Problem

K theory - important papers

Borel - Serre

Atiyah's power operations in K theory

Cartier - Bergman on $W_j$, structure of $H^n(A)$ as Hopf algebra

Bass - Milnor - Serre, $K_1$ algebraic

Trace + characteristic polynomials in the derived cat.

Motives + similarity of Groth. Galois theory + usual?

Polynomial equations in K theory

Atiyah character + cohomology spectral sequence
Jacobs, Some new results of Ergodic Theory, Jahresber. der Deutschen Math. 67 (1965) 149-182

Abramov

Kurewic

Kushnirenko, DAN 161 (1965) 37-38
Analytical problem. Assume $f$ expanding and show that similar estimates hold for?

$f : M \rightarrow M$

$f^* : \mathcal{C}_0(T^*_M) \leftrightarrow \mathcal{C}_0(T^*_M)$

$f$ does not act on $\Gamma(T_M)$ but only on $\Gamma(T^*_M)$

where $f^*$ has what kind of analytical properties?

\[
\|\omega\|_f = \sup_{x \in M} \frac{\|\omega(x)\|}{\|x\|} \quad x \text{ runs over } T_p(M).
\]

\[
\|f^*\omega\|_f = \sup_{x \in M} \frac{|f^*\omega(x)|}{\|x\|} \quad x \text{ runs over } T_p(M).
\]

\[
\|f^*\omega\|_f = \sup_{x \in M} \frac{|\omega(f_x x)|}{\|x\|} \quad f_* : T_p X \rightarrow T_{f_p X}
\]

\[
\|f^*\omega\|_f = \sup_{x \in M} \frac{\|\omega(f_x x)\|}{\|f_x x\|} \quad \text{assumed}
\]

\[
\geq \sup_{x \in M} \frac{\|\omega(f_x x)\|}{\|x\|} \quad \|f_x x\| > \lambda \|x\| \quad \text{all } x
\]

Therefore

\[
\|f^*\omega\|_f = \sup_{x \in M} \|f^*\omega\|_f \geq \sup_{x \in M} \|\omega\|_f \cdot \lambda = \lambda \|\omega\|_f.
\]

Thus we find that $f^*$ expands on $\mathcal{C}_0(T^*_M)$. How about $L^2$ estimates

\[
\|f^*\omega\|_{L^2} = \int \|f^*\omega\|_f^2 V \geq \lambda^2 \int \|\omega\|_f^2 V \leq \lambda^2 \int \|\omega\|_f^2 V
\]
To get the variance, compute:

\[ \sum_{x \in \hat{R}} \mathbb{E}_{\omega} \mathbb{E}_{\omega} \]

because

\[ \sum_{i} \sum_{x \in \hat{R}} \mathbb{E}_{\omega} \mathbb{E}_{\omega} \]

\[ U_i \rightarrow fU_i \]

\[ \mathcal{F}^{\ast} \mathcal{F} \]

\[ \int_{\mathbb{R}} \| \omega \|^2 \, f_{\mathcal{F}}(\mathcal{F}(\mathcal{F})) \, V = \int_{\mathbb{R}} \| \omega \|^2 \cdot \mathcal{F}_{\mathcal{F}}(\mathcal{F}) \, V \]

defined because \( f \) is a covering.

Thus, there is a function \( g \) non-zero such that

\[ \frac{f_{\mathcal{F}}V}{V} = g \]
$M$ compact smooth manifold
$T$ tangent bundle
$f$ diffeomorphism of $M.$

Assume $f$ expanding, i.e. there is a Riemannian metric $\|\cdot\|$ on $T$ such that $V_0 \in T$

$\|df(V_0)\| \geq c \lambda^n \|V_0\|$ \quad where $c > 0$, $\lambda > 1$

Ind. of $n.$

**Question:** Does there exist an absolute bound on the expansion, i.e. an estimate of the form

$\|df^n(V_0)\| \leq C \mu^n \|V_0\|$ \quad $\mu > 1$, all $n.$

Yes, this is clear because take $\mu = \|df\|.$

So now can look at the spectrum of $df$ vs. $df$ acting on $C^0(TM).$ **Question:** Is this spectrum the same for $C^k(TM)$?

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**Theorem:** If $A$ is a bounded linear transformation on a Banach space $V,$ then

$\|A\| \geq \sup \{ |\lambda| : \lambda \in \text{spec } A \} = \limsup_{n \to \infty} \|A^n\|$

**Proof:** $\text{spec } A = \{ \lambda \mid (A - \lambda)^{-1} \neq 0 \}.$

If $|\lambda| > \|A\|,$ then

$\frac{1}{A - \lambda} = \frac{1}{\lambda(1 - \frac{A}{\lambda})} = -\frac{1}{\lambda} \left(1 + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \cdots \right)$

Converges to $\lambda \notin \text{spec } A \Rightarrow \|A\| < \|A\|.$
$X$ compact topological space
$E$ vector bundle over $X$
$f$ acts on $E$ and on $X$.
Say $f$ expanding if for any norm $\| \cdot \|$ on $E$ we have
$\| f^n \| \geq c \| \cdot \|$ all $n$ ($c > 0, \lambda > 1$)

independent of choice of $\| \cdot \|$.

1. may modify $\| \cdot \|$ and $\lambda$ so that $c = 1$.

Let $1 < \mu < \lambda$ and $m \in \mathbb{N}$.

Choose $1 < \mu < \lambda$ and $m \in \mathbb{N}$.

Suppose $\| \cdot \| \geq \mu^n \| \cdot \|$ for $n \geq m$.

$\| f \| + \| f^2 \| + \ldots + \| f^n \| = N(\omega)$

Then $N(f^\omega) = N(\omega) = \| f_\omega \| - \| \omega \|$.

$\frac{N(f^\omega) - 1}{N(\omega)} = \frac{\| f_\omega \| - \| \omega \|}{N(\omega)} = 0$.
f acts on $E$ over $f \circ X$.
Then $f$ acts on $\Gamma(E^*)$.

$\omega \in \Gamma(E^*)$.

$\langle \omega, f^* \omega \rangle = \langle f(\omega), \omega \rangle$.

Assume $f(x) = x$. Then

3. $f$ expanding $\iff$ $f^*$ expands on $C^0(E^*)$.

