If $f: M \to M$ is expanding, then there is a metric on M and $\lambda > 1$ such that $|f(v)| \geq \lambda |v|$ $\quad v \in T_M$.

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<u>Theorem</u>: $f: M \to M$ expanding map. Then there is a measure μ on M such that $f^* \mu = d\mu$ where d is the degree of f.

<u>Proof</u>: Let $x_0 \in M$ and set

$$\mu(g) = \lim_{n \to \infty} \frac{1}{d^n} \sum_{y \in f^{-n} x_0} g(y)$$

We shall show that this limit exists. ~~Indeed given ε>0~~ Given $\varepsilon > 0$ let δ be chosen so that

$$\rho(x,y) < \delta \implies |g(x) - g(y)| < \varepsilon.$$

~~This~~ Then choose n so that $\lambda^{-n}(2 \operatorname{diam} M) < \delta$. ~~where~~ ~~$\lambda > 1$ is such that~~ Given $k \geq 0$ choose ~~there is~~ a simply-connected subset A containing $f^{-k} x_0$. Take union of geodesics joining to one point, to jiggle. For each $y \in f^{-n} x_0$ let $(f^{-n}A)_y$ be the connected component of $f^{-n}A$ containing y. Then

$$\operatorname{diam} (f^{-n}A)_y \leq \lambda^{-n} \operatorname{diam} M < \delta$$

In effect if u and v are two points of $(f^{-n}A)_y$, then $f^n u$ and $f^n v$ may be joined by an arc γ of length $\leq 2 \operatorname{diam} M$ and γ lifts to an arc γ' in $(f^{-n}A)_y$ joining u and v. But length $\gamma' \leq \lambda^{-n}$ length γ. Now

$$f^{-n-k}\{x_0\} = \coprod_{y \in f^{-n} x_0} (f^{-n} A)_y \cap f^{-n-k}\{x_0\}$$

and each of the pieces of this partition have d^k elements.

So

$$\frac{1}{d^{n+k}}\sum'_{y\in f^{-n-k}x_0} g(y) = \frac{1}{d^{n+k}}\sum'_{y\in f^{-n}x_0}\ \sum'_{z\in f^{-n-k}\{x_0\}\cap\{f^{-n}A\}_y} g(z)$$

$$\frac{1}{d^n}\sum'_{y\in f^{-n}x_0} g(y) = \frac{1}{d^{n+k}}\sum'_{y\in f^{-n}}\ \sum'_{z\in} g(y)$$

$$\left| \quad - \quad \right| \leq \frac{1}{d^{n+k}}\sum\sum' |g(z)-g(y)| < \varepsilon.$$

QED. Note that ~~if~~ if γ is an short arc joining two points x_0 and x_1, then the components of $f^{-n}\gamma$ are arcs of length $\leq \lambda^{-n}$ length γ. So if λ^{-n} diam $M<\delta$ we have,

$$\left|\frac{1}{d^n}\sum'_{y\in f^{-n}x_0} g(y) - \frac{1}{d^n}\sum_{y\in f^{-n}x_1} g(y)\right| \leq \frac{1}{d^n}\cdot\sum'\varepsilon = \varepsilon.$$

Thus the measure is independent of the choice of x_0. This shows that

$$\frac{1}{d^n} f_*^n g \xrightarrow{\text{uniformly}} \mu(g)$$

and hence if ν is any invariant measure of mass 1

$$\lim_n \nu\left(\frac{1}{d^n} f_*^n g\right) = \lim_n \nu\, g = \nu(\mu(g)) = \mu(g).$$

$\therefore$ measure is unique

Yuzvinskii, S.A.: Computing the entropy of a group of endomorphisms, Sibirskii Matematicheskii Zhurnal, 8 (1967), p 230-239
= Siberian Math Journal 8 (1967) p. 172–178.

Theorem: Let G be a torus and let T be an endomorphism of G. ~~[illegible]~~ and let $p(\lambda) = \det(\lambda - T^t)$ be the characteristic polynomial of T^t on $\hat{G}$. Then the entropy of T is

$$h(T) = \sum_{|\lambda_i| \geq 1}' \log |\lambda_i|$$

where λ_i are the eigenvalues of T. (roots of p)

More ~~[illegible]~~ generally ~~[illegible]~~ if G is such that $\hat{G}$ is a torsion free abelian group of finite rank (e.g. G solenoid) then

$$h(T) = \log s + \sum_{|\lambda_i| \geq 1}' \log |\lambda_i|$$

where s is the LCD of the coefficients of $p(\lambda) = \det(\lambda - T^t \otimes \mathbb{Q})$.

Definition of entropy: Let T be a measurable transformation of a probability space X. Let $H(\xi)$ be the entropy of a measurable partition ξ. and $H(\xi/\eta)$ the conditional entropy

$$h(T, \xi) = H\left(\xi / \bigvee_{k>0}^{\infty} T^{-k}\xi\right)$$

$$h(T) = \sup_{\xi} h(T, \xi)$$

Lemma: Let D be a dense ~~subgroup~~ subset of $\mathbb{R}^n$ and let μ be a measure on $\mathbb{R}^n$ such that for $\delta \in D$

$$(*) \qquad \mu(A+\delta) = c(\delta)\mu(A) \qquad \text{for all Borel sets } A$$

where $c(\delta)$ is a constant depending on δ. Then $c(\delta)$ is the restriction of a character of $\mathbb{R}^n$ to D and

$$\mu = a(c \cdot dx)$$

where dx is Lebesgue measure and a a constant.

Proof: $*$ $\Rightarrow$ for every continuous fn. with compact support φ on $\mathbb{R}^n$ that

$$\int \varphi(x-\delta)\,d\mu(x) = \int \varphi(x)\,d\mu(x+\delta) = c(\delta)\int \varphi(x)\,d\mu(x)$$

However φ is uniformly continuous so the LHS is uniformly continuous as δ runs over D. Therefore

$$\int \varphi(x-y)\,d\mu(x) = c(y)\int \varphi(x)\,d\mu(x)$$

where

$$c(y) = \lim_{\substack{\delta \to y \\ \delta \in D}} c(\delta).$$

i.e. $\mu(y+A) = c(y)\mu(A)$ from which ~~one~~ sees that ~~the null sets~~ c is a character and that the ~~null sets of μ are invariant under translation. By a result of Plessner $\mu = g \cdot dx$ where $g \in L^1(\mathbb{R}^n, dx)$, whence~~ μ/c is invariant under translation and therefore a multiple of Haar measure.

$\boxed{\text{Problem}}$ $\exists$ence of an invariant measure for an Anosov diffeomorphism.

~~Conjecture~~ First question is its uniqueness.

Not uniques at all e.g. take shift automorphism on Cantor subset coming from homoclinic pt. Look at any Bernoulli measure on Cantor set.

For standard toral diffeom. $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ showed that only Lebesgue measure was a product with respect to the eigenvalue ~~distribution~~ splitting. Same argument would work if eigenspaces are all 1-dimensional.

~~Thm~~ General case: f hyperbolic automorphism of a torus T

Suppose μ is an invariant measure on T which on $\tilde{T}$ is a product measure relative to the stable-unstable decomposition

Write $\mu = \mu_1 \cdot \mu_2$ on $\tilde{T}$.

Then $\boxed{\mu(A \times B) = \mu_1(A)\,\mu_2(B)}$

so

$$\mu(fA \times fB) = \mu_1(fA)\,\mu_2(fB) = \mu_1(A)\,\mu_2(B).$$

hence there is a constant c such that

$$\mu_1(fA) = c\,\mu_1(A)$$
$$\mu_2(fB) = \frac{1}{c}\,\mu_2(B)$$

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Moser's proof of structural stability.

