
Let \( f: X \to Y \) be an etale covering of degree \( d \).
Recall that we have defined a norm or determinant map

\[
\text{Norm}: U^g(X) \to U^{gd}(Y)
\]
as follows. Let \( \tilde{Y} \to Y \) be a principal \( G \)-bundle with \( G \) finite such that \( X \)
is the bundle associated to a set \( S \) on which \( G \) acts.
Then we have a diagram

\[
\begin{array}{ccc}
X & \leftarrow & \tilde{Y} \times S = \tilde{X} \\
\downarrow f & & \downarrow f \\
Y & \leftarrow & \tilde{Y}
\end{array}
\]

where the horizontal arrows are principal \( G \)-bundle maps.
Then the norm is the composition

\[
U(X) \sim U_G(\tilde{Y} \times S) \xrightarrow{\text{Norm}} U_G(\tilde{Y}) \sim U(Y)
\]

where the norm map is the composition

\[
U^g_G(\tilde{Y} \times S) \to U^{gd}_G(T_f \tilde{Y}) \xrightarrow{A^*} U^{gd}_G(\tilde{Y})
\]

the first map being associated to the functor

\[
Z \mapsto \operatorname{Maps}_S(S, Z) = T_f Z_S
\]

from \( G \)-manifolds over \( S \) to \( G \)-manifolds.

We shall now check that this definition is independent
of the choice of $\tilde{Y}$. In effect given another covering $\tilde{Y}$ with group $G_1$, we can dominate $\tilde{Y}$ and $\tilde{Y}'$ by $\tilde{Y} \times \tilde{Y}'$, with group $G_1 \times G_1$, and so assume that $G_1 \to G$ and that $\tilde{Y} = \tilde{Y}/N$, $N = \ker G_1 \to G$. Then one checks easily that

$$
\begin{align*}
U(X) & \twoheadrightarrow U_0(\tilde{Y} \times S) \twoheadrightarrow U_0(\tilde{Y}) \twoheadrightarrow U_0(Y) \twoheadrightarrow U(Y) \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
\[ Y = (X/Y)^d \text{ reg} = \text{subset of } X \times y \times y \cdots X \text{ d times} \]

with \( G = \Sigma_d \) acting on \( S = \{ 1, \ldots, d \} \). Then for \( s \in S \) we have

\[ \rho_s : Y \rightarrow X \]

and so \( \text{Norm}_f \alpha = [Z \rightarrow X] \) is represented by where

\[ \Pi \tilde{Z}_s \rightarrow \tilde{Z}_s \text{ etale} \rightarrow Z^d \]

\[ \text{cart.} \]

\[ Y \rightarrow \Delta \rightarrow \prod \tilde{Y} \rightarrow \text{etale} \rightarrow X^d \]

into \( \Sigma_d^* (\tilde{Y}) = U(Y) \). This means that

\[ \text{Norm}_f (\alpha) = \rho (Q_{d*} \alpha) \]

where \( \rho \) is the composition

\[ U_{d*} (X^d) \rightarrow U_{d*} ((X/Y)^d) \approx U(Y) \]

Consequences of this formula are

\[ \text{Norm}_f (\alpha \cdot \beta) = \text{Norm}_f \alpha \cdot \text{Norm}_f \beta \]
and the additivity

\[ \text{Norm}_f (\alpha + \beta) = \sum_{i+j=d} p(\text{ind}_{\Sigma_i \times \Sigma_j \rightarrow \Sigma_d} Q_i \alpha \otimes Q_j \beta) \]

The \(i,j\) term may be written as

\[ (g_{ij})^* (g_{ij})^-1 \{ Q_i \alpha \otimes Q_j \beta \} \]

\[ \{(X/Y)_{\text{reg}}^d \}/\Sigma_i \times \Sigma_j \]

\[ g_{ij} \]

\[ (X/Y)_{\text{reg}}^d \rightarrow X^i \times X^j \]

Here \(g_{ij} : X_{i,j} \rightarrow Y\) is the bundle of partitions of the fibers of \(f : X \rightarrow Y\) into a subset of \(i\) and a complementary subset of \(j\) points.

Consider the case \(i=1, j=d-1\). Then \(g_{1,d-1} \)

we may identify \(X_{1,d-1}\) with \(X\), \(g_{1,d-1}\) with \(f\), and \(g_{1,d-1}\) with \(\Delta\), and we get

\[
\text{Norm}_f (\alpha + t\beta) = \text{Norm}_f \alpha + f_* \left( \beta \cdot \mathcal{O}(f) \right) t + O(t^2)
\]

where \(\alpha \otimes [Z \rightarrow X]\) then \(\text{Cof}_f (\alpha) \in \mathcal{U}(X)\) is the element represented by
\[ (X/Y)_{\text{reg}} \xrightarrow{\sim} X \]

\[ X \xrightarrow{\sim} \{ (X/Y)_{\text{reg}} \}/\Sigma_{d-1} \]

\[ x \mapsto f^{-1}[f(x)] - \{x\} \]

Thus, \( \text{Off } \beta \) is the class represented by the manifold over \( X \) with fiber

\[ x \mapsto \prod Z_{g(x)}. \]

\[ x \in f^{-1}(f(x)) \]

\[ \text{or } x \neq x \]

Remarks:

1.) If \( \alpha \in \mathcal{U}(X) \) is represented by a proper smooth map \( Z \rightarrow X \), then \( \text{Norm } \alpha \) is represented by the manifold over \( Y \) with fiber

\[ y \mapsto \prod Z_x \]

\[ \text{ref } \{ y \} \]

Note that if \( Z = X \times Y \) with \( W \rightarrow Y \) smooth, this is not the same as \( W^d \), since there is no isomorphism between \( f^{-1}\{y\} \) and \( \{h_1, h_2, \ldots, h_d\} \).

2.) If \( \alpha \) and \( \beta \) are represented by proper smooth maps \( Z \rightarrow X \) and \( Z'' \rightarrow X \), then

\[ f(\text{ind } Z_x \times Z_y \rightarrow \Sigma \text{Id } (\alpha \otimes \beta)) \]

is represented by the manifold over \( Y \) whose fiber is
\[ y \rightarrow \prod_{I \subset f^{-1}(\{y\})} \left( \prod_{x \in I} Z^x_0 \times \prod_{x \in f^{-1}(\{y\}) - I} Z^x_1 \right) \quad \text{card } I = i \]

3.) The formula on page 4 in the box is analogous to the matrix formula

\[
\det (A + tB) = \det A + tr(A \cdot C_f B) t + O(t^2).
\]

4.) If we define the intermediate symmetric functions between trace and norm by

\[ \text{Norm}_f (1 + t \chi) = \sum_{j=1}^{d} t^j \sigma_j (\chi) \]

then

\[
\begin{cases}
\sigma_1 (\chi) = tr_f (\chi) = f_* \chi \\
\sigma_d (\chi) = \text{Norm}_f (\chi)
\end{cases}
\]

and the formula on page 4 gives

\[ \text{Norm}_f (\alpha + t \cdot f^* \beta) = \text{Norm}_f (\alpha) + t \beta \sigma_{d-1} (\alpha) \]

5.) We know that

\[ \text{Norm}_f (f^* \alpha) = \alpha^d \]
In general, if $x$ is represented by $Z \to Y$, then $\text{Norm}_f(f^*x)$ is represented by a path

$$(Z \times_Y X) \to Z^d$$

which yields

$$\text{Norm}_f(f^*x) = \tau(\mathcal{Q}_d x)$$

where $\tau$ is the composition

$$U_{\Sigma_d}(Y^d) \xrightarrow{\pi^*} U_{\Sigma_d^e}(\tilde{Y}) \cong U(Y).$$

This operation of forgetting $\Sigma_d$ by lifting to $\tilde{Y}$ differs from just forgetting the action. (See below for formula for $d=2$)

The situation in characteristic zero is given by the familiar formula

$$\text{Norm}_f(1-tx^{-1}) = \sum_{m=1}^{\infty} \frac{t^m}{m} f(x^m).$$

**Proof:** If $x \in U^{\omega}(X)$ we can form

$$c^{-1}x \in U^{\omega}(X)[[t]].$$
and
\[ \frac{d}{dt} e^{tx} = e^{tx} \cdot x \]

i.e.
\[ e^{(t+\epsilon)x} = e^{tx} (1 + \epsilon x) \]

\[ \epsilon \to 0 \]

Thus
\[ \text{Norm}_f(e^{tx}) = \varphi(t) \] satisfies
\[ \varphi(t+\epsilon) = \varphi(t) \text{ Norm}_f(1 + \epsilon x) \]
\[ = \varphi(t)(1 + f_*(x) \epsilon) \]

\[ \varphi'(t) = \varphi(t) f_*(x) \]

The solution of the differential equation is
\[ \varphi(t) = e^{tf_*(x)} \]

thus we have
\[ \boxed{\text{Norm}_f(e^{tx}) = e^{tf_*(x)}} \]

so as
\[ \frac{1}{1-tx} = e^{\sum \frac{t^m}{m!} x^m} \] we have
\[ \text{Norm}_f \left( \frac{1}{1-tx} \right) = e^{f_*(\sum \frac{t^m}{m!} x^m)} \]
\[ = e^{\sum \frac{t^m}{m!} f_*(x^m)} \]

as claimed.
October 7, 1969. Notes on orientations

Axioms for a class of oriented maps: A type of orientation \( O \) is a rule associating to each map of manifolds \( f \) a set \( O(f) \) and to each transversal cartesian square

\[
\begin{array}{c}
X' \xrightarrow{g} X \\
\downarrow f' \downarrow \downarrow f \\
Y' \xrightarrow{g} Y
\end{array}
\]

a map \( g^*: O(f) \rightarrow O(f') \)

such that the following conditions hold

1) (Transitivity). Given two transversal cartesian squares

\[
\begin{array}{c}
X'' \xrightarrow{h'} X' \xrightarrow{g'} X \\
\downarrow f'' \downarrow f' \downarrow \downarrow f \\
Y'' \xrightarrow{h'} Y' \xrightarrow{g'} Y
\end{array}
\]

then

\[
O(f) \xrightarrow{g^*} O(f')
\]

\[
\begin{array}{c}
gf \\
\downarrow h^*
\end{array}
\]

\[
O(f'') \quad \text{commutes}
\]

(Note once \( g \) trans. to \( f \), then \( gf \) trans. to \( f' \leftrightarrow h \) trans. to \( f' \))
2) (half exactness) Given \( f : X \to Y \) and \( Y = U \cup V \) where \( U, V \) are open, we have that

\[
\mathcal{O}(f) \longrightarrow \mathcal{O}(f|U) \times \mathcal{O}(f|V) \cup \mathcal{O}(f|U \cap V)
\]

is surjective.

3) (homotopy), \( \mathcal{O}(f) \xrightarrow{\sim} \mathcal{O}(f \times \text{id}_X) \)

Given a type of orientation \( \mathcal{O} \), let \( F(Y) \) be the bordism classes of pairs \( (f, x) \) where \( f \) is a proper map with target \( Y \) and \( x \in \mathcal{O}(f) \). Two such pairs \( (f, x) \) \( (f', x') \) are bordant if there is a proper map \( f^* : W \to Y \times \mathbb{R} \) and \( x'' \in \mathcal{O}(f^*) \) such that \( i^*_0 (f'', x'') = (f, x) \) and \( i^*_1 (f''', x') = (f', x') \).

I check that this is an equivalence relation. Reflexivity and symmetry are pretty clear. To prove transitivity suppose given bordisms \( W, W' \) joining \( f_0 : X_0 \to Y \) to \( f_1 : X_1 \to Y \) and \( f_2 \) to \( f_3 \). Then we can fit these bordisms together to get a \( W \) and it remains to orient \( W \). Now use 2); we have to check...
that \( x \) on \( W \) and \( x' \) on \( W' \) coincide over \( U \cap V \). But this is a product of \( f \times \text{id}_R \) so by 3), these orientations coincide. If over \( 1 \) they must coincide.

Given \( g : Y' \to Y \) define \( g^* : F(Y') \to F(Y) \) as follows. Given \([f : X \to Y, \alpha] \) move \( g \) transversal to \( f \) and form pull-back. This doesn't depend on choice of good map in the homotopy class of \( g \) and also doesn't depend on choice of representative.

Claim that \( F \) is half exact i.e. if \( U, V \) are two open subsets of \( Y \), then

\[
F(U \cup V) \to F(U) \times_{F(U \cap V)} F(V)
\]

is surjective. This is demonstrated geometrically as in the diagram below.

- A nice regular value of a \( C^\infty \) function \( \phi \) with
  
  \[
  \phi(U \cap V) = 0 \quad \phi(U \setminus V) = 1.
  \]
It follows that $F$ is ind-representable as a functor on the homotopy category of manifolds.

$$F(X) \cong \lim_{\alpha} [X, B\Sigma^n].$$

**Example:** Let $n$ be an integer $\geq 0$, and let $\mathcal{O}(f)$ be empty if $f$ is not an embedding, and if $f$ is an embedding, let $\mathcal{O}(f)$ be the equivalence classes of framings of $V_f$, two framings being equivalent if they are homotopic. Then $F(X) = \text{bordism classes of framed submanifolds of codimension } n$ and

$$F(X) = \{X, \Sigma^n\}.$$

This example shows that $F$ needn't have any group structure. It also generalizes to the case where framing is replaced by reduction to $(A_f)$ where $\xi : A \to BO(n)$ is a map. More precisely:

Let $E$ be a $n$-dimensional bundle over a space $X$. Fix $f : X \to A$. If $E$ is an $n$-dim. bundle over $X$, we define $E$ to be a $\xi$-structure on $E$ to be a map $\xi : X \to A$ together with an isomorphism $\varphi : E \cong f^*\xi$ and two such structures $(f, \varphi), (f', \varphi')$ are said to be equivalent if they are homotopic, i.e., $\exists h : X \times I \to A$ and $\psi : E \times I \cong h^*(\xi)$ with $h_0(\cdot, \cdot) = (f, \varphi)$ and $h_1(\cdot, \cdot) = (f', \varphi')$. 
Let $f: X \to Y$ be a finite covering of degree 2. It is Galois where $\Sigma_2$ interchanges points of the fiber. The norm map $\text{Norm}_f$ is thus the composition

$$U(X) \xrightarrow{\text{const}} U^\Sigma_2(X^2) \xrightarrow{U^\Sigma_2((X/Y)^2)} U^\Sigma_2(Y)$$

and since $(X/Y)^2 \xrightarrow{\text{pr}_1} X$ is a $\Sigma_2$-isomorphism, the norm carries $[\Sigma_2 \to X]$ into the equivariant class

$$\Sigma_2 \to X \xrightarrow{(\text{id}, \tau)} X^2$$

followed by descent to $Y$. By the additivity formula on page 4, we have

$$\text{Norm}_f(\alpha + \beta) = \text{Norm}_f\alpha + f_*(\alpha \cdot \tau^*\beta) + \text{Norm}_f\beta$$

Let $\mathcal{V}$ be the cohomology mod 2. Let $r \in \mathcal{V}(Y)$ be the Euler class of the line bundle over $Y$ given by $X$. Then the map

$$\mathcal{V}(Y) \xrightarrow{f^*} \mathcal{V}^\Sigma_2(X) \cong \mathcal{V}(Y)$$

is identity on $\mathcal{V}(Y)$ and sends $w$ to $r$. Hence from page 7 we
where, as customary,

\[ S_0 y = S_0^{n-1} y \quad \text{if} \quad y \text{ is of degree} \ n. \]

Now suppose that \( V = H \). Take the product of \( f \)
with \( \mathbb{R}^* \), let \( L \) be the canonical line bundle, and let \( x = c_1(L) \). If \( E \) is a vector bundle of
dimension \( n \) over \( X \), then

\[
\begin{align*}
C_{2n} (f^* E \otimes L) &= C_{2n} (f^* E) + C_{2n-1} (f^* E) \cdot x + \ldots + x^{2n} \\
\text{Norm}_f (c_n (E \otimes f^* L)) &= \text{Norm}_f (c_n E + c_{n-1} E \cdot f^* x + \ldots + f^* x^n) \\
&= \text{Norm}_f (c_n E) + f^* (c_n E \cdot (c_{n-1} E \cdot f^* x + \ldots + f^* x^n)) \\
&\quad + \text{Norm}_f (c_{n-1} E + \ldots + f^* x^{n-1}) + \text{Norm}_f (f^* x)
\end{align*}
\]

Now, \( \text{Norm}_f (f^* x) = x^2 + x \cdot r \) so one can eventually

expand out all of the terms, get a polynomial in \( x \)
whose coefficients give the classes \( c_q (f^* E) \). Thus \( c_q (f^* E) \)
can be expressed in terms of \( \text{Norms}, f^* \) and the Chern classes of \( E \).
For example, if \( E \) is a line bundle, we get

\[
c_2 (f^* E) + c_1 (f^* E) \cdot x + x^2 = \text{Norm} (c(E) + f^* x) =
\]
\[ \text{Norm}_f(c_1E) + f_*(c_1E) \cdot x + x^2 + x \cdot r \]

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\text{c}_1(f_*(E)) = f_*(c_1E) + r \\
\text{c}_2(f_*(E)) = \text{Norm}_f(c_2E)
\end{array}
\right.
\]

\[ \text{Miscellaneous remarks about the Norm:} \]

1. \[
\begin{aligned}
&f_*(\text{Norm}_f(\alpha)) = \alpha \cdot f_*(\alpha) \\
f_* f_*(\alpha) = \alpha + f_*(\alpha)
\end{aligned}
\]

Since the composition

\[ V_{Z_2}(x) \xrightarrow{f(x)} V(Y) \xrightarrow{f_*(x)} V(X) \]

is the same as forgetting the \( Z_2 \) action. These formulas generalize to any Galois covering.