Proof: $f$ expanding $\implies \|f \omega\| / \|\omega\| \geq \mu > 1$. so

$$\|f^* \omega\|_p = \sup_{\omega \in E^*} \frac{\langle f \omega, f^* \omega \rangle}{\|f \omega\| / \|\omega\|} = \sup_{\omega \in E^*} \frac{\langle f \omega, \omega \rangle}{\mu \|f \omega\| / \|\omega\|} \geq \mu \|\omega\|_p.$$

$\implies f^*$ expanding.

Conversely, suppose $f^*$ expands on $C^0(E^*)$, i.e.

$$\|f^* \omega\| \geq c^m \|\omega\|.$$

$$\|f^m \omega\|_p = \sup_{\omega \in C^0(E^*)} \frac{\langle f^m \omega, f^m \omega \rangle}{\|f^m \omega\|} = \sup_{\omega \in C^0(E^*)} \frac{\langle f^m \omega, f^m \omega \rangle}{\|f^m \omega\|}$$

$$\|f^m \omega\|_p = \langle \omega, (f^*)^m \omega \rangle \quad \text{where } \|\omega\|_{f^m_p} = \|\omega\| = 1$$
\[ f^n(e'+e'') = (f^n)'e' + (f^n)^{n-1}(ge'') + (f^n)^{n-2}(gf'e'') + \cdots + g(f^n)^{n-2}(gf^{n-2}e'') + (f^n)^{n-1}(gf^{n-1}e'') \]

Assume contracting
\[ ||f^n(e'+e'')|| \leq ||f^n||e'|| + C (||f^n|| + ||f^n||^2 + \cdots) ||e''|| \]
\[ \leq \lambda^n ||e'|| + \lambda^n ||e''|| + C n \lambda^{n-1} ||e''|| \]
\[ \leq \lambda^n ||e'|| + \left( \lambda^n + C n \lambda^{n-1} \right) ||e''||. \]

But if \( \lambda < \mu < 1 \)

Then \[ \frac{\lambda^n + C n \lambda^{n-1}}{\mu^n} = \left( \frac{\lambda}{\mu} \right)^n + C n \left( \frac{\lambda}{\mu} \right)^{n-1} \to 0. \]
\[ n \to \infty \]

Thus works for contracting and so for expanding in case that \( f' \) exists.

\[ E = E' \oplus E'' \]

\[ E_x \rightarrow E_{fx} \rightarrow E_{f^2x} \rightarrow E_{f^3x} \rightarrow \cdots \]

\[ || || \quad || \quad \quad || \quad \quad \quad || \quad \quad \quad || \]

Suppose that \[ \frac{||f^n(e'+e'')||}{||e'+e''||} \]
Proposition: Let \( f: X \to X \) be a map.

\[ O \to E' \to E \to E'' \to O \]

is a bundle over \( X \) on which \( f \) acts. Then if \( f \) expands on \( E' \) and \( E'' \), it expands on \( E \).

Proof: Let \( \tilde{X} = \text{lim}_{\leftarrow} \text{inverse of} \)

\[ X \xrightarrow{f} X \xrightarrow{\tilde{f}} \tilde{X} \xrightarrow{\text{lim}} X \]

And let \( \tilde{E} = \text{lim}_{\leftarrow} E \). Then \( \tilde{E}/\tilde{X} \) is a vector bundle. Now \( f \) yields an isomorphism \( \tilde{f} \) of \( \tilde{X} \) by

\[ \tilde{f} \{ x_n \} = \{ f(x_n) \} = \{ x_{n+1} \} \]

As diagram

\[ \begin{array}{ccc}
E & \xrightarrow{P_n} & E \\
\downarrow & & \downarrow \\
X & \xrightarrow{P_n} & X
\end{array} \]

commutes and \( P_n \) is surjective, it suffices for \( f \) to be expanding on \( E \) to show \( f \) expands on \( \tilde{E} \).

But then we are reduced to contractions, since there is
I have shown now that if \( f \) is an expanding diffeo of \( M \), then it's also expanding on \( AT^* \), \( J_b(1) \), \( J_b(T^*) \) etc., in which case \( f^* \) on \( A^*(X) \) is expanding.

**Question:** \( f^* : A^* \rightarrow A^* \) homotopy equivalence.

Under what conditions can we conclude that

\[
\cap(f^*)^n A^* \subset A^*
\]

is a homotopy equivalence.

\( \omega \in A^b \) \hspace{1cm} d\omega = 0.

\[
\omega_1 = \frac{1}{d} f^*_x \omega.
\]

\[
\omega_n = \left( \frac{1}{d} f^*_x \right)^n \omega.
\]
how much do we know about $g$?

It doesn't matter since if we take $g$ large $f^*$ will be expanding!

One can ask that $f_* V = V$. Invariant measure.

Example

$$S^1 \rightarrow S^1$$

$$2 \cdot dz = dz.$$

$$\sum_{i} \gamma_i (g^* dz)$$

can't representatives for $[\pi/\pi]$.

In the group case what can we say about the tangent bundle of $M$.

If there is no finite group part, then it is trivial because one may take the vector fields on $M$ and translate them commutes with the right $V$ translation, but not with $g$. So have finite group acting on $M_o$ with trivial tangent bundle. What can we do?

By general non-sense we get an element of $H^1(g; \mathfrak{gl}(C^\infty(M_o))^*)$. Note that $C^\infty(M_o)$ is a ring on which $G$ acts acyclically.

Suppose a finite $G$. $G$ acts nicely on a
Suppose \( M = G/\Gamma_0 \) is a nil-manifold and \( \Gamma/\Gamma_0 \) finite gp acting freely thereon. What forms \( f \) on \( M \) can be written in the form \((f^*)^n \) for all \( n \)? Do there exist any?

**Case 1:** \( \Gamma = \Gamma_0 \)

*Example:* \( 2 : S^1 \to S^1 \)

\[ \omega = \theta(z)dz \]

\[ 2^*\omega = 2\theta(2z)dz \]

Thus
\[ a(z)dz \in \text{Im}(2^* ) \quad \iff \quad a(z) = b(2z) \]
\[ a(\frac{z}{2^n}) \text{ defined} \]

\[ a(z) = a(z + \frac{1}{2^n}) \quad \text{all } n \]

\[ \iff a \text{ is constant.} \]

**Function on** \( G/\Gamma \quad \Gamma = \Gamma_0 \)

\[ f = \psi^* g \]

\[ f(x) = g(\psi x). \quad x \in (\Gamma + \chi) \]

Thus
\[ f(\chi x) = \psi(\chi x) = \psi x \]

\[ f(x, \chi) = g(\psi x)(\psi^* \psi) = g(\psi x) \psi^* \]

\[ R_{\psi^*} f \]
0 may modify \( \| \| \) and \( \lambda \) so that \( c \lambda^n > 1 \).

