October 6, 1969. More on norms and traces (see Sept 15, 1969).

Let $f: X \to Y$ be an étale covering of degree d. Recall that we have defined a norm or determinant map

$$\mathrm{Norm}_f : U^{2q}(X) \longrightarrow U^{2qd}(Y)$$

as follows. Let $\tilde{Y} \to Y$ be a principal G-bundle with G finite such that X is the bundle associated to a set S on which G acts. Then we have a diagram

$$\begin{tikzcd} X & \tilde{Y}\times S = \tilde{X} \arrow[l] \arrow[d, "\tilde{f}"] \\ Y \arrow[from=u, "f"'] & \tilde{Y} \arrow[l] \end{tikzcd}$$

where the horizontal arrows are principal G-bundle maps. Then the norm is the composition

$$U(X) \simeq U_G(\tilde{Y}\times S) \xrightarrow{\text{Norm}} U_G(\tilde{Y}) \simeq U(Y)$$

where the norm map is the composition

$$U_G^{2q}(\tilde{Y}\times S) \longrightarrow U_G^{2qd}\Big(\prod_S \tilde{Y}\Big) \xrightarrow{\Delta^*} U_G^{2qd}(\tilde{Y})$$

the first map being associated to the functor

$$Z \longmapsto \mathrm{Map}_{/S}(S, Z) = \prod_{s\in S} Z_s$$

from G-manifolds over S to G-manifolds.

We shall now check that this definition is independent of the choice of $\tilde{Y}$. In effect given ~~a~~ another covering $\tilde{Y}_1$ with group G_1 we can **dominate** $\tilde{Y}$ and $\tilde{Y}_1$ by $\tilde{Y}\times_Y \tilde{Y}_1$ with group $G \times G_1$ and ~~so~~ assume that $G_1 \twoheadrightarrow G$ and that $\tilde{Y} = \tilde{Y}_1/N$, $N = \mathrm{Ker}\, G_1 \to G$. Then one checks easily that

$$\begin{tikzcd} & U_G(\tilde{Y}\times S) \arrow[r] \arrow[d, equal] & U_G(\tilde{Y}) \arrow[d, equal] \arrow[dr, phantom, "\simeq"] & \\ U(X) \arrow[ur] \arrow[dr] & & & U(Y) \\ & U_{G_1}(\tilde{Y}_1\times S) \arrow[r] & U_{G_1}(\tilde{Y}_1) \arrow[ur, phantom, "\simeq"] & \end{tikzcd}$$

commutes so one sees the norm ~~depends~~ doesn't depends on $\tilde{Y}$.

~~This means that we can use a canonical $\tilde{Y}$ namely~~

~~$\tilde{Y} = (X/Y)^d_{reg}$ = subset of $X\times_Y X \times \cdots \times X$ d times where no two entries are the same~~

~~with $G = \Sigma_d$~~

We note that if $\alpha \in U(X)$ is represented by $[Z \to X]$ then $\mathrm{Norm}_f \alpha \in U(Y) \simeq U_G(\tilde{Y})$ is represented by

$$\begin{tikzcd} & \prod_{s\in S} \tilde{Z}_s \arrow[d] \\ \tilde{Y} \arrow[r, "\Delta"] & \prod_{s\in S} \tilde{Y} \end{tikzcd}$$

where $\tilde{Z} = Z \times_X (\tilde{Y}\times S) = \coprod_{s\in S} Z \times_{X \xleftarrow{s}} \tilde{Y}$ (Z pulled up to the s-sheet of $\tilde{X}$)

Now consider the canonical choice of $\tilde{Y}$, namely

$$\tilde{Y} = (X/Y)^d_{reg} = \text{subset of } X \times_Y X \times_Y \cdots X \ \ d \text{ times with all components distinct}$$

with $G = \Sigma_d$ acting on $S = \{1, \ldots, d\}$. Then for $s \in S$ we have

$$pr_s : \tilde{Y} \longrightarrow X$$

and so $\mathrm{Norm}_f\, \alpha$ $\quad \alpha = [Z \to X]$ is represented by where 1 goes under the maps

$$\begin{tikzcd}
 & \prod_{s \in S} \tilde{Z}_s \arrow[r, "\text{etale}"] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & Z^d \arrow[d] \\
\tilde{Y} \arrow[r, "\Delta"] & \prod_{s \in S} \tilde{Y} \arrow[r, "\text{etale}"] & X^d
\end{tikzcd}$$

into $U_{\Sigma_d}(\tilde{Y}) = U(Y)$. This means that

$$\boxed{\mathrm{Norm}_f(\alpha) = \rho\,(Q_d\,\alpha)}$$

where ρ is the composition

$$U_{\Sigma_d}(X^d) \longrightarrow U_{\Sigma_d}\left((X/Y)^d_{reg}\right) \simeq U(Y).$$

Consequences of this formula are

$$\boxed{\mathrm{Norm}_f(\alpha \cdot \beta) = \mathrm{Norm}_f\,\alpha \cdot \mathrm{Norm}_f\,\beta}$$

and the additivity

$$\mathrm{Norm}_f(\alpha+\beta) = \sum_{i+j=d} \rho\left(\mathrm{ind}_{\Sigma_i\times\Sigma_j\to\Sigma_d}\ Q_i\alpha\otimes Q_j\beta\right)$$

The i,j term may be written as

$$(g_{ij})_*(\gamma_{ij})^*\{Q_i\alpha\otimes Q_j\beta\}$$

$$\begin{array}{c} \{(X/Y)^d_{reg}\}/\Sigma_i\times\Sigma_j \\ \big\downarrow g_{ij} \\ Y \end{array}$$

$$(X/Y)^d_{reg} \xrightarrow{\ \gamma_{ij}\ } X^i\times X^j$$

Here $g_{i,j}: X_{i,j}\longrightarrow Y$ is the bundle of partitions of the fibers of $f: X\longrightarrow Y$ into a subset of i and a complementary subsets of j points.

Consider the case $i=1$, $j=d-1$. Then ~~$g_{1,d-1}$~~ we may identify $X_{1,d-1}$ with X, $g_{1,d-1}$ with f, and $\gamma_{1,d-1}$ with Δ, and we get

$$\boxed{\mathrm{Norm}_f(\alpha+t\beta) = \mathrm{Norm}_f\,\alpha + f_*\left(\beta\cdot \mathrm{Cof}_f(\alpha)\right)t + O(t^2)}$$

where if α ~~β~~ $= [Z\to X]$ then $\mathrm{Cof}_f(\alpha)\in U(X)$ is the element represented by

$$\begin{tikzcd} & Z^{d-1} \arrow[d] \\ (X/Y)^{d-1}_{reg} \arrow[r, hook] & X^{d-1} \end{tikzcd}$$

$$X \xrightarrow{\sim} \left\{(X/Y)^{d-1}_{reg}\right\}/\Sigma_{d-1}$$

$$x \longmapsto f^{-1}\{f(x)\} - \{x\}$$

Thus $\overbrace{\text{when } Z \to X \text{ is proper and smooth}}$ $\mathrm{Cof}_f\, \beta$ is the class represented by the manifold over X with fiber

$$x \longmapsto \prod_{\substack{x' \in f^{-1}f(x) \\ x' \neq x}} Z_{x'} .$$

Remarks:

1.) If $\alpha \in \mathcal{U}(X)$ is represented by a proper smooth map $Z \to X$, then $\mathrm{Norm}_f\, \alpha$ is represented by the manifold over Y with fiber

$$y \longmapsto \prod_{x \in f^{-1}\{y\}} Z_x$$

Note if $Z = X \times_Y W$ with $W \to Y$ smooth, this is not the same as W^d since there is no canonical isomorphism between $f^{-1}\{y\}$ and $\{1, 2, \dots, d\}$.

2.) If α and β are represented by proper smooth maps $Z' \to X$ and $Z'' \to X$, then

$$\rho\left(\mathrm{ind}_{\Sigma_i \times \Sigma_j \to \Sigma_d}\; Q_i \alpha \otimes Q_j \beta\right)$$

is represented by the manifold over Y whose fiber is

$$y \longmapsto \coprod_{\substack{I \subset f^{-1}\{y\} \\ \mathrm{card}\, I = i}} \left\{ \prod_{s \in I} Z'_s \times \prod_{s \in f^{-1}\{y\} - I} Z''_s \right\}$$

3.) The formula on page 4 in the box is analogous to the *matrix* formula

$$\det(A + tB) = \det A + \mathrm{tr}(A \cdot \mathrm{Cof}\, B)\, t + O(t^2).$$

4.) If we define the intermediate symmetric functions between trace and norm by

$$\mathrm{Norm}_f(1 + tx) = \sum_{j=1}^{d} t^j \sigma_j(x)$$

then

$$\begin{cases} \sigma_1(x) = \mathrm{tr}_f(x) = f_* x \\ \sigma_d(x) = \mathrm{Norm}_f(x) \end{cases}$$

and the formula on page 4 gives

$$\cancel{\mathrm{Norm}_f(f^*\alpha + t\beta) = \mathrm{Norm}_f(f^*\alpha) + t\beta\, \sigma_{d-1}(\beta)}$$

$$\mathrm{Norm}_f(\alpha + t \cdot f^*\beta) = \mathrm{Norm}_f(\alpha) + t\beta\, \sigma_{d-1}(\alpha)$$

5.) We know that

$$\mathrm{Norm}_f(f^*\alpha) \neq \alpha^d$$

in general. In effect if α is represented by $Z \to Y$ then $\mathrm{Norm}_f(f^*\alpha)$ is represented by a path

$$\begin{tikzcd}
 & (Z \times_Y X)^d \arrow[r] \arrow[d] & Z^d \arrow[d] \\
\tilde{Y} = (X/Y)^d_{reg} \arrow[r] \arrow[dr, "\pi"'] & X^d \arrow[r] & Y^d \\
 & Y \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

which yields

$$\mathrm{Norm}_f(f^*\alpha) = \tau(Q_d\alpha)$$

where τ is the composition

$$U_{\Sigma_d}(Y^d) \xrightarrow{\pi^*} U_{\Sigma_d}(\tilde{Y}) \simeq U(Y).$$

This operation of forgetting Σ_d action by lifting to $\tilde{Y}$ differs from just forgetting the action. (see below for formula for $d=2$)

The situation in characteristic zero is given by the familiar formula

Prop: $$\boxed{\mathrm{Norm}_f\,(1-t\alpha)^{-1} = e^{\sum_{m\geq 1}\frac{t^m}{m} f_*\{\alpha^m\}}}$$

Proof: If $\alpha \in U^{ev}(X)$ we can form

$$e^{t\alpha} \in U^{ev}(X)[[t]]$$

and

$$\frac{d}{dt} e^{t\alpha} = e^{t\alpha} . \alpha$$

i.e. $$e^{(t+\varepsilon)\alpha} = e^{t\alpha}(1+\varepsilon\alpha) \qquad \varepsilon^2 = 0$$

Thus $\mathrm{Norm}_f(e^{t\alpha}) = \varphi(t)$ satisfies

$$\varphi(t+\varepsilon) = \varphi(t)\, \mathrm{Norm}_f(1+\varepsilon\alpha)$$
$$= \varphi(t)(1+f_*\{\alpha\}\varepsilon)$$

$$\varphi'(t) = \varphi(t)\, f_*(\alpha)$$

The solution of the diff equation is

$$\varphi(t) = e^{t f_*(\alpha)}$$

thus we have

$$\boxed{\mathrm{Norm}_f(e^{t\alpha}) = e^{t f_*(\alpha)}}$$

~~so~~ as $\frac{1}{1-t\alpha} = e^{\sum \frac{t^m}{m}\alpha^m}$ we have

$$\mathrm{Norm}_f\left(\frac{1}{1-t\alpha}\right) = e^{f_*\left(\sum \frac{t^m}{m}\alpha^m\right)}$$
$$= e^{\sum_{m\geq 1} \frac{t^m}{m} f_*(\alpha^m)}$$

as claimed.

October 7, 1969. notes on orientations

Axioms for a class of oriented maps: A type of orientation $\mathcal{O}$ is a rule associating to each map of manifolds f a set $\mathcal{O}(f)$ and to each transversal cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

a map $$g^*: \mathcal{O}(f) \longrightarrow \mathcal{O}(f')$$

such that the following conditions hold

1) (transitivity). Given two transversal cartesian squares

$$\begin{tikzcd} X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y'' \arrow[r, "h"'] & Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

then

$$\begin{tikzcd} \mathcal{O}(f) \arrow[rr, "g^*"] \arrow[dr, "(gh)^*"'] & & \mathcal{O}(f') \arrow[dl, "h^*"] \\ & \mathcal{O}(f'') & \end{tikzcd}$$

commutes

(Note once g trans. to f, then gh trans to $f \Leftrightarrow h$ transv to f')

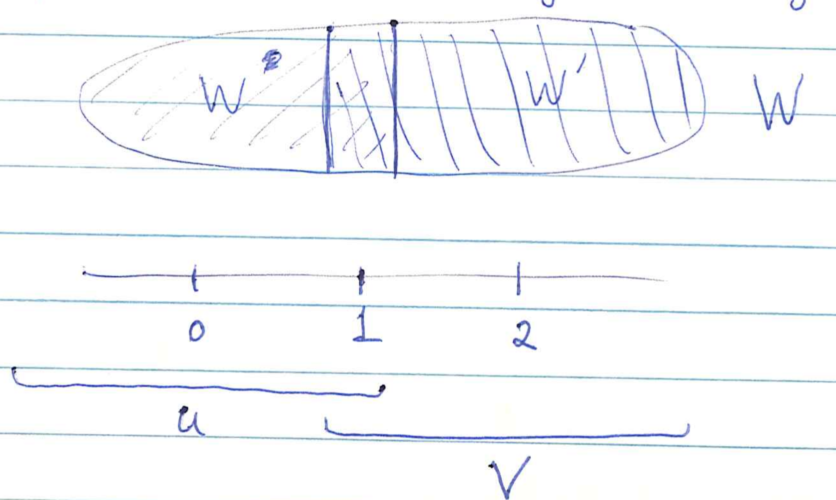
2) (half exactness) Given $f: X \to Y$ and $Y = U \cup V$ where U, V are open, we have that

$$\mathcal{O}(f) \longrightarrow \mathcal{O}(f|U) \times_{\mathcal{O}(f|U\cap V)} \mathcal{O}(f|V)$$

is surjective.