2. The norm is defined in terms of the external square \( Q_{ext} : V(X) \rightarrow V_{Z_2}(X^2) \). It unfortunately doesn't seem possible to define the norm using the internal squaring operations although this can be done for elements of the form \( f^*y \).

Recall the exact sequence of \( G \)-pairs for the bundle associated to \( X \) over \( Y \):

\[ V(Y) \xrightarrow{r} V(Y) \xrightarrow{f^*} V(X) \xrightarrow{f_*(x)} V(Y) \xrightarrow{r} \]

This shows that if \( \ker r \) is big, then the elements of form \( f^*y \) are few.
3.1) One knows in general for a group $G$ on $\{1, \ldots, d\}$ that there is a canonical isomorphism

$$H^*_G(x^d) \cong H^*(G, H(x)^{\otimes d})$$

provided that $H$ is cohomology with coefficients in a ring for which $H(x)$ is projective. In effect one can assume $X$ is a CW complex whence

$$H^*_G(x^d) = H^*(G, C^*(x^d)) = H^*(G, C^*(x) \otimes d).$$

By assumption on $H(x)$ there is a quasi-isomorphism $C^*(x) \rightarrow H^*(x)$ in the derived category of complexes of $A$-modules, hence

$$H^*(G, C^*(x) \otimes d) \rightarrow H^*(G, H(x)^{\otimes d}).$$

For $G = \mathbb{Z}_2$ this canonical isomorphism may be realized using $\mathcal{Q}$ext. In effect the Leray spectral sequence

$$E^2_{pq} = H^p(\mathbb{Z}_2, H^q(x^2)) \Rightarrow H^{p+q}_\mathbb{Z}_2(x^2)$$

degenerates since $E^2_{0q}$ is generated by $E^2_{0q}$ and $E^2_{10}$. This gives us an isomorphism of $gr H^*_\mathbb{Z}_2(x^2)$ with $E_2$. But using $\mathcal{Q}$ext: $H(x) \rightarrow H^*_\mathbb{Z}_2(x^2)$ we get a splitting of the edge homomorphism

$$\mathcal{Q}$ext(x) \quad \quad \nearrow_{\mathbb{Z}_2}$$

$$H^*_\mathbb{Z}_2(x^2) \quad \rightarrow \quad H^*(x^2) \otimes \mathbb{Z}_2 = \mathbb{P}_2(H(x))$$

and so we get
\[
\text{H}^*_{\mathbb{Z}_2}(X^2) \cong \Gamma_2^*(\text{H}(X)) \oplus \sum_{i \geq 0} \omega^i \text{H}(X)^{(2)}
\]

where the map sends \( \tau_2 x \) to \( Q_{\text{ext}}(x) \) and \( \omega^i \circ x^{(2)} \) to \( \omega^i \cdot Q_{\text{ext}}(x) \).

We have the following hierarchy:

\[
\text{external Steenrod operation } Q_{\text{ext}} : U(X) \to U_{\text{st}}^2(X)
\]

\[
\Downarrow
\]

Norm map for a covering

\[
\Downarrow
\]

\[
\text{internal Steenrod operation } Q_{\text{int}} : U(X) \to U_{\text{st}}^2(X)
\]

and it doesn't seem possible to reverse any of the arrows.

(Added Jan 29, 1970) The norm and \( Q_{\text{ext}} \) are actually equivalent by an argument similar to why proj.fails.

\[
\Rightarrow \text{ product formula.}
\]
October 9, 1969  On signs

The usual form of the projection formula

\[ f_x(f^x \cdot y) = x \cdot f^x y \]

asserting that \( f_x \) is a \( \Delta \)-module homomorphism is inconsistent with the sign convention since there should be a sign \((-1)^{\text{deg} f \cdot \text{deg} x}\). We shall now try to derive some consequences of ignoring this sign.

Suppose \( f : X \to Y \) and \( g : Z \to W \) are two oriented maps. Then there are two possible orientations for \( f \circ g \) as the two compositions in the square

\[
\begin{array}{ccc}
X \times Z & \xrightarrow{f \times id} & Y \times Z \\
\downarrow \text{id} \times g & & \downarrow \text{id} \times g \\
X \times W & \xrightarrow{f \times id} & Y \times W
\end{array}
\]

The two orientations differ by \((-1)^{\text{deg} f \cdot \text{deg} g}\).

From the geometrical point of view, the natural thing to do is to orient \( f \circ g \) as \((\text{id} \times g)(f \times \text{id})\), e.g., if \( f, g \) are framed embeddings, then one gets a framing for \( f \circ g \) by first taking the frame of \( f \) and then the frame of \( g \). So with this convention we calculate using standard projection formula

\[
(f \circ g)^* \nu = (f \circ g)^* (\text{pr}_1^* \nu \cdot \text{pr}_2^* \zeta)
\]
\[
= (-1)^{\deg f \cdot \deg g} \left( f \times id \right)_* \left( (id \times g)^* \right) \left( pr_1^* x \cdot pr_2^* z \right)
\]
\[
= (-1)^{\deg f \cdot \deg g} \left( f \times id \right)_* \left[ \left( pr_1^* x \cdot (id \times g)^* \right) pr_2^* z \right]
\]
\[
= (-1)^{\deg f \cdot \deg g} \left( f \times id \right)_* \left[ pr_1^* x \cdot pr_2^* g^* z \right]
\]
\[
= (-1)^{\deg f \cdot \deg g} \left( f \times id \right)_* \left[ pr_1^* x \cdot (f \times id)^* pr_2^* g^* z \right]
\]
\[
= (-1)^{\deg f \cdot \deg g} \left( f \times id \right)_* \left[ pr_1^* x \cdot pr_2^* g^* z \right] \cdot (-1)^{\deg g^* z \cdot \deg f}
\]
\[
= (-1)^{\deg f \cdot \deg g} pr_1^* f^_* x \cdot pr_2^* g^*_z
\]
\[
= (-1)^{\deg f \cdot \deg g} f^* x \otimes g^*_z.
\]

Observe that the other orientation for \( f \times g \) we wouldn't have a good sign, i.e. we would get \((-1)^{\deg g \cdot \deg f + \deg f \cdot \deg g}
\]

**Conclusion:** The following two sets of formulae are separately consistent from the point of view of signs:

\[
\begin{align*}
\left\{ \begin{array}{l}
\hat{f}_*(f^* x \cdot y) = \hat{x} \cdot \hat{f}_* y \\
(f \times g)_* = (id \times g)_* (f \times id)_*
\end{array} \right. \\
\left\{ \begin{array}{l}
(f \times g)_*(x \otimes y) = (-1)^{\deg f \cdot \deg g} (f \times g)_* (x \otimes y) \\
(f \times g)_*(x \otimes y) = (-1)^{\deg g \cdot \deg f} f^* x \otimes g^*_y
\end{array} \right. \\
\end{align*}
\]

The left is the standard one and perhaps more geometric since the Thom isomorphism is \( i_*(x) = x_*(\eta \times x) = \eta \times x \cdot i_* x \), but the right one seems more pleasant for the sign rules.
Remark: The projection formula implies the product axiom you used before! Here's the argument again using the sign-correct version of the projection formula:

\[ X \times Z \xrightarrow{f \times \text{id}} Y \times Z \xrightarrow{\text{id} \times g} Z \]

\[
\downarrow \text{id} \times g \quad \downarrow \text{id} \times g \quad \downarrow g \\
X \times W \xrightarrow{f \times \text{id}} Y \times W \xrightarrow{\text{id} \times g} W
\]

\[
\downarrow f \quad \downarrow \quad \downarrow \text{pt} \\
X \xrightarrow{f} Y \xrightarrow{g} pt
\]

\[
(f \times g)_* (x \otimes z) = (f \times \text{id})_* (\text{id} \times g)_* (pr_1^* x \cdot pr_2^* z)
\]

\[
= (f \times \text{id})_* ((\text{id} \times g)_* (\text{id} \times g)^* pr_1^* x \cdot pr_2^* z)
\]

\[
= (f \times \text{id})_* (pr_1^* x \cdot (\text{id} \times g)_* pr_2^* z) \cdot (-1)^{\deg g \cdot \deg x}
\]

\[
= (f \times \text{id})_* (pr_1^* x \cdot pr_2^* g_* z) \cdot (-1)''
\]

\[
= (f \times \text{id})_* (pr_1^* x \cdot (f \times \text{id})^* pr_2^* g_* z) \cdot (-1)''
\]

\[
= (f \times \text{id})_* (pr_1^* x) \cdot pr_2^* g_* z \cdot (-1)''
\]

\[
= (-1)^{\deg g \cdot \deg x} pr_1^* f_* x \cdot pr_2^* g_* z
\]

\[
= (-1)^{\deg g \cdot \deg x} f_* x \otimes g_* z
\]
Starting with the projection formula
\[ f_\ast(x \cdot f^\ast y) = f_\ast x \cdot y \]

one is forced to define the product \( f_\ast 1 \cdot g_\ast 1 \)
where \( f: X \to Y \) and \( g: Z \to Y \) are proper oriented maps
meeting transversally by
\[
\begin{array}{ccc}
X \times Y \times Z & \to & Z \\
\downarrow g & \downarrow f_\ast & \downarrow f_\ast \\
X & \to & Y \\
\end{array}
\]

\[ f_\ast 1 \cdot g_\ast 1 = f_\ast (1 \cdot f^\ast g_\ast 1) \]
\[ = f_\ast (g_\ast f^\ast 1) \]
\[ = f_\ast g_\ast 1 \]

Similarly, this forces us (if we want the formula \( (f \circ g)_\ast 1 = f_\ast 1 \circ g_\ast 1 \)) to define
\[
(f \times g)_\ast = (f \times \text{id})_\ast (\text{id} \times g)_\ast
\]

Thus the projection formula pins down the product:
\[
(\mu (f \times g)_\ast 1) = f_\ast 1 \circ g_\ast 1 = \mu (f_\ast 1 \cdot g_\ast 1) \circ \mu (f_\ast 1 \cdot g_\ast 1)
\]
\[ = (f \times \text{id})_\ast 1 \cdot (\text{id} \times g)_\ast 1 = (f \times \text{id})_\ast (\text{id} \times g)_\ast 1
\]
Let $G$ be an abelian compact Lie group, and let $U_G$ be the equivariant complex cobordism theory constructed by tom Dieck. I want to show there exists a localization $S^{n} U_G$ which is a universal theory satisfying the projective bundle theorem.

Suppose that $E$ is a vector bundle over $X$ (all with $G$ action) and that $E \cong L_1 \oplus \cdots \oplus L_n$ as a sum of $n$-line bundles. I claim then that $U_G(PE)$ is a free $U(X)$-module of rank $n$. In effect, proceeding by induction we have $E = L \oplus F$ and exact sequences

$$0 \to U(PL) \to U(PE) \to U(PF) \to U(PL) \to 0$$

and $i_*$ is injective on a direct summand since $PL \cong X$ and so $f_*i_* = id$. Thus $j^*$ is surjective and

$$U(PE) = U(PF) \oplus U(X)$$

so the claim follows by induction. Choose a basis $z_1, \ldots, z_n$ for $U(PE)$ as a $U(X)$-module, and write

$$e_i = \sum_{j=1}^{n} a_{ij} z_j \quad i = 0, 1, \ldots, n-1$$

and set

$$d_{e} = \det \{ a_{ij} \} \in U(X).$$
Note that changing basis alters $d_E$ by a unit. Now if $h$ is a theory for which the projective bundle theorem holds, then under the homomorphism $U \rightarrow h$, $d_E$ becomes a unit. Therefore the theory $S^{-1}U_G$ where $S = \{d_E | E \text{ runs over the representations of } G\}$ satisfies the projective bundle theorem for all equivariant bundles over a point and is clearly universal with this property.

Lemma: If $L_1, L_2$ are two line bundles over $X$, then there is an element $a \in S_U(G)$ with

$$e_a(L_1) - e_a(L_2) = a \cdot e_a(L_1 \otimes L_2^{-1})$$

(Also, $e_a(L_1 \otimes L_2^{-1}) = b (e_a(L_1) - e_a(L_2))$).

Proof: 

and let $f: X \rightarrow PV$, $f^* O_{PV}(1) \cong L_2$

and let $g: X \rightarrow PW$, $g^* O_{PW}(1) = L_1 \otimes L_2^{-1}$ be classifying maps for $L_1$ and $L_1 \otimes L_2^{-1}$, respectively. Then

$$(g \circ f)^* \{O_{PW}(1) \otimes O_{PV}(1)\} = L_1$$

so

$$e(L_1) = (g \circ f)^* e \{O_{PW}(1) \otimes O_{PV}(1)\}.$$
Now \( S_W^{\dagger} (PW \times PV) \) is generated as an \( SL(W) \)-algebra by \( e(\mathcal{O}_{PW}(1)) \otimes \mathbf{1} \) and \( \mathbf{1} \otimes e(\mathcal{O}_{PV}(1)) \) and in fact there is a formula

\[
e \left( \mathcal{O}_{PW}(1) \otimes \mathcal{O}_{PV}(1) \right) = \sum_{0 \leq k < \dim W, \ 0 \leq l < \dim V} a_{kl} e(\mathcal{O}_{PW}(1))^k \otimes e(\mathcal{O}_{PV}(1))^l
\]

where \( a_{kl} \) are uniquely determined elements of \( SL(W) \).

Consequently

\[
e(\mathcal{L}_1) = \sum_{0 \leq k < \dim W} a_{kl} e(\mathcal{O}_{PW}(1))^k \otimes e(\mathcal{L}_2)^l.
\]

Now we may suppose that \( W \) contains the trivial representation. This gives us a map \( \rho : W \to PW \) such that the map \( \rho : PV \to PW \times PV \) carries \( \mathcal{O}_{PW}(1) \otimes \mathcal{O}_{PV}(1) \) to \( \mathcal{O}_{PV}(1) \) and gives us the formula

\[
e(\mathcal{O}_{PV}(1)) = \sum_{0 \leq l < \dim V} a_{0l} e(\mathcal{O}_{PV}(1))^l
\]

By uniqueness

\[
a_{0l} = \begin{cases} 0 & l \neq 1 \\ 1 & l = 1. \end{cases}
\]

Thus

\[
e(\mathcal{L}_1) = \sum_{0 \leq k < \dim W} a_{kl} (e(\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}))^k \otimes e(\mathcal{L}_2)^l
\]

\[= e(\mathcal{L}_2) \mod e(\mathcal{L}_1 \otimes \mathcal{L}_2^{-1})
\]

proving the lemma.
Remark: The above proof amounts to using the "group law" which one gets when one has the projective bundle theorem for enough representations $V$ so that every $g$-line bundle is induced from $O_{pV}(1)$ for some $V$.

