SMALE NOTES QUILLEN

153

1) Differentiable dynamical systems Smale { Shub, Kopell }

2) Manifolds of sections of fiber bundles
non-linear analysis, calculus of variations { Palais, Uhlenbeck }

3) Transversality theory + applications

4) Theory of singularities (Mather) H. Levine

5) Morphogenesis (Thom)

6) "Can one hear the shape of a drum"
Differential topology invariants from
spectrum of geometric operators
In particular Reidermeister torsion for heat equation
I. Singer

7) Index theory for Fredholm operators – Koschorke.

Entropy of metric Autos. Rohlin 124 (1959) 980–983

Sinai Dokl. 124 (1959) 768–771

Prof- Quillen

Expanding Endomorphisms of Flat Manifolds

by

David Epstein and Michael Shub[1]

Let M be a compact differentiable manifold without boundary. A C^1-endomorphism $f: M \longrightarrow M$ is expanding if for some (and hence any) Riemannian metric on M there exist $c>0$, $\lambda>1$ such that $||Tf^m v|| \geq c\lambda^m||v||$ for all $v \in TM$ and all integers $m>0$. In this paper we show that any compact manifold with a flat Riemannian metric admits an expanding endomorphism. The classification of expanding endomorphisms, up to topological conjugacy, was studied in [3]. It is of interest not only abstractly but also because the inverse limit of an expanding endomorphism can be considered as an indecomposable piece of the non-wandering set of a diffeomorphism: see [4] and [5].

Preliminaries: We require some standard facts from differential geometry which may all be found in [6]. Let $E(n)$ denote the group of isometries of R^n. So $E(n)$ is the semi-direct product $O(n)\cdot R^n$, where $O(n)$ is the orthogonal group. We may consider a compact flat manifold as the orbit space R^n/Γ where Γ is a discrete uniform subgroup of $E(n)$. Such a group Γ is called a crystallographic or Bieberbach group. Two of the Bieberbach theorems on these groups are:

Theorem 1. (Bieberbach) If $\Gamma \subset E(n)$ is a crystallographic group then $\Gamma \cap R^n$ is a normal subgroup of finite index in Γ, and any minimal set of generators of $\Gamma \cap R^n$ is a vector space basis of R^n relative to which the $O(n)$-components of the elements of Γ have all entries integral.

1 This work was done while this author was partially supported by NSF GP-6868.

Theorem 2. (Bieberbach). Any isomorphism $f: \Gamma \longrightarrow \Sigma$ of crystallographic subgroups of $E(n)$ is of the form $\gamma \longrightarrow B\gamma B^{-1}$ for some affine transformation $B: R^n \longrightarrow R^n$.

Theorem 1 is as stated in [6;3.2.1], and Theorem 2 is as stated in the proof of [6,3.2.2]. Moreover $\Gamma/\Gamma\cap R^n$ is isomorphic to the holonomy group of M, [6;3.4.6]. Henceforth, we will write A for $\Gamma\cap R^n$ and F for $\Gamma/\Gamma\cap R^n$. The corresponding exact sequence is $0 \longrightarrow A \longrightarrow \Gamma \longrightarrow F \longrightarrow 0$; it will be called the exact sequence associated to M. Recall that an invertible affine map $B: R^n \longrightarrow R^n$ projects to an endomorphism of $M = R^n/\Gamma$ if the map $\gamma \longrightarrow B\gamma B^{-1}$ maps Γ into itself that is $B\Gamma B^{-1} \subset \Gamma$. The induced map on M is an expanding endomorphism if the eigenvalues of the linear part of B are all greater than one in absolute value; in which case the induced map on M is called an affine expanding endomorphism.

Construction of affine expanding endomorphisms.

We begin with examples of affine expanding endomorphisms of the n-torus, T^n. Consider T^n as R^n/Z^n where Z^n is the integral lattice. Let B_1 be an n by n matrix such that all the entries of B_1 are integers and all the eigenvalues of B_1 are greater than one in absolute value. B_1 may be thought of as a linear map $B: R^n \longrightarrow R^n$ such that $B(Z^n) \subset Z^n$. Thus considering Z^n as a group of translations operating on R^n, $B(Z^n) \subset Z^n$ and B defines an affine expanding endomorphism of T^n. Examples of such B's are provided by $k \cdot I_{R^n}$ where k is an integer not equal to -1, 0, or 1 and I_{R^n} is the identity map of R^n.

The torus, T^n, corresponds to $\Gamma = \Gamma\cap R^n = A$. We now consider the case where F has more than one element. The symbol $|F|$ denotes the order of F.

Notations: Let $B: R^n \longrightarrow R^n$ be an affine map. Then $B = L_B + v_B$ where L_B is a linear map and v_B denotes translation by the vector v_B.