3) (homotopy). $\mathcal{O}(f) \xrightarrow{\sim} \mathcal{O}(f \times id_{\mathbb{R}})$

Given a type of orientation $\mathcal{O}$, let $F(Y)$ be the bordism classes of pairs (f, α) where f is a proper map with target Y and $\alpha \in \mathcal{O}(f)$. Two such pairs (f, α) (f', α') are bordant if there is a proper map $f'': W \to Y \times R$ and $\alpha'' \in \mathcal{O}(f'')$ such that $i_0^*(f'', \alpha'') = (f, \alpha)$ and $i_1^*(f'', \alpha'') = (f', \alpha')$. I check that this is an equivalence relation. Reflexivity + Symmetry are pretty clear. To prove transitivity suppose given bordisms W, W' joining $f_0: X_0 \to Y$ to $f_1: X_1 \to Y$ and f_1 to f_2. Then we can fit these bordisms together to get a W



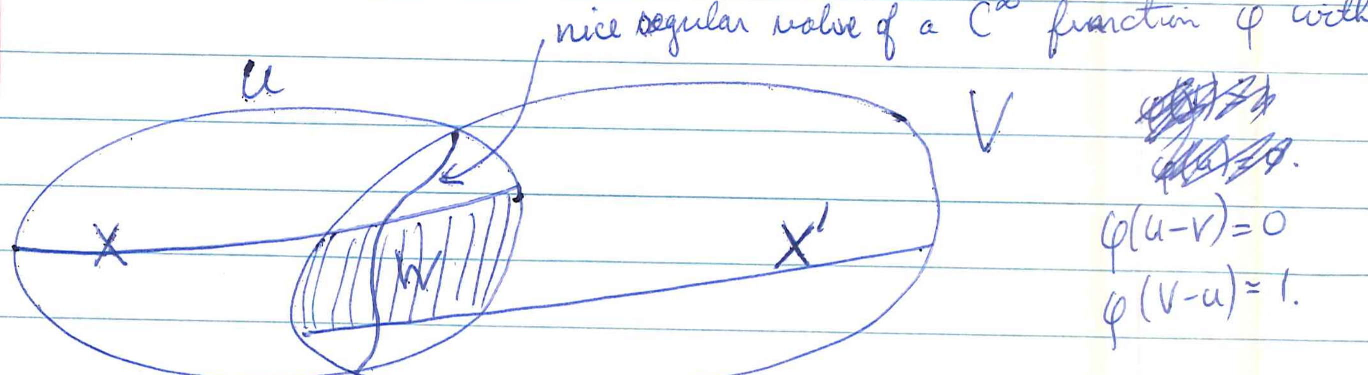
and it remains to orient W. Now use 2); we have to check that α on W and α' on W' coincide over $U \cap V$. But this is a product (There use proper) of $f_1 \times id_{\mathbb{R}}$ so by 3), as these orientations coincide over 1 they must coincide.

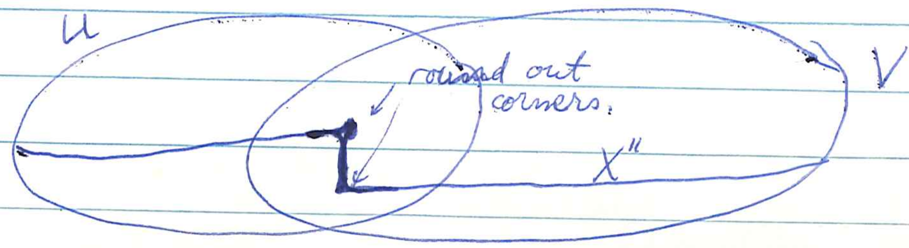
Given $g: Y' \longrightarrow Y$ define $g^*: F(Y') \longrightarrow F(Y)$ as follows. Given $[f: X \rightarrow Y, \alpha]$ move g transversal to f and form pull-back. This doesn't depend on choice of good map in the homotopy class of g and also doesn't depend on choice of representative.

Claim that F is half exact i.e. if U, V are two open subsets of Y, then

$$F(U \cup V) \longrightarrow F(U) \times_{F(U \cap V)} F(V)$$

is surjective. This is demonstrated geometrically as in the diagram below





It follows that F is ind-representable as a functor on the homotopy category of manifolds

$$F(X) \cong \varinjlim_{\alpha} [X, B_\alpha].$$

__Example:__ Let n be an integer ≥ 0 and let $O(f)$ be empty if f is not an embedding of codimension n and if f is an embedding of codimension n, let $O(f)$ be the equivalence classes of framings of ν_f, two framing being equivalent if they are homotopic. Then $F(X)$ = bordism classes of framed submanifolds of codimension n and

$$F(X) \cong [X, \Sigma^n]$$

This example shows that F needn't have any **group** structure. It also generalizes to the case where framing is replaced by reduction to (A, ξ) where $\xi: A \longrightarrow BO(n)$ is a map. More precisely:

~~ξ-structures where $\xi: A \to BO(n)$ is a map. Suppose that E is a n-dimensional bundle over X. Think of ξ~~

Let ξ be an n-dimensional (real) bundle over a space A. If E is an n-diml. bundle over X, we define a ξ-structure on E to be a map $f: X \longrightarrow A$ together with an isomorphism $\varphi: E \simeq f^*(\xi)$ and two such structures (f, φ), (f, φ') are said to be equivalent if they are homotopic, i.e. $\exists$ $h: X \times I \longrightarrow A$ and $\psi: E \times I \simeq h^*(\xi)$ with $i_0(h, \xi) = (f, \varphi)$ and $i_1(h, \xi) = (f', \varphi')$.

October 8, 1969. norm for a double covering

Let $f: X \to Y$ be a finite covering of degree 2. It is Galois where Σ_2 interchanges points of the fiber. The norm map Norm_f is thus the composition

$$\overset{ev}{U}(X) \xrightarrow{Q_{ext}} \overset{ev}{U}_{\mathbb{Z}_2}(X^2) \longrightarrow \overset{ev}{U}_{\mathbb{Z}_2}((X/Y)^2_{reg}) = \overset{ev}{U}(Y)$$

and since $(X/Y)^2_{reg} \xrightarrow{pr_1} X$ is a $\mathbb{Z}_2$-isomorphism, the norm carries $[Z \to X]$ into the equivariant class

$$\begin{array}{ccc} & & Z^2 \\ & & \downarrow \\ X & \xrightarrow{(id,\tau)} & X^2 \end{array}$$

followed by descent to Y. By the additivity formula on page 4 we have

$$\boxed{\mathrm{Norm}_f(\alpha+\beta) = \mathrm{Norm}_f\,\alpha + f_*(\alpha\cdot\tau^*\beta) + \mathrm{Norm}_f\,\beta}$$

~~Let~~ From now on we work with a theory V over $\mathfrak{N}$ eventually H, cohomology mod 2. Let $r \in V'(Y)$ be the Euler class of the line bundle over Y given by X. Then the map

$$\begin{array}{ccccc} V_{\mathbb{Z}_2}(Y) & \xrightarrow{f^*} & V_{\mathbb{Z}_2}(X) & \simeq & V(Y) \\ \| & & & & \\ V(Y)[w] & & & & \end{array}$$

is identity on $V(Y)$ and sends w to r. Hence from page 7 we get

$$\boxed{\mathrm{Norm}_f(f^*y) = \sum_{j\geq 0} (Sq_j\, y)\, r^j}$$

where as customary

$$Sq_j\, y = Sq^{n-j} y \quad \text{if } y \text{ is of degree } n.$$

Now suppose that $V = H$. Take the product of f with $\mathbb{RP}^\infty$, let L be the canonical line bundle, and let $x = c_1(L)$. If E is a vector bundle of dimension n over X, then

$$c_{2n}(f_*E \otimes L) = c_{2n}(f_*E) + c_{2n-1}(f_*E)\,x + \cdots + x^{2n}$$

$$\|$$

$$\mathrm{Norm}_f\left(c_n(E\otimes f^*L)\right) = \mathrm{Norm}_f\left(c_nE + c_{n-1}E\cdot f^*x + \cdots + f^*x^n\right)$$

$$= \mathrm{Norm}_f(c_nE) + f_*\left(\tau^*c_nE\cdot(c_{n-1}E\cdot f^*x + \cdots + f^*x^n)\right)$$

$$+ \mathrm{Norm}_f\left(c_{n-1}E + \cdots + f^*x^{n-1}\right)\cdot \mathrm{Norm}_f(f^*x)$$

Now $\mathrm{Norm}_f(f^*x) = x^2 + xr$ so one can eventually expand out all of the terms, get a polynomial in x whose coefficients give the classes $c_q(f_*E)$. Thus $c_q(f_*E)$ can be expressed in terms of Norms, f_* and the Chern classes of E. For example if E is a line bundle, we get

$$c_2(f_*E) + c_1(f_*E)\cdot x + x^2 = \mathrm{Norm}(c_1E) + f^*x) =$$

$$\mathrm{Norm}_f(c_1 E) + f_*(c_1 E)\cdot x + x^2 + xr \qquad \text{so}$$

$$\begin{cases} c_1(f_* E) = f_*(c_1 E) + r \\ c_2(f_* E) = \mathrm{Norm}_f(c_1 E) \end{cases}$$

Miscellaneous remarks about the Norm:

1.)
$$\boxed{\begin{aligned} f^*(\mathrm{Norm}_f(\alpha)) &= \alpha\cdot\tau^*\alpha \\ f^* f_*(\alpha) &= \alpha + \tau^*\alpha \end{aligned}}$$

since the composition

$$V_{\mathbb{Z}_2}(X) \simeq V(Y) \xrightarrow{f^*} V(X)$$

is the same as forgetting the $\mathbb{Z}_2$ action. These formulas generalize to any Galois covering.

2.) The norm is defined in terms of the external square $Q_{ext}: V(X) \longrightarrow V_{\mathbb{Z}_2}(X^2)$. It unfortunately doesn't seem possible to define the norm using the internal squaring operations although this can be done for elements of the form f^*y. Recall the exact sequence of Gysin for the line bundle associated to X over Y

$$V(Y) \xrightarrow{r} V(Y) \xrightarrow{f^*} V(X) \xrightarrow{f_*} V(Y) \xrightarrow{r} \cdots$$

This shows that if Ker r is big, then the elements of form f^*y are few.

3.) One knows in general for a permutation group G ~~acting~~ on $\{1, \dots, d\}$ that there is a canonical isomorphism

$$H^*_G(X^d) \cong H^*(G, H(X)^{\otimes d})$$

provided that H is cohomology with coefficients in a ring for which $H(X)$ is projective. In effect one can assume X is a CW complex whence

$$H^*_G(X^d) = H^*(G, C^\cdot(X^d))$$
$$= H^*(G, C^\cdot(X)^{\otimes d}).$$

By assumption on $H(X)$ there is a quasi-isomorphism $C^\cdot(X) \to H^\cdot(X)$ in the derived category of complexes of A-modules, hence

$$H^*(G, C^\cdot(X)^{\otimes d}) \simeq H^*(G, H^\cdot(X)^{\otimes d}).$$

For $G = \mathbb{Z}_2$ this canonical isomorphism may be realized using Q_{ext}. In effect the Leray spec. seq.

$$E_2^{pq} = H^p(\mathbb{Z}_2, H^q(X^2)) \Rightarrow H^{p+q}_{\mathbb{Z}_2}(X^2)$$

degenerates since E_2^{**} is generated by E_2^{0*} and E_2^{10}. This gives us an isomorphism of $\mathrm{gr}\, H_{\mathbb{Z}_2}(X^2)$ with E_2. But using $Q_{ext} : H(X) \longrightarrow H_{\mathbb{Z}_2}(X^2)$ we get a splitting of the edge homomorphism

$$\begin{tikzcd} Q_{ext}(x) & \gamma_2 x \arrow[l, mapsto] \arrow[dl, dashed] \\ H_{\mathbb{Z}_2}(X^2) \arrow[r, "\text{forget } \mathbb{Z}_2"'] & H(X^2)^{\mathbb{Z}_2} = \Gamma_2(H(X)) \end{tikzcd}$$

and so we get

$$\boxed{H^*_{\mathbb{Z}_2}(X^2) \cong \Gamma_2(H(X)) \oplus \sum_{i>0} w^i \otimes H(X)^{(2)}}$$

where the map sends $\gamma_2 x$ to $Q_{ext}(x)$ and $w^i \otimes x^{(2)}$ to $w^i \cdot Q_{ext}(x)$.

~~[illegible]~~ We have the following hierarchy

external Steenrod operation $\quad Q_{ext} : U(X) \to U_{\Sigma_d}(X^d)$

$\Downarrow$

Norm map for a covering

$\Downarrow$

internal Steenrod operation $\quad Q_{int} : U(X) \not\to U_{\Sigma_d}(X)$

and it doesn't seem possible to reverse any of the arrows. (added Jan 29, 1970) The norm and Q_{ext} are actually equivalent ~~[illegible]~~ by an argument similar to why proj. fmla. $\Leftrightarrow$ product formula.

<u>October 9, 1969</u> On signs

The usual form of the projection formula

$$f_*(f^*x \cdot y) = x \cdot f_* y$$

asserting that f_* is a left-module homomorphism is inconsistent with the sign convention since there should be a sign $(-1)^{\deg f \cdot \deg x}$. We shall now try to derive some consequences of ignoring this sign.

Suppose $f: X \to Y$ and $g: Z \to W$ are two oriented maps. Then there are two possible orientations for $f \times g$ as the two ~~[illegible] in the square~~ compositions in the square

$$\begin{array}{ccc} X\times Z & \xrightarrow{f\times id} & Y\times Z \\ \downarrow{\scriptstyle id\times g} & & \downarrow{\scriptstyle id\times g} \\ X\times W & \xrightarrow{f\times id} & Y\times W \end{array}$$

The two orientations differ by $(-1)^{\deg f \cdot \deg g}$

From the geometrical point of view the natural thing to do is to orient $f\times g$ as $(id\times g)(f\times id)$, e.g. if f, g are framed embeddings then one gets a framing for $f\times g$ by first taking the frame of f and then the frame of g. So with this convention we calculate ~~that~~ using standard projection formula

$$(f\times g)_*(x \otimes z) = (f\times g)_*(pr_1^* x \cdot pr_2^* z)$$

$$\cancel{\quad}$$

$$= (-1)^{\deg f.\deg g} (f\times id)_* \, (id\times g)_* \big((id\times g)^* pr_1^* x \cdot pr_2^* z \big)$$

$$= (-1)^{\deg f.\deg g} \, (f\times id)_* \big[pr_1^* x \cdot (id\times g)_* pr_2^* z \big]$$

$$= (-1)^{\deg f.\deg g} \, (f\times id)_* \big[pr_1^* x \cdot pr_2^* g_* z \big]$$

$$= (-1)^{\deg f.\deg g} \, (f\times id)_* \big[pr_1^* x \cdot (f\times id)^* pr_2^* g_* z \big]$$

$$= (-1)^{\deg f.\deg g} \, (f\times id)_* pr_1^* x \cdot pr_2^* g_* z \cdot (-1)^{(\deg g_* z).\deg f}$$

$$= (-1)^{\deg f.\deg z} \quad pr_1^* f_* x \cdot pr_2^* g_* z$$

$$= (-1)^{\deg f.\deg z} \quad f_* x \otimes g_* z.$$

Observe that for the other orientation for $f\times g$ we wouldn't have ~~...~~ a good sign, i.e. we would get $(-1)^{\deg g.\deg f + \deg f.\deg z}$.