Given $f, g \in \mathrm{Diff}(M)$.

~~$g \in$~~

want $h \in \mathrm{Homeo}(M) \ni$

$$hf = gh$$

Thus one consider the mapping

$$\mathrm{Diff}(M) \times \mathrm{Hom}^0(M, M) \longrightarrow \mathrm{Hom}$$

$$g \qquad h \qquad (hf, gh).$$

Show $f_* - 1$. We are **interested** in

$$f_*(X) - X \qquad\qquad ghf^{-1}$$

$$ghf^{-1} - h$$

$$h \rightsquigarrow ghf^{-1} - h$$

If g C^1 close to f differential still very reasonable. so implicit function may be used provided that $f_* - 1$ still invertible.

$$f_* - id$$

Same invariant sing. measure.

hence invariant measures are highly non-unique

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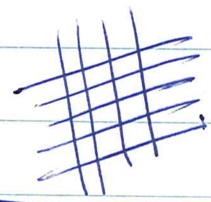
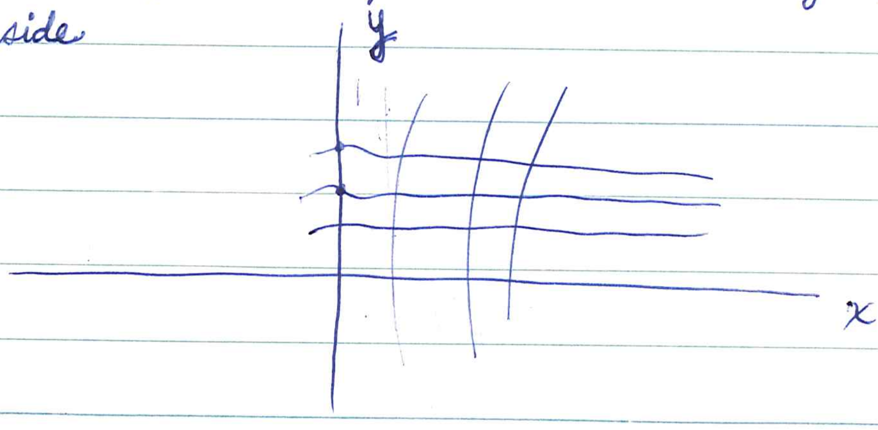
measure highly non-unique.

Example: A homoclinic point leads to a Cantor subset invariant what is a measure? linear functional on continuous functions? what is a measure which decomposes locally along the foliation.

Given an expanding transf. on $\mathbb{R}^n$ how many measures are invariant.



each leaf is ~~smooth~~ C^1 which means that it's the graph of a C^1 function. Similarly for leaves on the other side



$$\begin{cases} y = f(x,\alpha) \\ \alpha = f(0,\alpha) \end{cases}$$ and f is $\overbrace{C^1}^{\text{in } x}$ for each α.

will assume $f(x,0)=0$

Similarly we have $\begin{cases} x = g(y,\beta) \\ \beta = g(0,\beta) \end{cases}$ $g(0,y)=0.$

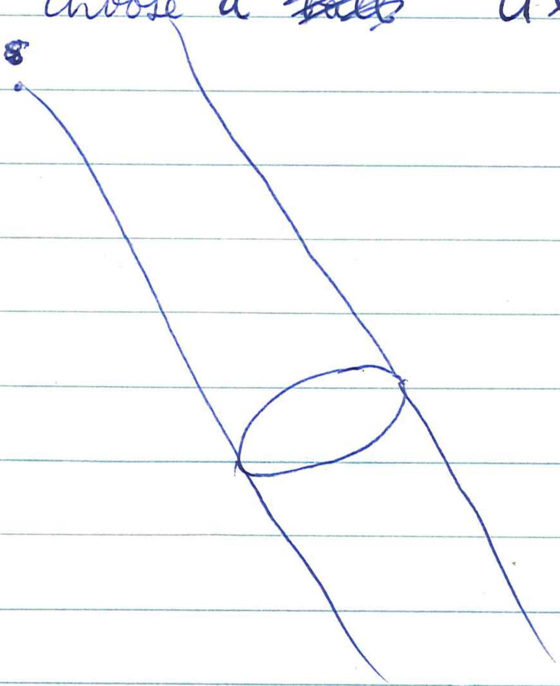
and g is C^1 in y for each β.

Lemma: Let μ be a measure on $X \times Y$, compact. μ is a product measure iff

$$\mu(f(x)\,g(y)) = \frac{\mu(f(x)) \cdot \mu(g(y))}{\mu(X \times Y)}$$

The non-compact case: Take μ on M and lift to G. μ is invariant under f and right translation by π_1 and μ is locally a product measure.

Now choose a ~~ball~~ strip $\bar{U} \times G^u$ where U is a bdd ball in G^s.



Question: In the general case all you are given is the foliations. Can you speak of a measure which locally decomposes as a product with respect to the foliations?

Problem: Suppose μ is a measure on M which is locally a product with respect to the foliations. Then does M have to come from Haar measure?

Lemma: Let μ be a measure on $X \times Y$. Then μ is a product measure if and only if

$$\mu(f(x)\,g(y)) = \mu(f(x)) \cdot \mu(g(y)).$$

Proof: Let $\mu_1 f = \dfrac{\mu(f(x))}{\mu(Y)}$ etc.

$$\mu = \mu_1 \times \mu_2$$

for functions of the form $f(x)\,g(y)$. But sums of these are dense.

This clearly

$$\iint_{X \times Y} f(x)\,g(y)\; d\mu_1(x)\; d\mu_2(y) = \mu_1(f)\,\mu_2(g).$$

where $\mu_1(f) = \dfrac{\mu(f(x))}{\mu(Y)}$

Therefore $G^s(px)$ and $G^u(px)$ are the generalized stable and unstable manifolds of x. Next we want to determine just exactly what a volume which is compatible with this foliation is!!!!! ~~It must be~~

Note that

Question for G do the stable and unstable foliations ~~[illegible]~~ give a global coordinate system.

Suppose a measure μ is locally decomposable. Is it globally decomposable?

Fundamental formula:

$$e^{-i\lambda u} P e^{i\lambda u} v \sim \sum_{J,j} \frac{1}{J!} \overbrace{\frac{\partial p_j}{\partial \xi^J}}^{(x,\, \xi_x)} \frac{\partial}{\partial y^J}\left(v e^{i h_x}\right)\Big|_{y=x}$$

here $\quad \xi_x = du(x) \quad$ and

$$u(y) = u(x) + \langle y-x, \xi_x \rangle + h_x(y).$$

<u>Thm</u>: (Mather) If non-periodic points are dense in M, then the spectrum of f_* on $C^0(TM)$ is rotationally symmetric.

<u>Cor</u>: If $1 \notin$ spectrum of f_*, then f_* is Anosov.

<u>Proof</u>: Let K_n = fixed points of f^n. Claim $\bigcap K_n$ has no interior. By Baire enough to show K_n has no interior. ~~If so choose n least and look at Int K_n as~~ Suppose n least $\ni$ K_n has interior. $n = 1$ impossible because $f_*^1 - id$ invertible $\therefore$ fixpoints are isolated + only finitely many. Choose $x \in K_n - \bigcup_{m \neq n} K_m$ and a small nbd U of x. Then $U, fU, \dots, f^{m-1}U$ are all ~~distinct~~ disjoint and take a bump vector field in U translate to obtain a non-zero invariant vector field. Thus $1 \in$ spectrum of f so we obtain a contradiction. So the non-periodic points are dense and by preceding theorem the spectrum doesn't meet the unit circle and therefore f is anosov.