We will prove the following theorem:

Theorem. Let M be a compact flat Riemannian manifold with associated exact sequence: $0 \longrightarrow A \longrightarrow \Gamma \longrightarrow F \longrightarrow 0$. Let $|F| > 1$ and let k be an integer greater than 0. Then there is an affine map $B: R^n \longrightarrow R^n$ such that $L_B = (k|F| + 1)\cdot I_{R^n}$ and B projects to an affine expanding endomorphism of M.

As an immediate and obvious corollary we have:

Corollary: Any compact flat Riemannian manifold is a non-trivial covering space of itself.

We proceed as follows: We look for a commutative diagram

$$(*) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \Gamma & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle I_F} & & \\ 0 & \longrightarrow & A & \longrightarrow & \Gamma & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

such that L is $(k|F| + 1)\cdot I_A$. For then, since L is injective, $f: \Gamma \longrightarrow \Gamma$ is a monomorphism. Thus, by Theorem 2 (Bieberbach), there is an affine transformation $B: R^n \longrightarrow R^n$ such that $f(\gamma) = B\gamma B^{-1}$ for $\gamma \in \Gamma$. Thus B projects to an endomorphism of M and $L_B|A = L = (k|F| + 1)I_A$. But by Theorem 1 (Bieberbach) A contains a vector space basis of R^n so $L_B = (k|F| + 1)\cdot I_{R^n}$.

Lemma 1: Given $(*)$ with L injective then $L_B|A = L$.

Proof: $A = \Gamma \cap R^n$, so if $a \in A$ we consider a as the translation $x \longrightarrow x + a$.

Now $B^{-1} = L_B^{-1} - L_B^{-1}(v_B)$. So $BaB^{-1}(x) = x + L_B(a)$ and $f(a) = BaB^{-1} = L_B(a)$.

We now show the existence of a diagram (*) with the required L's. A is considered as a left Γ module under conjugation. Since A is abelian the action of A on itself is trivial and thus the action of Γ on A induces an action of F on A. Under these conditions A^A, the elements of A left fixed under the action of A, equals A. $H^1(A,A)^\Gamma$, the Γ invariant elements of $H^1(A,A)$, is just $\mathrm{Hom}^\Gamma(A,A)$, the Γ module endomorphisms of A. (See [2] and [1, p. 190]). Thus the exact sequence in the remark [2, p. 130] becomes for this case:

(I) $0 \longrightarrow H^1(F,A) \longrightarrow H^1(\Gamma,A) \longrightarrow \mathrm{Hom}^\Gamma(A,A) \longrightarrow H^2(F,A) \longrightarrow H^2(\Gamma,A)$.

$H^1(\Gamma,A)$ is the group of all crossed homomorphisms $\psi: \Gamma \longrightarrow A$ (i.e. all functions satisfying $\psi(xy) = x\psi(y) + \psi(x)$ for $x,y \in \Gamma$) modulo the principal crossed homomorphisms (i.e. functions of the form $\psi(x) = xa - a$ for a fixed $a \in A$). The map $H^1(\Gamma,A) \longrightarrow \mathrm{Hom}^\Gamma(A,A)$ in the sequence is just the restriction map.

<u>Lemma 2</u>: There is a correspondence between crossed homomorphisms $\psi: \Gamma \longrightarrow A$ and diagrams (*), defined by $\psi(x) = f(x)x^{-1}$.

<u>Proof</u>: If

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & \Gamma \arrow[r, "p"] \arrow[d, "f"] & F \arrow[r] \arrow[d, "I_F"] & 0 \\ 0 \arrow[r] & A \arrow[r] & \Gamma \arrow[r, "p"] & F \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram, then $p(x) = p(f(x))$ for $x \in \Gamma$. So $f(x)x^{-1} \in \operatorname{Ker} p$ and there is a unique $a \in A$ such that $f(x)x^{-1} = a$.

Now $\psi(xy) = f(xy)(xy)^{-1} = f(x)f(y)y^{-1}x^{-1} = f(x)x^{-1}\, xf(y)y^{-1}x^{-1}$ which is in additive notation $\psi(x) + x\psi(y)$. On the other hand if $\psi\colon \Gamma \longrightarrow A$ is a crossed homomorphism then $f(x) = \psi(x)x$ defines a homomorphism

$f\colon \Gamma \longrightarrow \Gamma$; for $\psi(xy)xy = \psi(x)x\ \psi(y)x^{-1}xy = \psi(x)x\ \psi(y)y$ and

$f(x)x^{-1} = \psi(x)xx^{-1} = \psi(x) \in A$. So f induces the identity map on F. That is, the crossed homomorphism ψ corresponds to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & \Gamma & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle I_F} & & \\
0 & \longrightarrow & A & \longrightarrow & \Gamma & \longrightarrow & F & \longrightarrow & 0
\end{array}$$

where $f(x) = \psi(x)x$ and $L(a) = \psi(a) + a$ for $x \in \Gamma$ and $a \in A$.