Consequence 1: If $E = L_1 + \cdots + L_n$ is a split bundle over $X$, then $\mathcal{S}^{\mathcal{G}}$ satisfies the projective bundle theorem for $E$.

Proof: Use induction on $n$, putting $E = L + F$ where the theorem holds for $F$. There is an exact sequence

$$0 \to h(PF) \xrightarrow{k^*} h(PE) \xrightarrow{h^*} h(X) \to 0$$

so that $h(PE)$ has the basis $1$, $i^*1$, $i^*\xi$, $\ldots$, $i^*\xi^{n-2}$.

Recall the exact sequence

$$0 \to \mathcal{O}(-1) \to f^*E \to \mathcal{Q} \to 0$$

and the fact that $PF \subset PE$ is where $\mathcal{O}(-1) \subset f^*F = \ker \{f^*E \to f^*L\}$. Thus $PE$ is where the map $\mathcal{O}(-1) \to f^*E \to f^*L$ vanishes so

$$l_x 1 = c(f^*L \otimes \mathcal{O}(1)).$$
By the lemma (or rather a variant proved by the same method) we have that

\[ e(\theta(1) \otimes f^*L) = \alpha(\beta - e(f^*L')) \]

for some \( \alpha \in h(PE) \). Consequently \( h(PE) \) is generated by \( 1, (f^*x), \ldots, (f^*x)^{\gamma^2} \) where \( \gamma = e(f^*L') \), and hence \( h(PE) \) is generated by \( 1, \ldots, \bar{\gamma}^{n-1} \) as a \( h(pt) \)-module. As \( h(PE) \) is a free module of rank \( n \), it follows that \( 1, \ldots, \bar{\gamma}^{n-1} \) is a basis. \( \text{qed} \)

**Theorem:** \( S^{-1}U_G \) satisfies the projective bundle theorem.

**Proof:** Consider the spectral sequence

\[ E_2^{pq} = H^p(X/G, Gx) \rightarrow (S^{-1}U_G)^{\otimes q}(P(E|Gx)) \rightarrow (S^{-1}U_G)^{\otimes q}(PE) \]

For each \( x \in Gx \), \( E_2^{pq} \) splits and so by what we've just proved \( (S^{-1}U_G)^{\otimes q}(P(E|Gx)) \) is a free \( (S^{-1}U_G)^{\otimes q}(Gx) \) module with the correct basis.

\[ \Rightarrow (S^{-1}U_G)^{\otimes q}(x)^n \]

By comparison theorem the map on the abutment must be an isomorphism, \( \text{qed} \).
Conclusion: For $G$-abelian there is a universal Chern theory and moreover it is a localization of $U_1$ so it satisfies the exactness axiom.

Remark: The above considerations also hold for $N_G$ where $G$ is an elementary abelian 2-group (i.e. all irreducible real representations are 1-dimensional).

So now we wish to take up the case where $G$ is non-abelian. Let $D$ be a faithful representation of $G$ and let $F$ be the flag manifold of $D$. Then the isotropy groups of the $G$-action on $F$ are all abelian. Let $X$ be a $G$-space over which $D$ splits, i.e., for which there exists an equivariant map $f : X \to F$. Then the isotropy groups of $X$ are all abelian. Let $C$ be the full subcategory of all such $X$.

Given such an $X$, let $h(X) = S^{-1}_X U_0(X)$ where $S_x$ is the set of $d_E$ (page 2) where $E$ runs over the split vector bundles over $X$. I claim that if $X_1 \to X_2$ is any map in $C$ then

$$S^{-1}_{X_1} U(X_1) = S^{-1}_{X_2} U_0(X_2).$$

In effect since $D$ is a faithful representation we know that every representation of $G$ is contained in a direct sum of tensor powers of $D$ and $D^*$. Hence any representation is contained in a representation which splits
over $X$. This means that for spaces over $X$, such as $X_i$, there is a "group law" for Chern classes of line bundles in the theory $S^{-1}U_0(Y)$. Consequently, we know that for any split bundle $E$ over $X$, the projective bundle theorem holds for $S^{-1}U_0(E)$ and hence that $S^{-1}U_0(X)$ is already invertible in $S^{-1}U_0(X)$. This proves the claim and shows that $h$ satisfies the projective bundle theorem for spaces in $C$ and split bundles. However, since the isotropy groups of a space in $C$ are all abelian, it follows that $h$ satisfies the projective bundle theorem for all spaces in $C$ and all bundles.

Given any $G$-manifold $X$, we set

$$h(X) = \text{Ker} \{ h(X \times F) \to h(X \times F \times F) \}.$$

Then $h(X)$ is a functor from $G$-manifolds to graded rings endowed with a $G$-equivariant ring morphism for complex-oriented $G$-maps.

I claim this definition agrees with the old for a $G$-manifold in $C$. This means that I must show that

$$(*) \quad h(X) \to h(X \times F) \to h(X \times F \times F)$$

is exact if $X$ belongs to $C$. However, $h$ satisfies the projective bundle theorem, so

$$h(X \times F) = h(X)[t_1, \ldots, t_n] \big/ c_T(D) = \prod_{i=1}^n (T + t_i)$$

where $t_i = e(L_i)$ and $D = L_1 + \ldots + L_n$ is the canonical splitting of $D$ over $F$. Similarly

$$h(X \times F^2) = h(X)[t_1', \ldots, t_n', t_1'', \ldots, t_n''] \big/ c_T(D) = \prod_{i=1}^n (T + t_i)$$
Therefore we see that
\[ h(X \times F) \otimes h(X \times F) \xrightarrow{\sim} h(X \times F \times F) \]
and that \( h(X) \to h(X \times F) \) is faithfully flat. Thus the exactness of (x) follows by faithfully flat descent.

Next I claim that \( h \) as just defined for all \( G \)-manifolds satisfies the projective bundle theorem. Suppose \( E \) given over \( X \). Then we have

\[
\begin{array}{c}
h(X) \quad \to \quad h(X \times F) \quad \to \quad h(X \times F \times F) \\
\downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \sim \\
h(PE) \quad \to \quad h(PExF) \quad \to \quad h(PExFxF)
\end{array}
\]

where the map is given by the powers \( 1, \cdots, \xi^n \), \( \xi = e(G(1)) \). The cases come from the projective bundle theorem for spaces in \( C \), and the left arrow is an isomorphism by 5 lemmas.

It is clear that \( h \) is a universal equivariant theory satisfying the projective bundle theorem. Moreover \( h \) is exact since \( h(\cdot \times F) = S^1U_0(\cdot \times F) \) is exact and since \( h(\cdot) \) is a direct summand of \( h(\cdot \times F) \).

So we conclude

**Theorem:** There exists a universal equivariant cohomology theory with Thom class for equivariant bundles and which satisfies the projective bundle theorem.
To now denote by $\pi_G$ the universal theory to distinguish it from the $U_G$ of tom Dieck

\text{(first form)}

Corner-Floyd theorem: $K_G(\text{pt}) \otimes Q_G(x) \rightarrow K_G(x)$

\text{Proof:} We know that the canonical map

$$\mu: u_6(x) \rightarrow K_G(x)$$

is surjective. This is because $\mu(1-e(L^{-1})) = 1 - (1-L) = L$ and because if $f: \text{Flag}(E) \rightarrow X$ is the flag bundle of $E$ and $f^*E = L_1 + \cdots + L_n$, then $f^*E = \mu(z)$ for some $z \in U_6(\text{Flag}(E))$ and so

$$E = f_x(f^*E) = \mu(f_x z).$$

Let $Q(x) = K_G(\text{pt}) \otimes Q_G(x)$. Then $Q(x)$ satisfies the projective bundle theorem and

$$Q(x) \rightarrow K_G(x)$$

if $X$ is a product of projective spaces. This means that

$$c^Q(L_1 \otimes L_2) = c^Q(L_1) \cdot c^Q(L_2) - c^Q(L_1)c^Q(L_2)$$

and consequently the additive map

$$K_G(x) \rightarrow Q_G(x)$$

given on line bundles by

$$L \mapsto 1 - c^Q(L^{-1})$$
is a ring homomorphism. The composition

\[ K_g(x) \rightarrow \mathbb{Q}(x) \rightarrow K_g(x) \]

is evidently the identity on \( L \), hence in general. The composition the other way is the identity because \( \mathbb{Q}(x) \rightarrow \mathbb{Q}(x) \) and because

\[ \mathbb{Q}(x) \rightarrow \mathbb{Q}(x) \rightarrow K_g(x) \rightarrow \mathbb{Q}(x) \]

\[ e(L) \rightarrow e^q(L) \rightarrow 1-L^{-1} \rightarrow e^q(L) \]

preserves these classes and hence commutes with \( \xi \). Again, homomorphism (use here that \( 0 \rightarrow \mathbb{Q}(\mathbb{Q}(E)) \rightarrow \mathbb{Q}(P(E)) \rightarrow 0 \) is split exact). This proves the theorem.

**Corollary:** If \( G \) is abelian, then

\[ K_g(pt) \otimes \mathcal{U}_G(X) \twoheadrightarrow K_g(x) \]

**Proof:** Here \( \mathcal{G} = S^{-1}U_G \) and the elements of \( S \) get into units in \( K_g(pt) \). Precisely

\[ K_g(pt) \otimes \mathcal{G}(x) = K_g(pt) \otimes S^{-1}U_G(x) \]

\[ = S^{-1}K_g(pt) \otimes \mathcal{U}_G(X) = K_g(pt) \otimes \mathcal{U}_G(X) \]
Remarks: I don't know if the strong Conner-Floyd theorem holds for $G$ non-abelian. The problem comes from the fact that even after localizing so as to make $S^1(U(Fx)) \rightarrow S^1(U(F)) \rightarrow S^1(U(Fx))$ good for descent, I don't know how to carry the localization $U(F) \rightarrow S^1(U(F))$ down to something over a point which can be moved across the $S^1(U(Fx))$.

2. Above holds for $N_0$, where "abelian" now should be interpreted "elementary abelian" groups. Can you calculate your $h$ for $G = Z_3$? Here you have that $Z_3$ acts freely on the sphere of its own non-trivial representation and $N_0$ can be calculated by Tom Dieck's method. Is there a CF theorem relative to the map $N_0(X) \rightarrow H_0(X, Z_2)$?

3. In the course of the preceding arguments we have seen that any Chern theory on $G$-manifolds is determined by its restriction to the full subcategory of $G$-manifolds with only abelian isotropy groups. (slightly more amusing is $N_0$ with $G$ odd in which case this means: free $G$-manifolds.) One might ask whether in the case of $K_0$ theory one could restrict to cyclic groups? Maybe not but one at least restrict to hyper-elementary groups by an equivariant $K_0$-version of Brauer's theorem (Segal).

4. The theory $O_G$ has a generalized formal group law and it is now necessary to ask if there is such a universal law over $L_G$ and if $L_G \rightarrow O_G(pt)$ is an isomorphism.
First consider the universal unoriented Chern theory $h$ constructed from $N_G$ where $G$ is odd. Recall that $h(X) = S^*_k N_G(X)$ if $X$ is over $F$, the flag bundle of a faithful representation of $G$. As $G$ is a finite group of odd order, $F$ is a free $G$ manifold hence for a manifold $X$ we have

$$N_G(X) = N(X/G) = N(EG 	imes_G X).$$

But this satisfies the projective bundle theorem so $G$ is already invertible. Thus $h(X) \to N(X/G)$ for a free $G$-manifold in $C$ and so by descent

$$h(X) \sim N(EG 	imes_G X).$$

(usually I wouldn't be so happy about these infinite complexes except that it's OK here)

$$N(EG \times_G X) \cong H(EG \times_G X) \otimes N(pt)$$

$$\cong H(X)^G \otimes N(pt)$$

$$\cong \pi_1^G$$

Thus

$$h(X) \sim N(X)^G.$$
We can now ask whether $h(pt)$ has the universal generalized group law. If $L$ is a line bundle then an invariant Riemannian metric on $L$ gives an isomorphism $L^2 \cong 1$ so $c^1(L^2) = 0$. So if $G$ is abelian of odd order $c(L) = 0$ for all $G$-line bundles over a point. Hence the generalized group law will in fact be a ordinary group law and the universal ring is indeed $\mathcal{N}(pt)$ Thus the conjecture is true if $G$ is abelian and of odd order. But its true in general.

Proposition: Let $G$ be a finite group of odd order and let $h$ be a Chern theory on $G$-manifolds. Then

$$c^1_h(x) = 1 \quad \text{all} \quad x \in KO_G.$$ 

Proof: Consider the map

$$KO_G(X) \longrightarrow h(x)[[E]]^x$$

$$x \longmapsto c^1_h(x^2)$$

This is a homomorphism and hence is determined by its effect on line bundles. But

$$c^1_h(x^2L) = c^1_h(L^2) = 1 \quad \text{since} \quad L^2 \cong 1$$

Thus

$$c^1_h(x^2) = 1 \quad \text{all} \quad x \in KO_G(X).$$

Now, take $X = pt$ and use that $\psi^2$ is an auto, since $G$ is odd, qed.
Conclusion: If $G$ is a finite group of odd order and if $h$ is the universal unoriented Chern theory, then the generalized group law of $h$ is a formal group law and $h(pt) = \eta(pt)$.

Now suppose $G$ is a finite group and let $H$ be a subgroup of $G$. Let $h_0$ be the universal unoriented Chern theory. I claim that the element $\xi = [G/H \to pt]$ is a unit in $h_0(pt)$. By descent, i.e. exactness of

$$h_0(pt) \to h_0(F) \to h_0(F \times F)$$

$F$-flag manifold of a faithful representation, it suffices to show that it is invertible over $F$. By standard spectral sequence

$$E_2^{p,q} = H^p(F/G, G_x \to h^q_0(G_x)) \Rightarrow h_0^{p+q}(F)$$

it suffices to show $\xi$ is a unit over each $G$-orbit of $F$, that is, over $G/K$ where $K$ is an elementary 2-abelian subgroup of $G$. We will show $\xi$ is a unit in $h_0(G/K) = \eta_K(pt)$. So it's a question of considering the element $[G/H \to pt]$ as an element of $\eta_K(pt)$, and

$$[G/H \to pt] = \sum_{K \leq H} [K/K_1 \to pt]$$

Now the element $[K/K_1 \to pt]$ is invertible because as a $K$-set we have

$$K/K_1 \to \prod_{i} \mathbb{Z}_2$$

where $K$ acts on the $i$th factor by a homomorphism $K \to \mathbb{Z}_2$. 

For $K_i \leq K$
and this element is zero because it is a product of zero elements. The number of fixed points of $K$ on $G/H$ is odd since $G/H$ is odd. Thus in fact we see that

$$[G/H \to pt] = \delta [G:H] \cdot 1 = 1$$

in $\pi_1(pt)$.

This means that the resolution

$$G/H \times G/H \times G/H \cong G/H \times G/H \Rightarrow G/H \to pt$$

gives rise to an exact sequence of descent

$$h^*_G(x) \Rightarrow h^*(G/H \times x) \Rightarrow h^*(G/H \times G/H \times x) \Rightarrow \cdots$$

In other words, denoting by $S$ a system of representatives for the double cosets $HgH$ and

$$H \overset{\delta_g}{\longrightarrow} H, \quad \delta_g = \text{the inclusion}$$

then

$$h^*_G(x) \Rightarrow h^*_H(x) \Rightarrow \bigoplus_{g \in S} h^*_{H \cdot g \cdot H^{-1}}(x)$$

is exact. In particular

$$h^*_G(x) = h^*_H(x) \quad \text{if} \quad H \triangleleft G$$

which yields our earlier result when $H = e$. 
October 15, 1969:  

Let $G$ be a compact Lie group, let $X$ be a $G$-manifold, and let $\mathcal{F}$ be a local coefficient system on $X$ endowed with a compatible $G$-action. (This means that the stalk space $\mathcal{F}_x \to X$ is a map of $G$-spaces.) Then cohomology groups $H^*_G(X, \mathcal{F})$ are defined, say as the cohomology of the space $EG \times_G X$ with values in the local coefficient system $EG \times_G \mathcal{F}$.