**Proof.** Choose \( n \) such that \( c \lambda^n > 1 \). Then \( \| f'' o || > \| v || \). Let

\[
N(v) = \| v \|^2 + \cdots + \| f^n o \|^2 \quad (a \ new\ norm) .
\]

Then

\[
\frac{N(f o)}{N(v)} - 1 = \frac{N(f o) - N(v)}{N(v)} = \frac{\| f o \|^2 - \| v \|^2}{N(v)} > 0 \quad \text{unless} \ f o v = 0 .
\]

As \( B \) is compact conclude that this is bounded away from 0 so

\[
\frac{N(f o)}{N(v)} - 1 > \varepsilon > 0 .
\]

\[N(f o) > (1 + \varepsilon) N(v) .\]

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2. If \( 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0 \) is an exact sequence on which \( f \) acts, then

\( f \) exp, \( E \cong \text{exp} \) of \( E' \) and \( f \text{exp} \) on \( E'' \).

(\( \rightarrow \) clear).

**Proof.** Write \( E = E' \oplus E'' \) so

\[ f(e') = f e' \]

\[ f(e'') = f e'' + g e'' \]

where \( g : E'' \rightarrow E' \)

\[ f e'' \] is compact contracting map.
Suppose that \((A - \lambda)^{-1}\) exists for \(|\lambda| > c\).

Let \(f\) be a linear functional on \(L(WV)\). Then

\[
\frac{f(A - \lambda)}{\lambda} = \frac{1}{\lambda} \left( 1 + \frac{f(A)}{\lambda} + \frac{f(A^2)}{\lambda^2} + \cdots \right)
\]

holds for \(|\lambda| > \|A\|\) and is an equality of analytic function

so must have

\[
\frac{f(A^n)}{\lambda^n} \leq M, \quad \text{all } |\lambda| > c
\]

or

\[
f(A^n) \leq M, \quad |\lambda| > c.
\]

Thus by Banach-Steinhaus \(\|A^n\|/\|\lambda^n\|\) bounded \(A\)

\[
\frac{\|A^n\|}{|\lambda^n|} \leq C(\lambda), \quad |\lambda| > c.
\]

\[
\lim_{n \to \infty} \frac{\|A^n\|}{|\lambda|^n} \leq 1
\]

\[
\therefore \lim_{n \to \infty} \text{sup } \frac{\|A^n\|}{|\lambda|^n} \leq C.
\]

\[
|\lambda| > \lim_{n \to \infty} \text{sup } \frac{\|A^n\|}{|\lambda|^n}
\]

\[
\implies |\lambda| > \sqrt[n]{\|A^n\|} \quad \text{all } n > n_0
\]

\[
\implies > \frac{\|A^n\|}{|\lambda|^n} = \text{const}.
\]
A n-dimensional representation of \( G \)

\( G \) acts on \( S^V \sim S^{2n-1} \) and \( C_n(V) \) is the transgression class

\[ C_n(V) \in H^{2n}(G; \mathbb{Z}) \]

[chain]

Can you give a simple formula for \( C_n(V) \) by making suitable choices?

An element of \( H^{2n}(G; \mathbb{Z}) \) is an element function

\[ f : G^{2n} \to \mathbb{Z} \quad \Rightarrow \quad \delta f = 0. \]

Please we are trying to construct a section of \( S^V \)

over the \( 2n \)-skeleton. It is possible over the \( 2n-1 \)-skeleton.

This means that for each simplex of \( B_g \) of dim < 2n, it has a

covering simplex in \( S^V \).

\[ E_g \times V \]

\[ \Delta(\varrho) \to B_g \]

\[ \text{residue or trace of a transformation} \]

\( n \)-th Chern class

\( G \) acts on \( S^{2n-1} \) then it acts on the singulars

\[ \Rightarrow C_0(S^{2n-1}) \to C_0(S^{2n-1}) \to \mathbb{Z} \to 0 \]

which gives an element of \( \text{Ext}_G^{2n}(\mathbb{Z}, \mathbb{Z}) \cong H^{2n}_G(\mathbb{Z}) \)

which is \( C_n \).

Similarly, for the other Chern classes.
Theorem (Shub): Let $g$ be a point of $\text{proj}_{\text{point}}$ and $h$ be a second point. There exists a unique invariant $\tau$ such that $\tau(g,h)$ is the unique invariant $\tau(g,h)$.

The functor $\Pi_f$ from expanding maps with fixed to groups is fully faithful.

Theorem (Quillen): If $M$ is a compact smooth manifold with an expanding map $f$ on $M$, then for any continuous function $u$ on $M$ on $C^0(M)$,

$$\lim_{n \to \infty} I + f^n + \cdots + (f^n)^{n-1}$$

exists uniformly and converges to the measure $\mu$. This also works for $C^\infty$ or a slightly better than $C^2$. 

Problem: Define a metric on $M$ with the property that $d(f^{-m}x, f^{-m}y, f^m x, f^m y) = d(x, y)$ on the universal covering of $M$. Perhaps you need the notion of a geodesic joining two points of $M$.

stable
Fundamental problem: Define and solve generic partial differential operators.

Requirements of the definition:
1. Stability for lower order perturbations
2. Stability for variable coefficients
3. (Topological conjugacy)

Lots of questions which should be answered:
(a) Suppose we give a symbol sheaf on \( \mathbb{R}(T^*) \) flat over \( \Omega \). Then, what are the integrability conditions, e.g., integrability of the characteristic equations?
(b) Supposing the integrability conditions are satisfied, what about arbitrary variation of lower terms?

Hörmander's fundamental insight into the problem is the local character of the estimates and what to expect as one goes to the boundary.

I wish to prove exactness of the transpose sequence of compactly supported distributions. Therefore, I want a homotopy operator.