<u>Conclusion</u>: The following two sets of ~~...~~ formulae are separately consistent from the point of view of signs

$$\left\{\begin{array}{l} f_*(f^* x\cdot y) = x\cdot f_* y \\ (f\times g)_* = (id\times g)_*(f\times id)_* \\ (f\times g)_*(x\otimes y) = (-1)^{\deg f.\deg y}(f_* x\otimes g_* y) \end{array}\right. \qquad \left\{\begin{array}{l} f_*(x\cdot f^* y) = f_* x\cdot y \\ (f\times g)_* = (f\times id)_*(id\times g)_* \\ (f\times g)_*(x\otimes y) = (-1)^{\deg g.\deg x} f_* x\otimes g_* y \end{array}\right.$$

The left is the standard one and perhaps more geometric since the Thom isomorphism is $\quad i_*(x) = i_*(i^*\pi^* x) = \pi^* x\cdot i_* 1$, but the right one seems more pleasant for the sign rules.

<u>Remark:</u> The projection formula implies the product axiom you used before! Here's the argument again using the sign-correct version of the projection formula:

$$\begin{tikzcd}
X\times Z \arrow[r, "f\times id"] \arrow[d, "id\times g"] & Y\times Z \arrow[r] \arrow[d, "id\times g"] & Z \arrow[d, "g"] \\
X\times W \arrow[r, "f\times id"] \arrow[d] & Y\times W \arrow[r] \arrow[d] & W \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r] & pt
\end{tikzcd}$$

$$\begin{aligned}
(f\times g)_*(x\otimes z) &= (f\times id)_*(id\times g)_*\left(pr_1^*x\cdot pr_2^*z\right) \\
&= (f\times id)_*(id\times g)_*\left((id\times g)^*pr_1^*x\cdot pr_2^*z\right) \\
&= (f\times id)_*\left(pr_1^*x\cdot(id\times g)_*pr_2^*z\right)\cdot(-1)^{\deg g\cdot\deg x} \\
&= (f\times id)_*\left(pr_1^*x\cdot pr_2^*g_*z\right)\cdot(-1)^{''} \\
&= (f\times id)_*\left(pr_1^*x\cdot(f\times id)^*pr_2^*g_*z\right)(-1)^{''} \\
&= (f\times id)_*(pr_1^*x)\cdot pr_2^*g_*z\cdot(-1)^{''} \\
&= (-1)^{\deg g\cdot\deg x}\; pr_1^*f_*x\cdot pr_2^*g_*z \\
&= (-1)^{\deg g\cdot\deg x}\; f_*x\otimes g_*z.
\end{aligned}$$

Starting with the projection formula

$$f_*(x \cdot f^* y) = f_* x \cdot y$$

one is forced to define the product $f_* 1 \cdot g_* 1$ where $f: X \to Y$ and $g: Z \to Y$ are proper oriented maps meeting transversally by

$$\begin{tikzcd} X \times_Y Z \arrow[r, "f'"] \arrow[d, "g'"] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$$\begin{aligned} f_* 1 \cdot g_* 1 &= f_*(1 \cdot f^* g_* 1) \\ &= f_*(g'_* f'^* 1) \\ &= f_* g'_* 1 \end{aligned}$$

Similarly this forces us† (if we want the formula $(f \times g)_* 1 = f_* 1 \otimes g_* 1$) to define

$$\boxed{(f \times g)_* = (f \times id)_* (id \times g)_*}$$

Thus the projection formula pins down the product.

(† $(f \times g)_* 1 = f_* 1 \otimes g_* 1 = pr_1^* f_* 1 \cdot pr_2^* g_* 1$

$= (f \times id)_* 1 \cdot (id \times g)_* 1 = (f \times id)_* [(id \times g)'_* 1]$

$= (f \times id)_* (id \times g)_* 1$.)

October 13, 1969 — how to modify U_G to get a Chern theory.

Let G be an abelian compact Lie group, and let U_G be the equivariant complex cobordism theory constructed by tom Dieck. I want to show there exists a localization $S^{-1}U_G$ which is a universal theory satisfying the projective bundle theorem.

Suppose that E is a vector bundle over X (all with G action) and that E splits $E = L_1 + \cdots + L_n$ as a sum of n-line bundles. I claim then that $U_G(\mathbb{P}E)$ is a free $U(X)$- module of rank n. In effect proceeding by induction we have $E = L \oplus F$ and (long) exact sequences

$$\longrightarrow U(\mathbb{P}L) \xrightarrow{i_*} U(\mathbb{P}E) \xrightarrow{j^*} U(\mathbb{P}F) \xrightarrow{\delta} U(\mathbb{P}L) \longrightarrow \cdots$$

and i_* is injective on a direct summand since $\mathbb{P}L = X$ and so $f_* i_* = \mathrm{id}$. Thus j^* is surjective and

$$U(\mathbb{P}E) = U(\mathbb{P}F) \oplus U(X)$$

so the claim follows by induction. Choose a basis $z_1, \ldots, z_n$ for $U(\mathbb{P}E)$ as a $U(X)$-module and write

$$\xi^i = \sum_{j=1}^{n} a_{ij} z_j \qquad i = 0, 1, \ldots, n-1$$

and set

$$d_E = \det\{a_{ij}\} \in U(X).$$

Note that changing basis alters d_E by a unit. Now if h is a theory for which the projective bundle theorem holds, then under the homomorphism $U \longrightarrow h$ d_E becomes a unit. Therefore the theory

$$S^{-1}U_G \qquad \text{where} \qquad S = \left\{ d_E \,\middle|\, \begin{matrix} E \text{ runs over the} \\ \text{representations of } G \end{matrix} \right\}$$

~~is a~~ satisfies the projective bundle theorem for all equivariant bundles over a point (since G is abelian) ~~and is~~ clearly universal with this property.

<u>Lemma</u>: If L_1, L_2 are two line bundles over X, then there is an element $a \in S^{-1}U_G(X)$ with

$$e_\phi(L_1) - e_\phi(L_2) = a \cdot e_\phi(L_1 \otimes L_2^{-1})$$

(also $$e(L_1 \otimes L_2^{-1}) = b\left(e(L_1) - e(L_2)\right) \text{)}.$$

<u>Proof</u>: ~~For simplicity let $h = S^{-1}U_G$ to save writing, and let $p: h(X) \longrightarrow h(X)/\text{ideal generated by } e(L_1 \otimes L_2^{-1})$ be the canonical surjection.~~ Let $f: X \longrightarrow \mathbb{P}V$, $f^*\mathcal{O}_{\mathbb{P}V}(1) \cong L_2$ and let $g: X \longrightarrow \mathbb{P}W$, $g^*\mathcal{O}_{\mathbb{P}W}(1) = L_1 \otimes L_2^{-1}$ be classifying maps for L_1 and $L_1 \otimes L_2^{-1}$, respectively. Then

$$(g,f)^*\{\mathcal{O}_{\mathbb{P}W}(1) \otimes \mathcal{O}_{\mathbb{P}V}(1)\} = L_1$$

so

$$e(L_1) = (g,f)^* \, e\{\mathcal{O}_{\mathbb{P}W}(1) \otimes \mathcal{O}_{\mathbb{P}V}(1)\}.$$

Now $S^{-1}H_G(\mathbb{P}W \times \mathbb{P}V)$ is generated as an $S^{-1}H_G(pt)$-algebra by $e(\mathcal{O}_{\mathbb{P}W}(1))\otimes 1$ and $1 \otimes e(\mathcal{O}_{\mathbb{P}V}(1))$ and in fact there is a formula

$$e\{\mathcal{O}_{\mathbb{P}W}(1)\otimes \mathcal{O}_{\mathbb{P}V}(1)\} = \sum_{\substack{0\le k< \dim W \\ 0\le l< \dim V}} a_{kl}\, e(\mathcal{O}_{\mathbb{P}W}(1))^k \otimes e(\mathcal{O}_{\mathbb{P}V}(1))^l$$

where a_{kl} are uniquely determined elements of $S^{-1}H_G(pt)$. Consequently

$$e(L_1) = \sum a_{kl}\, e(L_1\otimes L_2^{-1})^k e(L_2)^l.$$

Now we may suppose that W contains the trivial representation. This gives us a map $pt \longrightarrow \mathbb{P}W$ such that the map $\mathbb{P}V \longrightarrow \mathbb{P}W \times \mathbb{P}V$ carries $\mathcal{O}_{\mathbb{P}W}(1)\otimes\mathcal{O}_{\mathbb{P}V}(1)$ to $\mathcal{O}_{\mathbb{P}V}(1)$ and gives us the formula

$$e(\mathcal{O}_{\mathbb{P}V}(1)) = \sum_{0\le l<\dim V} a_{0l}\, e(\mathcal{O}_{\mathbb{P}V}(1))^l$$

By uniqueness

$$a_{0l} = \begin{cases} 0 & l\neq 1 \\ 1 & l=1 .\end{cases}$$

Thus

$$e(L_1) = \sum a_{kl} (e(L_1\otimes L_2^{-1})^k)\, e(L_2)^l$$

$$\equiv e(L_2) \qquad \mathrm{mod}\ e(L_1\otimes L_2^{-1})$$

proving the lemma.

Remark: The above proof amounts to using the "group law" which one gets when one has the projective bundle theorem for enough representations V so that every G-line bundle is ~~[illegible]~~ induced from $\mathcal{O}_{\mathbb{P}V}(1)$ for some V.

Consequence 1: If $E = L_1 + \cdots + L_n$ is a split bundle over X, then $S^{-1}\mathcal{U}_G$ satisfies the projective bundle theorem for E.

Proof: Use induction on n putting $E = L + F$ where the theorem holds for F. There is an exact sequence ($h = S^{-1}\mathcal{U}_G$)

$$0 \longrightarrow h(\mathbb{P}F) \xrightarrow{\iota_*} h(\mathbb{P}E) \xrightarrow{j^*} h(X) \longrightarrow 0$$

so that $h(\mathbb{P}E)$ has the basis $1, i_* 1, \iota_* \bar{\xi}, \ldots, \iota_* \bar{\xi}^{n-2}$
$\bar{\xi} = c_1 \mathcal{O}_{\mathbb{P}F}(1) = \iota^* \xi$, $\xi = c_1 \mathcal{O}_{\mathbb{P}E}(1)$. Thus $h(\mathbb{P}E)$ has the basis

$$1 \quad \text{and} \quad (\iota_* 1)\, \xi^i \qquad 0 \le i \le n-2.$$

Recall the exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow f^*E \longrightarrow Q \longrightarrow 0$$

and the fact that $\mathbb{P}F \subset \mathbb{P}E$ is where $\mathcal{O}(-1) \subset f^*F = \mathrm{Ker}\{f^*E \longrightarrow f^*L\}$. Thus $\mathbb{P}F$ is where the map $\mathcal{O}(-1) \longrightarrow f^*E \longrightarrow f^*L$ vanishes so

$$\iota_* 1 = e(f^*L \otimes \mathcal{O}(1)).$$

By the lemma (or rather a variant proved by the same method) we have that

$$e(\mathcal{O}(1)\otimes f^*L) = a(\xi - e(f^*L^{-1}))$$

for some $a \in h(\mathbb{P}E)$. Consequently $h(\mathbb{P}E)$ is generated by $1, (\xi - f^*x), \dots, (\xi - f^*x)\xi^{n-2}$ where $x = e(\cancel{f}^*L^{-1})$, and hence $h(\mathbb{P}E)$ is generated by $1, \dots, \xi^{n-1}$ as a $h(pt)$-module. As $h(\mathbb{P}E)$ is a free module of rank n, it follows that $1, \dots, \xi^{n-1}$ is a basis. qed.

<u>Theorem</u>: $S^{-1}U_G$ satisfies the projective bundle theorem.

<u>Proof</u>: Consider spectral sequences

$$E_2^{pq} = H^p(\cancel{X}/G,\ Gx \longmapsto (S^{-1}U_G)^q(\mathbb{P}(E|Gx))) \Longrightarrow (S^{-1}U_G)^{p+q}(\mathbb{P}E)$$

For each x $E|Gx$ splits and so by what we've just proved $(S^{-1}U_G)^*(\mathbb{P}(E|Gx))$ is a free $(S^{-1}U_G)^*(Gx)$ module with the correct basis.

$$(E_2^{pq})' = H^p(X/G,\ Gx \longmapsto (S^{-1}U_G)^q(Gx)^n) \Longrightarrow (S^{-1}U_G)^{p+q}(X)^n$$

By comparison theorem the map on the abutment must be an isomorphism, qed.

<u>Conclusion</u>: For G-abelian there is a universal Chern theory and moreover it is a localization of $\mathcal{U}_G$ so it satisfies the exactness axiom.

<u>Remark</u>: The above considerations also hold for $\mathcal{N}_G$ where G is an elementary abelian 2-group (i.e. all irreducible real representations are 1-dimensional).

So now we wish to take up the case where G is non-abelian. Let D be a <u>faithful</u> representation of G and let F be the flag manifold of D. Then the isotropy groups of the G-action on F are all abelian. Let X be a G-space over which D splits, i.e. for which there exists an equivariant map $f: X \longrightarrow F$. Then the isotropy groups of X are all abelian. Let $\mathcal{C}$ be the full subcategory of all such X.

Given such an X, let $h(X) = S_X^{-1}\mathcal{U}_G(X)$ where S_X is the set of d_E (page 2) where E runs over the split vector bundles over X. I claim that if $X_1 \longrightarrow X_2$ is any map in $\mathcal{C}$ then

$$S_{X_1}^{-1}\mathcal{U}_G(X_1) = S_{X_2}^{-1}\mathcal{U}_G(X_1).$$

In effect since D is a faithful representation we know that every representation of G is contained in a direct sum of tensor powers of D and D^*. Hence any representation is contained in a representation which splits over X. This means that for ~~a~~ spaces Y over X, such as X_1, there is a "group law" for Chern classes of line bundles in the theory $S_X^{-1}U_G(Y)$. Consequently we know that for any split bundle E over X_1 the projective bundle theorem holds for $S_X^{-1}U_G(?)$ and hence that S_{X_1} is already invertible in $S_X^{-1}U_G(X_1)$ ~~[illegible]~~. This proves the claim and shows that ~~h~~ h satisfies the projective bundle theorem for spaces in $\mathcal{C}$ and split bundles. However since the isotropy groups of a space in $\mathcal{C}$ ~~[illegible]~~ are all abelian it follows (by spectral sequence argument (page 5)) that h satisfies the proj. bundle thm. for all spaces in $\mathcal{C}$ and all bundles.