Entropy for a measure preserving transformation.

X, μ measure space $\mu(X)=1$.
$T: X \to X$ measure preserving
take finite decomposition of X

$$X = \coprod_{i=1}^{h} A_i$$

$$\lim_{n\to\infty} -\frac{1}{n} \sum_{i_1,\dots,i_n=1}^{h} \mu(A_{i_1} \cap TA_{i_2} \cap \dots \cap T^{n-1}A_{i_n}) \log \mu(\cdot \text{---}). \quad = H \text{ partition}$$

always $\exists$.

need to reinterpret this formula!!!

Call partition R. Then X/R has a measure and we define a map

$$\Phi : X \xrightarrow{(pT^n)} \prod_{n\in\mathbb{Z}} (X/R)$$

Suppose that ~~we obtain the σ-field of meas. sets of X in the form~~ the σ-field of meas. subsets of X is gen. by $(T^m A_j)$. Then have a map of measure spaces

$$\Phi : X \longrightarrow Y$$

---

partition of unity $\quad \sum \rho_i = 1$
refined partition $\quad \sum_{i_1\cdots i_n} (\rho_{i_1}(x)\, \rho_{i_2}(Tx) \cdots \rho_{i_n}(T^n x))$

$$\boxed{\begin{array}{ll} E & \text{Riemannian + connection} \\ \downarrow p & \quad p \text{ smooth} \\ X & \text{Riemannian compact simply-connected.} \end{array}}$$

$$\begin{array}{c} e_0 \in E \\ \downarrow \\ x_0 \in X. \end{array}$$

The curvature? Given e_0 curvature is a map Λ^2

$$\boxed{T_e(E) \longrightarrow \Lambda^2 T_x(X)^* \otimes T_e(E)}$$

So we assume a bound on the curvature ~~along~~ for horizontal all curves of bdd. horizontal length issuing from e_0. Assume the curvature is small. Then I wish to find a flat section of E and I proceed a la Cech

I choose ~~an~~ a nice open covering U_i and nearly flat sections s_i according to the maximal tree

Lemma:
$$\Gamma(X, E) \longrightarrow \prod_i \Gamma(U_i, E) \rightrightarrows \prod_{i,j} \Gamma(U_i \cap U_j, E)$$

is an exact sequence of Banach manifolds.

~~[illegible]~~

Precise sense:

$$\Gamma(X, E) \longrightarrow \prod_i \Gamma(U_i, E) \rightrightarrows \prod_{i,j} \Gamma(U_i \cap U_j, E)$$

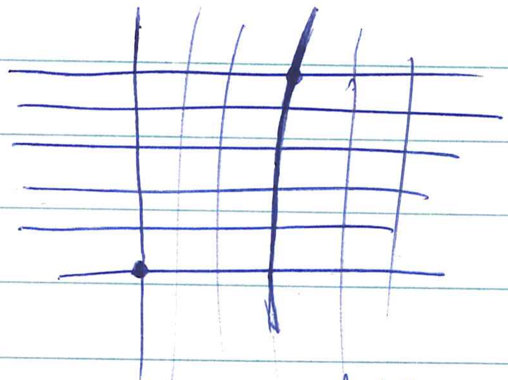
top exact as follows.

Proof:

$$\begin{tikzcd}
\mathrm{Hom}(T(X), T(E)) \arrow[d] \\
E \arrow[d] \\
X \arrow[uu, bend left]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(X, E) \arrow[r] \arrow[d, "D"] & \prod_i \Gamma(U_i, E) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} \Gamma(U_i \cap U_j, E) \\
\Gamma(X, \underline{\mathrm{Hom}}(T(X), T(E))) \arrow[r] \arrow[d, shift right, "D"'] \arrow[d, shift left] & \Theta & \\
\Gamma(X, \underline{\mathrm{Hom}}(\Lambda^2 T(X), T(E))) & &
\end{tikzcd}$$

Lemma 1: $\exists$ function $f(R)$ such that for all simply-connected compact Riemannian manifolds R two paths of length $< R$ may be joined by an arc of length $< f(R)$.

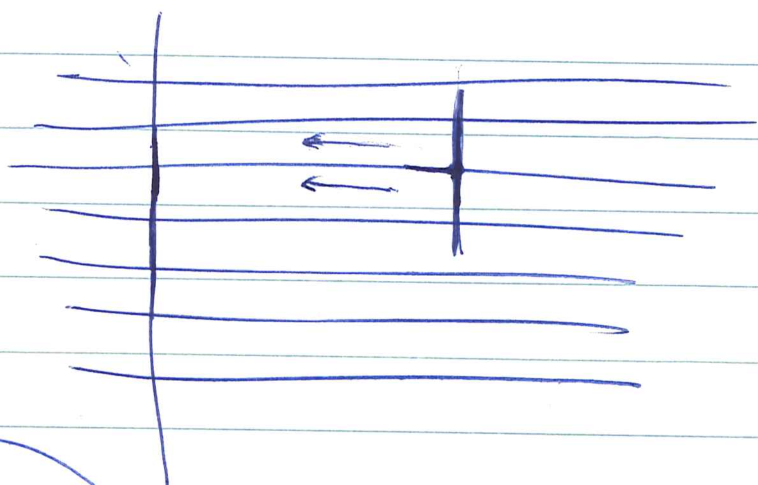
In two dimensions suppose we have an Anosov diffeo.



stable

unstable

$$y = \varphi(x)x^2 + \alpha$$



stable

Anosov claims that $\frac{\partial\varphi}{\partial y}$ continuous.

$$(x,y) \rightsquigarrow \varphi(x,y) = \alpha$$

for fixed x, ~~continuous~~ C^1 in y

and derivative in y

~~[illegible] C^1 in x.~~

$\frac{\partial\varphi}{\partial y}$ varies uniformly wrt x.

$y = g(x,\alpha)$ $\quad \frac{\partial g}{\partial\alpha}$ cont.

$\alpha = \varphi(x,y)$.

$\cancel{\frac{\partial\alpha}{\partial\varphi}}$ $\frac{\partial\alpha}{\partial y}$ cont.

$\therefore$ $\frac{\partial\varphi}{\partial y}(x,y)$ continuous

$$y = \varphi(\alpha)x^2 + \alpha$$

$$x' = x$$
$$y' = \varphi(y)x^2 + y$$

$$\varphi(2y) = \lambda\varphi(y)$$

$$\frac{\partial x'}{\partial x} = 1 \qquad \frac{\partial x'}{\partial y} = 0$$

$$y' \rightsquigarrow \varphi(2y)4x^2 + 2y =$$

$$\frac{\partial y'}{\partial x} = 2x\varphi(y) \qquad \frac{\partial y'}{\partial y} = 1 + \varphi'(y)x^2$$

$$[\varphi(2y)4 - \varphi(y)\cdot 2]x^2 + 2y'$$
$$4\lambda - 2$$

Jacobian is $1 + \varphi'(y)x^2$ which by suitable choice of φ may be made non-L' so that the transformation does not preserve Lebesgue null sets.