Suppose that $\rho: G \to U_n$ is a faithful representation. Then

$$H^*_G(X, \mathcal{F}) = H^*_U(U_n \times_G X, U_n \times_G \mathcal{F})$$

is the abutment of a spectral sequence with

$$E^2_{p,q} = H^p(BU_n, H^q(U_n \times_G X, U_n \times_G \mathcal{F})),$$

namely the Leray spectral sequence for the fibration

$$U_n \times_G X \to EU_n \times_G X \to BU_n.$$

Now $E^2_{p,q} = 0$ if $q > \dim (U_n \times_G X)$, so if we know that $H^*(U_n \times_G X, U_n \times_G \mathcal{F})$ is a finitely generated abelian group (which is the case if $X$ is compactifiable and $\mathcal{F}$ is a local coefficient system of finitely generated abelian groups) then we conclude that $E^{**}_{2}$ and hence $H^*_G(X, \mathcal{F})$ is a module of finite type over the ring $H^*(BU_n, \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \ldots, c_n]$. Thus it has a dimension, namely the dimension of its support.

In fact this definition is independent of the choice of $\rho$ since the $H^*(BU_n, \mathbb{Z})$-structure on $H^*_G(X, \mathcal{F})$ factors via
it is better to consider \( \text{dim}_p \) from the beginning.

The homomorphism

\[
p^*: H^e_{G}(B\mu_n, \mathbb{Z}) \rightarrow H^e_{G}(BG, \mathbb{Z}) = H^e_{G}(pt, \mathbb{Z}).
\]

Thus we see that \( H^e_{G}(X, \mathbb{F}) \) is a finite \( H^e_{G}(pt, \mathbb{Z}) \)-module and that

\[
\dim \{ H^e_{G}(X, \mathbb{F}) \} \leq \dim \{ H^e_{G}(pt, \mathbb{Z}) \}.
\]

We define the dimension of \( G \) to be the dimension of the ring \( H^e_{G}(pt, \mathbb{Z}) \).

This unfortunately is not quite right because if \( G = 1 \) then \( \dim H^e_{G}(pt) = \dim \mathbb{Z} = 1 \). A more reasonable definition is to consider the dimension of the scheme

\[
\text{Proj } H^e_{G}(pt, \mathbb{Z}) \quad (+1 \text{ see end.})
\]

since this depends only on the asymptotic features of the cohomology. However we still haven't achieved a really local situation yet since this 
\( \text{Proj} \) sits over \( \text{Spec } \mathbb{Z} \). Therefore if \( p \) is a prime number we define

\[
\text{dim}_p(G) = \dim \text{Proj} \left\{ H^e_{G}(pt, \mathbb{Z}) \otimes \mathbb{Z}_p \right\} + 1
\]

In virtue of the exact sequence

\[
0 \rightarrow H^e_{G}(pt, \mathbb{Z}) \otimes \mathbb{Z}_p \rightarrow H^e_{G}(pt, \mathbb{Z}_p) \rightarrow H^e_{G}(pt, \mathbb{Z}) \rightarrow 0
\]

we see that as both ends are finite type \( H^e_{G}(pt, \mathbb{Z}) \)-modules that

\[
\text{dim}_p(G) = \dim \text{Proj} \left\{ H^e_{G}(pt, \mathbb{Z}_p) \right\} + 1
\]
Of course, this dimension is also given by the degree of the Hilbert polynomial.

We also define \( \text{dim}_g(G) = \dim \text{Proj} \left( H^*_G(pt, \mathbb{Q}) \right) + 1 \)

Proposition: \( \dim(G) = \max \{ \dim_p(G), \dim_g(G) + 1 \} \).

Proof (?): This should be a result from algebraic geometry, the idea being that if \( f : X \rightarrow Y \) is a map of nice noetherian schemes of finite type then \( \dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{X,x} \otimes \mathcal{O}_{Y,y} \).

So to simplify life we work with \( \dim_p \) from now on and set \( H^*_G(X) = H^*_G(X, \mathbb{Z}_p) \), \( \dim X = \dim_p H^*_G(X) \).

Lemma: If \( X = U \cup V \), then

\[ \dim_p \mathcal{H}^*(X) \leq \max \{ \dim_p \mathcal{H}^*(U), \dim_p \mathcal{H}^*(V) \} \]

Proof: Have Mayer-Vietoris sequence

\[ H^{n-1}_G(U \cup V) \rightarrow H^*_G(X) \rightarrow H^*_G(U) \oplus H^*_G(V) \]

hence the support of \( H^*_G(X) \) as a sheaf on \( \text{Proj} H^*_G \) is contained in the union of that for \( U, V, U \cup V \).
Corollary: If \( X \) is a compactifiable \( G \)-manifold, then
\[
\dim_p(X) \leq \max_{x \in X} \dim(G_x).
\]

Proof: By a suitable induction from the corollary or directly by using the spectral sequence
\[
E_2^{p,q} = H^p(X/G, G_x \to H^q(G_x)) \Rightarrow H^{p+q}_G(X).
\]

Proposition:
\[
\dim_p(G) = \max_A \{ \dim_p(A) \}
\]
where \( p \) runs over all abelian subgroups of \( G \).

Proof: Let \( F \) be the flag manifold of a faithful representation of \( G \). Then
\[
H^*_G(\mathfrak{f}^*) \to H^*_G(F)
\]
so
\[
\dim_p G \leq \dim_p(F) \leq \max_{x \in F} \dim_p H^*_G(G/G_x)
\]
\[
= \max_{x \in F} \dim_p H^*_G(\mathfrak{f}^*) = \max_A \dim_p(A)
\]
where \( A \) runs over the abelian subgroups of \( G \).
Here we have used the fact that the dimensions of \( H^*_G(\mathfrak{f}^*) \)
as a ring and as an \( \mathfrak{g}_x \)-module are the same, which results from

Lemma: If \( A \to B \) finite, then \( \dim B \) as an \( A \)-module is same as \( \dim \) \( B \) as a ring.
Proof: \[ \dim B = \deg(n \mapsto B/m^\infty B) = \deg(n \mapsto B/m_\bar{B}^\infty B) \]

as \( mB \) and \( m_\bar{B} \) define the same topology.

The other direction comes from

\[ H^*_A(pt) = H^*_G(pt, \text{and} \rightarrow_G \mathbb{Z}_p) \]

which is a finite \( H^*_G(pt, \mathbb{Z}_p) \) module so

\[ \dim_p(A) \leq \dim_p(G) \]

if \( A \) is a subgroup of \( G \).

If \( A \) is a compact abelian Lie group, then

\[ A \cong T_k \times B \]

where \( T_k \) is a torus of dimension \( k \) and \( B \) is a finite abelian group. Then

\[ H^*_A(pt) \cong H^*_B(pt) \otimes \mathbb{Z}_p[x_1, \ldots, x_k] \quad \text{degree } x_i = 2 \]

\[ \dim_p(\mathbb{Z}/p^n\mathbb{Z}) = 0 \quad \text{if } p \text{ prime } \neq P \]

\[ \dim_p(\mathbb{Z}/p^n\mathbb{Z}) = \mathbb{Z}_p \]

Since

\[ H^*_B(pt, \mathbb{Z}_p) = \mathbb{Z}_p[w, \eta] \quad \dim w = 2 \quad \dim \eta = 1 \quad (p \text{ odd}) \]

Therefore if
\[
A = T_k \times B' \times \prod_{i=1}^{g} \mathbb{Z}/p^{n_i} \mathbb{Z} \quad 0 < n_1 \leq n_2 \leq \ldots \leq n_g
\]
then
\[
H_A^*(pt) = \mathbb{Z}_p[x_1, \ldots, x_k, w_1, w_2, \ldots, w_g, \eta_0]
\]
where \( \dim \eta_i = 1 \) and \( \dim w_i = \dim x_i = 2 \). Thus
\[
\dim A = k + g - 1 = \text{rank } (A) - 1
\]
so this shows we have wrong definition of dimension by one, so changing it we find the following result conjectured by Abijal:

"**Proposition:** \( \dim_p(G) = \text{rank of maximal elementary } p\text{-abelian subgroup of } G \) if \( p \) is a prime and
\[
\dim_q(G) = \text{maximal rank of a total subgroup of } G.
\]

"**Remarks:** For \( H^*_G(G, \mathbb{Z}_p) \) it is possible to descend from a situation with elementary \( p\text{-abelian isotropy group} \) to the following

"**Proposition:** Let \( E \) be a complex vector bundle of dimension \( n \) over \( X \) and let \( P \) be the associated principle \( U_n \) bundle. Let \( (\mathbb{Z}_p)^n \subset T_n \) be the kernel of multiplication by \( p \) and let \( f: Y \to X \) be the fibre bundle with \( Y = P/\mathbb{Z}_p \). Then \( f_*: H(Y) \to H(X) \) is surjective."
Proof: As the Gysin for the flag bundle map is surjective, we reduce to proving \( g_\ast \) surjective where \( g : P(\mathbb{Z}_p)^n \rightarrow P/T_n \). By an evident induction we reduce to proving that if \( P \) is a principal \( S^1 \) bundle, then \( h_\ast \) is surjective where \( h : P/\mathbb{Z}_p \rightarrow P/S^1 \). If \( L \) is the complex line bundle over \( X = P/S^1 \) associated to \( p \), then \( P/\mathbb{Z}_p \) is the sphere bundle of \( L^\otimes p \) and so we have the Gysin sequence

\[
0 \rightarrow H^0(X) \xrightarrow{h_\ast} H^0(P/\mathbb{Z}_p) \xrightarrow{h_\ast} H^0(Y) \rightarrow 0
\]

where \( e(L^\otimes p) = pe(L) = 0 \). This proves \( h_\ast \) is surjective. In fact taking \( P = ES^1 \) we have

\[
\begin{align*}
H^1(ES^1) & \xrightarrow{h^*} H^1(BZ_p) \xrightarrow{h^*} H^0(ES^1) \\
0 & \rightarrow \mathbb{Z}_p \beta \rightarrow \mathbb{Z}_p
\end{align*}
\]

showing that \( h_\ast \beta = 1 \) if \( \beta \) chosen correctly.

Example 1: \( G = O(n) \). Then the \( p \)-rank of \( G \) is \( \frac{n}{2} \) for \( p \) odd or \( 0 \) and \( n \) for \( p = 2 \) and this agrees well with the known formulae

\[
\begin{align*}
H^\ast(BO(n), \mathbb{Z}_2) &= \mathbb{Z}_2[w_1, \ldots, w_n] \\
H^\ast(BO(n), \mathbb{Z}_p) &= \mathbb{Z}_p[p_1, \ldots, p_{\frac{n}{2}}]
\end{align*}
\]

2) \( G = SO(n) \). Then the \( p \)-rank is \( \frac{n}{2} \) for \( p \) odd or \( 0 \) and \( n-1 \) for \( p = 2 \) which agrees with the formulae

\[
H^\ast(BSO(n), \mathbb{Z}_2) = \mathbb{Z}_2[w_2, \ldots, w_n]
\]
\[
H^*(B\text{SO}(\omega), \mathbb{Z}_p) = \prod_{p} \left[ \mathbb{Z}_p \left[ P_{\frac{1}{2}} \right] X \right] / \begin{cases} x = 0 & \text{if } n \text{ odd} \\ x^2 = P_{\frac{1}{2}} & \text{if } n \text{ even.} \end{cases}
\]
for \( p \) odd or zero.

Y. Segal's description of the Smith theorem: If \( \mathbb{Z}_p \) acts on a mod \( p \) homology sphere \( X \) with fixed space \( F \), then \( F \) is a mod \( p \) homology sphere.

Proof: Take

\[
\Rightarrow H^*_{\mathbb{Z}_p}(X, F) \Rightarrow \tilde{H}^*_{\mathbb{Z}_p}(X) \Rightarrow \tilde{H}^*_{\mathbb{Z}_p}(F) \Rightarrow \ldots
\]

and localize with respect to \( w = c_1(Q) \). Now \( H^*_{\mathbb{Z}_p}(X, F) \) is a module over \( H^*(X-F) \) which vanishes after a while as \( \mathbb{Z}_p \) acts freely on \( X-F \). Thus we get

\[
\tilde{H}^*_{\mathbb{Z}_p}(X)[w^{-1}] \Rightarrow \tilde{H}^*_{\mathbb{Z}_p}(F)[w^{-1}]
\]

Now the spectral sequence

\[
E^2_{p} = H^p(\mathbb{Z}_p, \tilde{H}^q(X)) \Rightarrow \tilde{H}^{p+q}_{\mathbb{Z}_p}(X)
\]

degenerates as we are assuming that \( X \) is a mod \( p \) homology sphere and it shows that \( \tilde{H}^*_{\mathbb{Z}_p}(X) \) is a free \( H^*(\mathbb{Z}_p) \) module with one generator. Hence

\[
\tilde{H}^*_{\mathbb{Z}_p}(F)[w^{-1}] = \tilde{H}^*(F) \otimes H^*(\mathbb{Z}_p)[w^{-1}]
\]

is a free \( H^*(\mathbb{Z}_p)[w^{-1}] \) module with one generator so \( F \) must be a mod \( p \) homology sphere, g.e.d.
The same theorem holds for $\mathbb{Z}_p^n$ by induction on $n$ and for $S^4$ acts on a rational homology sphere.

Remarks of Atiyah on the completion theorem $K_0(x) \otimes K(x) = K(x^0)$.

Suppose $G$ is a finite group. Recall that $R(G)^\wedge = \varprojlim R(G)/R(G)^n$ where $R(G)$ is the augmentation ideal. Now if $d = |G|$, then $d$ kills $R(G)/R(G)^n$ for $n > 0$. In effect, $i: 1 \to G$, then if $x \in R(G)$

$$dx = x \otimes 1 \cdot x \mod R(G)^2$$

$$= i_*(x^*x) = 0.$$ This implies that $R(G)^\wedge$ is a direct sum of a finite type module over the $p$-adic numbers $\mathbb{Z}_p$. In fact

**Theorem:** The $p$-primary component of $R(G)^\wedge$ is a free $\mathbb{Z}_p$-module whose rank is the number of conjugacy classes of $p$-singular elements in $G$.

First suppose $G$ is a $p$-group. Then we claim that the $R(G)$-adic and the $p$-adic filtrations define the same topology. We already know that if $|G| = p^n$ then $p^n R(G) \subset R(G)^2$. So $(p^n)^n R(G) \subset (p^n)^{n+1} R(G)^2 \subset R(G)^{n+1}$. For the other direction consider the embedding

$$R(G) \subset \mathbb{Z}[G]$$

given by associating to a virtual representation its values on the
conjugacy classes of $G$. Note that
$$R(G) \longrightarrow \mathbb{Z}[S]$$

since if $V$ is a representation of $G$ of dimension $n$, then
$$\text{tr}_V g = \sum \lambda_i$$ where $\lambda_i = n \rho_i$ and this goes into 1 in
the augmentation of $\mathbb{Z}[S]$. (Recall that $\mathbb{Z}[S]/\mathbb{Z} \sim \mathbb{Z}/p\mathbb{Z}$.)

Next by Artin-Rees we have that $\mathfrak{q} \mathfrak{n}$ with
$$p^{\mathfrak{q}} R(G) = \overline{R(G)} \cap p^{\mathfrak{n}+\mathfrak{q}} \mathbb{Z}[S]$$

But the $\mathbb{Z}[S]$-adic and the $p$-adic topology on $\mathbb{Z}[S]$
coincide so
$$p\overline{R(G)} \supset R(G) \cap p^{1+n} \mathbb{Z}[S]$$
$$\longrightarrow R(G) \cap (\mathbb{Z}[S]^g)^{\mathfrak{q}} \text{ some } g$$

proving the other inclusion of topologies. Thus from the claim we have that
$$\hat{R(G)} = \hat{R(G)} \otimes \hat{\mathbb{Z}_p}$$
is a free $\hat{\mathbb{Z}_p}$ module of rank $= \text{no. of non-trivial conjugacy}$
classes in $G$.