Given any G-manifold X we set

$$h(X) = \operatorname{Ker}\{h(X\times F) \rightrightarrows h(X\times F\times F)\}.$$

Then $h^*(X)$ is a functor from G-manifolds to graded rings endowed with Gysin homomorphism for complex-oriented proper G-maps. I claim this definition agrees with the old for a G-manifold in $\mathcal{C}$. This means that I must show that

$$(*) \qquad h(X) \longrightarrow h(X\times F) \rightrightarrows h(X\times F\times F)$$

is exact if X belongs to $\mathcal{C}$. However h satisfies the proj. bundle theorem, so

$$h(X\times F) \cong h(X)[t_1,\dots,t_n] \Big/ c_T(D) = \prod_{i=1}^{n}(T+t_i)$$

where $t_i = e(L_i)$ and $D = L_1+\dots+L_n$ is the canonical splitting of D over F. Similarly

$$h(X\times F\times F) \cong h(X)[t_1',\dots,t_n',t_1'',\dots,t_n''] \Big/ \begin{matrix} c_T(D) = \prod(T+t_i') \\ c_T(D) = \prod(T+t_i'') \end{matrix}$$

Therefore we see that

$$h(X\times F) \underset{h(X)}{\otimes} h(X\times F) \xrightarrow{\sim} h(X\times F\times F)$$

and that $h(X) \longrightarrow h(X\times F)$ is faithfully flat. Thus ~~that~~ the exactness of $(*)$ follows by faithfully flat descent.

Next I claim that h as just defined for all G-manifolds satisfies the projective bundle theorem. Suppose E given over X of dimension n. Then we have

$$\begin{tikzcd}
h(X)^n \arrow[r] \arrow[d] & h(X\times F)^n \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & h(X\times F\times F)^n \arrow[d, "\cong"] \\
h(\mathbb{P}E) \arrow[r] & h(\mathbb{P}E\times F) \arrow[r, shift left] \arrow[r, shift right] & h(\mathbb{P}E\times F\times F)
\end{tikzcd}$$

where the map is given by the powers $1, \ldots, \xi^{n-1}$, $\xi = c_1(\mathcal{O}(1))$. The isos. come from the projective bundle theorem ~~such~~ for spaces in $\mathcal{C}$, and the left arrow is an isomorphism by 5 lemma.

It is clear that h is a universal equivariant theory satisfying the projective bundle theorem. Moreover h is exact since $h(?\times F) = S_F^{-1} U_G(?\times F)$ is exact and since the functor $h(?)$ is a direct summand of $h(?\times F)$. So we conclude

<u>Theorem:</u> There exists a universal equivariant cohomology theory with Thom class for equivariant complex bundles and which satisfies the projective bundle theorem.

So now denote by Ω_G this universal theory to distinguish it from the U_G of tom Dieck

<u>Conner–Floyd theorem</u> (first form):
$$K_G(pt) \otimes_{\Omega_G(pt)} \Omega_G(X) \xrightarrow{\sim} K_G(X)$$

<u>Proof:</u> We know that the canonical map
$$\mu : U_G(X) \longrightarrow K_G(X)$$
is surjective. This is because $\mu(1-e(L^{-1})) = 1-(1-L) = L$ and because if $f : Flag(E) \to X$ is the flag bundle of E and $f^*E = L_1 + \cdots + L_n$, then $f^*E = \mu(z)$ for some $z \in U_G(Flag\, E)$ and so
$$E = f_*(f^*E) = \mu(f_* z).$$

Set $Q(X) = K_G(pt) \otimes_{\Omega_G(pt)} \Omega_G(X)$. Then $Q(X)$ satisfies the projective bundle theorem and
$$Q(X) \xrightarrow{\sim} K_G(X)$$
if X is a product of projective spaces. This means that
$$e^Q(L_1 \otimes L_2) = e^Q(L_1) + e^Q(L_2) - e^Q(L_1)e^Q(L_2)$$
and consequently the additive map
$$K_G(X) \longrightarrow Q(X)$$
given on line bundles by
$$L \longmapsto 1 - e^Q(L^{-1})$$

is a ring homomorphism. The composition

$$K_G(X) \longrightarrow Q(X) \longrightarrow K_G(X)$$

is evidently the identity on L hence in general. The composition the other way is the identity because ~~$U_G(X)$~~ $\Omega_G(X) \twoheadrightarrow Q(X)$ and because

$$\begin{array}{ccccccc} \Omega_G(X) & \longrightarrow & Q(X) & \longrightarrow & K_G(X) & \longrightarrow & Q(X) \\ e(L) & \longmapsto & e^Q(L) & \longmapsto & 1-L^{-1} & \longmapsto & e^Q(L) \end{array}$$

preserves Thom classes and hence commutes with Gysin homomorphism (may use here that $0 \to Q(X^E) \to Q(P(E+1)) \to Q(PE) \to 0$ is split exact). This proves the theorem.

Corollary: If G is abelian, then

$$K_G(pt) \otimes_{U_G(pt)} U_G(X) \xrightarrow{\sim} K_G(X)$$

Proof: Here $\Omega_G = S^{-1}U_G$ and the elements of S go into units in $K_G(pt)$. Precisely

$$K_G(pt) \otimes_{\Omega_G(pt)} \Omega_G(X) = K_G(pt) \otimes_{U_G(pt)} S^{-1}U_G(X)$$

$$= S^{-1}K_G(pt) \otimes_{U_G(pt)} U_G(X) = K_G(pt) \otimes_{U_G(pt)} U_G(X).$$

Remarks: 1. I don't know if the strong Conner-Floyd theorem holds for G non-abelian. The problem comes from the fact that even after localizing so as to make $S_0^{-1}U(pt) \longrightarrow S_0^{-1}U(F) \rightrightarrows S_0^{-1}U(F\times F)$ good for descent I don't know how to carry the localization $U(F) \longrightarrow S^{-1}U(F)$ down to something over a pt which can be moved across the $\otimes_{U(pt)}$.

2. Above holds for $\mathcal{N}_G$, where "abelian" now should be interpreted "elementary 2 abelian" group. Can you calculate your h for $G = \mathbb{Z}_3$? Here you have that $\mathbb{Z}_3$ acts freely on the sphere of its one non-trivial representation so $\mathcal{N}_G$ can be calculated by tom Dieck's method. Is there a C-F theorem relative to the map $\mathcal{N}_G(X) \longrightarrow H_G(X, \mathbb{Z}_2)$?

3. In the course of the preceding arguments we have seen that any Chern theory on G-manifolds is determined by its restriction to the full subcategory of G-manifolds with only abelian isotropy groups. (slightly more amusing is $\mathcal{N}_G$ with G odd in which case this means free G-manifolds!) One might ask whether in the case of K_G theory one could restrict to cyclic groups? Maybe not but one can at least retrict to hyperelementary groups by an equivariant K_G-version of Brauer's theorem (Segal).

4. The theory Ω_G has a generalized formal group law and it is now necessary to ask if there is such a universal law over L_G and if $L_G \to \Omega_G(pt)$ is an isomorphism.

October 14, 1969. some examples.

First consider the **universal unoriented Chern theory** h constructed from $\mathfrak{N}_G$ where G is odd. Recall that $h(X) = S_F^{-1}\mathfrak{N}_G(X)$ if X is over F, the flag bundle of a faithful representation of G. As G is a finite group of odd order F is a free G manifold hence for a manifold X in $\mathcal{C}$ we have

$$\mathfrak{N}_G(X) \cong \mathfrak{N}(X/G) \cong \mathfrak{N}(EG\times_G X).$$

But this satisfies the projective bundle theorem so S_F is already invertible. Thus $h(X) \xrightarrow{\sim} \mathfrak{N}(X/G)$ for a G-manifold in $\mathcal{C}$ and so by descent

$$h(X) \xrightarrow{\sim} \mathfrak{N}(EG\times_G X).$$

[Usually I wouldn't be so happy about these infinite complexes except that it's OK for H^* here)

$$\begin{aligned}\mathfrak{N}(EG\times_G X) &\cong H(EG\times_G X)\otimes \mathfrak{N}(pt)\\ &\cong H(X)^G \otimes \mathfrak{N}(pt)\\ &\cong \mathfrak{N}(X)^G\end{aligned}$$

Thus

$$\boxed{h(X) \xrightarrow{\sim} \mathfrak{N}(X)^G}$$

We can now ask whether $h(pt)$ has the universal generalized group law. If ~~[illegible]~~ L is a line bundle then an invariant Riemannian metric on L gives an isomorphism $L^{\otimes 2} \xrightarrow{\sim} 1$ so $e(L^{\otimes 2}) = 0$. So if G is abelian of odd order $e(L) = 0$ for all G-line bundles over a point. Hence the generalized ~~[illegible]~~ group law will in fact be a ordinary group law and the universal ring is indeed $\mathfrak{N}(pt)$. Thus the conjecture is true if G is abelian and of odd order. But its true in general.

<u>Proposition:</u> Let G be a finite group of odd order and let h be a unoriented Chern theory on G-manifolds. Then

$$c_t(x) = 1 \qquad \text{all} \quad x \in RO(G).$$

<u>Proof:</u> Consider the map

$$KO_G(X) \longrightarrow h(X)[[t]]^\times$$

$$x \longmapsto c_t(\psi^2 x)$$

This is a homomorphism and hence is determined by its effect on line bundles. But

$$c_t(\psi^2 L) = c_t(L^{\otimes 2}) = 1 \qquad \text{since } L^{\otimes 2} \simeq 1$$

Thus

$$c_t(\psi^2 x) = 1 \qquad \text{all} \quad x \in KO_G(X)$$

Now take $X = pt$ and use that ψ^2 is an auto. since G is odd. qed.

<u>Conclusion:</u> If G is a finite group of odd order and if h_G is the universal unoriented Chern theory, then the generalized group law of h is a formal group law (of height ∞) and $h_G(pt) = \mathfrak{N}(pt)$ has the universal such law.

Now suppose G is a finite group and let H be a Sylow 2 subgroup of G. Let h_G be the universal unoriented Chern theory. I claim that the element $\xi = [G/H \longrightarrow pt]$ is a unit in $h_G(pt)$. By descent, i.e. exactness of

$$h_G(pt) \longrightarrow h_G(F) \rightrightarrows h_G(F \times F)$$

$F=$ flag manifold of a faithful representation, it suffices to show that it is invertible over F. By standard spectral sequence

$$E_2^{pq} = H^p(F/G, Gx \longmapsto h_G^q(Gx)) \Longrightarrow h_G^{p+q}(F)$$

it suffices to show ξ is a unit over each G-orbit of F, that is, over G/K where K is an elementary 2-abelian subgroup of G. We will show ξ is a unit in $\mathfrak{N}_G(G/K) = \mathfrak{N}_K(pt)$. So its a question of considering the element $[G/H \to pt]$ as an element of $\mathfrak{N}_K(pt)$, and

$$[G/H \longrightarrow pt] = \sum_{KxH} [K/K \cap xHx^{-1} \longrightarrow pt]$$

Now the element $[K/K_1 \to pt]$ is zero for $K_1 < K$ because as a K-set we have

$$K/K_1 \xrightarrow{\sim} \prod_{i=1}^{g} \mathbb{Z}_2$$

where K acts on the ith factor by a homomorphism $K \xrightarrow{\varphi_i} \mathbb{Z}_2$.

and this element is zero because it is a product of zero elements. The number of fixed points of K on G/H is odd since G/H is odd. Thus in fact we see that

$$[G/H \longrightarrow pt] = [G:H] \cdot 1 = 1$$

in $\mathcal{N}_K(pt)$.

This means that the resolution

$$G/H \times G/H \times G/H \rightrightarrows G/H \times G/H \rightrightarrows G/H \longrightarrow pt$$

gives rise to an exact sequence of descent

$$h_G(X) \longrightarrow h_G(G/H \times X) \rightrightarrows h_G(G/H \times G/H \times X) \cdots$$

In other words denoting by S a system of ~~[illegible]~~ representatives for the double cosets HgH and

$$H \cap gHg^{-1} \underset{j_g}{\overset{i_g}{\rightrightarrows}} H, \quad i_g = \text{the inclusion}$$
$$j_g(x) = g^{-1}xg$$

then

$$h_G(X) \longrightarrow h_H(X) \underset{j^*}{\overset{i^*}{\rightrightarrows}} \prod_{g \in S} h_{H \cap gHg^{-1}}(X)$$

is exact. In particular

$$\boxed{h_G(X) = h_H(X)^{G/H} \quad \text{if } H \triangleleft G}$$

which yields our earlier result when $H = e$.

October 15, 1969: dimension (in the sense of Atiyah) of a compact Lie group

Let G be a compact Lie group, let X be a G-manifold, and let $\mathcal{F}$ be a local coefficient system on X endowed with a compatible G-action. (This means that the etale space $\tilde{\mathcal{F}} \to X$ is a map of G-spaces). Then cohomology groups $H^*_G(X, \mathcal{F})$ are defined, say as the cohomology of the space $EG \times_G X$ with values in the local coefficient system $EG \times_G \tilde{\mathcal{F}}$.

Suppose that $\rho : G \longrightarrow U_n$ is a faithful representation. Then

$$H^*_G(X, \mathcal{F}) = H^*_{U_n}(U_n \times_G X, U_n \times_G \mathcal{F})$$

is the abutment of a spectral sequence with

$$E_2^{pq} = H^p(BU_n, H^q(U_n \times_G X, U_n \times_G \mathcal{F})),$$

namely the Serre spectral sequence for the fibration

$$U_n \times_G X \longrightarrow EU_n \times_G X \longrightarrow BU_n .$$

Now $E_2^{pq} = 0$ if $q > \dim(U_n \times_G X)$, so if we know that $H^*(U_n \times_G X, U_n \times_G \mathcal{F})$ is a finitely generated abelian group (which is the case if X is compactifiable and $\mathcal{F}$ is a local coefficient system of finitely generated abelian groups) then we conclude that E_2^{**} and hence $H^*_G(X, \mathcal{F})$ is a module of finite type over the ring $H^*(BU_n, \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \ldots, c_n]$. Thus it has a dimension namely the dimension of its support.

In fact this definition is independent of the choice of ρ since the $H^*(BU_n, \mathbb{Z})$ module structure on $H^*_G(X, \mathcal{F})$ ~~one~~ factors via it is better to consider $\dim_p$ from the beginning

~~the~~ homomorphism

$$\rho^* : H^{ev}(BU_n, \mathbb{Z}) \longrightarrow H^{ev}(BG, \mathbb{Z}) = H^{ev}_G(pt, \mathbb{Z}).$$

Thus we see that $H^*_G(X, \mathcal{F})$ is a finite $H^{ev}_G(pt, \mathbb{Z})$ module and that

$$\dim\{H^*_G(X, \mathcal{F})\} \leq \dim\{H^{ev}_G(pt, \mathbb{Z})\}.$$

We might define the <u>dimension</u> of G to be the dimension of the ring $H^{ev}_G(pt, \mathbb{Z})$.

