}$. $\Theta_x(\varphi(x)) = \varphi(c_l(x))$ for $\varphi(x) \in Q(PV)$.

Thus one gets a map of bigebraes

$$Q(pt)[\hat{G}] \rightarrow T$$

and so a map of affine group schemes over Spec $Q(pt)$

$$\text{Spec } T \rightarrow G \times \text{Spec } Q(pt)$$

where we denote by $G$ the group scheme of multiplicative type over Spec $\mathbb{Z}$ whose character group is $\hat{G}$. Thus

$$G = \text{Spec } \mathbb{Z}[\hat{G}]$$
Proposition: If $A$ is a $\mathbb{Q}(pt)$-algebra, then an element of $(\text{Spec } T)(A)$ is the same as a natural ring homomorphism $K_g \rightarrow A \otimes_{\mathbb{Q}(pt)} \mathbb{Q}$.

Proof: An element $\mathfrak{g} \in (\text{Spec } T)(A)$ is a homomorphism $\mathfrak{g} : T \rightarrow A$ of $\mathbb{Q}(pt)$-algebras and may be identified with an element $\mathfrak{g}'$ of $A\{X\}^*$ such that $\Delta \mathfrak{g}' = \mathfrak{g}' \otimes \mathfrak{g}'$. This is the same as an operation $\text{Pic}_G \rightarrow (A \otimes_{\mathbb{Q}(pt)} \mathbb{Q})^*$ which is a homomorphism and hence the same as a ring homomorphism $K_g \rightarrow A \otimes_{\mathbb{Q}(pt)} \mathbb{Q}$.

Example 1: $G = \{e\}$, $Q = H_\ast(C,\mathbb{Z})$. Then $T = H_\ast(P^1,\mathbb{Z})$ is a divided power algebra on 1-generator $b$. If $A$ is a $\mathbb{Z}$-algebra, then a homomorphism $T \rightarrow A$ is the same as giving elements $a_i \in A$ verifying

$$a_i a_j = \frac{(i+j)!}{i! j!} a_{i+j}$$

or setting $q(X) = \sum a_i X^i$, $a_0 = 1$, we get an operation on line bundles $\hat{q}(L) = \sum a_i (\ell^\ast L)^i$ such that $\hat{q}(L_1 \otimes L_2) = \sum a_i (\ell_1^\ast L_1 + c_i^\ast L_2)^i$.

$$= \sum_{j,k} a_{j+k} \frac{(j+k)!}{j! k!} (\ell_1^\ast L_1)^j (c_i^\ast L_2)^k = \hat{q}(L) \hat{q}(L_2).$$
Remarks: If $F(x,y)$ is a formal group law over a ring $k$, then the bigebra $\mathcal{T}$ of distributions on the formal group

$$\mathcal{T} = \text{Hom}^+_k(k[[x]], k)$$

$k$-modules

is the coordinate ring of an affine group scheme over $k$. One checks that as predicted by Cartier duality, a point of

$\text{Spec } \mathcal{T}$

with values in a $k$-alg $A$ is the same thing as a homomorphism of the formal group into $\hat{G}_m$ over $\text{Spec } A$.

Note that this dual is quite different from the Tate $p$-divisible dual and the latter appears to be potentially more interesting.
2) \( G = \{e\}, \ K = K \). Then
\[
T' = \text{Hom}(K(p^n), Z) = \bigoplus \mathbb{Z}(D).
\]
The point of \( T \) with values in \( A \) is a formal power series
\[
\sum (a_n x^n) = (1+x)^a
\]
with all its coefficients in \( A \). Hence for \( A = \mathbb{Z} \) any \( a \) works and we get the operator \( \psi^a \).