Example: Fairly critical:
Suppose that we consider dist.

\[ D(\mathcal{E})^* \to D(\mathcal{E})^* \to \cdots \]

with compact support. We suppose this sequence to be exact in
Problem:
\[ T = \mathbb{C}^n \text{ coordinates } z_1, \ldots, z_n. \]

Let \( M \) be a finitely generated \( A = \mathbb{C}[T] \)-module. Then let \( \mathcal{F} \) be a coherent sheaf on \( \mathbb{P}(T) \) such that if \( \mathcal{F} \) comes from applying the associated sheaf \( M \) and there is an exact sequence

\[ 0 \rightarrow A \otimes V_2 \rightarrow A \otimes V_1 \rightarrow A \otimes V_0 \rightarrow M \]

of homogeneous maps. Then what?

Problem: Prove that to \( \mathcal{F} \) there is a number associated which measures some of the smoothness; this number is stable under change of metrics, and in small variations!
\[ f: \Omega \rightarrow \Omega \]
\[ p(A) = 0 \Rightarrow \mu(\varphi^{-1}A) = 0 \]
\[ A(f) = \frac{1}{n}(1 + \varphi + \ldots + (\varphi)^{n-1})f \text{ coni a.e. all } f \in L^1 \]
\[ \iff \frac{1}{n} \sum_{\ell=0}^{n-1} \rho(\varphi^{-\ell}(A)) \leq K \rho(A) \]
\[ \mathcal{E} K \in \mathcal{A} \]

\[ \mu = \frac{h \, d\theta}{2\pi} + \mu_5 \]

\[ \exp \left[ \log h \, \frac{d\theta}{2\pi} \right] \]  

\[ d(X_0, \mathbb{B}X^n) \]

\[ h(\alpha, T) = d(\alpha, \sqrt{T^{-n}} \alpha) \]

\[ n > 0 \]
Hadamard trick

Given \( P_1, \ldots, P_n \rightarrow Q = 0 \)

Look at polynomials

\[ P_1, P_2, \ldots, P_n, 1 - QT \]

Conclude that

\[ A_i(T) P_i + B(T)(1 - QT) = 1 \]

Set \( T = \frac{1}{Q} \)

\[ \sum_i A_i \left( \frac{1}{Q} \right) P_i = 1 \]

\[ \sum_i A_i P_i = Q^n \]

\[ Q_m A_i(T) \]

\[ Q^{k-1} \]

\( P_1, \ldots, P_n \) homogeneous of degree \( m \)

\( P_i = 0 \ \forall i \Rightarrow Q = 0 \), \( Q \) of degree \( k \)

\[ P_i \]

\[ S^{k-n} QT \]

\[ \sum_i A_i(T, S) P_i + B(T, S)(S^{k-n} - QT) = S^N \]

\[ Q^{\frac{m}{n}} \sum_i A_i(T) S^k P_i \]

Set \( S = 1 \).
Returning to our manifold, can we piece the formula for toral automorphisms?

Map $M \rightarrow$ simplex in Hilbert space from the partition.

What I want is to use scattering theory to factor my automorphism so as to get eigenvalues outside of the unit circle.

Prediction theory:

$H$ Hilbert space, $U$ unitary operator on $H$, $V$ cyclic vector

$D^+ = \langle u^n v^- : n \geq 0 \rangle$.

Assume $\bigwedge_{n \geq 0} u^n D^+ = 0$. $\bigwedge_{n \in \mathbb{Z}} u^n D^+ = H$.

and therefore if can represent

$\langle H, U, v^- \rangle = \langle L^2(S^1, \mu), z, 1 \rangle_{D^+}$ closure of helm. fun.

on the other hand I can also make

$\langle H, D^+, u \rangle = \langle L^2(S^1, \text{helms}), \text{helm}, z \rangle$.
Everything clearly works, and again we may define the entropy in the obvious way. All this works for a compact Hausdorff space and a measure of it.

**Theorem:** Entropy thus defined coincides with usual entropy.

**Proof:** Clear from continuity of $h(a, T)$ in $a$.

The point is that $h_n(a, T) = H(\mathcal{A}/ \mathcal{V}^n T^{-1} \mathcal{A})$ is monotone decreasing and continuous in $a$. Therefore?

$\forall a, b \in \mathcal{A}$, $d(a, b) = \sup_{x, y \in \mathcal{A}} |x - y|_T 

\text{hence}$

\[ d(a \cup a', b \cup b') \leq d(a, b) + d(a', b'). \]

So

\[ d(\mathcal{A} \cup T^{-n} \mathcal{A}, \mathcal{B} \cup T^{-n} \mathcal{B}) \leq (n+1) d(a, b) \]

\[ \frac{1}{n+1} H_{\mathcal{A}}(a \cup T^{-n} \mathcal{A}) - \frac{1}{n+1} H_{\mathcal{B}}(b \cup T^{-n} \mathcal{B}) \leq d(a, b) \]

\[ |h(a, T) - h(b, T)| \leq d(a, b). \]
Basic problem: Let $M$ be a compact $C^\infty$ manifold and let $f$ be an endomorphism of $M$ which leaves a smooth volume element $\omega$ invariant. Find a formula for the entropy of $f$ in terms of topological data of $f$ assuming that $f$ is structurally stable.

Can you replace measure partitions by partitions of unity?

Suppose $\sum_i f_i = 1$ is a partition of unity on $M$. Define its entropy with respect to $\omega$ to be the entropy of the simplicial complex associated to $\sum_i f_i$, the induced measure and the simplex decomposition.

Better $\sum_i f_i \omega = 1$

so define the entropy to be

$$\sum_i -f_i \omega \ln(f_i \omega) = H(\omega).$$

How about conditional entropy?