This Unfortunately is not quite right because if $G = 1$ then $\dim H^{ev}_G(pt) = \dim \mathbb{Z} = 1$. A more reasonable definition is to consider the dimension of the scheme

$$\operatorname{Proj} H^*_G(pt, \mathbb{Z}) \qquad (+1 \text{ see end.})$$

since this depends only on the asymptotic features of the cohomology. However we still haven't achieved a really local situation yet since this Proj sits over $\operatorname{Spec} \mathbb{Z}$. Therefore if p is a prime number we define

$$\dim_p(G) = \dim \operatorname{Proj}\{H^{ev}_G(pt, \mathbb{Z}) \otimes \mathbb{Z}_p\} + 1$$

In virtue of the exact sequence

$$0 \to H^{ev}_G(pt, \mathbb{Z}) \otimes \mathbb{Z}_p \longrightarrow H^{ev}_G(pt, \mathbb{Z}_p) \longrightarrow {}_pH^{ev}_G(pt, \mathbb{Z}) \to 0$$

we see that as both ends are finite type $H^{ev}_G(pt, \mathbb{Z})$ - modules that

$$\dim_p(G) = \dim \operatorname{Proj}\{H^{ev}_G(pt, \mathbb{Z}_p)\}. + 1$$

Of course this dimension is also given by the degree of the Hilbert polynomial

This is incorrect see remarks after page 6

$$P(q) = \operatorname{rank}_{\mathbb{Z}_p} H_G^{2q}(pt, \mathbb{Z}) \otimes \mathbb{Z}_p \qquad q >> 0$$

We also define $\dim_{\mathbb{Q}}(G) = \dim \operatorname{Proj}\{H_G^{ev}(pt, \mathbb{Q})\} + 1$ ~~Proposition~~

Proposition: $\dim(G) = \max_{p \text{ prime}} \{\dim_p(G), \dim_{\mathbb{Q}}(G) + 1\}$.

Proof. (?) This should be a result from algebraic geometry, the idea being that ~~the dimension of ... the fiber ... of a map $f: X \to Y$~~ if $f: X \longrightarrow Y$ is a map of nice noetherian schemes of finite type then $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} k(y)$.

So to simplify life we work with $\dim_p$ from now on and set $H_G^*(X) = H_G^*(X, \mathbb{Z}_p)$, $\dim X = \dim_p H_G^*(X)$.

Lemma: If $X = U \cup V$, then

$$\dim_p (X) \leq \max\{\dim_p (U), \dim_p H_G^*(V), \dim_p (U \cap V)\}.$$

Proof: Have Mayer-Vietoris sequence

$$H_G^{*-1}(U \cap V) \longrightarrow H_G^*(X) \longrightarrow H_G^*(U) + H_G^*(V)$$

hence the support of $H_G^*(X)$ as a sheaf on $\operatorname{Proj} H^*(G)$ is contained in the union of that for $U, V, U \cap V$.

Corollary: If X is a compactifiable G-manifold, then

$$\dim_p(X) \leq \max_{x\in X} \dim(Gx).$$

Proof: By a suitable induction from the corollary or directly by using the spectral sequence

$$E_2^{pq} = H^p(X/G,\ Gx \longmapsto H_G^q(Gx)) \Longrightarrow H_G^{p+q}(X).$$

Proposition:

$$\dim_p(G) = \max_A \{\dim_p(A)\}$$

where p runs over all abelian subgroups of G.

Proof: Let F be the flag manifold of a faithful representation of G. Then

$$H_G^*(pt) \hookrightarrow H_G^*(F)$$

so

$$\dim_p G \leq \dim_p(F) \leq \max_{x\in F} \dim_p H_G^*(G/G_x)$$

$$= \max_{x\in F} \dim_p H_{G_x}^*(pt) = \max_A \dim_p(A)$$

where A runs over the abelian subgroups of G.
Here we have used the fact that the dimension of $H_H^*(pt)$ as a ring and as a $H_G^*(pt)$ module are the same which results from

Lemma: If $A \to B$ finite (with A local noetherian), then $\dim B$ as an A-module is same as $\dim B$ as a ring.

Proof: $\dim_{A\,mod} B = \deg(n \mapsto B/m_A^n B) = \deg(n \mapsto B/m_B^n)$

as $m_A B$ and m_B define the same topology.

The other direction comes from

$$H_A^*(pt) = H_G^*(pt, \mathrm{ind}_{A\to G} \mathbb{Z}_p)$$

which is a finite $H_G^{ev}(pt, \mathbb{Z}_p)$ module so

$$\dim_p(A) \leq \dim_p(G)$$

if A is a subgroup of G.

this only works if $[G:A] < \infty$ and in general one should use just that $H_A^*(pt) = H_G^*(G/A)$

If A is a compact abelian Lie group, then

$$A \simeq T_k \times B$$

where T_k is a torus of dimension k and B is a finite abelian group. Then ~~$H_A^*(pt)$ is a finite $H_B^*(pt)$ module~~

$$H_A^*(pt) \cong H_B^*(pt) \otimes \mathbb{Z}_p[x_1, \ldots, x_k] \qquad \text{degree } x_i = 2$$

$$\dim_p(\mathbb{Z}/q^n\mathbb{Z}) = 0 \qquad q \text{ prime} \neq p$$

$$\dim_p(\mathbb{Z}/p^n\mathbb{Z}) = 1$$

Since

$$H^*_{\mathbb{Z}/p^n\mathbb{Z}}(pt, \mathbb{Z}_p) = \mathbb{Z}_p[w, \eta] \qquad \begin{array}{l}\dim w = 2\\ \dim \eta = 1\end{array} \qquad (p \text{ odd})$$

Therefore if

(checked OKAY also for $p=2$)

B' finite order prime to p

$$A = T_k \times B' \times \prod_{i=1}^{q} \mathbb{Z}/p^{n_i}\mathbb{Z} \qquad 0 < n_1 \leq n_2 \leq \cdots \leq n_q$$

then

$$H_A^*(pt) = \mathbb{Z}_p[x_1, \ldots, x_k, w_1, \eta_1, \ldots, w_q, \eta_q]$$

where $\dim \eta_i = 1$ and $\dim w_i = \dim x_i = 2$. Thus

$$\dim A = k + q - 1 = \operatorname{rank}_{\mathbb{Z}_p}({}_pA) - 1$$

so this shows we have wrong definition of dimension by one, so changing it we find the following result conjectured by Atiyah.

<u>Proposition:</u> $\dim_p(G)$ = maximal rank of an elementary p-abelian subgroup of G.

if p is a prime and

$\dim_Q(G)$ = maximal rank of a toral subgroup of G.

<u>Remarks:</u> For $H_G^*(\ , \mathbb{Z}_p)$ it is possible to descend from a situation with elementary p-abelian isotropy groups in virtue of the following

<u>Proposition:</u> Let E be a complex vector bundle of dimension n over X and let P be the associated principle U_n bundle. Let $(\mathbb{Z}_p)^n \subset T_n$ be the kernel of multiplication by p and let $f: Y \longrightarrow X$ be the fibre bundle with $Y = P/(\mathbb{Z}_p)^n$. Then $f_*: H(Y) \longrightarrow H(X)$ is surjective.

<u>Proof</u>: As the Gysin for the flag bundle map is surjective, we reduce to proving g_* surjective where $g: P/(\mathbb{Z}_p)^n \longrightarrow P/T_n$. By an evident induction we reduce to proving that if P is a principal S^1 bundle, then h_* is surjective where $h: P/\mathbb{Z}_p \longrightarrow P/S^1$. If L is the complex line bundle over $X = P/S^1$ associated to p, then $P/\mathbb{Z}_p$ is the sphere bundle of $L^{\otimes p}$ and so we have the Gysin sequence

$$\xrightarrow{0} H^q(X) \xrightarrow{h^*} H^q(P/\mathbb{Z}_p) \xrightarrow{h_*} H^{q-1}(X) \xrightarrow{0}$$

where $e(L^{\otimes p}) = p\,e(L) = 0$. This proves h_* is surjective. In fact taking $P = ES^1$ we have

$$\begin{array}{ccccccc} H^1(BS^1) & \xrightarrow{h^*} & H^1(B\mathbb{Z}_p) & \xrightarrow{h_*} & H^0(BS^1) & \longrightarrow & 0 \\ \| & & \| & & \| & & \\ 0 & & \mathbb{Z}_p\beta & & \mathbb{Z}_p & & \end{array}$$

showing that $h_*\beta = 1$ if β is chosen correctly.

<u>Examples</u> 1) $G = O(n)$. Then the p-rank of G is $[\frac{n}{2}]$ for p odd or 0 and n for $p=2$ and this agrees well with the known formulae

$$H^*(BO(n), \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \dots, w_n]$$

$$H^*(BO(n), \mathbb{Z}_p) = \mathbb{Z}_p[p_1, \dots, p_{[\frac{n}{2}]}]$$

2) $G = SO(n)$. Then the p-rank is $[\frac{n}{2}]$ for p odd or 0 and $n-1$ for $p=2$ which agrees with the formulas

$H^*(BSO(n), \mathbb{Z}_2) = \mathbb{Z}_2[w_2, \dots, w_n]$ ~~$H^*(BSO(n$~~

$$H^*(BSO(n), \mathbb{Z}_p) = \mathbb{Z}_p[p_1, \ldots, p_{[\frac{n}{2}]}, \chi] \Big/ \begin{cases} \chi = 0 & \text{if } n \text{ odd} \\ \chi^2 = p_{[\frac{n}{2}]} & \text{if } n \text{ even.} \end{cases}$$

for p odd or zero

G. Segal's description of the Smith theorem: If $\mathbb{Z}_p$ acts on a mod p homology sphere X with fixed space F, then F is a mod p homology sphere.

Proof: Take

$$\longrightarrow H^*_{\mathbb{Z}_p}(X,F) \longrightarrow \tilde{H}^*_{\mathbb{Z}_p}(X) \longrightarrow \tilde{H}^*_{\mathbb{Z}_p}(F) \longrightarrow \cdots$$

and localize with respect to $w = c_1(\eta)$. Now $H^*_{\mathbb{Z}_p}(X,F)$ is a module over $H^*_{\mathbb{Z}_p}(X-F)$ which vanishes after a while as $\mathbb{Z}_p$ acts freely on $X-F$. Thus we get

$$\tilde{H}^*_{\mathbb{Z}_p}(X)[w^{-1}] \xrightarrow{\sim} \tilde{H}^*_{\mathbb{Z}_p}(F)[w^{-1}]$$

Now the spectral sequence

$$E_2^{pq} = H^p(B\mathbb{Z}_p, \tilde{H}^q(X)) \Longrightarrow \tilde{H}^{p+q}_{\mathbb{Z}_p}(X)$$

degenerates ~~vanishes~~ as we are assuming that X is a mod p homology sphere, and it shows that $\tilde{H}^*_{\mathbb{Z}_p}(X)$ is a free $H^*(B\mathbb{Z}_p)$ module with one generator. Hence

$$\tilde{H}^*_{\mathbb{Z}_p}(F)[w^{-1}] = \tilde{H}^*(F) \otimes_{\mathbb{Z}_p} H^*(B\mathbb{Z}_p)[w^{-1}]$$

is a free $H^*(B\mathbb{Z}_p)[w^{-1}]$ module with one generator so F must be a mod p homology sphere. q.e.d.

The same theorem holds for $\mathbb{Z}_{p^n}$ by induction on n and for S^1 actions on a rational homology sphere.

Remarks of Atiyah on the Completion theorem $K_G(X)^\wedge \xrightarrow{\sim} K(X_G)$.

Suppose G is a finite group. Recall that $R(G)^\wedge = \varprojlim_n R(G)/\bar{R}(G)^n$ where $\bar{R}(G)$ is the augmentation ideal. Now if $d = |G|$, then d kills $\bar{R}(G)^n/\bar{R}(G)^{n+1}$ for $n>0$. In effect if $i: 1 \longrightarrow G$, then if $x \in \bar{R}(G)$

$$dx \equiv \iota_* 1 \cdot x \qquad \text{mod } \bar{R}(G)^2$$
$$= \iota_*(\iota^* x) = 0.$$

This implies that $\bar{R}(G)^\wedge$ is a direct sum for each $p|d$ of a finite type module over the p-adic numbers $\hat{\mathbb{Z}}_p$. In fact

Theorem: The p-primary component of $\bar{R}(G)^\wedge$ is a free $\hat{\mathbb{Z}}_p$ module whose rank is the number of conjugacy classes of non-trivial p-singular elements in G.

First suppose G is a p-group. Then we claim that the $\bar{R}(G)$-adic and the p-adic filtration define the same topology. We already know that if $|G| = p^a$ then $p^a \bar{R}(G) \subset \bar{R}(G)^2$ so $(p^a)^n \bar{R}(G) \subset (p^a)^{n-1} \bar{R}(G)^2 \subset \bar{R}(G)^{n+1}$. For the other direction consider the embedding

$$R(G) \hookrightarrow \mathbb{Z}[\zeta]^{\mathcal{S}} \qquad \zeta = e^{2\pi i \frac{1}{p^a}}$$

$\mathcal{S}$ = set of conjugacy classes

given by associating to a virtual representation its values on the conjugacy classes of G. Note that

$$\overline{R(G)} \hookrightarrow \overline{\mathbb{Z}[\zeta]}^{\,g}$$

since if V is a representation of G of dimension n, then $\mathrm{tr}_V\, g = \sum \lambda_i$ where $\lambda_i = \zeta^{a_i}$ and this goes into 1 in the augmentation ideal of $\mathbb{Z}[\zeta]$. (Recall that $\mathbb{Z}[\zeta]/\overline{\mathbb{Z}[\zeta]} \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$). Next ~~by~~ Artin-Rees we have that $\exists\, N$ with

$$p^n \bar{R}(G) \supset \bar{R}(G) \cap p^{n+N}\, \overline{\mathbb{Z}[\zeta]}^{\,g}$$

But the $\overline{\mathbb{Z}[\zeta]}$-adic and the p-adic topology on $\mathbb{Z}[\zeta]$ coincide so

$$p\bar{R}(G) \supset \bar{R}(G) \cap p^{1+N}\, \overline{\mathbb{Z}[\zeta]}^{\,S}$$

$$\supset \bar{R}(G) \cap \left(\overline{\mathbb{Z}[\zeta]}^{\,g}\right)^{S} \qquad \text{some } g$$

$$\supset \bar{R}(G)^{g}$$

proving the other inclusion of topologies. Thus from the claim we have that

$$\bar{R}(G)^{\wedge} \simeq \bar{R}(G) \otimes \hat{\mathbb{Z}}_p$$

is a free $\hat{\mathbb{Z}}_p$ module of rank $=$ no. of non-trivial conjugacy classes in G.