3) Assume that \( \left[ c_i(x) - c_i(x') \right] \in Q(pt)^* \) for \( x \neq x' \). Then by the Chinese remainder theorem, for \( A \in Q(pt) \)-alg,
\[
A[x]/(T(x-c_i(x))^a x) \cong \prod_x A[x]/(x-c_i(x))^a x
\]
so
\[
A[x] = \prod_x A[[x-c_i(x)]]
\]

\begin{equation*}
\text{equivalently,}
\begin{align*}
D(A) &= \prod_x D_x(A) \\
&= D_x(A)
\end{align*}
\end{equation*}
for the * operation

where \( D_x(A) = \{ a \mid \exists n \in \mathbb{N} \ (a - c_i(x))^n = 0 \} \). Then in fact as abelian group functors we have
\[
D(A) \cong \hat{G} \times D_0(A)
\]
where \( D_0(A) \) is endowed with a formal group law in the usual sense. Hence as topological Hopf algebras we have
\[
Q(pt)[x] = (Q(pt) \hat{G} \otimes Q(pt))[[x]]
\]
To show that $Q(G)[X] \cong Q(G) \hat{\otimes} Q(G)[X]$ as topological Hopf algebras.

Example: let $Q(F_2) = F_2$, let $G = \mathbb{Z}/2\mathbb{Z}$ and $c_1: (\mathbb{Z}/2\mathbb{Z})^\wedge \rightarrow F_2$ be the only possible additive isomorphism. Then for any $F_2$-algebra $A$ we have

$$D(A) = \{ a \mid \exists n \ (a^2 - a)^n = 0 \}.$$ 

Let the group law be given by

$$a \ast b = a + b.$$ 

Observe this is legitimate since

$$(a + b)^2 - (a + b) = (a^2 - a) + (b^2 - b)^n$$

and since $a - c_1(x_1)$ nilpotent order $n$, $b - c_1(x_2)$ nilpotent of order $m \Rightarrow a + b - c_1(x_1 x_2) = (a - c_1(x_1)) + (b - c_1(x_2))$ nilpotent of order $n + m - 1$.

$$D(A) = \hat{\mathbb{G}}_a(A) \times (\mathbb{Z}/2\mathbb{Z})(A)$$

$a \mapsto ((a^2 - a), \lim_n a^{2^n})$

where $\mathbb{Z}/2\mathbb{Z}$ is the constant group scheme

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow G_a \rightarrow G_a \xrightarrow{x \mapsto x^2 - x}$$

The inverse isomorphism is given by $(x, y) \mapsto (x + x^2 + x^4 + \ldots + y)$.

It follows that as topological Hopf algebras we have

$$F_2 \{ X \} \cong F_2[Y][[X]] \otimes F_2[[T]]/(T^2 - T)$$

$\Delta Y = Y \otimes 1 + \Theta Y$  

$\Delta T = T \otimes 1 + \Theta T$

Note that $F_2 \mathbb{Z}/2\mathbb{Z} = F_2 \mathbb{Z}/2\mathbb{Z} = F_2 1 \otimes F_2 X$  

$X^2 = X$  

$\Delta X = X \otimes 1 + \Theta X$
and therefore

\[ F_2[[T]]/(T^2 - 1) \cong F_2^{\mathbb{Z}/2} \]

just as we conjectured.

Let \( k = \mathbb{Q}(\sqrt{2}) \) to simplify writing, and let

\[ 1 = \sum_{x \in G} \delta_x \]

be the decomposition of 1 with respect to the isomorphism

\[ k\{x\} \cong \prod_x k[[x - c(x)]] \]

As an idempotent in \( k[[x]] \) is determined by its reduction modulo the ideal \( (y) \), it follows that \( \delta_x \) is the unique idempotent element of \( k\{x\} \) such that

\[ \delta_x(c_1(x)) = \begin{cases} 1 & x' = x \\ 0 & x' \neq x \end{cases} \]

Interpreting elements of \( k\{x\} \) as natural transformations \( D(A) \to A \) we have that

\[ \delta_x : D(A) \to A \]

is the unique natural transformation such that

\[ \delta_x(a)^2 = \delta_x(a) \]

\[ \delta_x(c_1(x')) = \begin{cases} 1 & x = x' \\ 0 & x \neq x' \end{cases} \]

This argument clearly also shows that for any \( k \)-algebra \( k' \)
\( \delta_x \) in \( k\{x\} \) is characterized by the same properties. Thus
we have
\[ \delta_X(a \ast x) = \sum_{x_1 \ast x_2 = x} \delta_{x_1}(a) \cdot \delta_{x_2}(x) \]
for \( a \in D(k') \) since as natural transformations of \( x \in D(A) \rightarrow A \) for any \( k' \)-algebra \( A \), they are both idempotent and have the same values for \( x = c_i(x) \), by a similar argument which shows
\[ \delta_X(a \ast c_i(x')) = \delta_X(x')^{-1}(a) \].