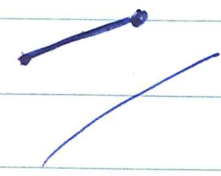
$X \rightsquigarrow 2X$

$y \rightsquigarrow$

Homsets (M, M) Banach manifold
and $\varphi \mapsto f\varphi f^{-1}$ is a C^∞ mapping. For $\varphi = \mathrm{id}$ we ~~consider functions~~ get a hyperbolic fixed point since derivative is

$$\cancel{X \mapsto Df \circ X \circ f^{-1}}$$

$$\boxed{X \longmapsto f_* X}$$

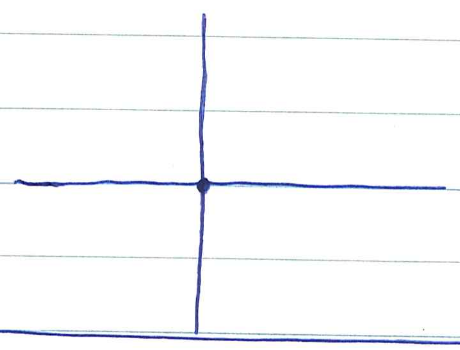


Thus there are stable and unstable submanifolds! Let

$$\varphi \in W^s \quad \text{so that} \quad f^m \varphi f^{-m} \xrightarrow[\text{unif.}]{} \mathrm{id} \qquad m \to \infty.$$

In particular if $x \in M$, then the function

$$\boxed{\text{Generalized stable + unstable manifolds}}$$



non-smoothness example

$$\boxed{y = \varphi(\alpha) x^2 + \alpha}$$

C^1 in x for α fixed.
yet not C^1 in α.

~~arrange~~ Arrange $\varphi(\alpha)$ to be increasing so get a foliation not smooth.

Then we may consider the ~~[illegible]~~ Banach space V of bounded ~~sections~~ of $T(M)$.
and let U be the open neighborhood of those sections sufficiently near to 0.

To look at maps $M \xrightarrow{\varphi} M$ near enuf to id_M we want x and φx to lie in stable manifold i.e.

$$d(f^m x, f^m \varphi x) \longrightarrow 0 \qquad m \longrightarrow \infty$$

want this to be "uniform" in x, i.e. setting $y = f^m x$

$$d(x, f^m \varphi f^{-m} x) \longrightarrow 0$$

$$\text{or} \qquad f^m \varphi f^{-m} \longrightarrow \varphi \qquad \text{unif.}$$

Thus we ~~[illegible]~~ have conjugation by f on $\mathrm{Hom}(M,M)$

$$\begin{array}{ccc} \mathrm{Bddsec}\,T_\varepsilon(M) & \overset{E}{\simeq} & U_\varepsilon\,\mathrm{set\,Hom}(M,M) \subset \mathrm{set\,Hom}(M,M) \circlearrowleft f \circ f^{-1} \\ & & \downarrow \\ T_{\varepsilon'}(M) & & U_{\varepsilon'} \end{array}$$

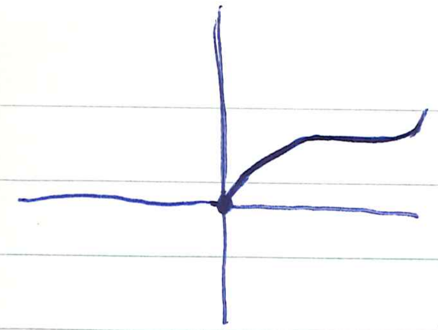
$\varphi \rightsquigarrow f \varphi f^{-1}$

$x \quad \varphi x$

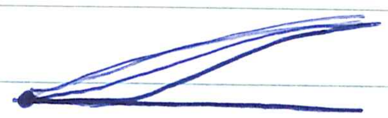
$x \quad f \varphi f^{-1} x$

Thus if X is a bounded section of $T(M)$ of size $< \varepsilon$ X defines a map $M \to M$ by

$$x \rightsquigarrow \exp(X_x).$$



$f(v, \varphi v) = (f$

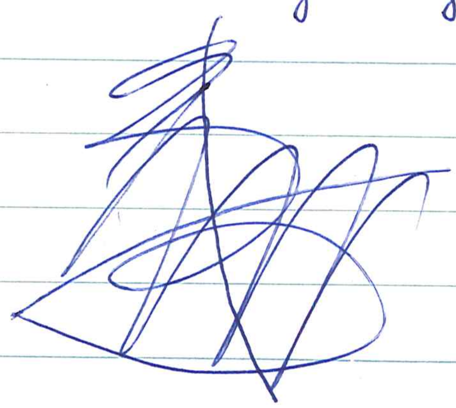


ordinary differential equations

$$\dot{x} = f(x,t)$$

$$x(t) = x(0) + \int_0^t f(x(t),t)\,dt.$$

Application to the generalized stable manifold theorem:



M Anosov and we consider bounded mappings $M \longrightarrow M$. which are (close ?) to id.

Choose an exponential map ~~[illegible]~~ $T_\varepsilon(M) \xrightarrow{E} M \times M$.

$$E : \underset{(x,v)}{T_\varepsilon(M)} \xrightarrow{\sim} \underset{(x,\ \exp v)}{U_\varepsilon(\Delta)}$$

Review stable manifold thm.

V Banach space

$0 \ni U \xrightarrow{f} V \qquad f,\ C^1$ and $f(0)=0$.

$df_0 : V \longrightarrow V$ hyperbolic

$\Longrightarrow$ there exists a C^1 submanifold W^s_{loc} of U passing thru 0 whose tangent plane is E^s and invariant under f and carried into 0 under iterates of f.

Proof: f C^1 means that

$$f(y)-f(x) = f'(x)(y-x) + o(\|y-x\|) \qquad y,x \in U.$$

where f' as a map

$U \longrightarrow \mathrm{Hom}(V,V)$ is continuous

$V = V^u \oplus V^s \qquad f'(0)$ expands on V^u + contracts on V^s.

Let $B = B_\varepsilon V^u \times B_\varepsilon V^s \qquad B_\varepsilon$ = closed ε ball.

and we consider maps $\varphi : B_\varepsilon V^u \longrightarrow B_\varepsilon V^s$

The point is to choose ε so small that if we identify φ with $\mathrm{gr}\,\varphi$ ~~is well defined.~~ then $f\,\mathrm{gr}\,\varphi = \mathrm{gr}(f_*\varphi)$ is well defined.

Conclude therefore that the splitting is not left invariant. so instead we try a right invariant splitting and a right invariant metric on G. If G^s is the stable subgroup corresponding to the stable ~~subg~~ subalgebra $\mathfrak{g}^s$ then $p(G^s x)$ is the stable manifold through px. For example if $y \in G^s$, then

$$d(p(f^m yx), pf^m x) \leq d(f^m y \cdot f^m x, f^m x) = d(f^m y, 1) \to 0.$$

First question: Does the foliation give a general splitting

$$G \simeq G^s \times G^u \;?$$

ie to show that any point $g \in G$ may be written

$$\left.\begin{aligned} g &= g_1 g_2 \\ &= g_1' g_2' \end{aligned}\right| \text{ unique + exists} \qquad \begin{aligned} &g_1' \in G^s \quad g_2' \in G^u \\ &g_1 \in G^u \quad g_2 \in G^s \end{aligned}$$

Then

$$g \mapsto (g_2, g_2').$$

gives

$$G \simeq G^s \times G^u.$$

Then

$$f(g) \mapsto (fg_2, fg_2')$$

G nilpotent Lie group, simply-connected
f hyperbolic automorphism of G
π uniform discrete subgroup of G invariant under f.