~~Secondly let H be a Sylow p subgroup of G.~~

Suppose now that G ~~is~~ an arbitrary finite group and let H be a p-Sylow subgroup of G. By essentially the argument on page 9 one sees that

$$i^* : \varprojlim_n \bar{R}(G)/\bar{R}(G)^n_{(p)} \longrightarrow \varprojlim_m \bar{R}(H)/\bar{R}(H)^m_{(p)}$$

is injective. In other words ~~the~~

$$\bar{R}(G)^{\wedge}_{(p)} \hookrightarrow \bar{R}(H)^{\wedge}$$

which shows the former is a free $\hat{\mathbb{Z}}_p$ modules. For general nonsense reasons the image is invariant under the action of the normalizer N of H and this shows that rank $\bar{R}(G)^{\wedge}_{(p)} \leq$ number of non-trivial conjugacy classes of p-singular elements of G, by virtue of the well-known

(incorrect because not true that if x, y are conj. in G, then the P conj classes of x, y are conj)

Lemma: Let S_1, S_2 be subsets of the Sylow subgroup H of G which are invariant under the conjugation action of H, and suppose S_1 and S_2 are conjugate in G. Then they are already conjugate in the normalizer N of H in G.

Proof: Let $xS_1x^{-1} = S_2$ and let $G' =$ normalizer of S_2 in G so that $H \subset G'$ and also $xHx^{-1} \subset G'$. Then H and xHx^{-1} being two Sylow subgroups of G', there is a $g \in G'$ with $gxHx^{-1}g^{-1} = H$ or $gx \in N$, and now $gxS_1x^{-1}g^{-1} = S_2$.

~~Thus it appears that Atiyah's theorem asserts that~~

$$\cancel{\bar{R}(G)^{\wedge}_{(p)} \xrightarrow{\sim} \left(\bar{R}(H)^{\wedge}\right)^{N}}$$

also incorrect

~~In effect we know the image is a direct summand and that~~

$$\cancel{\operatorname{Hom}_{\mathbb{Z}_p}\left(\bar{R}(H)^{\wedge}, \mathbb{Q}_p[t]\right) \to \operatorname{Hom}_{\mathbb{Z}_p}\left(\bar{R}(H), \mathbb{Q}_p[t]\right)}$$

October 17, 1969: Complex orientations of a map

In the following we will work with the category of C^∞ G-manifolds where G is a compact Lie group. We consider only G-manifolds which may be embedded equivariantly as a closed submanifold of some representation of G (this implies the set of orbit types of the manifold is finite and probably the converse is true). From now on everything will be equivariant ~~without~~ unless ~~there is no mention~~ stated otherwise.

~~Let E be a complex bundle over Y with projection $p: E \to Y$. We consider the set of pairs (i, J) where $i: X \to E$ is a closed embedding such that $pi = f$ and where J is a complex structure on ν_i. An isotopy of such pairs is a family (i_t, J_t) $t \in R$ such that~~

Let $f: X \longrightarrow Y$ be a map of manifolds and let $p: E \to Y$ be a complex vector bundle over Y. We consider pairs (i, J) where $i: X \longrightarrow E$ is an embedding with $pi = f$ and where J is a complex structure on the normal bundle ν_i of i. Two such pairs are called <u>isotopic</u> if they are the restriction for two values $a, b \in R$ of a pair for the product map $f \times id: X \times R \longrightarrow Y \times R$ and the product bundle $p \times id: E \times R \longrightarrow Y \times R$. Let $\mathcal{O}(f, E)$ be the isotopy classes of such pairs.

~~If $E \to F$ is an injective map of complex vector bundles over Y, then there is an ob-~~ An injective map $u: E \to F$ of complex vector bundles over Y induces a map $u_*: \mathcal{O}(E) \to \mathcal{O}(F)$ and if u and u' are isotopic injections, i.e. homotopic through injections, then $u_* = u'_*$. We define the set of complex orientations of f to be

$$\mathcal{O}(f) = \varinjlim_{E} \mathcal{O}(f, E)$$

where the limit is taken over the filtering category $I(Y)$ whose objects are the complex vector bundles over Y and ~~with maps~~ whose morphisms are the isotopy classes of injections. ~~Product bundles $Y \times V$ where V is a complex bundle over a point (i.e. a complex representation of G) are cofinal~~ Observe that if $g : Y \longrightarrow Z$ is a map then the functor $F \longmapsto g^*F$

$$g^* : I(Z) \longrightarrow I(Y)$$

is cofinal, so that we may take the inductive limit over $I(Z)$. In particular if $g : Z \longrightarrow pt$, then since a vector bundle over a point is just a complex representation of G and since any two injections $V \rightrightarrows W$ of complex representations are isotopic, the category $I(pt)$ is equivalent to the ordered set of isomorphism classes of complex representations of G.

(Lemma: G compact Lie group, V, W two complex representations of G. Then any two injections $V \rightrightarrows W$ are homotopic.

Proof: $\quad Inj(V, W) = \prod_{s \in \hat{G}} Inj(V_s, W_s)$

where sub-s denotes part of V purely associated to s. As

$$V_s = Im \left\{ Hom_G(E_s, V) \otimes_{\mathbb{C}} E_s \longrightarrow V \right\}$$

~~so~~ we have isomorphisms $V_s = \mathbb{C}^r \otimes E_s$, $W_s = \mathbb{C}^s \otimes E_s$ and so

$$\mathrm{Inj}(V_s, W_s) \cong \mathrm{Inj}(\mathbb{C}^r, \mathbb{C}^s) = \frac{Gl(s,\mathbb{C})}{\left\{\begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}\right\}}$$

which is connected as $Gl(s,\mathbb{C})$ is connected.)

As an example take $G = 1$ whence

$$\mathcal{O}(f) = \varinjlim_n \mathcal{O}(f, Y \times \mathbb{C}^n)$$

One knows ~~that~~ for $n \gg \dim X$ (2 index $f + 1$?) that there is one isotopy class of embedding of X into $Y \times \mathbb{C}^n$, and that the normal bundles of two such embeddings are ~~canonically~~ isomorphic unique up to homotopy type. Thus $\mathcal{O}(f, Y \times \mathbb{C}^n)$ is isomorphic to the set of ~~isomorphism~~ isotopy classes of complex structures on ν_i for a given i.

Operations on complex orientations:

<u>Pullback</u>: Suppose that $g: Y' \to Y$ is transversal to $f: X \to Y$ so that ~~there is~~ we can form the fibre product

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}.$$

Then there is an induced map

$$g^* : \mathcal{O}(f, E) \longrightarrow \mathcal{O}(f', g^*E)$$

coming from the fact that the square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle i'} & & \big\downarrow{\scriptstyle i} \\ g^*E & \xrightarrow{g''} & E \end{array}$$

is transversal and so there is a canonical isomorphism

$$\nu_{i'} = (g')^* \nu_i$$

<u>Composition:</u> Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps and let E, F be complex bundles on Z. Then there is a map

$$(*) \qquad \mathcal{S}(f, g^*E) \times \mathcal{S}(g, F) \longrightarrow \mathcal{S}(gf, F \oplus E)$$

defined as follows. Given $i' : Y \hookrightarrow F$ over g with complex structure J' on $\nu_{i'}$ and $i : X \hookrightarrow g^*E = Y \times_Z E$ over f with complex structure J on ν_i, consider the embedding i'' which is the composition

$$X \xrightarrow{i} Y \times_Z E \xrightarrow{i' \times_Z id} F \times_Z E = F \oplus E$$

Then $\nu_{i''}$ fits into an exact sequence

$$0 \longrightarrow \nu_i \longrightarrow \nu_{i''} \longrightarrow p^*\nu_{i'} \longrightarrow 0$$

where $p = pr_1 : F \times_Z E \longrightarrow F$, and hence up to isotopy has a unique complex structure J'' such that the above is an exact sequence of complex vector bundles. The map (*) is defined by sending (i, J) and (i', J') to (i'', J'').

started Oct. 15
<u>October 18, 1969:</u>

Let p be a fixed prime. Recall that the (p-) dimension of G is defined to be the dimension of the $\mathbb{Z}_p$-algebra $H^{ev}(G, \mathbb{Z}_p)$ which is of finite type. This dimension is the maximal rank of an elementary abelian p-subgroup by Oct. 15

Observe that if $A \subset G$ is an elem. ab. p-subgp of rank r, then there is a homomorphism

$$H^{ev}_G(pt) \longrightarrow H^{ev}_A(pt) \longrightarrow H^{ev}_A(pt)/\text{nilpotents} = \mathbb{Z}_p[w_1, \dots, w_r]$$

(this is for p odd; for $p=2$ it's easier), which is finite and hence the kernel $\mathfrak{p}_A$ is a prime ideal in $H^{ev}_G(pt)$ which is of dimension r, i.e. defines ~~an irreducible subvariety of Spec~~ a point of Spec $H^{ev}_G(pt)$ of dimension r. Thus we get a map

$$\left\{\begin{matrix}\text{Conjugacy classes of} \\ \text{elem. ab. } p \text{ subgrps} \\ \text{of rank } r \text{ in } G\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{prime ideals} \\ \text{in Spec } H^{ev}_G(pt) \text{ of} \\ \text{dimension } r\end{matrix}\right\}$$

which we want to understand.

Note that $H^{ev}_G(pt)/\mathfrak{p}_A \hookrightarrow \mathbb{Z}_p[w_1, \dots, w_r]$ so the variety defined over $\mathbb{Z}_p$ by $\mathfrak{p}_A$ is of dimension r and rational which tends to suggest very strongly that it might be ~~the an orbit~~ the closure of an orbit of the action of the Steenrod algebra.

unirational i.e. image of a rational variety

Let $R = H^{ev}_G(pt)$ and let $\mathfrak{p}$ be a <u>minimal</u> prime ideal of R so that $R_\mathfrak{p}$ is an Artin ring. If X is a nice G-space set

$$\chi_\mathfrak{p}(X) = \text{length}\left\{H^{ev}_G(X)_\mathfrak{p}\right\} - \text{length}\left\{H^{odd}_G(X)_\mathfrak{p}\right\}$$

Then clearly we have

Lemma 1: If $X = U \cup V$ where U and V are nice G-invariant open sets, then

$$\chi_p(X) = \chi_p(U) + \chi_p(V) - \chi_p(U \cap V).$$

(More generally we can replace p ~~that~~ by an arbitrary prime ideal in R and consider only spaces X for which $H_G(X)_p$ is a R_p module of finite length. Then $\chi_p(X)$ is defined and there is a generalization of the lemma.)

Lemma 2: If p is minimal, if $B \supseteq A$ are elementary sub-p-groups of G, ~~then~~ and if $H^*(A)_p \neq 0$, then $A = B$.

~~Proof: ... $H^*(B)_p$ ... If $B < A$ ... p_B for dimension reasons, and so $H^*(A)_p \neq 0$. we have~~

Proof: If $H^*(A)_p \neq 0$, then $(R/p_A)_p \neq 0$ and so $p_A \subset p$. By minimality $p = p_A$. If $B \supsetneq A$, then $p_A > p_B$ by dimension reasons, so p not minimal. qed.

~~A corollary of this lemma is that if X is a space with only p elementary abelian isotropy groups, then~~

Proposition: Let X be a compactifiable G-manifold with elementary p-abelian isotropy groups. Let $A_1, \ldots, A_k$ be representatives for the elementary p-abelian subgroups A of G with $p_A = p$ under conjugation. Then

$$H^*_G(X)_p = \prod_{i=1}^{k} H^*_G(G\times_{N_i} X^{A_i})_p$$

where N_i is the normalizer of A_i.

Proof: We consider the stratification of X into its orbit types which are finitely many in number. Note that if A is elem p-abelian with $p_A = p$, then by lemma 2 A is a maximal elem. p-ab. subgp. of G, and therefore every point of X^A has isotropy group A. This implies that the strata ~~those~~ with orbit type G/A is

$$G\times_N X^A$$

where N is the normalizer of A in G. Let U be ~~the~~ an open invariant subset of X not containing any of the orbits types G/A_i. Then by Mayer-Vietoris and induction one sees that $H_G(U)_p = 0$ and hence again by Mayer-Vietoris that

$$H_G(X)_p = H_G\Big(\coprod_i G\times_{N_i} X^{A_i}\Big)_p$$

proving the proposition.

Corollary: If p is a minimal prime of $H^{ev}_G(pt)$, then there exists as maximal elementary p-abelian subgroup A of G with $p = p_A$.

Proof: If not then ~~the~~ for the p-flag manifold F of a faithful representation of G we have $H_G(F)_p = 0$ hence $H_G(pt)_p = 0$ which is nonsense.

Some calculations:

Lemma: Let $R \longrightarrow S$ be a local homomorphism of local noetherian rings with finite residue fields extension $k_R \longrightarrow k_S$. Then a finite length S-module M is also of finite length as an R-module and

$$\mathrm{length}_R(M) = [k_S : k_R]\,\mathrm{length}_S(M).$$

In effect if M is a simple S-module, $M \simeq k_S$ and this has length $[k_S : k_R]$. Both sides are additive functions of M.

~~We apply this where $R \to S$ is the localization of the map $H^{ev}_G(pt) \longrightarrow H^{ev}_A(pt)$ with respect to $\mathfrak{p}_A$, and where $M = H^{odd}_A(pt)$ and we find that $\chi_{\mathfrak{p}_A}(G/A) =$~~

Let's change notation slightly and put

$$\ell^{+}_{G,\mathfrak{p}}(X) = \mathrm{length}_{H^{ev}_G(pt)_{\mathfrak{p}}}\left(H^{ev}_G(X)_{\mathfrak{p}}\right)$$

$$\ell^{-}_{G,\mathfrak{p}}(X) = \mathrm{length}_{H^{ev}_G(pt)_{\mathfrak{p}}}\left(H^{odd}_G(X)_{\mathfrak{p}}\right)$$

provided these are defined. Let

$$\chi_{G,\mathfrak{p}}(X) = \ell^{+}_{G,\mathfrak{p}}(X) - \ell^{-}_{G,\mathfrak{p}}(X).$$

Let $A \longrightarrow G$ be homomorphism of compact groups, let $\mathfrak{q}$ be a prime in $H^{ev}_A(pt)$ and let $\mathfrak{p}$ be the inverse image in $H^{ev}_G(pt)$.

As $H^{ev}_G(pt) \longrightarrow H^{ev}_A(pt)$ is ~~a~~ finite ~~[illegible]~~ if $A \subset G$ it follows that

$$(*) \qquad H^{ev}_G(pt)_{\mathfrak{p}} \longrightarrow H^{ev}_A(pt)_{\mathfrak{q}}$$

can be taken as the map $R \to S$ in the lemma. Thus if X is an A-space for which ~~[illegible]~~ $l^{\pm}_{A,\mathfrak{q}}(X)$ is defined we have

$$l^{\pm}_{G,\mathfrak{p}}(G \times_A X) = r \cdot l^{\pm}_{A,\mathfrak{q}}(X)$$

where r is the degree of the residue field extension of $(*)$.