Therefore
\[ \Delta \delta_X = \sum_{x_1 \otimes x_2 = x} \delta_{x_1} \otimes \delta_{x_2} \]
In other words we have a map of topological Hopf algebras
\[ \hat{\mathbb{G}} \rightarrow \mathbb{K}\{X\} \]
\[ f \rightarrow \sum f(x) \delta_X \]
which is a section of the map in the opposite direction given by \( \varphi \rightarrow (x \mapsto \varphi(c_i(x))) \).

I claim we have the following split exact sequence of abelian group functors
\[ 0 \rightarrow D_0(A) \xrightarrow{i} D(A) \xrightarrow{s} \hat{\mathbb{G}}(A) \rightarrow 0 \]
where \( D_0(A) \) is the set of nilpotent elements of \( A \) with group law \( \ast \), \( i \) in the inclusion, \( \hat{\mathbb{G}} \) is the group scheme associated to the group \( \hat{\mathbb{G}} \), i.e.
\[ \hat{G}(A) = \{ \text{partitions} \quad 1 = \sum \alpha_x, \quad \alpha_x \in A, \quad \alpha_x \alpha_{x'} = 0 \quad x \neq x' \} \]

\[ = \text{Hom(} \text{Spec} A, \hat{G}) = \text{locally constant functors on Spec} A \text{ with values in} \hat{G} \]

and where

\[ \pi(x) = \left( 1 = \sum \delta_x(a) \right) \]

\[ \delta(x) = \sum_\alpha \delta_{\alpha}(c(x)) \]

\[ S(1 = \sum \alpha_x) = \sum_\alpha \alpha_x c(x). \]

\( S \) is well-defined because in fact if \( z = \sum \alpha_x c(x) \), we have

\[ \pi(z - c(x)) = 0 \quad \forall \alpha_x \neq 0 \]

as one sees locally on Spec \( A \). Given a partition \( 1 = \sum \alpha_x \)

\[ \pi: A \to \prod \alpha_x \pi \]

\[ \delta_x \left( \sum \alpha_x c(x) \right) = \delta_x \left( c(x) \right) = \{ 0 = \alpha_x \} \quad \text{in} \ A \]

for each \( x ' \Rightarrow \pi S = 0 \). Finally we calculate \( \text{ker} \pi \)

\[ = \{ a \mid \delta_x(a) = 0 \quad \text{for all} \ x \neq 1 \} \]

But this means that the map \( k[x] \to A \) sending \( x \) to \( a \) has \( \delta_x \) in its kernel

for \( x \neq 1 \), hence it factors through \( k[[x]] \to A \), which is true

iff \( a \) is nilpotent.

Thus from the exact sequence we have

\[ D(A) \cong D_0(A) \times \hat{G}(A) \]

or an isomorphism of topological Hopf algebras.
\[ k[x] \sim \hat{k} \otimes k[x] \]

where \( \Delta X = F(x \otimes 1, 1 \otimes x) \) is an ordinary formal group law.

We can now read off the group-like elements in \( k[x] \). First of all a group-like in \( \hat{k} \) is the same as a homomorphism \( \hat{G} \to k^* \), or the same as a point of the algebraic group \( G \) with values in \( k \).

