Lambek 2

Standard Construction $(\mathcal{A}, T, \eta, \mu)$

$$\eta : 1 \to T \qquad \mu : T^2 \to T$$

$$\mu(\eta T) = 1 = \mu(T\eta)$$

$$\mu(\mu T) = \mu(T\mu)$$

Defn: Category Trip Objects are standard const.

map $H : (\mathcal{A}, T, \eta, \mu) \to (\mathcal{B}, T, \eta, \mu)$

$$HT = TH \qquad H\eta = \eta H \qquad H\mu = \mu H$$

Claim

$$\text{Trip} \underset{F}{\overset{\mathcal{U}}{\rightleftarrows}} \text{Cat}$$

has an adjoint.

Deduction system $D(\mathcal{X})$ ~~$(\mathcal{A}, T, \eta, \mu)$~~

$\mathcal{X}$ any category

terms: 1. objects of $\mathcal{X}$

2. if A is a term, then $T(A)$ is a term

formulas $A \to B$

rules of inference + axioms

$id_A : A \to A$

$h(A) : A \to T(A)$

$m(A) : T^2(A) \to T(A)$

$$\frac{A \to B \quad B \to C}{A \to C}$$

$D_f : X \to Y$ $\forall$ map in $\mathcal{X}$

$$\frac{A \xrightarrow{p} B}{TA \xrightarrow{Tp} TB}$$

The converse of lemma does not hold

$$\frac{T(A) \to T(A) \quad T(A) \to T^2(A)}{T^2(A) \to T^2(A)} \not\equiv T^2(A) \to T^2(A).$$

i.e. $(\eta T)\mu \neq 1_{T^2}$ in $F(\mathcal{X})$

otherwise equal for all standard constructions

A <u>standard construction</u> is idempotent $\Leftrightarrow$ this holds

For idemp triples: two proofs are equiv $\Leftrightarrow$ same gene

Construction of the adjoint F:

$F(\mathcal{X})$: ~~objects~~ objects = terms of $D(\mathcal{X})$
maps = equivalence classes of proofs.

$$1_A = [1_A]$$
$$[P][Q] = [PQ]$$
$$T[P] = [T(P)]$$
$$\eta(A) = [h(A)]$$
$$\mu(A) = [m(A)]$$
$$\lambda(x) = X$$
$$\lambda(f) = [D_f]$$

Equivalence ~~class~~ relation = smallest equiv. reln $\ni$

$$\frac{P \equiv Q \quad R \equiv S}{PR \equiv QS.}$$

$$P\, id_A \equiv id_B\, P \qquad \text{if} \qquad P : A \to B.$$

T functor $\quad T(1_A) \equiv 1_{T(A)}$

$$T(PQ) \equiv T(P)\,T(Q)$$

η natural $\quad T(P)\,h(A) \equiv h(B)\,P.$

μ natural

triple identities

λ_X functor $\qquad D_{1_X} \equiv 1_X \qquad D_{fg} \equiv D_f\, D_g$

Clearly everything goes

Lawvere claims $\quad F(\mathcal{X}) = \mathcal{X} \times \Delta(1)$

Is Trip tripleable over Cat?

Beck's conditions $\quad (\mathcal{a}, \mathcal{U})$ is triplable iff

1. $\mathcal{U}$ reflects isos $\quad$ ($\mathcal{U}f$ iso $\Rightarrow f$ iso.)
2. Suppose $A_1 \rightrightarrows A_2 \ni \quad \mathcal{U}(A_1) \rightrightarrows \mathcal{U}(A_2) \xrightarrow{b} B$

a contractible cokernel, then $\exists$ cokernel $A_1 \rightrightarrows A_2 \xrightarrow{a} \mathbf{A} \ni$
$\mathcal{U}(A) \cong B \qquad \mathcal{U}(a) \cong b. \quad$ + more

contractible means $\quad 2$ extra maps + 4 equations

Paré has shown that contractible may be replaced by "absolute"

ie. $B_1 \rightrightarrows B_2 \longrightarrow B$ is an absolute cokernel provided

$H(B_1) \rightrightarrows H(B_2) \longrightarrow H(B)$ cokernel for all functors $H : \mathcal{B} \longrightarrow ?$

<u>Proposition</u>: λ is an embedding

a) Given $P : X \longrightarrow Y \quad \exists\, f : X \longrightarrow Y \ni \quad D_f \equiv P$

b) $D_f \equiv D_g \Longrightarrow f = g.$

<u>Defn</u>: The generality of a proof P in $D(\mathcal{X})$ = the set of all functors $2 \longrightarrow \mathcal{X} \ni \exists P'$ in $D(2) \;\&\; H(P') = P$

$$2 = \quad \underset{0}{\bullet} \longrightarrow \underset{1}{\bullet}$$

<u>Lemma</u>: Equivalent proofs have the same generality

Proof of b): Let $H : 2 \longrightarrow X \qquad H(0) = X \quad H(1) = Y \quad H(\to) = f$

$\therefore \quad H \in Gen(D_f)$

$\Longrightarrow H \in Gen(D_g) \qquad$ by lemma

$\Longrightarrow \exists\, \varphi \in 2 \ni \quad H(\varphi) = g$

but $\quad \varphi = (\longrightarrow)$ must go to f. $\therefore f = g$. $\quad \boxed{\text{go back}}$