Then f induces a map on $M = G/\pi$, which we denote by f and which is Anosov, because

$$TM = G \times_\pi \mathfrak{g} \qquad \begin{array}{ccc} G\times\mathfrak{g} & \longrightarrow & G\times_\pi \mathfrak{g} \\ \downarrow & & \downarrow \\ G & \xrightarrow{p} & G/\pi \end{array}$$

because the splitting is invariant under π and because f contracts on $G \times_\pi \mathfrak{g}^s$

What is the tangent bundle of a homogeneous space?

$$G \times_H \mathfrak{g}/\mathfrak{h}$$

$$T_{eH}(G/H) = \mathfrak{g}/\mathfrak{h}.$$

$$0 \to T_e H \longrightarrow T_e G \longrightarrow T_e(G/H) \longrightarrow 0$$

$$G \times T_{eH}(G/H) \xrightarrow{\varphi} T(G/H).$$

$$\varphi(gh \times v) = \varphi(g, hv)$$

τ = structure type = index sets for families of predicates + constants.

<u>structure</u> $\mathfrak{A}$ = set A

indexed family of predicates

indexed family of constants. (elements of A).

<u>language L_τ of the structure</u> ~~equals~~ = sentences made up from variables, predicates, and constants.

If $\phi \in L_\tau$, then $\mathfrak{A} \models \phi$ means that ϕ is true in $\mathfrak{A}$ when free variables vary overall elements of A.

elementary extension $\mathfrak{A} \subset \mathfrak{B}$ such that if $a_1, \ldots, a_n \in A$, and $\phi \in L_\tau$ has free variables $x_1, \ldots, x_n$, then

$$\mathfrak{A} \models \phi(a_1, \ldots, a_n) \iff \mathfrak{B} \models \phi(a_1, \ldots, a_n).$$

<u>Fundamental triviality:</u> $\mathfrak{A} \subset$ (ultraproduct of $\mathfrak{A}$) is an elementary extension.

<u>Example:</u> ultra-product of fields is a field.

$$\forall x \,(x \neq 0 \Longrightarrow \exists y \ni yx = 1).$$

$$\forall x \,\exists y \,(x = 0 \text{ or } yx = 1)$$

If true in each $\mathfrak{A}_i$ true in $\prod' \mathfrak{A}_i$, since given x_i choose y_i with $(x_i = 0$ or $y_i x_i = 1)$. Then

$$I = \{i \mid x_i = 0\} \cup \{i \mid y_i x_i = 1\}$$

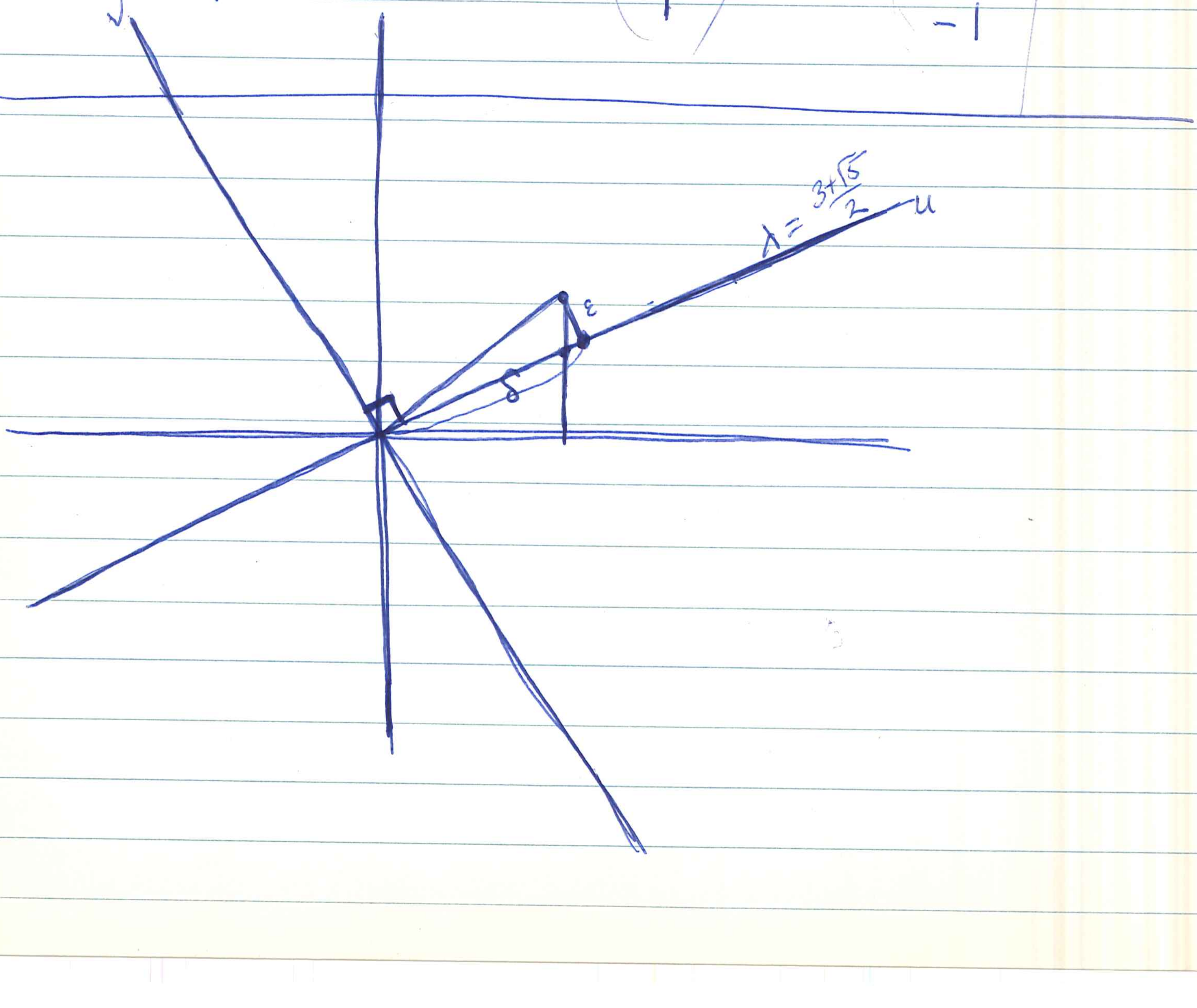
So one belongs to the ultra filter $\Rightarrow$ $\vec{x} = 0$ or $\vec{y}\vec{x} = 1$.

try to construct an invariant non-Lebesgue measure for toral diffeomorphisms.

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad \lambda^2 - 3\lambda + 1 = 0$$

$$\lambda = \frac{3 \pm \sqrt{5}}{2}$$

$$\begin{matrix} \frac{1+\sqrt{5}}{2} & -1 \\ -1 & \frac{-1+\sqrt{5}}{2} \end{matrix} \qquad \begin{pmatrix} \frac{-1+\sqrt{5}}{2} \\ 1 \end{pmatrix} \qquad \begin{pmatrix} \frac{+1+\sqrt{5}}{2} \\ -1 \end{pmatrix}$$



To construct an invariant product measure to be given by distribution fn.

$$F(u)\, G(V)$$

F monotone continuous strictly increasing

So want

$$F(\lambda u)\, G\left(\frac{v}{\lambda}\right) = F(u)\, G(v)$$

$$\left[F(u+\delta) - F(\delta)\right]\left[G(v+\varepsilon) - G(\varepsilon)\right] = F(u)\, G(v).$$

$$\left\{ \begin{aligned} &\frac{F(\lambda u)}{F(u)} = c \\ &\frac{F(u+\delta) - F(\delta)}{F(u)} = K \end{aligned} \right.$$

$$\boxed{\begin{aligned} &F(\lambda u) = cF(u) \\ &F(u+\delta) = F(\delta) + KF(u) \end{aligned}}$$

from these equations we get

$$F(n\delta) = \frac{K^{n+1} - 1}{K-1} F(\delta).$$

$$\therefore \quad F(\lambda^q n\delta) = c^q \frac{K^{n+1} - 1}{K-1} F(\delta).$$

$\lambda > 1$

Let $n \to \infty$ $q \to -\infty$ in such a way that $\lambda^q n\delta \to A$.