<u>Proof of the Theorem:</u> I_A is obviously a Γ module endomorphism of A. Since $|F| \cdot v = 0$ for all $v \in H^2(F,A)$; see [6; p. 236], $k|F| \cdot I_A \in \mathrm{Hom}^{\Gamma}(A,A)$ which is sent to 0 in $H^2(F,A)$ by the map in (I). Thus by the exactness of (I), there is a crossed homomorphism $\psi\colon \Gamma \longrightarrow A$ such that considered as a crossed homomorphism $\psi\,|A = k|F| \cdot I_A$. Thus $f(x) = \psi(x)x$ restricts to $L\colon A \longrightarrow A\colon L(a) = (k|F| + 1) \cdot I_A$.

David Epstein
Warwick University and University of California, Berkeley

Michael Shub
University of California, Berkeley and Brandeis University, Waltham, Mass.

References

1. Cartan, H. and Eilenberg, S. - Homological Algebra - Princeton University Press, Princeton, New Jersey, 1956.

2. Hochschild, G and Serre, J. P. - Cohomology of Group Extensions - Trans. Amer. Math. Soc., vol. 74, 1953, pp. 110-134.

3. Shub, M. - Endomorphisms of Compact Differentiable Manifolds, to appear.

4. Smale, S. - Differentiable Dynamical Systems - Bull. Amer. Math. Soc., Nov. 1967.

5. Williams, R. F. - One Dimensional Non-Wandering Sets, to appear.

6. Wolf, J. - Spaces of Constant Curvature, McGraw Hill, New York, 1967.

Entropy

Claim that

$$(Af)(x) = \sum_{y \neq x} f(y)$$

commutes with expectation operators

$$(E_\pi A f)(x) = \frac{1}{\mu(P)} \int_P Af \qquad x \in P$$

$$= \frac{1}{\operatorname{card} P} \sum_{y \in P - \{x\}} f(y)$$

$$(A E_\pi f)(x) = \sum_{y \neq x} (E_\pi f)(y)$$

$$= \sum_Q \sum_{y \in Q} \frac{1}{\operatorname{card} Q} \sum_{z \in Q} f(z)$$

$$(Af)(x) = n \int f - f(x).$$

clearly commutes since

$$A = n \int$$

$$E_\varepsilon \qquad E_0$$
$$\| \qquad\quad \int$$
$$id$$

$$E_0 E_p = E_p E_0 = 0.$$

iden

perfectly clear?

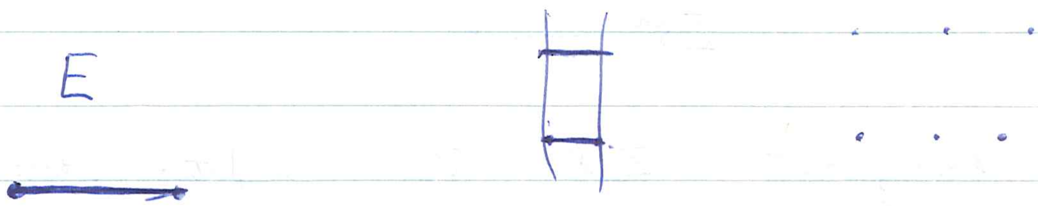
If P, Q independent then

$$E_P E_Q = E_{P\wedge Q} = E_Q E_P$$

$\cancel{E_P(\chi_A f)}$

What operator commutes with all E_P?

E



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

$$a+b = c+d.$$

$$\begin{matrix} x+y & x \\ x+y & y \end{matrix}$$

$$E \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} \underline{a+b} & \underline{a+b} \\ \underline{c+d} & \underline{c+d} \end{pmatrix} = \begin{pmatrix} \underline{a+c} & \underline{b+d} \\ \underline{a+c} & \underline{b+d} \end{pmatrix}$$

$b=c$
$a=d$

first claim is that $\underline{G} \to F_{\underline{G}}$

$$\begin{tikzcd} s(\text{pro-}p\text{-gps}) \arrow[r] \arrow[d] & \text{pro-}p\mathcal{H} \arrow[d] \\ Ho(s\text{-}p\text{gp.}) \arrow[ur] \arrow[r, dashed, "\approx"'] & (\text{pro-}p\mathcal{H})^b \end{tikzcd}$$

clear because ~~one gets a flat isomorphism~~ Artin-Mazur show that

$f: \mathcal{X} \to \mathcal{Y}$ is a flat isom

$\iff H^*(\mathcal{Y}) \xrightarrow{\sim} H^*(\mathcal{X})$.

which means an isom when X has only finitely many homotopy groups.

~~[illegible]~~ Suppose $X \overset{Ob}{\in} p\mathcal{H}$, then obtain

---

Thus $E_Q E_P = E_P E_Q = E_P \qquad P \leq Q$.

Now examine the algebra of operators on the fns. obtained in this way!