Suppose now that $G$ is an arbitrary finite group
and let $H$ be a $p$-Sylow subgroup of $G$. By essentially the
argument on page 9 one sees that

$$\text{im } \overline{R(G)} / R(G) \longrightarrow \text{im } \overline{R(H)} / R(H)$$
is injective. In other words the
\[ \overline{R(G)}(p) \hookrightarrow \overline{R(H)} \]
which shows the former is a free \( \mathbb{Z}_p \) module. For general reasons it is easy to show that the image is invariant under the action of the normalizer \( N \) of \( H \) and this shows that \( \text{rank} \, \overline{R(G)}(p) \leq \text{number of non-trivial conjugacy classes of } p \)-subgroups (elements of \( G \) by virtue of the well-known lemma: let \( S_1, S_2 \) be subsets of the Sylow subgroup \( H \) of \( G \) which are invariant under conjugation action of \( H \), and suppose \( S_1 \) and \( S_2 \) are conjugate in \( G \). Then they are already conjugate in the normalizer \( N \) of \( H \) in \( G \).

Proof: Let \( x S_1 x^{-1} = S_2 \) and let \( G' = \text{normalizer of } S_2 \) in \( G \). So that \( H \leq G' \) and also \( x H x^{-1} \leq G' \). Then \( H \) and \( x H x^{-1} \) being two Sylow subgroups of \( G' \), there is a \( g \in G' \) with \( g x H x^{-1} g^{-1} = H \) or \( g x e N \), and \( g x S_1 g^{-1} = S_2 \).

Thus it appears that Atiyah's theorem asserts that
\[ \overline{R(G)}(p) \sim (\overline{R(H)}(p))^N \]
also incorrect

[Diagram]
October 17, 1969: Complex orientations of a map

In the following we will work with the category of $C^\infty$ $G$-manifolds where $G$ is a compact Lie group. We consider only $G$-manifolds which may be embedded equivariantly as a closed submanifold of some representation of $G$ (this implies the set of orbit types of the manifold is finite and probably the converse is true). From now on everything will be equivariant unless stated otherwise.

Let $\mathcal{E}$ be a complex bundle over $Y$ with projection $p : E \to Y$. We consider the set of pairs $(i, J)$ where $i : X \to E$ is a closed embedding such that $pi = f$ and where $J$ is a complex structure on $i_!$. An isotopy of such pairs $(i, J)$ is a family $(i_t, J_t)_{t \in \mathbb{R}}$ such that

\[ p : E \to Y \]

Let $f : X \to Y$ be a map of manifolds and let $\mathcal{E}$ be a vector bundle over $Y$. We consider pairs $(i, J)$ where $i : X \to E$ is an embedding with $pi = f$ and where $J$ is a complex structure on $i_!$, the normal bundle $i_!$ of $i$. Two such pairs are called isotopic if they are the restriction for two values $a, b \in \mathbb{R}$ of a pair for the product map $(f \times id) : X \times \mathbb{R} \to Y \times \mathbb{R}$ and the product bundle $pxid : E \times \mathbb{R} \to Y \times \mathbb{R}$. Let $\mathcal{I}(E)$ be the isotopy classes of such pairs.

If $F \to E$ is an injective map of complex vector bundles over $Y$, then there is an injective map $u : E \to F$ of complex vector bundles over $Y$ induces a map $u_* : \mathcal{I}(E) \to \mathcal{I}(F)$ and if $u$ and $u'$ are isotopic injections, i.e. homotopic through injections, then $u_* = u'_*$. We define
the set of complex orientations of \( f \) to be

\[
O(f) = \lim_{E} O(f, E)
\]

where the limit is taken over the category of complex vector bundles over \( Y \) and whose morphisms are the isotopy classes of injections. Notice that the fiber product bundles \( \times V \rightarrow Y \rightarrow Z \) over a point (i.e., a complex representation of \( G \)) are cofinal. Observe that if \( g : Y \rightarrow Z \) is a map then the functor \( F \mapsto g^* F \)

\[
g^* : I(Z) \rightarrow I(Y)
\]

is cofinal, so that we may take the inductive limit over \( I(Z) \). In particular, if \( g : Z \rightarrow Y \), then since a vector bundle over a point is just a representation of \( G \) and since any two \( G \)-injections \( V \cong W \) of complex representations are isotopic, the category \( I(G) \) is equivalent to the ordered set of isomorphism classes of complex representations of \( G \).

(Lemma: \( G \) compact Lie group, \( V, W \) two complex representations of \( G \). Then any two injections \( V \cong W \) are homotopic.)

Proof: \( \text{Im} \big( \text{Inj}(V, W) \big) = \bigoplus_{G} \text{Inj}(V, W) \)

where suffix denotes part of \( V \) purely associated to \( s \). As

\[
V_s = \text{Im} \bigg\{ \text{Hom}(E_0, V) \otimes_{E_0} E_s \rightarrow V \bigg\}
we have isomorphisms \( V_o = \mathbb{C}^n \otimes E_2 \), \( W_o = \mathbb{C}^n \otimes E_3 \).

and so

\[
\text{Inj}(V_o, W_o) \cong \text{Inj}(\mathbb{C}^n, \mathbb{C}^n) = \frac{\text{GL}(\mathbb{C}, \mathbb{C})}{\left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right\}}
\]

which is connected as \( \text{GL}(\mathbb{C}, \mathbb{C}) \) is connected.

As an example, take \( G = 1 \) whence

\[
O(f) = \lim_{n \to \infty} O(f, Y \times \mathbb{C}^n)
\]

One knows that for \( n \gg \dim X \) (2 Inf \( f + 1 \)?) that there is an isotopy class of embedding of \( X \) into \( Y \times \mathbb{C}^n \), and that the normal bundles of two such embeddings are \( \text{consistently isomorphic} \) unique up to homotopy type. Thus \( O(f, Y \times \mathbb{C}^n) \) is isomorphic to the set of isotopy classes of complex structures on \( V_2 \) for a given \( f \).

Operations on complex orientations:

Pullback: suppose that \( g : Y' \to Y \) is transversal to \( f : X \to Y \) so that \( X \times_Y Y' \to X \) we can form the fibre product

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

Then there is an induced map
\[ g^* : \mathcal{O}(f_* E) \longrightarrow \mathcal{O}(f'_* g^* E) \]

coming from the fact that the square

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow i' & & \downarrow i \\
\mathcal{O}(g^* E) & \xrightarrow{g^*} & \mathcal{O}(E)
\end{array}
\]

is transversal and so there is a canonical isomorphism

\[ \psi_x = (g')^* \psi_x \]

Composition: Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \) be maps

and let \( E, F \) be complex bundles on \( Z \). Then there is a map

\[ \psi : \mathcal{O}((f, g^* E)) \times \mathcal{O}(g^* F) \longrightarrow \mathcal{O}(gf^* F \oplus E) \]

defined as follows: Given \( X : Y \longrightarrow F \) over \( g \) with

complex structure \( J_{gf} \) on \( Y \), and \( i : X \longrightarrow g^* E = Y_x E \) over \( f \)

with complex structure \( J \) on \( X \), consider the embedding \( i^* \)

which is the composition

\[ X \xrightarrow{i} Y \times X \xrightarrow{g \times \text{id}} F \times E \longrightarrow F \oplus E \]

Then \( \psi_x \) fits into an exact sequence
\[ 0 \rightarrow \nu_i \rightarrow \nu'' \rightarrow p^*\nu' \rightarrow 0 \]

where \( p = p_{r_1} : F \times_{Z} E \rightarrow F \), and hence up to isotopy has a unique complex structure \( J'' \) such that the above is an exact sequence of vector bundles. The map (\( \star \)) is defined by sending \((i, J)\) and \((i', J')\) to \((i'', J'')\).
Let $p$ be a fixed prime. Recall that the $(p)$-dimension of $G$ is defined to be the dimension of the $\mathbb{Z}_p$-algebra $H^0(G, \mathbb{Z}_p)$ which is of finite type. This dimension is the maximal rank of an elementary abelian $p$-subgroup by Oct. 15.

Observe that if $A \subset G$ is an elementary abelian $p$-subgroup of rank $r$, then there is a homomorphism

$$H^0_G(\text{pt}) \longrightarrow H^0_A(\text{pt}) \longrightarrow H^0_A(\text{pt})/\text{subsets} = \mathbb{Z}_p[\omega_1, \ldots, \omega_r]$$

(this is for $p$ odd; for $p=2$ it is easier), which is finite and hence the kernel $pA$ is a prime ideal in $H^0_G(\text{pt})$ which is of dimension $r$, i.e., defines a point $[A]$ in $\text{Spec } H^0_G(\text{pt})$ of dimension $r$. Thus we get a map

$$\text{Conjugacy classes of \{elem. ab. \ p-subgroups\} of rank } r \in G \longrightarrow \{\text{prime ideals in } \text{Spec } H^0_G(\text{pt}) \text{ of dimension } r\}$$

which we want to understand.

Note that $H^0_G(\text{pt})/pA \longrightarrow \mathbb{Z}_p[\omega_1, \ldots, \omega_r]$ so the variety over $\mathbb{Z}_p$ by $pA$ is of dimension $r$ and rational which tends to suggest very strongly that it might be the closure of an orbit of the action of the Brauer algebra.

Let $R = H^0_G(\text{pt})$ and let $p$ be a minimal prime ideal of $R$ so that $R_p$ is an Artin ring. If $X$ is a $G$-space set

$$X_p(X) = \text{length } \{H^0_G(X)_p\} - \text{length } \{H^0_G(X)_p\}$$

Then clearly we have
Lemma 1: If $X = U \cup V$ where $U$ and $V$ are $G$-invariant open sets, then

$$X_p(X) = X_p(U) + X_p(V) - X_p(U \cap V).$$

(More generally we can replace $p$ by an arbitrary prime ideal in $R$ and consider only spaces $X$ for which $H_{*}(X)_p$ is a $R_p$ module of finite length. Then $X_p(X)$ is defined and there is a generalization of the lemma.)

Lemma 2: If $p$ is minimal, $p$ is a prime ideal in $G$, and $H^*(A)_p = 0$, then $A = B$.

Proof: If $H^*(A)_p \neq 0$, then $(R/pA)_p \neq 0$ and so $pA < p$. By minimality $p = pA$. If $B \geq A$, then $pA > pB$ by dimension reasons, so $p$ not minimal, qed.

Proposition: Let $X$ be a compactifiable $G$-manifold with elementary $p$-abelian isotropy groups. Let $A_1, \ldots, A_k$ be representatives for the elementary $p$-abelian subgroups of $G$ with $pA_i = p$ under conjugation. Then
\[ H^*_G(x)_p = \prod_{i=1}^{k} H^*_G(G \times_{N_i} x_{A_i})_p \]

where \( N_i \) is the normalizer of \( A_i \).

**Proof:** We consider the stratification of \( X \) into its orbit types, which are finitely many in number. Note that if \( A \) is \( p \)-abelian with \( p_A = p \), then by lemma 2 \( A \) is a maximal elem. \( p \)-ab. subgp. of \( G \), and therefore every point of \( X^A \) has isotropy group \( A \). This implies that the strata with orbit type \( G/A \) is

\[ G \times_{N} X^A \]

where \( N \) is the normalizer of \( A \) in \( G \). Let \( U \) be the open invariant subset of \( X \) not containing any of the orbits types \( G/A_i \). Then by Mayer-Vietoris and induction one sees that \( H_6(U)_p = 0 \) and hence again by Mayer-Vietoris that

\[ H_6(X)_p = H_6(\prod_{i} G \times_{N_i} x_{A_i})_p \]

proving the proposition.

**Corollary:** If \( p \) is a minimal prime of \( H^*_G(pt) \), then there exists a maximal elementary \( p \)-abelian subgroup \( A \) of \( G \) with \( p = p_A \).

**Proof:** If not then for the \( p \)-flag manifold \( F \) of a faithful representation of \( G \) we have \( H_6(F)_p = 0 \) hence \( H_6(pt)_p = 0 \) which is nonsense.
Some calculations:

Lemma: Let $R \to S$ be a homomorphism of local noetherian rings with residue fields extension $k_R \to k_S$. Then a finite length $S$-module $M$ is also of finite length as an $R$-module and

$$\text{length}_R(M) = [k_S : k_R] \cdot \text{length}_S(M).$$

In effect if $M$ is a simple $S$-module, $M = k_S$ and this has length $[k_S : k_R]$. Both sides are additive functions of $M$.

\[\text{We apply this where } R \to S \text{ is the localization of the map } H^w_G(pt) \to H^w_A(pt) \text{ with respect to } \mathfrak{p}, \text{ and where } M = H^\text{ord}_A(pt) \text{ and we find that }\]

\[x_{\mathfrak{p}}(G/A) = \text{...}\]

Let's change notation slightly and put

$$l^+_{G, \mathfrak{p}}(X) = \text{length} \left( H^w_G(pt) \mod \mathfrak{p} \right) \left( H^w_G(X) \right)$$

$$l^-_{G, \mathfrak{p}}(X) = \text{length} \left( H^w_G(pt) \mod \mathfrak{p} \right) \left( H^\text{ord}_G(X) \right)$$

provided these are defined. Set

$$x_{G, \mathfrak{p}}(X) = l^+_{G, \mathfrak{p}}(X) - l^-_{G, \mathfrak{p}}(X).$$

Let $A \to G$ be homomorphism of compact groups, let $\mathfrak{p}$ be a prime in $H^w_A(pt)$ and let $\mathfrak{p}$ be the inverse image in $H^w_G(pt)$. 
As $H_x^w(\text{pt}) \to H_A^w(\text{pt})$ is finite if $A \subset G$ it follows that

\[(x) \quad H_x^w(\text{pt})_p \to H_A^w(\text{pt})_p\]

can be taken as the map $R \to S$ in the lemma. Thus if $X$ is an $A$-space for which $l_{A,p}^{\pm}(X)$ is defined we have

\[l_{G,p}^{\pm}(G \times A X) = R \cdot l_{A,p}^{\pm}(X)\]

where $R$ is the degree of the residue extension of $(x)$. Apply this where $A$ is an elementary $p$-abelian subgroup of $G$ and where $p_A$ is the ideal of nilpotent elements, whence $p = p_A$. We find that

\[X_{G,p_A}(G/A) = R \cdot X_{A,p}(\text{pt}).\]

To calculate the latter suppose rank $A = k$ and embed $A$ inside a torus $T$ of rank $b$ and apply the lemma to the map

\[H_x^w(\text{pt})_0 \to H_A^w(\text{pt})_0\]

Letting $s$ be the residue extension degree we find that

\[\text{length} H_x^w(\text{pt})_0 \left( H_A^w(\text{pt})_0 \right) = s \cdot \text{length} H_A^w(\text{pt})_0 \left( H_A^w(\text{pt})_0 \right)\]
Case 1: \( p \) odd. Then 
\[ H^*_A(\text{pt}) \cong \Lambda[\eta_1, \ldots, \eta_r] \otimes_{\mathbb{Z}} \mathbb{Z}_p[\omega_{\eta_1}, \ldots, \omega_{\eta_r}] \]
so
\[ H^w_T(\text{pt}) \xrightarrow{\sim} H^w_A(\text{pt})/\wp \]
so \( s = 1 \). But \( H^+_A(\text{pt}) \) is a free \( H^w_T(\text{pt}) \) module of rank \( 2^{r-1} \) so
\[ \text{length}_{H^w_T(\text{pt})/\wp}(H^+_A(\text{pt})/\wp) = \text{length}_{H^w_A(\text{pt})/\wp}(H^+_A(\text{pt})/\wp) = 2^{r-1} \]
so
\[ \chi_{A, \wp}(\text{pt}) = 0. \]

Case 2: \( p = 2 \). Then 
\[ H^*_A(\text{pt}) = \mathbb{Z}_2[\eta_1, \ldots, \eta_r] \]
and the image of \( H^*_T(\text{pt}) \) in \( \mathbb{Z}_2[\eta_1^2, \ldots, \eta_r^2] \). Here \( \wp = p = 0 \)
and localizing \( H^w_A(\text{pt}) \) with respect to \( p \) is the same as localizing \( H^w_T(\text{pt}) \) with respect to \( \wp \), since \( H^w_T(\text{pt}) \) would be a finite extension of \( \mathbb{Z}_2(\eta_1, \ldots, \eta_r) \) and hence a field. Thus
\[ [H^w_A(\text{pt})/\wp, H^w_T(\text{pt})/\wp] = 2^{r-1} = 0 \] and
\[ \text{length}_{H^w_A(\text{pt})/\wp}(H^+_A(\text{pt})/\wp) = 1, \]
so again
\[ \chi_{A, \wp}(\text{pt}) = 0. \]
Corollary: Suppose $X$ is a compactifiable $G$-manifold such that all isotropy groups are elementary abelian. If $p$ is a minimal prime of $H^*_G(pt)$, then

$$e_{G,p}(X) = 0.$$ 

Conclusion: This Euler characteristic isn't very interesting.