Apply this where A is an elementary p abelian subgroup of G and where $\mathfrak{q}$ is the ideal of nilpotent elements, whence $\mathfrak{p} = \mathfrak{p}_A$. We find that

$$\chi_{G,\mathfrak{p}_A}(G/A) = r \cdot \chi_{A,\mathfrak{q}}(pt).$$

~~On the other hand if we embed A as the points of order p of a torus (T), then taking G = T, we have p = the 0 ideal, and hence so~~

To calculate the latter suppose rank $A = k$ and embed A inside a ~~[illegible]~~ torus T of rank k and apply the lemma to the map

$$H^{ev}_T(pt)_0 \longrightarrow H^{ev}_A(pt)_{\mathfrak{q}}.$$

Letting s be the residue extension degree we find that

$$\text{length}_{H^{ev}_T(pt)_0}\left(H^{\pm}_A(pt)_{\mathfrak{q}}\right) = s \cdot \text{length}_{H^{ev}_A(pt)_0}\left(H^{\pm}_A(pt)_{\mathfrak{q}}\right)$$

Case 1: p odd. Then $H^*_A(pt) \simeq \Lambda[\eta_1, \dots, \eta_r] \otimes \mathbb{Z}_p[\omega_1, \dots, \omega_r]$
so

$$H^{ev}_T(pt) \xrightarrow{\sim} H^{ev}_A(pt)/\mathfrak{q}$$

so $s = 1$. But $H^{\pm}_A(pt)$ is a free $H^{ev}_T(pt)$ module of rank 2^{r-1} so

$$\text{length}_{H^{ev}_T(pt)_0}\left(H^{\pm}_A(pt)_{\mathfrak{q}}\right) = \text{length}_{H^{ev}_A(pt)/\mathfrak{q}}\left(H^{\pm}_A(pt)_{\mathfrak{q}}\right) = 2^{r-1}$$

so

$$\boxed{\chi_{A,\mathfrak{q}}(pt) = 0.}$$

Case 2: $p = 2$. Then $H^*_A(pt) = \mathbb{Z}_2[\eta_1, \dots, \eta_r]$ and the image of $H^*_T(pt)$ is $\mathbb{Z}_2[\eta_1^2, \dots, \eta_r^2]$. ~~It follows that the residue field extension (for $\mathfrak{p} \neq \mathfrak{q}$) is~~ Here $\mathfrak{q} = \mathfrak{p} = 0$ and localizing $H^{ev}_A(pt)$ with respect to $\mathfrak{p}$ is the same as localizing with respect to $\mathfrak{q}$ since $H^{ev}_A(pt)_{\mathfrak{p}}$ would be a finite extension of $\mathbb{Z}_2(\eta_1, \dots, \eta_r)$ and hence a field. Thus

$$\left[H^{ev}_A(pt)_{\mathfrak{q}}, H^{ev}_T(pt)_{\mathfrak{p}}\right] = 2^{r-1} = s$$

and

$$\text{length}_{H^{ev}_A(pt)_{\mathfrak{q}}}\left(H^{\pm}_A(pt)_{\mathfrak{q}}\right) = 1,$$

so again

$$\boxed{\chi_{A,\mathfrak{q}}(pt) = 0.}$$

<u>Corollary:</u> Suppose X is a compactifiable G-manifold such that all isotropy groups are elementary p-abelian. If $\mathfrak{p}$ is a minimal prime of $H_G^{ev}(pt)$, then

$$\chi_{G,\mathfrak{p}}(X) = 0.$$

<u>Conclusion:</u> This Euler characteristic isn't very interesting.

Here's a modification if $p=2$. Let $\mathfrak{p}$ be a minimal prime of $H_G^*(pt)$ and set for a compatifiable G-manifold X

$$l_{G,\mathfrak{p}}(X) = \mathrm{length}_{H_G(pt)_\mathfrak{p}}\left(H_G(X)_\mathfrak{p}\right)$$

and observe that modulo 2 this is an additive function of X e.g. given $Y \subset X$ we have an exact sequence of $H_G^*(pt)$-module

$$\begin{tikzcd}
H_G^*(X,Y) \arrow[r] & H_G^*(X) \arrow[d] \\
 & H_G^*(Y) \arrow[ul, dashed]
\end{tikzcd}$$

whence

$$l_{G,\mathfrak{p}}(X) \equiv l_{G,\mathfrak{p}}(X,Y) + l_{G,\mathfrak{p}}(Y) \qquad (\mathrm{mod}\ 2)$$

in virtue of

<u>Lemma:</u> Let

$$\begin{tikzcd}
M' \arrow[r] & M \arrow[dl] \\
M'' \arrow[u] &
\end{tikzcd}$$

be an exact triangle of objects of finite length in an abelian category

Then $$\ell(M') + \ell(M'') = \ell(M) \pmod 2.$$

Proof. Let K^a, C^a denote ~~kernels & cokernels~~ image and coimage at M^a so that

$$\ell(M') + \ell(M) + \ell(M'') = \ell(K') + \ell(C') + \ell(K) + \ell(C) + \ell(K'') + \ell(C'')$$
$$= 2\big(\ell(K') + \ell(K) + \ell(K'')\big).$$

~~For p odd there is something similar, namely if we restrict to spaces X with finite abelian groups for isotropy groups, then we know that~~ $\cancel{\chi_p(X) = 0}$ ~~and hence~~

$$\cancel{\ell^+_{G,p}(X) = \ell^-_{G,p}(X)}$$

~~will behave additively modulo 2.~~

~~Let G be a compact group and let p be a minimal prime in~~ $\cancel{H^{ev}_G(pt)}$.

$I_G(X)$ ordered set of G torus

page 16 $\pi_0(X^{(H)})_N \xrightarrow{\sim} \pi_0(\overline{X^{(H)}})_N$

23 when is $H_H(X) \cong H_G(X) \otimes_{H_G} H_G(G/H)$

29 localization thm.

October 20, 1969.

Let G be a compact Lie group and let p be a fixed prime number. I want to determine the spectrum of $H_G^{ev}(pt)$, the equivariant cohomology with coefficients $\mathbb{Z}_p$.

Let A be an elementary abelian p-group of rank r. Then for p odd

$$H_A^*(pt) = \Lambda(A^*) \otimes S(A^*)$$

where A^* is considered as a vector space over $\mathbb{Z}_p$ and Λ (resp. S) denotes exterior algebra (resp. symmetric algebra). Thus

$$H_A^{ev}(pt) = \Lambda^{ev}(A^*) \otimes S(A^*)$$

and as ~~A~~ $\Lambda^{ev}(A^*)$ is an Artin ring it follows that

$$H_A^{ev}(pt)/\text{nilideal} \simeq S(A^*).$$

For $p=2$ we have

$$H_A^*(pt) = S(A^*)$$

so in either case we have that

$$\operatorname{Spec}\{H_A^{ev}(pt)\} \cong \text{the affine space over } \operatorname{Spec} \mathbb{Z}_p \text{ associated to the vector space } A \text{ over } \mathbb{Z}_p.$$

We now consider the action of the Steenrod algebra $\mathcal{A}$

on ~~[illegible]~~ $H^{ev}_A(pt)$. Note that if A is embedded in a torus of the same rank, then there is an isomorphism

$$H^*_T(pt) \xrightarrow{\cong} H^*_A(pt)/\text{nilideal}.$$

The two-sided ideal generated by the Bockstein β acts trivially on the left, since BT is torsion-free. It's not true that the $\mathcal{a}$ action on $H^*_A(pt)$ passes to the quotient by the nil-ideal, but the above facts are justification for feeling that only the action of the algebra of reduced powers matters in questions about prime ideals.

So from now on we let $\mathcal{a}$ denote the algebra of reduced powers. Recall that it has a basis of $\mathbb{Z}_p$ given by P_α where

$$P_{\underline{t}}(x) = \sum t^\alpha P_\alpha(x) \qquad \underline{t} = t_1, t_2, \dots$$

and where $P_{\underline{t}}$ is the multiplicative operation given on elements in H^2 by

$$P_{\underline{t}}(x) = \sum_{i\geq 0} t_i x^{p^i} \qquad t_0 = 0.$$

It is clear that if $B \subset A$ then the prime ideal $\mathfrak{p}_B$ which is the kernel of

$$H^{ev}_A(pt) \longrightarrow H^{ev}_B(pt)/\mathfrak{n}_B$$

$\mathfrak{n}_B$ = nil ideal, is stable under the action of $\mathcal{a}$ ie.

$$P_{\underline{t}}(\mathfrak{p}_B) \subset \mathfrak{p}_B[[\underline{t}]]$$

and I would now like to prove the converse. It amounts to the following:

<u>Proposition</u>: Let A be a vector space over $\mathbb{Z}_p$ and let $\mathfrak{p}$ be a (homogeneous) prime ideal of $S(A^*)$ stable under the action of $\mathcal{A}$. Then $\mathfrak{p} = \mathrm{Ker}\{S(A^*) \longrightarrow S(B^*)\}$ for some subspace B of A.

<u>Proof</u>: Choose a basis for A, ~~and~~ regard $\mathrm{Spec}(A^*)$ as affine r space over $\mathrm{Spec}\,\mathbb{Z}_p$ and regard $\mathfrak{p}$ as giving an irreducible subscheme $Z \subset \mathrm{Spec}\,A^*$. Let k be an algebraically closed field over $\mathrm{Spec}\,\mathbb{Z}_p$ and let $x = (x_1, \dots, x_r) \in k^r$ be a geometric point of Z, i.e. $f(x_1, \dots, x_r) = 0$ for all $f \in \mathfrak{p}$. Since $\mathfrak{p}$ is stable under $\mathcal{A}$, it follows that if $t_1, \dots, t_n$ are indeterminates, then

$$f(X + t_1 X^{(p)} + \dots + t_n X^{(p^n)}) \subset \mathfrak{p}[[t_1, \dots, t_n]] \cap S(A^*)[\underline{t}]$$

$$= \mathfrak{p}[\underline{t}]$$

where $X^{(p)} = (X_1^p, \dots, X_r^p)$, and consequently

$$f(x + t_1 x^{(p)} + \dots + t_n x^{(p^n)}) = 0.$$

Therefore we see that if $x \in Z_k$, then the subspace spanned by $x, x^{(p)}, x^{(p^2)}, \dots\dots$ also is contained in Z_k. It is necessary now to determine the rank of this subspace.

Let $V = \{(a_1, \dots, a_r) \in k^r \mid \sum a_i x_i^{p^q} = 0 \text{ for all } q \geq 0\}$. Then

$$\mathrm{rank}\,\{x_i^{p^q}\}_{1 \leq i \leq r,\ q \geq 0} = r - \dim_k V.$$

~~Recall V is generated by its minimal elements i.e. $(a_1, \dots, a_r) \in V$ with support $= \{ i \mid a_i \neq 0 \}$ minimal and with at least one $a_i = 1$. As k is perfect each a_i has a unique p^{th} root, so~~

$$\cancel{\sum_{i=1}^{n} a_i^{1/p} x_i^{p^{g-1}} = 0 \quad \text{all } g \geq 1}$$

~~hence by minimality $a_i^{1/p} = a_i$ for all i. Thus V is generated by its elements with all coefficients in $\mathbb{Z}_p$ and this means that V is pretty small.~~

Now by descent it is known that a ~~[illegible]~~ subspace V of k^r stable under $x \longmapsto x^{(p)}$ is defined over $\mathbb{Z}_p$. Therefore we see that $x \in Z_k \Longrightarrow Z_k \supset$ smallest subspace of k^r defined over $\mathbb{Z}_p$ and containing x. So ~~[illegible]~~ take x to be a generic point of Z_k. ~~[illegible]~~ Then Z is the smallest subvariety over $\mathbb{Z}_p$ containing x and hence is contained in the smallest linear subspace defined over $\mathbb{Z}_p$ containing x, showing these two are the same. q.e.d.

<u>Conclusion:</u> Let A be an elementary p-abelian group. ~~[illegible]~~ If $\mathfrak{p}$ is a homogeneous prime ideal of $H_A^{ev}(pt)$ which is stable under the action of the algebra of reduced Steenrod powers, then there is a unique subgroup B of A such that $\mathfrak{p}$ is the inverse image of $\mathfrak{r}_B$ under the restriction homomorphism

$$H_A^{ev}(pt) \longrightarrow H_B^{ev}(pt) .$$

So now let G be a compact Lie group. To each elementary p subgroup A of G define the prime ideal $\mathfrak{p}_A$ in $H^{ev}_G(pt)$ as the inverse image of $\mathfrak{r}_A$ under the restriction homomorphism $H^{ev}_G(pt) \longrightarrow H^{ev}_A(pt)$. It is clear that $\mathfrak{p}_A$ is homogeneous and stable under the action of $\mathfrak{a}$ and that $\mathfrak{p}_A$ depends only on the conjugacy class of A in G. We can now state the basic conjectures.

Conjecture: The map $A \longmapsto \mathfrak{p}_A$ establishes an ordered preserving bijection between the set of conjugacy classes of elementary p-subgroups of G and $\mathfrak{a}$-invariant homogeneous prime ideals in $H^{ev}_G(pt)$.

We shall prove the map is surjective. First some lemmas:

Lemma 1: Let X be a nice G-space and let $u \in H^{ev}_G(X)$. Then u is nilpotent iff u restricted to each orbit of X is nilpotent.

Proof: Let t be an indeterminate of degree $= -\deg u$ and consider the operator of multiplying by $1+tu$ on $H^*_G(Y)[t]$ for all G-spaces over Y. Then $1+tu$ is a unit on each fiber and hence for all Y by the spectral sequence, whence u is nilpotent.

Lemma 2: If X is a nice G-space, then the map

$$H^{ev}_G(X) \longrightarrow H^0(X/G,\ Gx \longmapsto H^{ev}_G(Gx)) = E_2^{0,\mathrm{even}}$$

induces a homeomorphism of spectra.

Proof: We know that the kernel consists of nilpotent elements, so it suffices† to prove that if $z \in E_2^{0,ev}$, then for some n, z^{p^n} comes from $x \in H_G^{ev}(X)$. So let $X = U_1 \cup \cdots \cup U_N$ where each U_j is a tubular neighborhood of an orbit, and proceed by induction on j to construct x over $U_1 \cup \cdots \cup U_j$. This reduces us to the case where we are given $x' \in H_G^{ev}(V)$ and $x'' \in H_G^{ev}(U)$ coinciding with z^{p^a} on each orbit in $U \cup V$. Then $x' - x''$ in $H_G^{ev}(U \cap V)$ restricts to zero on each orbit, so is nilpotent by lemma 1. Then $\exists\, b \ni$

$$(x')^{p^b} - (x'')^{p^b} = (x' - x'')^{p^b} = 0$$

in $H_G^{ev}(U \cup V)$ and this by Mayer-Vietoris means $\exists\, x \in H_G(U \cup V)$ restricting to $(x')^{p^b}$ and $(x'')^{p^b}$. Thus x restricts to $z^{p^{a+b}}$ and the induction step is clear.

~~[illegible]~~

† One knows that $E_2^{0,ev}$ is a finite $H_G^{ev}(pt)$-module (see below) hence the map $\operatorname{Spec}\{H_G^{ev}(X)\} \longleftarrow \operatorname{Spec}\{E_2^{0,ev}\}$ is closed by Cohen-Seidenberg. But what we show proves that it is 1-1 onto, hence a homeomorphism.