$E_P(\chi_A f)$ on P_j $\qquad E_P(\chi_A f) = E_P(\chi_A) \cdot E_{P \cap A}(f)$.

$$= \frac{1}{\mu(P_j)} \int_{P_j} \chi_A f = \frac{1}{\mu(P_j)} \int_{A \cap P_j} f$$

for each pt. $z \in$

what operators commute with expectation operators?

n=3.

$$\begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}\begin{pmatrix} f(x) \\ f(y) \\ f(z) \end{pmatrix} = \begin{matrix} \frac{1}{2}(f(x)+f(y)) \\ \frac{1}{2}(f(x)+f(y)) \\ \frac{1}{3}f(z) \end{matrix}$$

$$\begin{pmatrix} a & b & e \\ b & a & c \\ e & c & a \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a & b & e \\ b & a & c \\ e & c & a \end{pmatrix}$$

$$\begin{pmatrix} a+b+e & a+b+e & a+b+e \\ b+a+c & b+a+c & b+a+c \\ e+c+a & e+c+a & e+c+a \end{pmatrix} = \begin{pmatrix} a+b+e & b+a+c & e+c+a \\ a+b+e & b+a+c & e+c+a \\ a+b+e & b+a+c & e+c+a \end{pmatrix}$$

$e=c$
$b=c$
$b=e$

$$\begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$$

only operators wh

$$\begin{pmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{pmatrix}$$

Rest of simplicial profinite groups paper

(A) Homotopy theory of simplicial profinite groups.

show axioms for closed simplicial model category are satisfied

free simplicial profinite groups

(B.) Comparison with Artin-Mazur theory

~~In other ...~~

Part B analysis.

Let $p\mathcal{H}$ be the category

objects – ptd. connected CW cxs. $\ni$ $\pi_q X$ finite p-gps. $\forall q$.

maps – homotopy classes of continuous maps.

also objects are reduced simplicial ~~sets~~ finite satisfying extension conditions + having $\pi_q X$ a p-gp. $\forall q$.

~~For each simplicial pro-p gp. G~~

pro-p-$\mathcal{H}$ = filtered inverse systems of objects in $p\mathcal{H}$

= functors $F: p\mathcal{H} \longrightarrow (\text{sets})$ which are of the form

$$F(X) = \varinjlim_{i \in I} \operatorname{Hom}_{p\mathcal{H}}(X_i, X)$$

where I is a filtering cat and $i \longmapsto X_i$ is a contrav. functor.

Given a simplicial pro-p-gp. $\underline{G}$ define

$$F_{\underline{G}}(X) = \varinjlim_{u} \operatorname{Hom}_{p\mathcal{H}}(\overline{W}(\underline{G}/\underline{u}), X).$$

invariant forms Hodge theory e.g.

$$f^* \omega = \lambda \omega$$

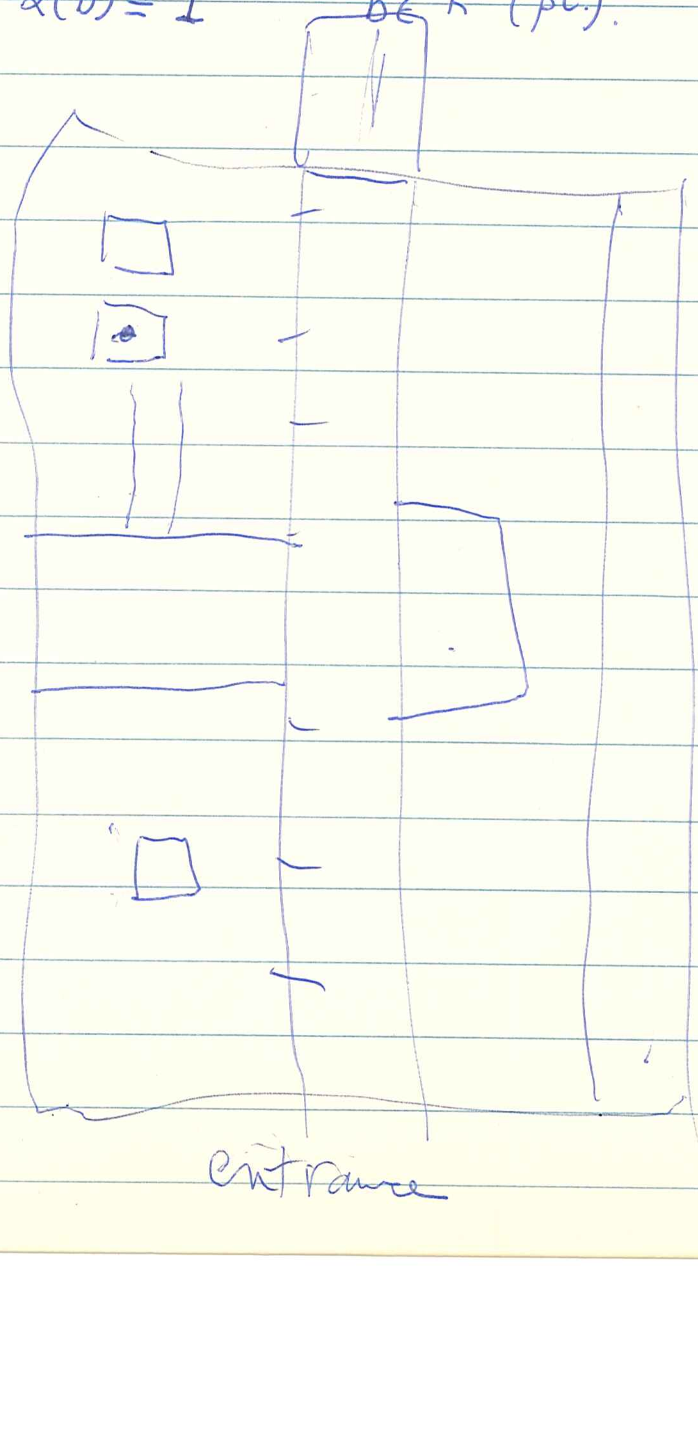
consequently one can look for the spectrum of