Here's a modification if $p = 2$. Let $p$ be a minimal prime of $H^*_G(pt)$ and set for a compactifiable $G$-manifold $X$

$$e_{G,p}(X) = \text{length } H^*_G(pt)_p \left( H^*_G(X)_p \right)$$

and observe that modulo 2 this is an additive function of $X$. E.g., given $Y < X$ we have an exact sequence of $H^*_G(pt)$-modules

$$0 \rightarrow H^*_G(X,Y) \rightarrow H^*_G(X) \rightarrow H^*_G(Y) \rightarrow 0$$

whence

$$e_{G,p}(X) \equiv e_{G,p}(X,Y) + e_{G,p}(Y) \pmod{2}$$

in virtue of

**Lemma:** Let

$$M'' \longrightarrow M' \longrightarrow M$$

be an exact triangle of objects of finite length in an abelian category...
Then \[ l(M') + l(M'') = l(M) \pmod{2} \].

Proof. Let \( K, C \) denote image and coimage at \( M' \) so that

\[
l(M') + l(M) + l(M'') = l(K') + l(C') + l(K) + l(C) + l(K'') + l(C'')
\]

\[
= 2(l(K') + l(K) + l(K'')).
\]

For \( p \) odd, there is something simpler, namely if we restrict to spaces with finite abelian groups for matching factors, then we know that \( \chi_p(x) = 0 \) and hence

\[
e_{G,p}(x) = e_{G,p}(x)
\]

will behave additively \pmod{2}.\]
Let $G$ be a compact Lie group and let $p$ be a fixed prime number. I want to determine the spectrum of $H^*_G(pt)$, the equivariant cohomology with coefficients $\mathbb{Z}_p$.

Let $A$ be an elementary abelian $p$-group of rank $r$. Then for $p$ odd

$$H^*_A(pt) = \Lambda(A^*) \otimes S(A^*)$$

where $A^*$ is considered as a vector space over $\mathbb{Z}_p$ and $\Lambda$ (resp. $S$) denotes exterior algebra (resp. symmetric algebra). Thus

$$H^*_G(pt) = \Lambda^{ev}(A^*) \otimes S(A^*)$$

and as $\Lambda^{ev}(A^*)$ is an Artin ring it follows that

$$H^*_G(pt)/nilideal \cong S(A^*)$$

For $p = 2$ we have

$$H^*_A(pt) = S(A^*)$$

so in either case we have that

$$\text{Spec } H^*_A(pt) \cong \text{the affine space over Spec } \mathbb{Z}_p$$

associated to the vector space $A^*$ over $\mathbb{Z}_p$.

We now consider the action of the Steenrod algebra $A$.
on $H^w_A(pt)$. Note that if $A$ is embedded in a
torus of the same rank, then there is an isomorphism

$$H^*_T(pt) \cong H^*_A(pt)/\text{nilideal}.$$  

The two-sided ideal generated by the Bockstein $\beta$ acts trivially
on the left, since $BT$ is torsion-free. It's not true that
the $A$ action on $H^*_A(pt)$ passes to the quotient by the nil-ideal,
but the above facts are justification for feeling that only
the action of the algebra of reduced powers matters in
doubts about prime ideals.

So from now on we let $A$ denote the algebra of
reduced powers. Recall that it has a basis of $\mathbb{Z}_p$ given
by $P_t$ where

$$P_t(x) = \sum_i t^i P_i(x)$$  \quad $t = t_1, t_2, \ldots$

and where $P_t$ is the multiplicative operation given on elements
in $H^2$ by

$$P_t(x) = \sum_{i \geq 0} t^i x^i$$  \quad $t_0 = 0$.  

It is clear that if $B < A$ then the prime ideal $P_B$
which is the kernel of

$$H^w_A(pt) \longrightarrow H^w_B(pt)/\mathfrak{n}_B$$  

$\mathfrak{n}_B =$ nil-ideal, is stable under the action of $A$, i.e.

$$P_t(P_B) \subset P_B[\{t_i\}]$$
and I would now like to prove the converse. It amounts to the following:

**Proposition.** Let \( A \) be a vector space over \( \mathbb{Z}_p \) and let \( \mathfrak{p} \) be a prime ideal of \( S(A^*) \) stable under the action of \( A \). Then \( \mathfrak{p} = \text{Ker}\{S(A^*) \to S(B^*)\} \) for some subspace \( B \) of \( A \).

**Proof.** Choose a basis for \( A \) and regard \( \text{Spec}(A^*) \) as affine \( A \)-space over \( \text{Spec} \mathbb{Z}_p \) and regard \( \mathfrak{p} \) as giving an irreducible subscheme \( Z \subset \text{Spec} A^* \). Let \( K \) be an algebraically closed field over \( \text{Spec} \mathbb{Z}_p \) and let \( x = (x_1, \ldots, x_n) \in k^n \) be a geometric point of \( Z \), i.e. \( f(x_1, \ldots, x_n) = 0 \) for all \( f \in \mathfrak{p} \). Since \( \mathfrak{p} \) is stable under \( A \), it follows that if \( t_1, \ldots, t_n \) are indeterminates, then

\[
f(x + t_1 x_1^{(p)} + \cdots + t_n x_n^{(p)}) < p[[t_1, \ldots, t_n]] \cap S(A^)[[t]]
\]

where \( x^{(p)} = (x_1^{(p)}, \ldots, x_n^{(p)}) \)

and consequently

\[
f(x + t_1 x_1^{(p)} + \cdots + t_n x_n^{(p)}) = 0.
\]

Therefore we see that if \( x \in Z_k \), then the subspace spanned by \( x, x^{(p)}, x^{(p)} \), \ldots also is contained in \( Z_k \). It is necessary now to determine the rank of this subspace.

Let \( V = \{(a_1, a_2) \in k^2 | \sum a_i x_i^{(p)} = 0 \text{ for all } g \geq 0\} \).

Then

\[
\text{rank } \{x_i^{(p)} \}_{i=1}^{r} = n - \dim_k V.
\]
Recall $V$ is generated by its minimal elements i.e. $(a_i, \ldots, a_n) \in V$ with support $\{i \mid a_i \neq 0\}$ minimal and with at least one $a_i = 1.$ As $k$ is perfect each $a_i$ has a unique $p$th root, so
\[
\sum_{i=1}^n a_i^{\frac{1}{p}} x_i^b = 0 \quad \text{all } b \geq 1
\]
hence by minimality $a_i^{\frac{1}{p}} = a_i$ for all $i.$ Thus $V$ is generated by its elements with all coefficients in $\mathbb{F}_p,$ and this means that $V$ is pretty small.

Now by descent it is known that a subspace $V$ of $k^n$ stable under $x \mapsto x^{(p)}$ is defined over $\mathbb{F}_p.$ Therefore we see that $x \in \mathbb{Z}_k \Rightarrow \mathbb{Z}_k \supseteq$ smallest subspace of $k^n$ defined over $\mathbb{F}_p$ and containing $x.$ So take $x$ to be a generic point of $\mathbb{Z}_k.$ Then $Z$ is the smallest subvariety over $\mathbb{F}_p$ containing $x$ and hence is contained in the smallest linear subspace defined over $\mathbb{F}_p$ containing $x,$ showing these two are the same. q.e.d.

**Conclusion:** Let $A$ be an elementary $p$-abelian group. If $\mathfrak{p}$ is a prime ideal of $H^0_A(\text{pt})$ which is stable under the action of the algebra of reduced Steinberg powers, then there is a unique subgroup $B$ of $A$ such that $\mathfrak{p}$ is the inverse image of $\mathfrak{m}_B$ under the restriction homomorphism
\[
H^0_A(\text{pt}) \twoheadrightarrow H^0_B(\text{pt}).
\]
So now let $G$ be a compact Lie group. For each elementary $p$-subgroup $A$ of $G$, define the prime ideal $\mathfrak{p}_A$ in $\mathcal{H}_G^u(p)$ as the inverse image of $\mathfrak{m}_A$ under the restriction homomorphism $\mathcal{H}_G^u(pt) \rightarrow \mathcal{H}_A^u(pt)$. It is clear that $\mathfrak{p}_A$ is stable under the action of $A$ and that $\mathfrak{p}_A$ depends only on the conjugacy class of $A$ in $G$. We can now state the basic conjecture:

**Conjecture:** The map $A \mapsto \mathfrak{p}_A$ establishes an bijection between conjugacy classes of elementary $p$-subgroups of $G$ and $A$-invariant homogeneous prime ideals in $\mathcal{H}_G^u(pt)$.

We shall prove this map is surjective. First, some lemmas:

**Lemma 1:** Let $X$ be a nice $G$-space and let $u \in \mathcal{H}_G^u(X)$. Then $u$ is nilpotent iff $u$ restricted to each orbit of $X$ is nilpotent.

**Proof:** Let $t$ be an indeterminate of degree $-\deg u$ and consider the operator of multiplying by $1 + t u$ on $\mathcal{H}_G^u(Y)[t]$ for all $G$-spaces over $Y$. Then $1 + t u$ is a unit on each fiber and hence for all $Y$ by the spectral sequence, whence $u$ is nilpotent.

**Lemma 2:** If $X$ is a nice $G$-space, then the map

$$\mathcal{H}_G^u(X) \rightarrow H^0(X/G, Gx \mapsto \mathcal{H}_G^u(Gx)) = E^2,$$
induces a homeomorphism of spectra.

Proof: We know that the kernel consists of nilpotent elements, so it suffices to prove that if \( z \in E^{0, \infty}_{2} \) then for some \( n, z^{n} \) comes from \( x \in H^{\infty}_{G}(X) \). Let \( X = U_{1} \cup \ldots \cup U_{N} \) where each \( U_{j} \) is a tubular neighborhood of an orbit, and proceed by induction to construct \( x \) over \( U_{1} \cup \ldots \cup U_{j} \). This reduces us to the case where we are given \( x' \in H^{\infty}_{G}(V) \) and \( x'' \in H^{\infty}_{G}(U) \) coinciding with \( z^{p} \) on each orbit in \( U \cup V \). Then \( x' - x'' \) in \( H^{\infty}_{G}(U \cup V) \) restricts to zero on each orbit, so is nilpotent by lemma 4. Then \( \exists b \geq 0 \)

\[
(x')^{p} - (x'')^{p} = (x' - x'')^{p} = 0
\]

in \( H^{\infty}_{G}(U \cup V) \) and this by Mayer-Vietoris means \( \exists x \in H^{\infty}_{G}(U \cup V) \) restricting to \( (x')^{p} \) and \( (x'')^{p} \). Thus \( x \) restricts to \( z^{p} \) and the induction step is clear.

One knows that \( E^{0, \infty}_{2} \) is a finite \( H_{G}^{\infty}(pt) \)-module (as below) hence the map \( \text{Spec} \{ H_{G}^{\infty}(X) \} \leftarrow \text{Spec} \{ E^{0, \infty}_{2} \} \) is closed by Cohen-Steinberg. But what we show proves that it is 1-1 onto, hence a homeomorphism.

Interpretation of \( H^{\infty}(X/G, Gx) \mapsto H^{\infty}_{G}(Gx) \). Consider the category whose objects are homotopy classes of maps \( Y \rightarrow X \) of \( G \)-spaces where \( Y \) is transitive and where a map
Interpretation of $H^0(\pi; G_x \to H^*_G(G_x))$: Let $O \leq X$ be a $G$-orbit. Then there is a tubular neighborhood $U$ of $O$ which is $G$-invariant and so there is a map

$$H^*_G(O) \xrightarrow{\sim} H^*_G(U) \longrightarrow H^*_G(O')$$

for any orbit $O'$ sufficiently near $O$. $H^0(\pi; G_x \to H^*_G(G_x))$ is the set of functions $\lambda$ with assign to each orbit $O \in X/G$ an element $\lambda(O) \in H^*_G(O)$ in such a way as to be compatible with the specialization homomorphism $H^*_G(O) \rightarrow H^*_G(O')$ whenever $O'$ is near to $O$.

Now consider the stratification of $X$ into orbit types, and refine the stratification so that all of the strata are connected. The set of strata is finite and is the union of orbits $G \cdot O$. The set of strata is finite. Choose a base point $* \in O$ in each of the minimal (i.e., closed) strata and denote the finite set by $x_1, \ldots, x_n$. Then we have that

$$E^0_* = H^0(\pi; G_0 \to H^*_G(G_0)) \hookrightarrow \bigoplus_{i=1}^n H^*_G(G_{x_i})$$

which proves that $E^0_*$ is a finite $H^*_G(pt)$-module. The image seems a bit difficult to describe in general, since the fundamental groups of the strata come into play.
Lemma 3: Let \( \Lambda \) act on a ring \( R \). If \( \mathfrak{p} \) is a prime ideal in \( R \), then the largest \( \Lambda \)-stable ideal contained in \( \mathfrak{p} \) is a prime ideal.

Proof: The largest \( \Lambda \)-stable ideal contained in \( \mathfrak{p} \) is
\[
\mathfrak{o}_\mathfrak{p} = \{ x \in R \mid \varphi(x) \in \mathfrak{p}[[t]] \}.
\]
In effect, \( \mathfrak{o}_\mathfrak{p} \) is an ideal contained in \( \mathfrak{p} \) which clearly contains any \( \Lambda \)-stable ideal, and \( \mathfrak{o}_\mathfrak{p} \) itself is \( \Lambda \)-stable because
\[
\varphi\left( \varphi(x) \right) = \varphi \left( \varphi(x) \right) \in \mathfrak{p}[[t]]
\]
where
\[
\sum v_i x^p_i = \sum u_i x^p_i \cdot \sum u_i x^p_i.
\]
But \( \mathfrak{o}_\mathfrak{p} \) is prime since if \( xy \in \mathfrak{o}_\mathfrak{p} \), then
\[
\varphi\left( \varphi(xy) \right) = \varphi \left( \varphi(xy) \right) \cdot \varphi \left( \varphi(y) \right) \in \mathfrak{p}[[t]]
\]
and one of these must be in \( \mathfrak{p}[[t]] \) as this is a prime ideal.

Corollary of lemmas 2 \& 3: The map
\[
\begin{align*}
H^0_G(X) & \longrightarrow H^0(X/G) \\
0 & \longrightarrow H^0_G(O)
\end{align*}
\]
induces a bijection of \( \Lambda \)-invariant homogeneous prime ideals.

Proof: For any \( \Lambda \)-ring homomorphism \( \varphi: R \to R' \), let \( \mathfrak{p} \subseteq R \) be an \( \Lambda \)-invariant prime. Then we
know that there is a \( p' \) in \( R' \) such that \( p' \cap R = p \). By lemma 2, \( p' = \{ x \in R' \mid H^k x \cap p' \neq \emptyset \} \) is a prime ideal of \( R' \) containing \( p \) and contained in \( p' \). As \( R \to R' \) is finite it follows that \( q' = p' \), so \( p \) comes from an invariant ideal.

Next stage consists of choosing an embedding of \( G \) into a unitary group \( U_n \) and letting \( X = \) the flag manifold \( U_n / T_n \). Then

\[
H^*_G(\text{pt}) \to H^*_G(X)
\]

is locally free of rank \( (n! \cdot n) \). In fact if \( V \) is the faithful representation of \( G \) given by \( G \to U_n \) and if

\[
c_i(V) = \sum_{k=1}^n k^i c_i(V)
\]

is the total Chern polynomial of \( V \), then

\[
H^*_G(X) \cong H^*_G(\text{pt}) \left[ X_1, \ldots, X_n \right] / \left( \sigma_i(x) = c_i(V), \ldots, \sigma_n(x) = c_n(V) \right)
\]

where the \( \sigma_i \) are the elementary symmetric functions of the \( X \)'s.

Let \( F^j \) be the stratification of \( R \) into \( G \)-connected components of the submanifolds of pure orbit type and let \( x_j \in F^j \) be a base point. Then we know that the composition

\[
H^*_G(\text{pt}) \to H^*_G(X) \to E^{o*} \to T^* H^*_G(x_j)
\]
is a finite morphism whose kernel is a nilideal, hence the corresponding map on $\text{Spec}$'s is surjective by Cohen - Seidenberg and even surjective on the set of invariant prime ideals by the argument used in the proof of the corollary to lemmas 2+3. Now the isotropy group of any point is abelian (and even elementary $p$-abelian provided we work with the $p$-flag manifold $U_n/p^j$ for which $H^1_G(U_n/p^j)$ is still locally free), so we conclude the following.