Thus $q \sim \dfrac{\log \frac{A}{\delta} - \log n}{\log \lambda}$

so $c^{\delta} \sim n^{\frac{\log c}{\log \lambda}} \cdot \text{const.}$

But if $K > 1$ $\quad \dfrac{K^{n+1} - 1}{K-1}$ goes to inf. exponentially

Contradiction. $\therefore K = 1$

$$F(n\delta) = n F(\delta)$$

$$F(\lambda^{\delta} n \delta) = c^{\delta} n \, F(\delta).$$

again by monotonicity $\lambda = c \Rightarrow F(x) = \frac{x}{\delta} F(\delta).$

So this method will not yield an F.

metric

$|nx| \leq n|x|$

$\|x+y\| \leq \|x\| + \|y\|$

too much to ask for + $\|fx\| = 2\|x\|$.

Assume I can find a ball nbd. B of 0 (section) in TV such that

$B = f^{-1} f B$

And that $fB \supset \lambda B$ $\quad \lambda > 1$

Now will show how to define a metric on

$$\begin{cases} B \text{ symmetric} \\ B \cdot B \subset fB \\ B = f^{-1} f B \end{cases}$$

Special case: f contraction on a vector space V.

B ball.

$\underline{fB} \subset B$

convex body

take a ball of radius $\varepsilon > 0$

so lines thru O

so $\quad f\dot{B} \quad \dot{B}$

$\frac{1}{2}\dot{B} \quad \dot{B}$

in this way get a homeo of $V \longrightarrow V$
such that f is mult by $\frac{1}{2}$.

---

define $\quad \chi(v) = \begin{cases} 0 & v \notin B \\ 1 & v \in B \end{cases}$

---

metric

$\{\rho(x) \le 1\} = B_1$

$$fB_1 = B_2$$

Now suppose that B is a compact nbd. of Δ in $M\times M$ such that

(i) B symmetric,

(ii) $B\cdot B\subset fB$

(iii) $B = f^{-1}\{fB\}$.

Then there is a unique function $\rho: B \to R_{\geq 0}$ $\ni$

$$\begin{cases} \rho(x,y)+\rho(y,z)\geq\rho(x,z) & (x,y),(y,z),(x,z)\in B. \\ \rho(x,y)=\rho(y,x) & (x,y)\in B. \end{cases}$$

$$\left\{ \rho(fx,fy) = 2\rho(x,y) \right. \quad \text{if } (fx,fy)\in B. \quad (\Rightarrow (x,y)\in f^{-1}B\subset f^{-1}fB=B)$$

$$f^{-n}B = \{(x,y)\in B \mid \rho(x,y)\leq \tfrac{1}{2^n}\}.$$

$$\underbrace{f^{-r}B\cdot f^{-s}B}_{X\ \cdot}\ \overset{?}{=}\ \underbrace{f^{-s}B\cdot f^{-r}B}_{Y}$$

?

symmetric.

$$X\cdot Y = \{(x,z)\mid \exists y \text{ with } (x,y)\in X \text{ \& } (y,z)\in Y\}.$$

$$(X\cdot Y)^t = \{(z,x)\mid \exists y \text{ with } (z,y)\in Y^t \text{ \& } (y,x)\in Z^t\}.$$

$$= Y^t.X^t. = Y\cdot X$$

x, y, z

$f^rx, f^ry\in B$

$f^sy, f^sz\in B.$

Defn: A "good" metric on a manifold M is one ρ such that $\forall \varepsilon > 0$

$$\rho(x,y) = \sum_{i=0}^{n-1} \rho(x_i, x_{i+1})$$

for some sequence $x = x_0, \ldots, x_n = y$ with $\rho(x_i, x_{i+1}) < \varepsilon$ all i.

Example: Any Riemannian metric on a smooth manifold M which is complete.

Claim that if ρ is a metric on M which is "good" and satisfies

$$\rho(x,y) < 1 \Longrightarrow \rho(fx, fy) = 2\rho(x,y)$$

then the good metric on $\tilde{M}$ coinciding with $\rho(px, py)$ when x and y are close satisfies

$$\begin{cases} \rho(\gamma x, \gamma y) = \rho(x,y) \\ \rho(fx, fy) = 2\rho(x,y). \end{cases} \qquad \gamma \in \pi$$

Proof: Suppose that $B_x(\varepsilon)$ is a ball for each $x \in \tilde{M}$. $\forall\, \varepsilon \le 1$ and let $U_\varepsilon = \bigcup_x x \times \tilde{B}_{px}(\varepsilon)$ in $\tilde{M}$. Now define d on $\tilde{M}$ by

$$d_\varepsilon(x,y) = \inf \sum \rho(px_i, px_{i+1})$$

where the inf is taken over all sequences $x = x_0, \text{———}, x_n = y$

such that x_i, x_{i+1} lie in U_ε. Claim d_ε independent of ε. Clearly

$$d_\varepsilon(x,y) \le d_{\varepsilon'}(x,y) \qquad \text{if } \varepsilon \geq \varepsilon'$$

since d_ε has more sequences. On the other hand given

$$x = x_0, x_1 \cdots \qquad x_n = y$$

with $\rho(px_i, px_{i+1}) < \varepsilon$

~~we may subdivide this sequence without changing total length.~~

we may write

$$px_i = py_{i0}, \cdots, py_{ij_i} = px_{i+1}$$

where $\rho(px_i, px_{i+1}) = \sum \rho(py_{ij}, py_{ij+1})$

and $py_{ij}, py_{ij+1} \in U_{\varepsilon'}$.

Thus $d_\varepsilon \geq d_{\varepsilon'}$.

$$d_\varepsilon(\gamma x, \gamma y) = d_\varepsilon(x,y)$$

since γ auto. of M + preserves M_ε.

$\cancel{d_\varepsilon(px, py)}$

$$d(fx, fy) = d_\varepsilon(fx, fy) = d_{\varepsilon}(x,y) \qquad \text{clear.}$$

$d(x,y) = d(f^{-m}\gamma f^{m}x, f^{-m}\gamma f^{m}y)$

all m.

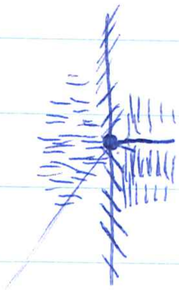
$\Rightarrow$ G locally compact + isotropy group is compact.