$$\varprojlim_{r} \pi_*\left(G/\Gamma^p_r G\right) \overset{?}{=} \pi_*(G)\otimes \mathbb{Z}_p$$

X locally compact ~~[illegible]~~, then $K(X) = \mathrm{Ker}\, K(X^+) \to K(\infty)$

$$K(X) \underset{\alpha}{\overset{\beta}{\rightleftarrows}} K^{-2}(X)$$

1) α functorial
2) α $K(X)$ module hom.
3) $\alpha(b) = \overset{\otimes}{1}$ $\quad b \in K^{-2}(pt.)$



entrance

Increasing family of partitions i.e.

$$P_\lambda \le P_\mu \quad \text{if} \quad \lambda \le \mu$$

to each partition π we have a expectation operator

$$(E_\pi f)(y) = \int_{\pi_y} f$$

$$P \le Q \qquad \underline{E_Q E_P} = E_P$$

$$E_P E_Q = E_P$$

$$(E_P E_Q f)(x) = \int_{P_x} \underline{(E_Q f)} = \sum_{\substack{Q_i \in P_j \\ \ni x}} \mu(Q_i) \int_{Q_i} f$$

$$(E_Q f) = \int_{Q_i} f \quad \text{on } Q_i.$$

$$E_P E_Q f = \frac{1}{\mu(P_j)} \int_{P_j} E_Q f \quad \text{on } P_j = \bigvee Q_i$$

$$= \frac{1}{\mu} \sum \cancel{\mu(Q_i)} \int_{Q_i} f$$

Problems.

Lattices

subsets	μ	fns.	$\int$
subspaces	dim	operators	tr
partitions	H	$\boxed{\quad\cdots\quad}$	

analogue of a

Given a function f get an increasing family of ~~measures~~ subsets

$$A_\lambda = \{x \mid f(x) \le \lambda\}$$

with

$$\bigcup_\lambda A_\lambda = X \qquad \text{real fn. } f.$$

$$\bigcap_\lambda A_\lambda = \phi$$

Similarly given an operator A real i.e. self ~~adj~~ adjoint get subspace

$$E_\lambda \qquad \text{partition of } 1.$$

Therefore the analogue for partitions is to take an increasing family of partitions

$$P_\lambda$$

and what can one do to these par

which case ~~it is absolutely critical~~ there should be a homotopy operator. I propose to define such a homotopy operator as a pseudo-quasi-differential-operator. By necessity such an operator will expand supports slightly.

In the case of a 2-dim$^{\underline{l}}$ manifold it follows from Mike's thm.

Start with a fun.

Problem: To each transformation $f^{-n}\gamma f^{n}$ and measurable set A we can consider

$$\text{dist}\,(A, f^{-n}\gamma f^{n}A)$$

$$\|$$

$$d^{n}\,\text{dist}\,(f^{n}A, \gamma f^{n}A)$$

Problem is to show that the set of γ's for which this is true is a subgroup of finte index.

$$d^{n}\,\text{dist}\,(f^{n}A, \gamma_1\gamma_2 f^{n}A) \le d^{n}\{\text{dist}\,(f^{n}A, \gamma_1 f^{n}A) + \text{dist}\,(\gamma_1 f^{n}A, \gamma_1\gamma_2 f^{n}A)\}.$$

hence is a subgroup.

normal?

$$\text{dist}\,(f^{n}A, \gamma\gamma_1\gamma^{-1}f^{n}A)$$

$$\|$$

$$\text{dist}\,(\gamma^{-1}f^{n}A, \gamma_1\gamma^{-1}f^{n}A) \qquad f^{n}\boxed{f^{-n}\gamma^{-1}f^{n}A}$$

$$\text{dist}\,(f^{n}\underbrace{(f^{-n}\gamma^{-1}f^{n})}A, \gamma_1 f^{n}\underbrace{(f^{-n}\gamma^{-1}f^{n})}A)$$