Proposition: Let $A_j, j = 1, \ldots, N$ be representatives for the conjugacy classes of maximal abelian (resp. maximal elementary $p$-abelian) subgroups of $G$. Then the morphism

$$H^1_G(pt) \xrightarrow{\bigoplus} \bigoplus_j H^1_{A_j}(pt)$$

is finite and has a nilpotent kernel. Consequently

$$\bigoplus_j \text{Spec } H^1_{A_j}(pt) \longrightarrow \bigoplus_j \text{Spec } H^1_G(pt)$$

is surjective and also

$$\bigoplus_j \{\text{Spec } H^1_{A_j}(pt)\}^\alpha \longrightarrow \{\text{Spec } H^1_G(pt)\}^\alpha$$

where the superscript $\alpha$ denotes homogeneous $\alpha$-invariant ideals.
Note that any maximal abelian subgroup appears as an isotropy group of a minimal stratum of $K$, and hence there are only finitely many conjugacy classes.

**Corollary:** The set of homogeneous $G$-invariant prime ideals in $H^*_G(X)$ for a nice $G$-space is finite.

**Proof:** $H^*_G(X) \hookrightarrow H^*_G(X \times F)$ and the latter has the same invariant spectrum as $H^b((X \times F)/G, O) \rightarrow H^*_G(O)$ which embeds into a finite product $\prod H^*_A(pt)$ whose invariant spectrum is finite by our earlier calculations.

---

We can be more precise, namely any invariant prime in $H^*_G(X)$ is the pull back of $\eta_A$ under a map $G/A \rightarrow X$.

$$H^*_G(X) \rightarrow H^*_G(G/A) = H^*_A(pt).$$
Let $X$ be a nice G-space. We can generalize the conjecture about the invariant primes in $H^*_G(pt)$ on page 5 as follows:

Let $I_G(X)$ be the ordered set of invariant prime ideas in $H^*_G(X)$. Let $J_G(X)$ be the ordered set associated to the following category: The objects are pairs $(A,f)$ where $A$ is an elementary profinite subgroup of $G$ and where $f \in [G/A,X] \cong \pi_0(X^A)$, and $\text{Hom}((A,f),(B,g)) = \{ \lambda \in [G/A,G/B] : g\lambda = f \}$, or equivalently such that

\[
\begin{array}{ccc}
G/A & \xrightarrow{f} & X \\
\downarrow \lambda & & \downarrow g \\
G/B & \xrightarrow{\gamma} & X
\end{array}
\]

is commutative. By definition $J_G(X)$ is the ordered set associated to the pre-ordered set of pairs $(A,f)$ with the relation $(A,f) \geq (B,g) \iff \exists \text{ map } (A,f) \rightarrow (B,g)$. Note that if $(A,f) \geq (B,g)$ and $(B,g) \geq (A,f)$, then $(A,f)$ and $(B,g)$ are isomorphic in the above category, since if any map $G/A \rightarrow G/A$ is an isomorphism (otherwise we would have $A > xAx^{-1} > x^2Ax^{-2} > \cdots$ contradicting the fact that compact Lie groups satisfy the d.c.c. on closed subgroups) (see remark 3 below).

Define

\[
\begin{array}{ccc}
\rho : J_G(X) & \rightarrow & I_G(X) \\
(A,f) & \mapsto & \rho(A,f)
\end{array}
\]

by associating to $f : G/A \rightarrow X$ the pull-back of $M_A$ under
The map $H^*_G(X) \xrightarrow{f^*} H^*_G(G/A) \cong H^*_A(pt)$. It is immediate that if

$$(A,f) \geq (B,g) \quad \text{i.e.} \quad \exists \quad \frac{G/A}{f} \xrightarrow{g} X$$

then

$$p(A,f) \geq p(B,g),$$

and so $p$ is a map of ordered sets.

**Conjecture:** $p$ is an isomorphism of partially-ordered sets.

**Remarks:**
1. According to our earlier work, the map $p$ is surjective.
2. It suffices to prove the conjecture when $G = U_n$ for all $n$, since one easily sees that

$$I_G(X) \cong I_U(U \times G X) \quad \text{since} \quad H^*_U(U \times G X) \cong H^*_G(X)$$

$$J_G(X) \cong J_U(U \times G X)$$

if $G \leq U$. The last isomorphism comes from the fact that

$J_G(X)$ depends on the category of $G$-spaces over $X$ which are transitive and have elementary abelian isotropy groups, and this category is equivalent to the corresponding category of $U$ spaces over $U \times_G X$.

3. As a set $J_G(X)$ is given by
\[ J_G(X) = \left\{ (A, \lambda) \mid \text{A elem. subg. of } G \right\} / G \text{ action} \]

where \( g(A, \lambda) = (gAg^{-1}, g\lambda) \) and \( g\lambda \) denotes the image of \( \lambda \) under the map \( g : X^A \to X^{gA^{-1}} \). This follows from the "note" sentence on page 12 since an isomorphism of \( \lambda : G/A \to X \) and \( \mu : G/B \to X \) comes from an element \( g \in G \times A = gBg^{-1} \) and \( g\mu = \lambda \).

Suppose \( G \) is abelian. Then

\[ J_G(X) = \bigcap_{A \in G} \pi_0(X^A)_G \]

and \( (A, \lambda) \preceq (B, \mu) \iff A \supseteq B \) and \( \lambda \mapsto \mu \) under the map \( \pi_0(X^A)_G \to \pi_0(X^B)_G \).

Note that if \( G \) is connected, then \( \pi_0(X^A)_G = \pi_0(X^A) \).

Now suppose \( G \) is a torus \( T \), and let \( X \) be a nice compact \( T \) space. Replace \( X \) by \( X \times (T/\mathcal{P}) \); I claim this doesn't affect either \( J_T \) or \( \pi_T \). Indeed

\[ H_T(X \times (T/\mathcal{P})) = H_T(X) \left[ e_{\mathcal{P}} \right] \]

and \( \text{dim} e_{\mathcal{P}} = 1 \), so the spectrum doesn't change. Also if \( A \) is an elementary abelian subgroup of \( T \), then \( A \subset \mathcal{P} \), so \( (T/\mathcal{P})^A = (T/\mathcal{T}) \) and

\[ \pi_0((X \times T/\mathcal{P})^A) = \pi_0(X^A \times T/\mathcal{T}) = \pi_0(X^A) \]
Thus to prove the equality of $J_T(X)$ and $I_T(X)$ I may suppose that all isotropy groups of $X$ are contained in $p_T$.

I use induction on the number of isotropy groups of $X$. Let $A_0$ be maximal among the isotropy groups, so that $X^{A_0}$ is a minimal stratum of $X$. Now

$$J_T(X^{A_0}) = \coprod_{A \in \text{Isol}(X) \setminus A_0} \pi_0\left(X^{A_0} \setminus (X^{A_0} \setminus X^{A_0})\right)$$

Lemma: If $A$ is an isotropy group of $X$, let $X^{(A)} \subset X^A$ be the open submanifold of points with isotropy group $A$. Then $\pi_0\left(X^{(A)}\right) \simeq \pi_0\left(X^A\right)$.

Proof: If $p$ is odd, this is because the strata are of even codimension, hence removing $X^B$ from $X^A$ for $B > A$ doesn't disconnect $X^A$. If $p = 2$, this argument doesn't work, however by removing one $X^B$ at a time we reduce to showing that if $X^A$ is connected so is $X^A \setminus X^B$ for $B > A$. The problem occurs when $X^B$ is a hyperplane in $X^A$, whence $B/A$ acts non-trivially in the normal direction. I can assume that $A = 0$ and $B = \mathbb{Z}_2$ since $T/A$ acts faithfully on $X^A$. So we are reduced to showing that if a torus $T$ acts faithfully on a connected manifold $X$, then it doesn't disconnect $X$. To take a normal vector $v$ on one side of $Y$ and a path in $T$ joining...
Lemma: Let a compact Lie group $G$ act on a manifold $X$, let $H$ be a closed subgroup of $G$, and let $N$ be the normalizer of $H$ in $G$. Let $X^H$ be the submanifold of $X$ consisting of those points with isotropy group equal to $H$ and let $Y$ be the closure of $X^H$ in $X$. Then 
\[ \pi_0(X^H)_N \rightarrow \pi_0(Y)_N \]

Proof: We may assume that $H = 1$, $G = N$. Then $X^H$ is the open submanifold of $X$ where $N$ acts freely. Now, $Y$ is the union of those components of $X$ on which $X$ acts freely somewhere. Let $y \in Y$; the bad set near $y$ is the union of submanifolds finitely many in number, hence the good points are open and dense near $y$. To replace $X$ by $Y$ and assume $X^H$ is dense in $X$, clearly $\pi_0(X^H) \rightarrow \pi_0(Y)$ is surjective. To prove injectivity I can suppose that $Y$ is connected. If $X^H$ is disconnected, then the bad set contains submanifolds of codimension 1, necessarily of the form $Y^A$ where $A = \mathbb{Z}_2$. Since at a generic point of $Y^A$, $A$ must act faithfully on the isotropy space, normal space. Now I can remove the bad submanifolds of codimension 2 from $Y$. If I want to and suppose the bad set is a submanifold of codimension 1. Now the thing to note is that two points reflecting each other through $Y^A$ are conjugate under $A$.
and therefore represent the same element in $\pi_0(X^{(H)})_N$. Therefore modulo the $N$ action, $Y^A$ and its conjugates don't disconnect $Y$.

q.e.d.

If $A \in \text{Iso}(X)$ and $A \neq A_0$, then $(X - X_{A_0})^A = X^A - X_{A_0} \cap X^A = X^A$ since $X_{A_0} \cap X^A \subset X_{A_0}^A$ and $A_0A > A_0$.

On the other hand if $A = A_0$, then $(X - X_{A_0})^A = X^A - X_{A_0}$.

Since $X_{A_0}$ might contain whole components of $X$, we must be careful.
October 25, 1969:

We are trying to prove that \( J_G(X) \rightarrow J_G(X) \) and we will now check the reduction to the case where \( G \) is a torus. Let \( F \) be the flag manifold of a faithful complex representation of \( G \). Let's assume, as we may, that \( G \) is a unitary group, and that \( F = G/T \). One knows that

\[
H_G(X) \rightarrow H_G(X \times F)
\]

is a free finite map and hence that the induced map on spectra is surjective and universally open. Moreover we know that

\[
H_G(X \times F \times F) \cong H_G(X \times F) \otimes_{H_G(X)} H_G(X \times F)
\]

and that for two rings over a third \( k \) that the map

\[
\text{Spec } (R \otimes_k S) \rightarrow (\text{Spec } R) \times_{\text{Spec } k} (\text{Spec } S)
\]

of topological spaces is surjective. Therefore we get a diagram

\[
\begin{align*}
\text{Spec } (R \otimes_k R) & \twoheadrightarrow \text{Spec } (R \times \text{Spec } R) \\
P_1 \downarrow & \quad \downarrow P_2 \\
\text{Spec } R & \twoheadrightarrow \text{Spec } R \\
f & \downarrow \text{Spec } \text{surj} \\
\text{Spec } k & \quad \text{Topological space}
\end{align*}
\]

where \( k = H_G(X) \), \( R = H_G(X \times F) \). It follows that \( \text{Spec } k \) is the quotient
of Spec \( R \) by the pre-equivalence relation defined by \( p_1 \) and \( p_2 \). Now we know already that as the maps \( f_i \) are finite, the inverse image of an invariant prime consists of invariant primes. Thus we get a diagram

\[
\begin{array}{c}
\mathcal{I}_G(X \times F, F) \xrightarrow{\text{surjective}} \\
\downarrow \quad \downarrow p_1 \quad \downarrow p_2 \\
\mathcal{I}_G(X \times F) \xrightarrow{\text{open surjective}} \\
\mathcal{I}_G(X)
\end{array}
\]

which proves that \( \mathcal{I}_G(X) \) is the topological space quotient of \( \mathcal{I}_G(X \times F) \) by the couple \( (p_1, p_2) \). As the topology and order are equivalent, it follows that \( \mathcal{I}_G(X) \) is the quotient as ordered sets of \( \mathcal{I}_G(X \times F) \) by \( (p_1, p_2) \).

Now we check the descent for \( \mathcal{J}_G \). Recall that if \( A_1, \ldots, A_N \) are representatives for the conjugacy classes of elementary \( p \)-abelian subgroups of \( G \) and if \( N_i \) is the normalizer of \( A_i \), then as a set

\[
\mathcal{J}_G(X) = \prod_{i=1}^N \mathcal{P}_G(A_i)_{N_i}
\]

To show that

\[
\mathcal{J}_G(X \times F, F) \Rightarrow \mathcal{J}_G(X \times F) \Rightarrow \mathcal{J}_G(X)
\]

is exact in the category of sets, it clearly suffices to consider
the fiber coming from $A_i$ and to show that
$$\pi_0((X \times F)^{A_i})_{N_i} \rightarrow \pi_0((X \times F)^{A_i})_{N_i} \rightarrow \pi_0(X^{A_i})_{N_i}$$
is exact. As dividing by a group is a left adjoint it suffices to show that without the $N_i$ is exact. By $X \rightarrow X^{A_i}$ and $X \rightarrow \pi_0(X)$ commute with products hence we must see that
$$\pi_0(X^{A_i}) \times \pi_0(F^{A_i}) \times \pi_0(F^{A_i}) \rightarrow \pi_0(X^{A_i}) \times \pi_0(F^{A_i}) \rightarrow \pi_0(X^{A_i})$$
is exact. This is true provided $F^{A_i} \neq \emptyset$ which is clear since $A_i$ is an abelian group and hence leaves some flag invariant.

It remains to check that the diagram $\lambda$ is exact in the category of ordered sets, or equivalently that if $\varphi: J_0((X \times F)) \rightarrow S$ is a map of partially ordered sets factoring through $J_0((X \times F)) \rightarrow J_0(X)$, then $\varphi: J_0(X) \rightarrow S$ is compatible with the ordering.

It will suffice to show that if $(A, \lambda) \geq (B, \mu)$ in $J_0(X)$ then each can be lifted to $J_0((X \times F))$ preserving the order.

We may suppose that $B \supset A$ and that

$\begin{array}{ccc}
G/A & \xrightarrow{\lambda} & X \\
\downarrow{\lambda} & \searrow \mu & \\
G/B & \rightarrow & X
\end{array}$

is homotopy commutative or equivalently that $\lambda$ under the map $\pi_0(X^B) \rightarrow \pi_0(X^A)$. But choose a point $P \in \pi_0(F^B)$ and let $Q$ be the corresponding
point in $\Pi_0(F^A)$ whence

$$
\begin{array}{c}
G/A \xrightarrow{\lambda \times \eta} X \times F \\
\downarrow \mu \times P \\
G/B
\end{array}
$$

is homotopy commutative. Thus we have lifted $(A, \lambda \times \eta)$ to $(B, \mu \times P)$ and are finished.

Since we can assume $G = U$, $F = G/T$ and since

$$
J_G(X \times G/T) = J_T(X)
$$

$$
I_G(X \times G/T) = I_T(X)
$$

we reach the following.

**Conclusion:** To prove $J_G \Rightarrow I_G$ it suffices to consider the case where $G$ is a torus.
October 27, 1969

Proof that \( I_G(X) \cong I_G(X) \) when \( G \) is a torus.