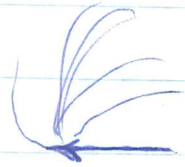
$$\sum_i a_i \frac{\partial}{\partial x_i} = \sum_{i,j} a_i \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}$$

$$= \sum_{i,j} a_i \frac{x_i}{r^3}(-2x_j)\frac{\partial}{\partial y_j} + \sum_{j} a_j \frac{1}{r^2}\frac{\partial f}{\partial y_j}$$

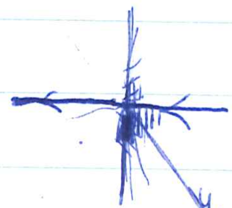
Conclude this method won't work!

$$\neq \sum_{i,j} a_i\left(-\frac{2x_i x_j}{r^3} + \delta_{ij}\frac{1}{r^2}\right)\frac{\partial}{\partial y_j}$$





$$\therefore \quad \frac{a_i}{r} \longrightarrow 0$$



$y_1 = \frac{x_1}{r^2}$ $\quad y_2 = \frac{x_2}{r^2}$

~~(1, y)~~ $\quad (1, x_2)$

$$\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$$

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1}\left(\frac{1}{r^2} - \frac{2x_1x_2}{r^3}\right) + \frac{\partial}{\partial y_2}\left(-\frac{2x_1x_2}{r^3}\right).$$

$$\frac{\partial}{\partial y_1}\left(\text{---} - \frac{2x_1x_2^2}{r^3}\right)$$

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial y_1}\left(-\frac{2x_1x_2}{r^3}\right) + \frac{\partial}{\partial y_2}\left(\frac{1}{r^2} - \frac{2x_1x_2}{r^3}\right)$$

$$(f^{-m}\gamma f^m)(0) = f^{-m}\gamma 0 \in K$$

$$\frac{r}{1} = \frac{r-a}{\sqrt{1-a^2}}$$

can I bound df.

$$(1-a^2)r^2 = (r-a)^2$$

$f^{-m}\gamma f^m$

$$\cancel{r^2} - a^2r^2 = \cancel{r^2} - 2ar + a^2$$

v

$$a^2r^2 - 2ar + a^2 = 0. \qquad ar^2 - 2r + a = 0.$$

$$\cancel{r = \frac{2a \pm \sqrt{4a^2 - 4a^4}}{2}}$$

$$a = \frac{2r}{r^2+1} \sim \frac{2}{r}$$

$$x \longmapsto \frac{x}{|x|^2} = y.$$

$$\frac{dy}{dx} = \frac{1}{r^2}\frac{dx}{dx} - \frac{2}{r^3}\frac{dr}{dx}$$

$$= \frac{1}{r^2} -$$

$$\frac{d}{dx} = \frac{d}{dy} \cdot \frac{dy}{dx}$$

$$= \qquad \|f^{-m}\gamma f^m v\|$$

f expands.

$$\|fv\| \geq \lambda \|v\|$$

$$y_i = \frac{x_i}{r^2}$$

$$\frac{\partial f}{\partial x_i} = \sum_j \frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial x_i} = \sum_j -2\frac{x_j}{r^3}\frac{\partial f}{\partial y_j} + \frac{1}{r^2}\frac{\partial f}{\partial y_i}$$

Thus $\sum_i a_i \frac{\partial f}{\partial x_i} \rightsquigarrow \sum_j (-2)\frac{x_j}{r^3} a_i \frac{\partial f}{\partial y_j}$

$$X = a_i \frac{d}{dx^i}$$

$$\begin{cases} \frac{d}{dt} \varphi_t^j(p) = a_j(\varphi_t^1(p), \dots, \varphi_t^n(p)) \\ \varphi_0^j(p) = p^j . \end{cases}$$

~~existence thm. $\Rightarrow$ $\varphi_t : M \times \mathbb{R} \to M$~~

$\underbrace{e^{tX}}\, p$ — cont. in t, p + derivative is also.

$\boxed{\text{Thus } X \mapsto e^X \text{ is a well-defined map}}$

$C^0(TM) \longrightarrow \text{Aut } M$

both Banach manifolds and so by the implicit function thm. it is OKAY. / ~~Interesting~~ consequences

Aut M

M' endowed with ~~the~~ a metric from M.

~~Check geodesics~~

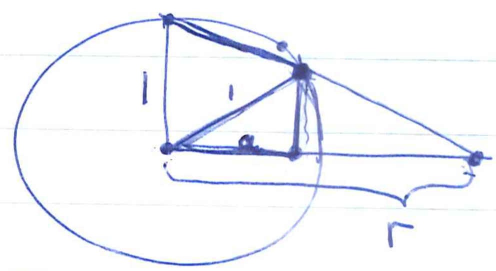
To define $\text{Aut}(M')$ and $C^0(TM)$ by growth conditions so that the map exp is defined.

idea: Consider $M' \cup \{\infty\} = \hat{M}$ n sphere and extend the differential structure to $\hat{M}$, if possible.

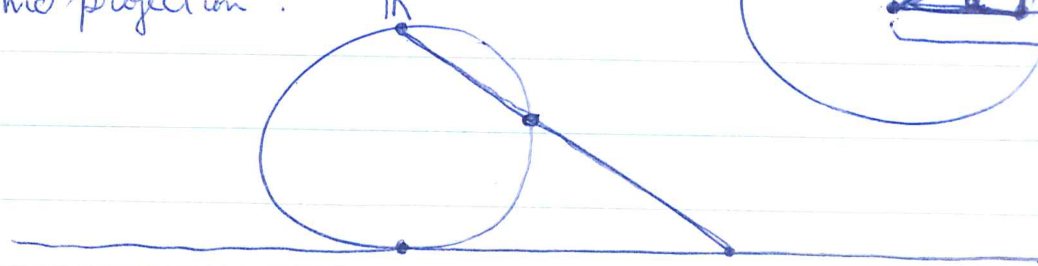
Suppose

X vector field on $\mathbb{R}^n$ $\ni$ $|X(x)| \leq C\,|x|$.

Then is the vector field complete.



stereographic projection :



If $a = 2^{-r_1} + 2^{-r_2} + \cdots + 2^{-r_n}$

Let $\rho(x, y) = \inf |c|$

where c is a finite sequence of pts. of M

$x = x_0, \ldots, x_n = y$

and $|c| = \sum_{i=0}^{n} |x_i, x_{i+1}|$

and $|x, y| = \begin{cases} \infty & (x, y) \notin B. \\ \frac{1}{2^n} & n \text{ largest} \ni (f^n x, f^n y) \in B. \end{cases}$

Conjecture: V finite dim v.s. over $\mathbb{R}$

f linear expansion

B ~~closed~~ compact neighborhood of 0 such that

(i) $B = -B$

(ii) $B + B \subset fB$.

Then there is a cont. function $|\ |: V \to \mathbb{R}_{\geq 0}$ $\ni$

$|x+y| \leq |x| + |y|$ $\qquad |x| = |-x|$.

$|fx| = 2|x|$

$B = \{x ; |x| \leq 1\}$.

~~Proof~~ the conditions are necessary.

maybe we must generalize (ii) to

$$\star \begin{cases} f^{r_1}B + f^{r_2}B + \cdots + f^{r_n}B \subseteq f^m B. \\ \text{if} \quad 2^m \geq 2^{r_1} + \cdots + 2^{r_n}. \end{cases}$$

Use induction on n.

Assume $r_1 \leq r_2 \leq \cdots \leq r_n$

if $r_i = r_{i+1}$

then $f^{r_i}B + f^{r_{i+1}}B = f^{r_i}(B+B) \subseteq f^{r_i+1}B$.

so it suffices to prove $\star$, when the r_i are distinct.

clearly

$$2^{r_1} + \cdots + 2^{r_{n-1}} < 2^{r_n}$$

so by induction $f^{r_1}B + \cdots + f^{r_{n-1}}B \subseteq f^{r_n}B + f^{r_n}B \subset f^{r_n+1}B \subseteq f^m B$.

Uniqueness of $| \, |$:

Suppose $|x| = r$.