We have shown that f has an invariant measure μ which we may lift upstairs to $\tilde{M}$. Upstairs it is not invariant however satisfies

$$\mu(f^{-1}A) = \frac{1}{d}\mu(A).$$

Now suppose we can choose a partition finite of M with the property that $f^{-1}a \leq a$. In other words

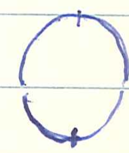
If it is also true that $\bigvee f^{-n}a = 0$ partition, then clearly

$$H(f,a) = H(a/\bigvee_{n>0} f^{-n}a)$$

$$= \lim_{n\to\infty} H(a/f^{-n}a) \qquad \text{no.}$$

$f^{-1}a$ independent

Then are independent



$$H(a \vee f^{-1}a / f^{-1}a) = H(a)$$

$$H(a \vee f^{-1}a \vee \cdots \vee$$

Then $H(f) = H(a)$.

Mike Shub claims

suppose (M,f) expanding map and $(\bar{M},\bar{f})$ is another such that with suitable choice of fixpts

$$\pi_1 f \simeq \pi_1 \bar{f}.$$

Then there is a ! homeomorphism ~~joining~~ of f and $\bar{f}$ preserving basepts. The proof **consists** of ~~looking at the continuous m~~ choosing a continuous map from M to $\bar{M}$ which is in the right homotopy class and then smoothing it out by f.

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & \bar{M} \\ \downarrow f & & \downarrow \bar{f} \\ M & \xrightarrow{\varphi} & \bar{M} \end{array}$$ homotopy commutative.

$$\tilde{M} \xrightarrow{\varphi} \bar{\tilde{M}}$$

$\varphi(x\gamma) = \varphi(x)\gamma.$

$\varphi(f(x)) \sim \bar{f}(\varphi(x)).$

replace φ by

$$(\bar{f}^{-1}\varphi f)(x\gamma) = \bar{f}^{-1}([\varphi(fx)](f\gamma))$$

$$= [(\bar{f}^{-1}\varphi f)(x)]\cdot\gamma$$

Then $\bar{f}^{-1}\varphi f$ is smoother

Clearly $(\bar{f})^{-1}\varphi f^{-1}$ choose φ nice.

$\bar{f}^{-1}\varphi f$ $\varphi f \sim \bar{f}\varphi$ Γ equivariant homotopy.

Problem The group I am after is the closure in the topological group of homeom. of $\tilde{M}$.

[illegible] cont. maps.

On universal covering there is a metric + each f γ is uniformly continuous in this metric.

x, y join by a curve of length $\leq d(x,y)+\varepsilon$

$$\Rightarrow d(fx, fy) \leq \lambda\, d(x,y) \qquad \lambda \text{ max. expansion coeff. on } M.$$

Also $$d(x\gamma, y\gamma) = d(x,y).$$

Therefore consider closure in the space of uniform cont. maps. $M \to M$. ?

$$\varphi \in \mathrm{Maps}(M, M) \quad \ni \quad d(\varphi x, \varphi y) \leq C_\varphi d(x,y).$$

Problem is to show that

$$\frac{d\left(f^{-n}\gamma f^n x,\, f^{-n}\gamma f^n y\right)}{d(x,y)} \leq C$$

independent of n, γ.

Can you characterize the sheaf of solutions of a good linear differential equation?

A. Every point x has a system of neighborhoods U such that $H^q(U, \mathcal{S}) = 0 \quad q > 0$.

B. There is a resolution of $\mathcal{S}$ by vector bundles

$$0 \to \mathcal{S} \to \underline{E}^0 \to \underline{E}^1 \to \cdots$$

C. $\mathcal{S}$ is a sheaf of Frechet Montel spaces and for every open nbd U of x there is a V such that

Fairly important if U is a small smooth ~~convex~~ ball around x then $H^q(U, \mathcal{S}) = 0$.

Typical theorem (Peetre) is that any $\varphi : \underline{E} \to \underline{F}$ is a differential operator.

160

Mackey's presentation of statistical mechanics

A homogeneous body – to be thought of as gas in a cylinder.

V_A volume of A

T_A temperature. Mackey wants T to be thought of as belonging to an ordered set. ~~Assume some parametrization of T as follows for $G_A(T,V)$ such that The idea~~ Experimentally one notes that if A and B are bodies at temperatures T_A and T_B and if placed in contact they approach an equilibrium temperature $E_{A,B}(T_1,T_2)$ and that there is a numerical function $H_A(T_1,T_2)$ called the quantity of heat required to move A from temperature T_1 to temperature T_2 such that

$$H_A(T_1, E_{A,B}(T_1,T_2)) + H_B(T_2, E_{AB}(T_1,T_2)) = 0$$

$$H_A(T_1,T_2) + H_A(T_2,T_3) = H_A(T_1,T_3).$$

If A_1 and A_2 are of same substance

$$\frac{H_{A_1}(T_1,T_2)}{m_{A_1}} = \frac{H_{A_2}(T_1,T_2)}{m_{A_2}}$$

masses

Except that in all this V doesn't change. So actually we have

$$H_A^\circ(T_1, V)$$

$$H_A(T_1, T_2; V) = H_A(T_2, T_0; V) - H_A(T_1, T_0; V)$$

set $$C_A(T, V) = \frac{\partial H_A}{\partial T}(T, T_0; V).$$

$C_A(T,V)$ called specific heat of A at temp T and Volume V

Note that it depends on a parameterization of temperature.