Thus

**Proposition:** 1) If \( c_1(x) - c_1(x') \in \mathbb{Q}(pt)^* = k^* \) for \( x \neq x' \), then to each element \( g \in G(k) \), there is a unique natural ring homomorphism \( \Theta_g : k_G \to \mathbb{Q} \) such that

\[
\Theta_g(x) = x(g) \quad \text{in } \mathbb{Q}(pt)
\]
\[
\Theta_g(O(1)) = 1 \quad \text{if } O(1) \text{ is the canonical line bundle on } P^n \text{ with trivial } G \text{-action and } k \subseteq k_G.
\]

2) If in addition \( k \subseteq \mathbb{Q} \), then there is a unique operation \( ch_1 : k_G \to \mathbb{Q} \) such that

\[
ch_1(x) = 1 \quad \text{in } \mathbb{Q}(pt)
\]
\[
ch_1(O(1)) = 1 + \lambda c_1(O(1)) + \ldots
\]

if \( O(1) \) is the line bundle on \( P^n \) with trivial \( G \)-action.

3) Under the hypotheses of 1) + 2) any ring homomorphism \( k_G \to \mathbb{Q} \) is the extension of the homomorphism...
$\Theta_g \cdot ch_x : \text{Pic}_0 \rightarrow \mathbb{Q}$ given by

$$(\Theta_g \cdot ch_x)(L) = \Theta_g(L) \cdot ch_x(L).$$

Problem: Can you generalize your old result that $\Omega \otimes \mathbb{Q}$ is calculable in terms of $K \otimes \mathbb{Q}$ to obtain the localization $\Omega \otimes \mathbb{Q} \left[ \frac{c_i(x) - c_i(x')} {x \neq x'} \right]$, from $K$.

Note that the corresponding localization of $K$ and that of $\mathbb{Q}$ might be zero, e.g., $G = \mathbb{Z}/G\mathbb{Z}$, $R(G) = \mathbb{Z}[G]/(G^{G-1})$.

$$R(G) \otimes \mathbb{Q} = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}[\alpha] \times \mathbb{Q}[\alpha]$$

With $\alpha = 1$, $T \rightarrow T$, $T \rightarrow T^2$. In any case, one gets that different powers $T^i$ becomes the same in each factor hence

$$R(G)[\mathbb{T}^{-1}, \mathbb{T}^i]_{i \neq 0, 1},$$

Note that $K \otimes \mathbb{Q} \left[ \frac{1} {x - x'} \right]_{x \neq x'} = K \left[ \frac{1} {x - x'} \right]_{x \neq x'} \otimes \mathbb{Q}$ is identically zero for the finite group $(\mathbb{Z}/2\mathbb{Z})^2$, since $R(G) = \text{group ring} \mathbb{Q}[(\mathbb{Z}/2\mathbb{Z})^2] = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ and in each factor two different elements of $(\mathbb{Z}/2\mathbb{Z})^2$ coincide.
Let $Q$ be a theory as above such that $c_1(x) - c_1(x') \in Q(pt)^*$ and $Q \subseteq Q(pt) = k$. Then the "formal group law" is known once one knows the character or equivalently the logarithm for the formal group induced on $D_0$. One has necessarily that

$$\ell(x) = \sum_{n \geq 0} P_{n-1} \frac{x^n}{n}$$

where $P_{n-1} = [P_{n-1}]$ in $k$. In effect by the projective bundle formula one has

$$P_{n-1} = \text{res} \left[ \frac{d\ell(z)}{z^n} \right].$$

(Question: Does $w$ always exist for these formal groups?)

Thus to the theory $Q$ we have the invariants

$$\begin{cases} c_1(x) \in Q(pt) \quad, \quad c_1(1) = 0 \\ P_n \in Q(pt) \quad, \quad P_n = 1 \end{cases}$$

which makes reasonable the following

**Conjecture:** $\Omega_Q(pt) \left[ \frac{1}{c_1(x) - c_1(x')} \right] \otimes Q \cong Q[P_n, c_1(x), \frac{1}{c_1(x) - c_1(x')}]$

for $n > 0$, $x \neq 1$, $x \neq x'$.

To prove this conjecture we shall have to construct a theory $Q$ with $Q(pt)$ the given candidate.