**Proposition 1:** Let \( A \) be an elementary abelian subgroup of \( G \).

Then there is an isomorphism

\[ H_G(X, f_A) \cong H_G(X^A, f_A) \]

**Proof:** We have to show that \( H_G(X^A, f_A) = 0 \), as the \( H_0(X, X^A) \) is a \( H_G(X, X^A) \) module, it suffices to prove that \( H_G(X, f_A) = 0 \) if \( X^A = \emptyset \). By use of the spectral sequence

\[ E_2^{pq} = H^p(X/G, \Omega) \Rightarrow H^q_G(X, f_A) \]

we must show that if \( G/B \) is elementary abelian and \( (G/B)^A = \emptyset \Rightarrow \) \( A \neq B \Rightarrow B \cap A < A \). Now by the formula for the cohomology of an elementary \( p \)-group, the cohomology square

\[ \begin{array}{ccc}
A \cap B & \rightarrow & B \\
\downarrow & & \downarrow \\
A & \rightarrow & G
\end{array} \]

gives rise to a cocartesian square.
But \((G/B)^A = \emptyset \iff A \not\subseteq B \iff B \cap A < A\). Let \(X\) be a non-trivial character of \(A\) which is trivial on \(B\cap A\) and extend \(X\) to a character of \(G/B\). Then \(e(X) \in H^2_G(\text{pt})\) vanishes in \(H^2_B(\text{pt})\) and restricts to a generator of \(H^2_A(\text{pt})/\mathfrak{m}_A\). Thus \(e(X) \not\in \mathfrak{p}_A\), so \(H_G(G/B)_{\mathfrak{p}_A} = 0\).

**Proposition 2.** If \(A\) is an elementary abelian subgroup of a torus \(G\), then the map

\[
\rho: H^*_G(X) \otimes_{H^*_G(\text{pt})} H^*_G(\text{pt})/\mathfrak{p}_A \to H^*_A(X)
\]

induces a homomorphism of spectra.

**Proof.** We will show that the kernel is nilpotent and that every element in the right comes from the left after being raised to a sufficiently high power of \(p\). First we show that \(\rho\) is consistent with \(\rho_{G/H}\). Writing \(X = U_1 \cdots U_n\), where the \(U_i\) are subtori, we reduce to checking this when \(X = G/H\) and that it's true for \(X = U_1 \cup U_2\) if it is true for \(U_1\) and \(U_2\). For an abelian compact Lie group \(H\) we have that \(H^*_H(\text{pt})/\mathfrak{m}_H = S(\hat{\mathbb{A}}/p\hat{\mathbb{A}})\), so

\[
H^*_G(G/H) \otimes_{H^*_G(\text{pt})} (H^*_G(\text{pt})/\mathfrak{p}_A)/\text{nilpotents} = S(\hat{\mathbb{A}}/p\hat{\mathbb{A}}) \otimes_{S(\hat{G})} S(\hat{A})
\]

\[
= S(\hat{\mathbb{A}}/p\hat{\mathbb{A}} + \hat{\mathbb{G}}) = S(\hat{\mathbb{H}}/\hat{\mathbb{A}})
\]
and it suffices to check that

\[ H^*_H(A_{pt}) \longrightarrow H^*_A(G/H) \]

induces a homeomorphism of spectra. But \( A/\text{An}H \) acts freely on \( G/H \) so

\[ \text{Spec} \left\{ H^*_A(G/H) \right\} = \text{Spec} \left\{ H^0(G/\text{AH}, \otimes A \times H \to H^*_A(A \times H)) \right\} \]

\[ = \text{Spec} \left\{ H^0(G/\text{AH}) \otimes H^*_A(\text{pt}) \right\} \]

\[ = \text{Spec} \left\{ H^*_A(\text{pt}) \right\} \]

since \( G/\text{AH} \) is connected. Therefore we see that \( \phi \) is a homeomorphism if \( X = G/H \).

More generally suppose that \( G/H \) acts freely on \( X \).

Then

\[ \text{Spec} \left\{ H^*_G(X) \right\} = \text{Spec} \left\{ H^0(X/G, Gx \to H^*_G(Gx)) \right\} \]

\[ = \text{Spec} \left\{ H^0(X/G) \otimes H^*_H(\text{pt}) \right\} \]

Thus

\[ \text{Spec} \left\{ H^*_G(X) \otimes (H_G(\text{pt})/\mathcal{P}_A) \right\} = \text{part of Spec } H^*_G(X) \text{ over } \mathcal{P}(\mathcal{P}_A) \]

\[ = \text{part of } H^0(X/G) \otimes H^*_H(\text{pt}) \text{ over } \mathcal{P}(\mathcal{P}_A) \]

\[ = \pi_0(X/G) \times \text{Spec } H^*_H(\text{pt}) \otimes (H^*_G(\text{pt})/\mathcal{P}_A) \]

\[ = \pi_0(X/G) \times \text{Spec } H^*_H(\text{pt}) \]

\[ = \text{Spec } \left\{ H^0(X/G) \otimes H^*_H(\text{pt}) \right\} = \text{Spec} \left\{ H^0(X/A, A \times A \to H^*_A(A \times A)) \right\} \]
Thus we have checked the proposition if there is a single orbit type.

Next we want to use induction on the number of orbit types, so suppose $X = U \cup V$, and consider the square

$$(X) \quad \begin{array}{c}
\mathbb{H}_G(X)/\mathbb{F}_A \\
\downarrow
\end{array} \begin{array}{c}
\{\mathbb{H}_G(U)/\mathbb{F}_A, \mathbb{H}_G(U \cup V)/\mathbb{F}_A\} \\
\downarrow
\end{array} \begin{array}{c}
\mathbb{H}_A(X) \\
\rightarrow
\end{array} \begin{array}{c}
\mathbb{H}_A(U) \times \mathbb{H}_A(U \cup V) \\
\rightarrow
\end{array} \begin{array}{c}
\mathbb{H}_A(V) \\
\end{array}$$

We claim that the horizontal arrows induce isomorphisms of spectra. This is clear for the bottom by Mayer-Vietoris. For the top, first let $x \in \mathbb{H}_G(X)$ and suppose that $x/\mathbb{F}_A \equiv 0 \mod \mathbb{F}_A$. Let $K$ and $I$ be the kernel and image of the map

$$\mathbb{H}_G(X) \rightarrow \mathbb{H}_G(U) + \mathbb{H}_G(V).$$

By Artin-Rees, there is some $k$ such that $I \cap \mathbb{F}_A^{n+k}(\mathbb{H}_G(U) + \mathbb{H}_G(V)) \subseteq I \mathbb{F}_A^k$. Thus for $p^a > k$ we have that $(x^p, y^p) \in I \mathbb{F}_A^k$. Since

$$K/\mathbb{F}_A^{n+k} \rightarrow \mathbb{H}_G(X)/\mathbb{F}_A^n \rightarrow I/\mathbb{F}_A^n \rightarrow 0$$

is exact if follows that there is some $y \in K$ with $x^p \equiv y \equiv 0 \mod \mathbb{F}_A$ and so the kernel of the upper row of $(X)$ is nilpotent.

Now suppose give $u \in \mathbb{H}_G(U)$ and $v \in \mathbb{H}_G(V)$ such
that $\mu_1 \mid (uv) \equiv v \mid (uv)$ mod $p A$. Consider the exact sequence

$$H_6(x) \rightarrow H_6(u) + H_6(v) \rightarrow C \rightarrow 0$$

$$C \rightarrow H_6(uv)$$

By Artin-Rees $C \cap p^{n+k} H_6(uv) \subset p^n C$. Thus

$$(u^{p^{n+k}} - v^{p^{n+k}}) = (u/v^{p^{n+k}})^{p^k} \subset p^n C$$

and so from the exactness of

$$H_6(x)/p \rightarrow H_6(u)/p + H_6(v)/p \rightarrow C/p \rightarrow 0$$

we see $\exists x \in H_6(x)$ s.t.

$$x \mid u \equiv u^{p^k} \quad \text{mod } p$$

$$x \mid v \equiv v^{p^k} \quad \text{mod } p$$

Therefore, the cokernel of the top row of $(\ast)$ is killed.

By assumption for $u, v$ and $uv$ the right hand vertical arrow has its kernel + cokernel killed by a power of Frobenius. By diagram chasing it must be so for the first vertical arrow, which concludes the proof of proposition 2.

Remark: In view of the fact that I don't yet understand decomposition of a $G$-manifold, it is worth remarking that in the above argument 1) to get nilpotency $\mu_1$ of the kernel one
doesn't need to know anything about $U_n V$. To get surjectivity, one needs only the nilpotency assertion for $U_n V$.

Proof that $I_G(X) \sim I_G(X)$ as sets, when $G$ is a torus. We calculate the fiber over $p_A \in H^*_G(kt)$. To show that

$$P_{p_A}(X) \sim \text{Spec } H^*_G(X) \otimes_{H^*_G(kt)} k(p_A).$$

But

$$H^*_G(X) \otimes_{H^*_G(kt)} k(p_A) \sim H^*_A(X_A) \otimes_{H^*_A(kt)} k(p_A)$$

$\sim H^*_G(X) \otimes_{H^*_G(kt)} k(p_A)$ by prop 1.

$$\sim \left( H^*_G(X) \otimes_{H^*_G(kt)} \frac{H^*_G(kt)}{p_A} \right) \otimes \frac{H^*_A(kt)}{p_A}$$

so by proposition 2 this has the same spectrum as

$$H^*_A(X_A) \otimes_{p_A}.$$

But as $A$ acts trivially on $X^A$, there is a single orbit type.
\[ \text{Spec } \mathcal{H}_A(X^A) = \pi_0(X^A) \times \text{Spec } \mathcal{H}_A(pt). \]

Thus
\[ \text{Spec } \left( H_g(X) \otimes_{H_g(pt)} k(\mathfrak{p}_A) \right)^{\mathfrak{p}_A} = \pi_0(X^A). \]
as claimed.

It remains to check the ordering. Suppose \( \mathfrak{p}, \mathfrak{p}' \) are invariant primes in \( H_g(X) \) coming from \( \lambda \in \pi_0(X^A) \) and \( \lambda' \in \pi_0(A') \), respectively, and that \( \mathfrak{p} < \mathfrak{p}' \). Then \( \mathfrak{p}_A < \mathfrak{p}'_A \) in \( H_g(pt) \) and so we know that \( A > A' \) and that \( X^A < X'^A \). We must show that \( \lambda \sim \lambda' \). We may replace \( X \) by \( X'^A \) without affecting the spectrum containing \( \mathfrak{p}_A \) since
\[ H_g(X)_{\mathfrak{p}_A} = H_g(X'^A)_{\mathfrak{p}_A} \]
by prop. 1.

But now if we can decompose \( X'^A \) into its connected components affecting a corresponding decomposition of the spectrum of \( H_g(X'^A) \), then it is then clear that if \( \lambda \) were not in the component of \( \lambda' \), then we could not have \( \mathfrak{p} < \mathfrak{p}' \).
Localization theorem: Let $G$ be a compact Lie group acting on a nice $G$-space $X$ and let $A$ be an elementary $p$-abelian subgroup of $G$ with associated prime ideal $p_A$ in $H(G)$. Then

$$H^*_G(X)_{p_A} \cong H^*_G(GX^A)_{p_A}$$

Proof: It suffices to show that if $X^A = \emptyset$ then $H^*_G(X)_{p_A} = 0$, by the Atiyah–Segal argument. It suffices to show that there is no invariant prime ideal in $H^*_G(X)$ whose inverse image in $H^*_G(pt)$ is contained in $p_A$. In effect $H^*_G(X)$ is a finite $H^*_G(pt)$ module so if $H^*_G(X)_{p_A} \neq 0$ we know, by Nakayama, that there is a prime in $H^*_G(X)$, necessarily invariant, whose inverse image in $H^*_G(pt)$ is $p_A$. By our theorem $p$ is represented by a homotopy class of $*$.

\[ G/A' \to X \]

and the fact that the inverse image of $f$ in $H^*_G(pt)_{p_A}$ is contained in $p_A$ means that $f$ is a diagram

\[ \begin{array}{ccc}
G/A' & \to & X \\
\uparrow & & \downarrow \\
G/A & \to & pt
\end{array} \]
so \( X^A \neq \emptyset \), a contradiction.

---

Basic special case: Invariant prime ideals in \( H_G^*(pt) \) correspond to conjugacy classes of elementary \( p \)-abelian subgroups of \( G \).

Further questions:

1) do these arguments work in \( K_0 \) theory
2) structure of the orbits of the \( A \)-action: Thus what is the image of the map

\[
\begin{array}{c}
H_G(pt) \\
\longrightarrow \end{array} \left( \frac{H_A(pt)}{r_A} \right)^N
\]

---

Example: \( \text{Spin}(n) \). Any maximal elementary \( 2 \)-abelian subgroup \( A \) of \( \text{Spin}(n) \) contains \( \text{Ker} \pi \) and hence is the inverse image \( \pi^{-1}(\pi(A)) \), where \( \pi: \text{Spin}(n) \to \text{SO}(n) \) is the natural map. Up to conjugation we may suppose that \( \pi(A) \) is contained in the diagonal matrices \( D \). \( \pi(A) \) is an isotropic subspace of \( D \) for the bilinear form defining the extension

\[
\begin{array}{c}
0 \\
\longrightarrow \end{array} \mathbb{Z}_2 \longrightarrow \pi^* D \longrightarrow D \longrightarrow 0
\]

Two subgroups of \( D \) are conjugate in \( \text{SO}(n) \) if they are already conjugate under the Weyl group \( W \). So one concludes that conjugacy classes of maximal elementary \( 2 \)-subgroups of \( \text{Spin}(n) \) are in correspondence with \( W \) conjugacy classes of maximal isotropic subspaces of \( D \).

Let $H$ be a Sylow $p$-subgroup of $G$, so that

$$H^*_G(pt) \longrightarrow H^*_H(pt)$$

is injective. The corresponding map of spectra

$$\text{Spec } \{H^*_H(pt)\} \overset{a}{\longrightarrow} \text{Spec } \{H^*_G(pt)\}$$

is surjective, due to the fact that non-conjugate elementary abelian subgroups of $H$ may become conjugate in $G$.

*fusion?
October 28, 1967

Legals version of Smith theory leads to the following theorem everywhere of positive dimension.

Theorem: Let \( Z_p \) act on a compact manifold \( M \). Assume either that \( p = 2 \) or that \( p \) is odd and \( M \) is oriented. Then the fixed submanifold cannot be a single point.

Proof: Assume that there is at least one fixed point and let \( i : \text{pt} \rightarrow M \) be the inclusion. Let \( f : M \rightarrow \text{pt} \).

By assumption, \( f \) is equivariantly oriented if \( p \) is odd. In any case, there is a Borel homomorphism \( f_* \) and hence by transitivity one for \( i \) in equivariant cohomology \( H^*_0(M) = H^*(E_G \times M, \mathbb{Z}_p) \).

Thus we have

\[
\begin{array}{c}
H_0(\text{pt}) \xrightarrow{f_*} H_0(M) \xrightarrow{f_*} H_0(\text{pt}) \\
\end{array}
\]

with \( i^* f_* = id \)

and \( f_* i_* = id \)

and \( f_*(f^* x) = f_* 1 \cdot x = 0 \)

since \( f_* 1 \in H^{-n}_0(\text{pt}) = 0 \) for dimensional reasons.

Now consider the triangle

\[
\begin{array}{c}
\rightarrow H^*_G(M, M^G) \rightarrow H^*_G(M) \rightarrow H^*_G(M^G) \rightarrow \ldots
\end{array}
\]

and localize with respect to the generator \( w \in H^*_0(\text{pt}) \). We get that
$H_6^*(M)[w^{-1}] \cong H_6^*(M^G)[w^{-1}]$

But the left side has rank at least 2 over $H_6^*(pt)[w^{-1}]$ because of the sequences (v). Hence $M^G$ cannot consist of a single point.

Suppose now that the fixed submanifold $M^G$ consists of isolated points and let us apply the localization procedure for restricting to the fixed submanifold.

\[ M^G \xrightarrow{i} M \]
\[ f^* : f_! \rightarrow f_* \]
\[ f^* \mathbb{C}(\mu_f) = \sum P \]

where if the eigenvalues $\lambda_j$ in the normal direction at $P$ are $\lambda^1, \ldots, \lambda^n$, then

\[ \nu_P = \prod_{j=1}^n \frac{1}{\lambda_j} \left( \eta \lambda_j \right)^{-1} = \left[ \left( \prod_{j=1}^n \frac{1}{\lambda_j} \right) \cdot w \cdot w^{-1} \right]^{-1} \]

Remark: For the situation of the theorem, note that the Euler characteristic of $H_6(M)$ as an $H_6(pt)$ module is defined and that $\chi(H_6(M)) = \chi(H(pt))$ using the spectral sequence and the fact that $G$ acts