(change notation from $H_A(T,V)$ to $H_A(V,T)$).

<u>First law of therm.</u> says there is a fn. $U_A(V,T)$ such that

$$U_A(V, T_1) - U_A(V, T_2) = H_A^\circ(V, T_1) - H_A^\circ(V, T_2)$$

and if (V_1, T_1) and (V_2, T_2) lie on an adiabatic curve then

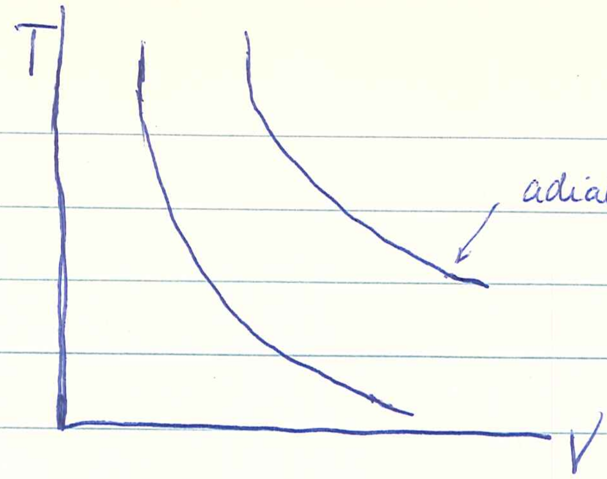
$U_A(V_1, T_1) - U_A(V_2, T_2) = -$ total mechanical energy needed to move system from V_1, T_1 to V_2, T_2

$U_A(V,T)$ is the <u>internal energy of the system A</u> at volume V and temperature T. Thus

~~$\frac{\partial U_A}{\partial T} = C_A$~~ specific heat.

$$\boxed{\frac{\partial U_A}{\partial T} = C_A}$$

Picture



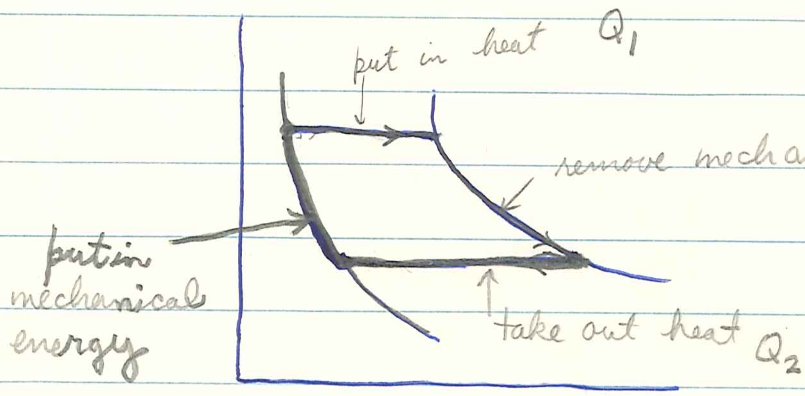
$$\lambda_A = \frac{dT}{dV} \quad \text{along adiabatic curve}$$

Then λ_A and U_A determine everything.

Mackey introduces <u>*state*</u> *or pressure* <u>*fns.*</u>

$$P_A = - \left\{ \begin{array}{l} \text{rate of change of } U_A \text{ wrt } V \text{ along} \\ \text{adiabatics} \end{array} \right\}$$

<u>Carnot</u> cycle



$$\text{efficiency} = \frac{Q_1 - Q_2}{Q_1} = \alpha(T_1, T_2).$$

2nd law of thermodynamics

160

Mackey claims $\exists\,!$ way of parameterizing temperature up to a mult. const. $\ni$

$$\alpha(T_1, T_2) = \frac{T_1 - T_2}{T_1}$$

in which case the 2nd law becomes

2nd Law: of therm.

$$\boxed{\frac{dU_A + p_A\, dV}{T} = dS_A}$$

some S_A

S_A called entropy

$$F_A = U_A - TS_A$$

Helmholtz free energy

determines everything else.