Given $\sum_i g_i = 1$ and $\sum_j g_j = 1$

$$H(\theta / \varphi) = \sum_{i,j} -\ln \left( \frac{\omega(p \cdot g_i)}{\omega(g_i)} \right) \cdot \omega(p \cdot g_i)$$
Old definition:

\[ h(a, T) = \lim_{n \to \infty} \frac{1}{n} H(a \cup T^{-1} a \cup \cdots \cup T^{-n} a) \]

\[ H(a \cup T^{-1} a) = H(a) + H(T^{-1} a) \]

\[ H(a \cup T^{-1} a \cup T^{-2} a) = H(T^{-1} a \cup T^{-2} a) + H(\frac{T^{-1} a}{T^{-2} a}) \]

\[ = H(T^{-2} a) + H(T^{-2} a / T^{-3} a) + H(\frac{T^{-3} a}{T^{-4} a}) \]

Now that \( H(T^{-1} a / T^{-n} a) = \sum_{i=0}^{n-1} -\ln \frac{\mu(T^{-i} a \cap T^{-i} a)}{\mu(T^{-i} a)} \), true for finite partitions

\[ H(\frac{T^{-1} a \cup T^{-2} a \cup \cdots \cup T^{-n} a}{T^{-n} a}) = \sum_{i=0}^{n-1} H(\frac{T^{-i} a}{T^{-i} a}) \]

and as these are decreasing quantities the limit exists.

\[ h(a, T) = \lim_{n \to \infty} H(\frac{a}{T^{-1} a \cup \cdots \cup T^{-n} a}) \]
Question: Suppose $\sum_{i=1}^{\infty} p_i = 1$ for all $p_i > 0$.

\[ \sum_{i=1}^{\infty} -p_i \ln p_i < \infty. \]

No: First take $\sum_{i=1}^{\infty} 2^i = 1$. Then divide each $2^i$ up into $2^{2^i}$ pieces, whence

\[ \sum_{j} -p_j \ln p_j = \sum_{i} \frac{1}{2^i} \ln 2^i = \sum_{i} \ln 2 = \infty. \]

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If the ultimate partition has finite entropy all transformations have entropy 0. 
*Generalizations*

Entropy is a measure of measuring partitions which tends monotonically toward infinity as the partition gets finer. Another fundamental property is its convexity, i.e.

\[ H(a/e_1) = H(\text{ave}_1) - H(e_1) \leq H(\text{ave}_2) - H(e_2) \]

\[ h(a, T) = h(a/V_{n>0}^{-1}a) = H(V_{n>0}^{-1}a/V_{n>0}^{-1}a) \]

\[ = H(B | T^{-1}B) \quad \text{not rigorous because } H(B) = \infty \]

and \( T^{-1}B \subset B \).

\[ h(a, T) = H(B) - H(T^{-1}B) \]

Why is this the same as the old definition?
Brumer's talk:

\([K: \mathbb{Q}] < \infty\). EK units of K. There are various topologies on \(E_K = \mathbb{G}_m(A)\) where A = integers of K.

- Congruence topology - nbhd basis: \(\{u \in E_K | u \equiv 1 \bmod \mathfrak{o}_K\}\).
- Profinite topology - \(\mathfrak{n} \in E_K\).

Theorem of Chevalley: Two topologies are the same.

Brumer's problem: To show that if \(\mathfrak{p}\) is a prime number, then the following two topologies are the same:

- \(p\)-congruence topology - nbhd basis: \(\{u \in E_K | u \equiv 1 \bmod p^n\}\)
- Equivalence induced topology from \(\mathbb{G}_m(A) \rightarrow \mathbb{G}_m(A)\)

where \(\mathfrak{A} = \lim \frac{A}{p^nA} = \bigcap_{\mathfrak{p}} A_{\mathfrak{p}}\).

- \(p\)-topology - \(p^n E_K\).

Brumer's theorem: If K is real and abelian, then the \(p\)-congruence is finer than the \(p\)-topology on \(E_K\).

He proves this by taking a large \(p\)-adic field \(\Omega\), containing \(K\), and considering the map

\[L: E_K \otimes \Omega \rightarrow \Omega[G]\]

\[L(u \otimes x) = \sum (x \cdot \log \sigma) \cdot u\]

where \(\log \sigma\) is the \(p\)-adic logarithm. By a result of Minkowski, there is a unit \(u\) generating \(E_K \otimes \Omega\) as an \(\Omega[G]\) module. One assumes \(L\) not injective and get a relation
Wienstein tells that Avez has shown that the entropy of a
diffeomorphism of a compact manifold $M$ is $\leq \dim M$ (max change in
$n-1$ dimensional area), at least provided it leaves a smooth measure
on $M$ invariant.

Let $X, \mu$ be a probability space. — Boolean $\sigma$-algebra with
a trace. — Commutative $W^*$ algebra with a trace,

distinguished positive
linear functional.

From a partition of $1$
Entropy

\((\Omega, \mu)\) probability space.

If \(A = A_1 \cup \cdots \cup A_r\) is a partition of \(\Omega\), we set

\[
H(A) = - \sum_i \mu(A_i) \ln \mu(A_i)
\]

Entropy of the partition \(A\). Equivalently,

\[
H(A) = \int - \ln \left( \text{measure of fibers} \right) \, d\mu(y)
\]

\(y \in \Omega / A\)

Conditional entropy of a finite partition \(A\) w.r.t another \(B\)

\[
H(A/B) = ?
\]

Weinstein says that \(- \log \mu(A_i)\) is the amount of information one gets from knowing the point \(y\) in \(A_i\), so that \(H(A)\) is the expectation of the amount of information offered by the partition.

Hence

\[
H(A/B) = \sum_j \left[ - \sum_i \frac{\mu(A_i \cap B_j)}{\mu(B_j)} \ln \left( \frac{\mu(A_i \cap B_j)}{\mu(B_j)} \right) \right] \mu(B_j)
\]

amount of information obtained from having a point in \(A \cap B_j\) when we already know its in \(B_j\).
\[ H(A \mid B) = H(A \lor B) - H(B) \]

\[ a \leq B \Rightarrow H(A \mid B) = 0. \]

Claim that

\[ d(A, B) = H(A \mid B) + H(B \mid A) \]

is a metric on the set of finite partitions.

\[ d(A, B) + d(B, C) = 2H(A \lor B) - H(A) - H(B) + 2H(B \lor C) - H(B) - H(C) \]

\[ H(A \lor B \lor C) = H(A \lor B \mid C) + H(C) \]

\[ = H(A \mid B \lor C) + H(B \lor C). \]

\[ d(A, B) = \left\{ \sum - \left\{ \ln \left( \frac{\mu(A \cap B_i)}{\mu(B_i)} \right) + \ln \left( \frac{\mu(A \cap B_i)}{\mu(A)} \right) \right\} \mu(A \cap B_i \cap C_k) \right\} \]

\[ + \]

\[ d(B, C) = \left\{ \sum - \left\{ \ln \left( \frac{\mu(B_j \cap C_k)}{\mu(B_j)} \right) + \ln \left( \frac{\mu(B_j \cap C_k)}{\mu(C_k)} \right) \right\} \mu(B_j \cap C_k) \right\} \]
If $T^{-1}$ automorphism of $(\Omega, \mu)$, enough to have $T^*_\mu = \mu$