~~Choose~~ Given m choose $a \ni$

$$\frac{a}{2^m} \leq r$$

then $a \leq 2^m |x|$

and write

$$a = 2^{r_1} + \cdots + 2^{r_n}$$

$$0 \leq r_1 < r_2 < \cdots < r_n.$$

Thus

$$|f^m x| = 2^m |x| > 2^{r_1} + \cdots + 2^{r_n}$$

so $f^m x \notin f^{r_1}B + \cdots + f^{r_n}B.$

doesn't work!

Urysohn procedure:

$$0 \subset f^{-2}B \subset f^{-1}B \subset f^{-1}B + f^{-2}B \subset B$$

Thus if $r = 2^{-r_1} + 2^{-r_2} + \cdots 2^{-r_n}$

$$0 \leq r_1 < r_2 < \cdots$$

So

G nilpotent Lie group, simply-connected
f hyperbolic auto. of G
π uniform discrete subgroup, invariant under f

~~Problem~~: Then f induces an automorphism f of $G/\pi = M$ and f is Anosov. What are the stable and unstable foliations of M? It should be the left translation of the splitting at the origin. Thus

$$f^m(xG^s) = f^m x(\underset{G^s}{\underset{\parallel}{f G^s}}) \qquad p: G \to G/\pi$$

so

$G^s x$ ~~xG^s~~ is the generalized stable manifold thru

$$p(xG^s)$$

let $y \in G^s$ ~~so~~ that $f^m y \to 0$. as $m \to \infty$

then

$$f^m p(xy) = p(f^m x \cdot f^m y)$$

~~and dist $(p f^m x, p(f^m x \cdot f^m y)) \leq$ dist $(f^m x, f^m x \cdot f^m y)$~~

$d(pf^m x, p(f^m y \cdot f^m x)) \leq \text{dist}(f^m x, f^m y\, f^m x) = \qquad = \text{dist}(0, f^m y) \to 0.$

Thus if we choose a right inv. metric on G we see that ~~$p(xG^s)$~~ $p(G^s x)$ is the generalized stable manifold passing thru x.

If $\quad a = 2^{r_1} + 2^{r_2} + \cdots \; 2^{r_n}$

$$r_1 > r_2 > r_3 > \cdots > 2^{r_n}$$

Set

$$B_a = f^{r_1}B + f^{r_2}B + \cdots + f^{r_n}B. \qquad \text{compact nbd of } 0.$$

Clearly if $\quad a' = 2^{r_1'} + 2^{r_2'} + \cdots$

★ shows that $\quad a < a' \implies B_a \subset B_{a'}.$

$$B_a + B_{a'} \subseteq B_{a+a'} \qquad \text{clear.}$$

more basic than ★.

$$fB_a = B_{2a} \qquad \text{clear.}$$

$$-B_a = B_a. \qquad B_1 = B.$$

~~This~~ Note that $\quad B_a = \bigcap_{a' > a} B_{a'}$

since $\quad B$

$$a < \frac{1}{2^m} \implies B_a \subset f^{-m}B.$$

Thus unique function

$$|x| = \inf\{a \mid x \in B_a\}.$$

~~[illegible]~~ invariant measure ie $f^*\mu =$

$$(f^*\mu)(A) =$$

$$\int g(f_*\mu) = \mu(g\circ f) = d\cdot\mu(g).$$

$\int$

$$g = \chi_A$$

$$\cancel{f_*\mu = d\cdot\mu} \quad X$$

$$g\circ f = \chi_{f^{-1}A}$$

$$\int_X f_*\mu = \int_X \mu = 1$$

$f^*\mu$

$$\mu(f^{-1}A) = \mu(A).$$

$$(f_*\mu)(A) = \mu(f^{-1}A) = \mu(A).$$

$$f_*\mu = \mu. \qquad \int g(f_*\mu) = \int (f^*g)\mu$$

Bochner's theorem: G locally compact abelian group.
a function f on G is the Fourier transform of a measure μ
$\Longleftrightarrow$ it is positive definite ie

$f(x_i - x_j)$ positive definite matrix
where $x_1, \dots, x_n$ are dist. pts.

In my case I have a measure μ on $\mathbb{R}^2/\Lambda$
where Λ is the lattice generated by $1,1$ and $1,\lambda^{-1}$.

~~If $\check{\Lambda}$ is the dual lattice, then~~

$$T^\wedge = (R^2/\Lambda)^\wedge = \Lambda' \quad \text{where } \Lambda' \text{ is the dual lattice}$$
$$= \xi \in \mathbb{R}^2 \quad (\xi, \Lambda) \subset 2\pi\mathbb{Z}.$$

Suppose μ is my positive measure. Then

$$\mu(\xi) = \int e^{2\pi i \xi} \mu \qquad \xi \in \Lambda'$$

is a function on $T^\wedge$ which is positive definite.

$$\mu(A^t \xi) = \mu(\xi)$$

150

$$\boxed{\forall \vec{x}\ \exists \vec{y}\ \forall \vec{z}\quad F(\vec{x}, \vec{y}, \vec{z})}$$

given x_i choose $y_i \ni \forall z_i$

Thus must prove that

$$\underline{F(\vec{x}, \vec{y}, \vec{z}) \Longleftrightarrow \overset{\exists A}{F(x_i, y_i, z_i)}} \quad \text{all } i \in A \in \mathcal{U}.$$

$\mathcal{U}$ ultrafilter.

~~$F \equiv F_0 \vee F_1 \vee F_2 \vee \cdots \vee F_n$.~~

~~where $F_i = \bigwedge_j F_{ij}$~~

~~and~~ Look at F's for which ~~true~~

OKAY for $\begin{cases} \text{elementary predicates} \\ \cup \\ \cap \end{cases}$

$I \qquad \bigcap L_r$

$I \subset L$

$\Rightarrow \underline{I_r \subset L_r}$.

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$x \in \mathfrak{M}$ $\quad x \in R \Longleftrightarrow \phi(x).$

$x \subset R \Longleftrightarrow \psi(x).$

Check carefully

$$X_n = X$$

$$p^n_{n-1} : X_n \longrightarrow X_{n-1}$$

$$p^n_{n-1}(x) = fx.$$

$$\tilde{X} = \varprojlim X_n = \{ (x_n) \in \prod_{n=0}^{\infty} X_n \mid \forall n > 0 \quad fx_n = x_{n-1} \}.$$

$$\downarrow \tilde{f}$$

$$\tilde{X}$$

$$\tilde{f}(x_n) = (y_n)$$

$$y_n = fx_n \qquad \forall n \geq 0$$

$$n \geq 1 \qquad f(y_n) = f^2 x_n = f x_{n-1} = y_{n-1}.$$

Claim $\tilde{f}$ injective.

If $\tilde{f}(x_n) = \tilde{f}(y_n)$

$\Longrightarrow \forall n \geq 0 \quad fx_n = fy_n \Longrightarrow \forall n \geq 1 \quad x_{n-1} = y_{n-1} \Longrightarrow (x_n) = (y_n).$

Claim $\tilde{f}$ onto.

Given $(y_n)_{n \geq 0}$ with $f(y_n) = y_{n-1}$ $n \geq 1$ let

$$x_n = \begin{cases} y_{n-1} & n \geq 1 \\ f(y_0) & n = 0 \end{cases}$$

If $n \geq 1$, $fx_n = f(y_{n-1}) = \begin{cases} y_{n-2} & n \geq 2 \\ f(y_0) & n = 1 \end{cases} = \begin{cases} x_{n-1} & n \geq 2 \\ x_0 & n = 1 \end{cases} = x_{n-1}$

Thus it ~~works~~ $\in \tilde{X}$. $\tilde{f}(x_n) \stackrel{?}{=} (y_n)$ i.e. $y_n = f(x_n)$ no.