Statistical mechanics

Hamiltonian system (M, Ω, H). A state is a probability measure $\rho\, dV$ on M. Randomness of the state is

$$-\int \rho \log \rho \, dV \qquad dV = \Omega^n.$$

Maximize this subject to the condition

$$\int H \rho\, dV = E$$

and find

$$\rho = Ae^{-\frac{H}{B}} \qquad A, B \text{ constants}$$

$$\int \rho \, dV = 1 \implies A = \frac{1}{\int e^{-H/B} dV}.$$

One then argues that $B = kT$ +

$$U_A(T) = \frac{\int H e^{-H/kT} \cancel{\rho}\, d\,\mathrm{Vol}}{\int e^{-H/kT} d\,\mathrm{Vol}} \quad \cancel{\&} + \quad p_A(T) = \frac{\int \frac{\partial H}{\partial V} e^{-H/kT} d\,\mathrm{Vol}}{\int e^{-H/kT} d\,\mathrm{Vol}}.$$

One then ~~checks~~ calculates that

$$\frac{dU + p\,dV}{T} = dS \qquad S = kT \frac{\partial}{\partial T} \log P_A(T) + k \log P_A(T)$$

where $\quad P_A(T) = \int e^{-H/kT} d\,\mathrm{Vol}. \quad$ partition fn.

One then replaces P_A by a measure β on $\mathbb{R}$

$$P(T) = \int e^{-x/kT} d\beta(x).$$

hence partition fn. is Laplace ~~transforms~~ transform of a pos. measure

$$\beta(E) = \int_{H^{-1}(E)} d\,\mathrm{Vol}. \quad + \quad \beta = H_*(d\,\mathrm{Vol})$$

Quantum mechanical analogue of a prob. dist. in phase space is a measure on the set of projections in Hilbert space + most general one is of the form $F \mapsto tr(F\Lambda)$ where Λ pos. s.a. op. of trace class, + trace = 1. Again to maximize

$$-Tr(\Lambda \ln \Lambda)$$

subject to $Tr(\Lambda) = 1$ + $Tr(\Lambda H) = E$.

Again

$$\Lambda = Ae^{-H/B} \qquad A, B \text{ const.}$$

$\therefore$ B the ! soln of $E = \dfrac{Tr(He^{-H/B})}{Tr(e^{-H/B})}$.

Again $B = kT$

$$P(T) = Tr(e^{-H/kT}).$$

$$P(T) = \sum e^{-E_j/kT} = \int e^{-x/kT}\, d\beta$$

<u>Classification of m.p. transf.</u>

(H, R, T) ergodic separable.

$Z = \{ S \subset R \mid S$ asa subalg. with $\underline{H}(S) < \infty \}$.

Review Mackey's theory.

<u>disjointness</u> <u>fix C^* algebra</u>

$\mathrm{Hom}(V,W) = 0 \iff$ no common subrep.

$\boxed{\text{primary}}$ cannot be split into disjoint parts.

$Z\,\mathrm{Hom}(V,V) = \mathbb{C}\,\mathrm{id}$ $\Updownarrow$

$\left.\begin{matrix}(1-E)\varphi = \varphi = \varphi E \\ + Z = \mathbb{C}\,\mathrm{id}\end{matrix}\right\} \Rightarrow \varphi = 0.$

<u>irred.</u> $\mathrm{Hom}(V,V) = \mathbb{C}\,\mathrm{id}$.

$U \subset V \subset U$.

ergodic m.p. transf. (H, R, T). R masa T unitary.

therefore T leaves measure class alone, e.g. $H = L^2(X,\mu)$

$R = L^\infty(X,\mu)$ μ some measure. Then $T(1) = f$ is a function on X

R has maximal ideal space X, R = Cont. fns. on X

If v a cyclic vector for R, then let $\mu(f) = (fv, v)$ measure μ on X. Now let

$$\nu(f) = (fTv, Tv) = (T^{-1}fTv, v)$$

ν and μ are absolutely continuous because $\exists\ g \in L^2(X,\mu) \ni gv = Tv$. So

$$\nu(f) = (fTv, Tv) = (fgv, gv) = f|g|^2 v, v$$

$$= \mu(f|g|^2).$$

$$\therefore\ \nu = |g|^2\mu.$$

Question: When does $\exists$ invariant measure? Not necessarily, I think. Existence of an invariant measure in same measure class is probably a hopeless cohomology question!!!

(H, R, T) Hilbert space, R masa, T unitary $TRT^{-1} = R$

assume only a single invariant v. (ergodicity)

Morphism

X compact space. measure class, T auto. of X.

$f: X \longrightarrow Y$ commute with T

Mackey's Oxford Notes: An outline.

1. Introduction

2.+3. finite group representations

4. Preliminaries concerning infinite groups + measure spaces. Standard and analytic Borel spaces. An analytic Borel group with a left (or right) non-trivial invariant measure class is locally compact

5. Compact groups + Peter-Weyl

6. Loc. comp. abelian gps. Hahn-Hellinger theory.

7. Direct integrals of unitary reps. in the general case. Borel Hilbert bundle. Borel structure on $\hat{G}$.

8. Primary reps + von-Neumann Murray factors. Primary representations, quasi-equivalence

9. Hilbert G bundles, imprimitivity, induced reps.

10. Unitary reps of semi-direct products

11. ——— general group ext.

12. Irred. unitary reps of compact simple gps

13. ——— un- ———

14. Ergodic theory

15. Prob. theory.