Then

$$H \left( \bigvee_{n \geq 0} T^{-n}a \big/ \bigvee_{n > 0} T^{-n}a \right) = d \left( \bigvee_{n \geq 0} T^{-n}a, \bigvee_{n > 0} T^{-n}a \right)$$

is calculated how?

$$H \left( \frac{B}{T^{-1}B} \right) \quad T^{-1}B \leq B$$

$$H(B) - H(T^{-1}B)$$

$$H(B) ?$$

$$H(a)$$

$$H(a \vee B/c) = H(a/c) + H(a/bv/c)$$

$$B \geq c \quad \Rightarrow \quad H(a/b) \leq H(a/c)$$

$$H(a/b) + H(b/c) = H(a \vee B) - H(B) + H(bv/c) - H(c)$$

$$= H(a \vee B/c) + H(c/b)$$

$$H(a/b) + H(b/c) = H(a \vee B) - H(B) + H(bv/c) - H(c)$$

$$= H(a \vee B/c) - H(a \vee B/c) + H(bv/c) - H(c) - H(B)$$

$$=$$
\[ H(a/c) = H(a \cap c) - H(c) \]

then

\[ H(a/b) + H(b/c) = [H(a) - H(b)] + [H(b) - H(c)] = H(a/c). \]

Suppose we work with measurable sets and

\[ d(A,B) = \mu(A-B) + \mu(B-A) = \mu(A \cup B) - \mu(A \cap B). \]

Then triangle inequality

\[ = \mu(A) + \mu(B) - 2\mu(A \cap B) = 2\mu(A \cup B) - \mu(A) - \mu(B). \]

\[ d(A,B) + d(B,C) = \mu(A \cup B) - \mu(A \cap B) + \mu(B \cup C) - \mu(B \cap C) \]

\[ \mu(A \cup B) + \mu(B \cup C) - (\mu(A \cup C) + \mu(B)) \geq 0 \]

\[ \mu(A \cup B \cup C) - \mu(B \cup (A \cap C)) - \{\mu(A \cup B \cup C) - \mu(B \cup (A \cup C))\} \]
3.4.

\[ d(A, B) = 2\mu(A \cup B) - \mu(A) - \mu(B) \]
\[ d(B, C) = 2\mu(B \cup C) - \mu(B) - \mu(C) \]
\[ d(A, C) = 2\mu(A \cup C) - \mu(A) - \mu(C) \]

\[ d(A, B) + d(B, C) - d(A, C) = 2\left[ \mu(A \cup B) + \mu(B \cup C) - \mu(A \cup C) - \mu(B) \right] \]

\[ = 2\left[ \mu(A \cup B \cup C) + \mu(B \cup (A \cup C)) - \mu(A \cup B \cup C) - \mu(B) \right] \geq 0. \]

\[ d(A, B) = 2H(A \cup B) - H(A) - H(B) \]
\[ d(B, C) = 2H(B \cup C) - H(B) - H(C) \]
\[ d(A, C) = 2H(A \cup C) - H(A) - H(C) \]

\[ d(A, B) + d(B, C) - d(A, C) = 2\left[ H(A \cup B) + H(B \cup C) - H(A \cup C) - H(B) \right] \]

\[ H(A \cup B) + H(B \cup C) = H(A \cup B \cup C) + H( \cdots ) \]

\[ H(A \cup B) + H(B) = H(A \cup B \cup C) - H(B \mid A \cup C) + H(B) \]

\[ H(B \mid A \cup C) + H(A \mid C) = H(A \cup B \mid C). \]

\[ H(A \cup B) + H(B \cup C) = H(A \cup B \cup C) - H(C / A \cup B) + H(B) + H(C / B) \]
\[
\begin{cases}
H(A \cap B) + H(B \cap C) \geq H(A \cup B \cup C) + H(B) \\
H(A \cup C) + H(B) \leq H(A \cup B \cup C) + H(B)
\end{cases}
\]

because

\[H(A \cup B) + H(B \cup C) = H(A \cup B \cup C) - H(C \setminus A \cup B) + H(C/B) + \beta(B)\]

\[H(C/A)\]

\[H(C/C_1) \leq H(C/C_2)\]

if \( C_1 \geq C_2 \) by convexity of \(-x \log x\)

so this proves a mequality
Prediction Theory

\[ H, U, \mathbf{a} \]

Bochner \( \Rightarrow \) \( \mu \) on \( S' \) \( \Rightarrow \) \( \mu(z^n) = \langle U^n a, a \rangle \)

whence \( H = L^2(S', \mu) \)
\( U \) \( \mathbf{a} \) \( 1 \)

Let \( D^+ \) be closed subspace of \( H \) gen. by \( U^n a \) \( n \geq 0 \).

\( D^+ \) \( \subset \) closure of polys in \( z \) in \( L^2(S', \mu) \).

We assume that \( D^+ \subset H \) in which case we look at

\[ H = \bigcap_{n \geq 0} U^n D^+ \oplus H_1 \]

Assume that \( H_1 = 0 \). In this case we may project \( 1 \) onto \( D^* = U D^+ \oplus N_1 \) getting \( 1 = \sum_1, f \geq (1-f)+f \).

Then

\[ H = \sum_{n \geq 0} C z^n f. \text{ and } f \text{ closure of polynomials.} \]

Can define map \( L^2(S^{1/2}, \rho) \rightarrow L^2(S', \mu) \) by

\[ f \rightarrow f \]

Thus \( \langle z^K f, z^K f \rangle = \int_0^{2\pi} f^2 \|f\|^2. \quad k \geq e. \)
Assume \( \nu \) generates

Then \( \bigcap_{n>0} \nu^n D^+ = \emptyset \) \( \iff \mu \) absolutely cont. with respect to Lebesgue measure.

(\( \implies \)) Choose \( \nu = \nu_0 + \nu_1 \) \( \nu_1 \in D^+ \) \( \nu_0 \bot D^+ \). Then

\[
D^+ = C \nu_0 \oplus \nu D^+
\]
\[
= C \nu_0 + C \nu_0 + \cdots + C \nu_0 + \nu^{n-1} D^+
\]

\( \nu_0 \) generates \( \mathcal{H} \)

so measure from \( \nu_0 \) is absolutely cont. w.r.t. \( \frac{dz}{2\pi i} \)

Thus \( \mu = \mu_\mathcal{S} + h \frac{dz}{2\pi i} \)

\( \nu_0 \to \mathcal{H} = \bigcap_{n>0} \nu^n D^+ \oplus \mathcal{H} \)

where \( \mathcal{H} \) has \( \bigcap \)
and so therefore

\[ |f|^2 \, d\mu = \frac{|f|^2}{2\pi} \]

i.e.

\[ d\mu = \frac{|f|^2}{2\pi} \frac{1}{|f|^2} \, d\theta \]

In particular, if \( f \neq 0 \) a.e.

\[ \frac{1}{|f|^2} \in L^1(S^1, d\theta) \]

Since we get an isomorphism.

Therefore

\[ \frac{1}{f} = g \text{ is a fn in } L^2(S^1, d\theta) \]

which is the boundary values of a holomorphic function.

in the interior. \[ \therefore g \in H^2(S^1) \]

So now we need the theorem which says that

where \( g \in H^2 \)

\[ \int \text{Log } h \, d\theta < \infty \]

Take

\[ \int \text{Log } h = 2 \text{ Re } \text{Log } g \]

Thus, want

\[ \text{Log } h = k + \overline{k} \]

Then, indeterminate case occurs when if we write \( \mu = \mu_0 + K \frac{d\theta}{2\pi} \),

we have \( \int \text{Log } h \, d\theta < \infty \).
Question: What is the distance from $z^{-1}$ to closure of analytic functions?

Want $\|f\|^2 = \int |f|^2 \, d\mu$

$\lambda \xrightarrow{\frac{1}{f}} 1$

$\lambda \xrightarrow{1} f$

$\lambda \xrightarrow{\left(\frac{1}{f} - 1\right)} 1 - f$

$\int \frac{1}{|f|^2} \, d\theta = 0$

I am after

$1 - \|f\|^2 \int \frac{1}{|f|^2} \, d\theta = \int d\mu = \|f\|^2$. Assume $\|f\|^2 = 1$

$\|f\|^2 \int h \frac{d\theta}{2\pi} = 1$

Where $d\mu = \|f\|^2 \, h \frac{d\theta}{2\pi}$

Up to a constant

$\alpha^2 \int h \frac{d\theta}{2\pi} = 1$

$\alpha = \|f\|$
\[ d\mu = \frac{||f||^2}{2\pi} h d\theta \quad \Rightarrow \quad \int_{0}^{2\pi} h d\theta = 1 \]

\[ h = \frac{1}{||f||^2} = |g|^2. \]

\[ \int \left( \frac{1}{f} - 1 \right) \frac{d\theta}{2\pi} = 0 \quad \Rightarrow \quad \int (g - 1) d\theta = 0 \]

\[ g = 1 + a_1 z + a_2 z^2 + \ldots \quad \Rightarrow \quad g = e^p \]

\[ 2\Re p = \log h. \]

\[ \int |g|^2 d\theta = \int \left( 1 + a_1 z + a_2 z^2 + \ldots \right)^2 d\theta = \int 1 + a_1^2 + a_2^2 + \ldots d\theta \]

\[ \frac{d\mu}{d\theta} = \frac{||f||^2 \cdot h}{2\pi} \]

\[ d\mu = h \frac{d\theta}{2\pi} = |g|^2 \frac{d\theta}{2\pi} \]

Unique up to constant of absolute value 1, normalize by requiring \( g(0) \) positive. Then

\[ g = ||f||g \]

\[ \int g \frac{d\theta}{2\pi} = ||f||. \]
\[
\int \frac{d\theta}{2\pi} = g(0).
\]

\[
\log g(0) = g = e^k
\]

\[
|g|^2 = h
\]

\[
g = e^k
\]

\[
2k = \log h
\]

\[
k(0) \text{ real}
\]

\[
k(0) = \frac{1}{2}
\]

\[
2k(\theta) = \int \log h \, \frac{d\theta}{2\pi} \quad \text{harmonic fun.}
\]

\[
q(0) = e^{2k(0)} = e^{\int \log h \, \frac{d\theta}{2\pi}}
\]

\[
\|f\| = e^{\frac{1}{2} \int \log h \, \frac{d\theta}{2\pi}}
\]

\[
\|f\|^2 = e^{\int \log h \, \frac{d\theta}{2\pi}}
\]
A kind of continuous entropy

In other words, given the probability distribution

$$h \cdot \frac{d\phi}{2\pi}$$

from our stochastic process, the distance

$$d(a, u_{D^+})^2 = e^{\int \log h \cdot \frac{d\phi}{2\pi}}$$

$$\|a\| = 1$$

$$h \in L^1 \Rightarrow (\log h)^+ \in L^1$$

so

$$-\infty \leq \int \log h \cdot \frac{d\phi}{2\pi} < \infty.$$ 

A basic problem: Given a measure $h \cdot \frac{d\phi}{2\pi}$, why is

$$\int \log h \cdot \frac{d\phi}{2\pi}$$

a good animal?

Can you use this somehow to calculate entropy?

(work in the space of partitions $L^2(X, \mu)$ rather than in the Hilbert space $L^2(X, \mu)$)

First point is that a partition is like a measurable set, so we need to embed partitions into some linear or quasi-linear manifold of sorts so the entropy appears as a kind of linear functional. Thus two partitions $A$ and $B$ are said to be independent if...
\[ \mu(A_i \cap B_j) = \mu(A_i) \cdot \mu(B_j) \]

in which case the entropies add:

\[ H(A \cup B) = H(A) + H(B) \]

Make an algebra out of the free abelian group generated by a point by

\[ \frac{\mathbb{Z}}{2\mathbb{Z}} \]

\[ (f \cdot g)(x, y) = \sum_{x \equiv y} f(x, z) \cdot g(z, y) \]

given a partially ordered set it is a category so one can form its group ring free abelian group generated by intervals with obvious multiplication. It is a non-commutative ring.

Given a partition it defines an operator on measurable functions namely conditional expectations. A projection operator satisfying the Reynolds identity \[ E(fg) = Ef \cdot Eg. \] Consider the algebra generated by these conditional expectations. What does this mean for partitions of unity.
problems

moduli space for Anosov diffeomorphisms + expanding maps.

Problem: Let $f: M \to M$ be an expanding map and let $f_t$ be a $t$-parameter deformation. According to Shub, $f_t$ is conjugate to $f$ by a $t$-homeomorphism.

$$h_t f h_t^{-1} = f_t$$

differentiate at $t = 0$ so get

$$f_t'(x) - x = Y_t$$

where

$$Y_t = \frac{d}{dt} f_t(m) \bigg|_{t=0}$$

now the point is that $f_t - 1$ is invertible on the

$$\frac{d}{dt} (h_t f h_t^{-1}) \bigg|_{t=0} = X f - df \circ X = \frac{d}{dt} f_t \bigg|_{t=0}$$

$$f_t(m) = h_t(f(h_t^{-1}m))$$

$$Y_{fm} = X_{fm} - df \circ X_m$$
Suppose we work with autors. Then this can be written

\[ Y = X - f^*_X = (1 - f^*_X)X. \]

and we must be able to solve the equation.

**Problem:** In general you want to consider only very smooth variations \( Y \) and you want to solve for \( X \). So the obvious thing is to see if the space \( \ker 1 - f^*_X \) is of finite codimension in the space of all \( Y \). Therefore it seems desirable to have the image of \( 1 - f^*_X \) closed and in fact of finite codimension in the space \( \text{set of } C^\infty \text{ sections of } T \).

Let's consider expanding maps on \( S^2 \). Then given \( Y \) we want to solve \( Y_{f_m} = X_{f_m} - df(X_m) \) for \( X \). Suppose

\[ Y = a(z) \frac{d}{d\theta} \quad \text{a periodic} \]

\[ f(z) = \frac{dz}{d\theta} = 2 \frac{d}{d\theta} \]

\[ a(2z) \frac{d}{d\theta} = b(2z) \frac{d}{d\theta} - 2b(z) \frac{d}{d\theta} \]

Therefore we have to solve the equation

\[ a(2z) = b(2z) - 2b(z) \]

in periodic functions.
Do over your calculations carefully

\[ M = S^1 \quad f(z) = z^2 \]

\[ f_t(m) = h_t(f(h_t^{-1}m)) \]

\[ \frac{d}{dt} \bigg|_{t=0} Y_m t + f(m) = f_t(m) \]

\[ Y_m = \underbrace{X_{f_m}}_{\text{fixed point}} - df(X_m) \]

\[ Y_m \in T_{f_m}(M) \]

\[ Y \in \Gamma(f^*T) \]

\[ Y = f^*X - df(X) \]

\[ a(z) \left( \frac{d}{d\theta} \right) z^2 = b(z) \left( \frac{d}{d\theta} \right) z^2 - 2b(z) \left( \frac{d}{d\theta} \right) z^2 \]

To solve

\[ a(z) = b(z^2) - 2b(z) \]

\[ \Sigma a_n z^n = \Sigma b_n z^{2n} - \Sigma 2b_n z^n \]
Write 
\[ a(z) = \sum_{n=0}^{\infty} a_n z^n \]
\[ b(z) = \sum_{n=0}^{\infty} b_n z^n \]

\[ a(z^2) \]
\[ b(z^2) \]
\[ \sum_i a_n z^{2n} = \sum_i b_n z^n - 2b_n z^n \]

\[ \therefore a_n = b_n - 2b_{2n} \]

Therefore 
\[ n \text{ odd} \implies b_n = 0. \]

Thus given \( a_n \) can solve for \( b_n \) as follows:

\[ a_1 = -2b_2 \]
\[ a_2 = b_2 - 2b_4 \]
\[ a_4 = b_4 - 2b_8 \]

\[ b_2 = -\frac{1}{2} a_1 \]
\[ b_4 = -\frac{1}{2} a_2 + \frac{1}{2} b_2 \]
\[ b_8 = -\frac{1}{2} a_4 - \frac{1}{4} a_2 - \frac{1}{8} a_1 \]
\[ 2^{-m} B \left\{ \frac{2^k (2^{g-1})^{m+1} - 1}{2^{g-1} - 1} \right\} = O(2^{-m}). \]
\[
\begin{align*}
\left\{ \begin{array}{ll}
a_n &= -2b_n & \text{if } n \text{ odd} \\ 
2^n a_n &= b_{n/2} - 2b_n & \text{if } n \text{ even}
\end{array} \right.
\]

\[b_k = (-1)^k a_k \quad \text{if } k \text{ odd.}\]

\[b_{2m} = -\frac{1}{2} a_{2m} + \frac{1}{2} b_{2m-1}\]

\[
\begin{aligned}
b_{2m} &= \left[ (\pm \frac{1}{2})_a a_{2m} + \left( \begin{array}{c}
2m \n\end{array} \right) a_{2m-1} \right. + \ldots + \left( \begin{array}{c}
2m \n\end{array} \right)^{m-1} a_k \bigg] \\
&\quad \text{if } k \text{ odd.}
\end{aligned}
\]

Therefore, there is a 1-1 correspondence between \(a\) and \(b\) sequences.

Can you estimate size of \(b\) sequence in terms of the \(a\) sequence?

Suppose

\[|a_N| \leq C \left( \frac{N}{\ln N} \right)^8\]

\[
-2b_{2m} = \sum a_{2m} + \frac{1}{2} a_{2m-1} + \ldots + \frac{1}{2m} a_1
\]

\[2|b_{2m}| \leq C \left\{ (2^m)^{-8} + \frac{1}{2} (2^{m-1})^{-8} + \ldots + \frac{1}{2m} \right\}\]

\[2|b_{2m}| \leq C \left( \frac{2^{-m}}{2^{-m} \cdot (8-1)} \right) \quad \tau_m
\]

\[2^{-m_8} \left\{ 1 + 2^{-1} + \ldots + 2^{-m (g-1)} \right\}\]