Interpretation of $H^0(X/G, Gx \longmapsto H_G^{ev}(Gx))$. Consider the category whose objects are ~~maps~~ homotopy classes of maps

$$Y \to X$$

of G spaces where Y is transitive and where a map

Interpretation of $H^0(X/G, Gx \mapsto H^*_G(Gx))$: Let $\mathcal{O} \subset X$ be a G-orbit. Then there is a tubular neighborhood $\mathcal{U}$ of $\mathcal{O}$ which is G-invariant and **so** there is a map

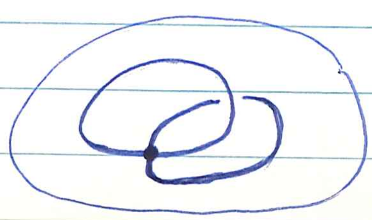
$$H^*_G(\mathcal{O}) \simeq H^*_G(\mathcal{U}) \longrightarrow H^*_G(\mathcal{O}')$$

for any **orbit** $\mathcal{O}'$ sufficiently near $\mathcal{O}$. $H^0(X/G, Gx \mapsto H^*_G(Gx))$ $=$ the set of functions λ with assign to each orbit $\mathcal{O} \in X/G$ an element $\lambda(\mathcal{O}) \in H^*_G(\mathcal{O})$ in such a way as to be compatible with the specialization homomorphism $H^*_G(\mathcal{O}) \longrightarrow H^*_G(\mathcal{O}')$ whenever **$\mathcal{O}'$ is near to $\mathcal{O}$.**

Now consider the stratification of X **into** orbit types, and refine the **stratification** so that all of the strata are connected. ~~The set of strata is finite and is the~~ ~~[illegible] of injective G-maps Y → X~~ ~~[illegible] The set~~ The set of strata is finite. Choose a base point in each of the minimal (i.e. closed) strata and **denote** the finite set by $x_1, \dots, x_n$. Then we have that

$$E_2^{0*} = H^0(X/G, \mathcal{O} \mapsto H^*_G(\mathcal{O})) \hookrightarrow \prod_{i=1}^{n} H^*_G(Gx_i)$$

which proves that E_2^{0*} is a finite $H^{ev}_G(pt)$-module. The image seems a bit difficult to describe in general, since the fundamental groups of the strata come into play



Lemma 3: Let $\mathfrak{a}$ act on a ring R. If $\mathfrak{p}$ is a prime ideal in R, then the largest $\mathfrak{a}$-stable ideal contained in $\mathfrak{p}$ is a prime ideal.

Proof: The largest $\mathfrak{a}$-stable ideal contained in $\mathfrak{p}$ is

$$\mathfrak{q} = \{x \in R \mid P_{\underline{t}}(x) \in \mathfrak{p}[[\underline{t}]]\}.$$

In effect ~~this~~ $\mathfrak{q}$ is an ideal contained in $\mathfrak{p}$ ~~and~~ which clearly contains any $\mathfrak{a}$-stable ideal, and $\mathfrak{q}$ itself is $\mathfrak{a}$-stable because

$$P_{\underline{t}}(P_{\underline{u}}x) = P_{\underline{v}}(x) \in \mathfrak{p}[[\underline{u}]][[\underline{t}]] \quad \text{where}$$

$$\sum v_i X^{p^i} = \sum t_i X^{p^i} \circ \sum u_i X^{p^i}.$$

But $\mathfrak{q}$ is prime since if $xy \in \mathfrak{q}$, then

$$P_{\underline{t}}(xy) = P_{\underline{t}}(x) \cdot P_{\underline{t}}(y) \in \mathfrak{p}[[\underline{t}]]$$

and one of these must be in $\mathfrak{p}[[\underline{t}]]$ as this is a prime ideal.

Corollary of lemmas 2 + 3: The map

$$H_G^{ev}(X) \longrightarrow H^0(X/G, \mathcal{O} \longmapsto H_G^*(\mathcal{O}))$$

induces a ~~[illegible]~~ bijection of the set $\mathfrak{a}$-invariant homogeneous prime ideals.

Proof: ~~If~~ Let this ring homomorphism be denoted $R \to R'$ and let $\mathfrak{p} \subset R$ be an ~~[illegible]~~ invariant prime. Then we know that there is a $\mathfrak{p}'$ in R' such that $\mathfrak{p}' \cap R = \mathfrak{p}$. By lemma 2, $\mathfrak{q} = \{x \in R' \mid P_t x \in \mathfrak{p}'[[t]]\}$ is a prime ideal of R' containing $\mathfrak{p}$ and contained in $\mathfrak{p}'$. As $R \to R'$ is finite it follows that $\mathfrak{q} = \mathfrak{p}'$, so $\mathfrak{p}$ comes from an invariant prime ideal.

Next stage consists of choosing an embedding of G into a unitary group U_n and letting $X = F \times$ the flag manifold U_n/T_n. Then

$$H^*_G(pt) \longrightarrow H^*_G(\overset{F}{\cancel{X}})$$

is locally free of rank $(n!)$. In fact if V is the faithful representation of G given by $G \hookrightarrow U_n$ and if

$$c_t(V) = \sum_{i=1}^{n} t^i c_i(V)$$

is the total Chern polynomial of V, then

$$H^*_G(\overset{F}{\cancel{X}}) \cong H^*_G(pt)[X_1, \dots, X_n] \Big/ \big(\sigma_1(X) = c_1(V), \dots, \sigma_n(X) = c_n(V)\big)$$

where the σ_i are the elementary symmetric functions of the X's.

Let $\{F_j \;\; j = 1, \dots, N\}$ be the minimal components of the stratification of F into G-connected components of the submanifolds of pure orbit type and let $x_j \in F_j$ be a base point. Then we know that the composition

$$H^*_G(pt) \hookrightarrow H^*_G(F) \longrightarrow E_2^{0*} \hookrightarrow \prod_j H^*_G(Gx_j)$$

is a finite morphism whose kernel is a nilideal, hence the corresponding map on Spec's is surjective by Cohen – Seidenberg and even surjective on the set of invariant prime ideals by the argument used in the proof of the corollary to lemmas 2+3. Now the isotropy group of any point is abelian (and even elementary p-abelian provided we work with the p-flag manifold $U_n/{}_pT_n$ for which $H_G(pt) \hookrightarrow H_G(U_n/{}_pT_n)$ is still locally free), so we conclude the following

<u>Proposition:</u> Let A_j $j=1,\dots,N$ be representatives for the conjugacy classes of maximal abelian (resp. maximal elementary p-abelian) subgroups of G. Then the morphism

$$H^*_G(pt) \longrightarrow \prod_j H^*_{A_j}(pt)$$

is finite and has a nilpotent kernel. Consequently

$$\coprod_j \operatorname{Spec} H^{ev}_{A_j}(pt) \longrightarrow \operatorname{Spec} H^{ev}_G(pt)$$

is surjective and also

$$\coprod_j \{\operatorname{Spec} H^{ev}_{A_j}(pt)\}^a \longrightarrow \{\operatorname{Spec} H^{ev}_G(pt)\}^a$$

where the superscript a denotes homogeneous a-invariant ideals.

Note that any maximal abelian subgroup appears as an isotropy group of a minimal ~~[illegible]~~ stratum of F, and hence there are only finitely many conjugacy classes.

Corollary: The set of homogeneous α-invariant prime ideals in $H_G^{ev}(X)$ for a nice G-space X is finite.

Proof: $H_G^{ev}(X) \hookrightarrow H_G^{ev}(X\times F)$ and the latter has the same invariant spectrum as $H^0((X\times F)/G, \mathcal{O} \mapsto H_G^{ev}(\mathcal{O}))$ which embeds into a finite product $\prod_i H_{A_i}(pt)$ whose invariant spectrum is finite by our earlier calculations.

We can be more precise, namely any invariant prime in $H_G^*(X)$ is the pull back of ν_A ~~[illegible]~~ under a map

$$G/A \longrightarrow X$$

$$H_G^*(X) \longrightarrow H_G^*(G/A) = H_A^*(pt).$$

October 24, 1969

Let X be a nice G-space. We can generalize the conjecture about the invariant primes in $H^*_G(pt)$ on page 5 as follows.

Let $I_G(X)$ be the ordered set of invariant prime ideals in $H^*_G(X)$. Let $J_G(X)$ be the ordered set associated to the following category: The objects are pairs (A,f) where A is an elementary p-abelian subgroup of G and where $f \in [G/A, X] \cong \pi_0(X^A)$, and $\mathrm{Hom}((A,f),(B,g)) = \{\lambda \in [G/A, G/B] \ni g\lambda = f$, or equivalently such that

$$\begin{tikzcd} G/A \arrow[dr, "f"] \arrow[d, "\lambda"'] & \\ G/B \arrow[r, "g"'] & X \end{tikzcd}$$

is commutative$\}$. By definition $J_G(X)$ is the partially ordered set associated to the pre-ordered set of pairs (A,f) with the relation $(A,f) \geq (B,g) \iff \exists$ map $(A,f) \longrightarrow (B,g)$. Note that if $(A,f) \geq (B,g)$ and $(B,g) \geq (A,f)$, then (A,f) and (B,g) are isomorphic in the above category, since any map $G/A \longrightarrow G/A$ is an isomorphism (otherwise we would have $A > xAx^{-1} > x^2Ax^{-2} > \cdots$ contradicting the fact that compact Lie groups satisfy the d.c.c. on closed subgroups.) (see remark 3 below)

Define

$$\boxed{\begin{array}{rl} \rho : J_G(X) & \longrightarrow I_G(X) \\ (A,f) & \longmapsto p(A,f) \end{array}}$$

by associating to $f: G/A \to X$ the pull-back of v_A under the map $H^*_G(X) \xrightarrow{f^*} H_G(G/A) \cong H^*_A(pt)$. It is immediate that if

$$(A,f) \geqslant (B,g) \quad \text{i.e. } \exists \quad \begin{tikzcd} G/A \arrow[d, "\lambda"] \arrow[r, "f"] & X \\ G/B \arrow[ur, "g"'] & \end{tikzcd}$$

then

$$p_{(A,f)} \supset p_{(B,g)},$$

and so ρ is a map of ordered sets.

<u>Conjecture:</u> ρ is an isomorphism of partially-ordered sets.

<u>Remarks:</u> 1. According to our earlier work the map ρ is surjective

2. It suffices to prove the conjecture when $G = U_n$ for all n, since one easily sees that

$$I_G(X) \cong I_U(U \times_G X) \qquad \text{since } H^*_U(U \times_G X) \cong H^*_G(X)$$

$$J_G(X) \cong J_U(U \times_G X)$$

if $G \subset U$. The last isomorphism comes from the fact that $J_G(X)$ depends on the category of G-spaces over X which are transitive and have elementary abelian isotropy groups, and this category is equivalent to the corresponding category of U spaces over $U \times_G X$.

3. As a set $J_G(X)$ is given by

$$\boxed{J_G(X) = \left\{ (A,\lambda) \;\middle|\; \begin{array}{l} A \text{ elem. } p \text{ ab subgrp of } G \\ \text{and } \lambda \in \pi_0(X^A) \end{array} \right\} \Big/ G \text{ action}}$$

where $g(A,\lambda) = (gAg^{-1}, g\lambda)$ and $g\lambda$ denotes the image of λ under the map $g: X^A \longrightarrow X^{gAg^{-1}}$. This follows from ~~the~~ the "note" sentence on page 12 since an isomorphism of $\lambda: G/A \longrightarrow X$ and $\mu: G/B \longrightarrow X$ comes from an element $g \in G \ni A = gBg^{-1}$ and $g\mu = \lambda$.

Suppose G is abelian. Then

$$J_G(X) = \coprod_{A \subset_p G} \pi_0(X^A)_G$$

and $(A,\lambda) \leq (B,\mu) \iff A \supset B$ and $\lambda \mapsto \mu$ under the map $\pi_0(X^A)_G \longrightarrow \pi_0(X^B)_G$

Note that if G is connected, then $\pi_0(X^A)_G = \pi_0(X^A)$.

Now suppose G is a torus T, and let X be a nice compact T space. Replace X by $X \times (T/_pT)$; I claim this doesn't affect either J_T or I_T. Indeed

$$H_T(X \times (T/_pT)) = H_T(X)[\tau_1, \cdots, \tau_k] \qquad \dim \tau_i = 1$$

so the spectrum doesn't change. Also if A is an elementary abelian subgroup of T, then $A \subset_p T$, so $(T/_pT)^A = (T/_pT)$ and

$$\pi_0\{(X \times T/_pT)^A\} = \pi_0(X^A \times T/_pT) = \pi_0(X^A).$$

Thus to prove the equality of $J_T(X)$ and $I_T(X)$ I may suppose that all isotropy groups of X are contained in ${}_pT$. I use induction on the number of isotropy groups of X. Let A_0 be maximal among the isotropy groups, so that X^{A_0} is a minimal stratum of X. Now

$$J_T(X - X^{A_0}) = \coprod_{\substack{A \in \mathrm{Iso}(X) \\ A \neq A_0}} \pi_0\left((X - X^{A_0})^A\right)$$

$$J_T(X^{A_0}) = \pi_0(X^{A_0})$$

Lemma: If A is an isotropy group of X, let $X^{(A)} \subset X^A$ be the open submanifold of points with isotropy group A. Then $\pi_0(X^{(A)}) \xrightarrow{\sim} \pi_0(X^A)$.

Proof. If p is odd, this is because the strata are of even codimension, hence removing X^B from X^A for $B > A$ doesn't disconnect X^A. If $p = 2$, this argument doesn't work, however by removing one X^B at a time we reduce to showing that if X^A is connected so is $X^A - X^B$ for $B > A$. The problem occurs when X^B is a hyperplane in X^A whence B/A acts non-trivially in the normal direction. I can assume that $A = 0$ and $B = \mathbb{Z}_2$ since T/A acts faithfully on X^A. So we are reduced to showing that if a torus T acts faithfully on a connected manifold X, then ~~an invariant~~ a hypersurface of the form $Y = X^B$, $B \simeq \mathbb{Z}_2 \subset T$ doesn't disconnect X. So take a normal vector v on one side of Y and a path α_t in T joining digression:

Lemma: Let a compact Lie group G act on a manifold X, let H be a closed subgroup of G, and let N be the normalizer of H in G. Let $X^{(H)}$ be the open submanifold of X^H ~~where isotropy~~ consisting of those points with isotropy group equal to H and let Y be the closure of $X^{(H)}$ in X^H. Then

$$\pi_0(X^{(H)})_N \xrightarrow{\sim} \pi_0(Y)_N$$

Proof: We may assume that $H = 1$, $G = N$. Then $X^{(H)}$ is the open submanifold of X where N acts freely. Now Y is the union of those components of X on which X acts freely somewhere; ~~in effect if $y \in Y$, then y has arbitrarily near points with trivial isotropy groups, hence~~ ~~the isotropy representation at y is faithful.~~ because Y is open in X. Indeed let $y \in Y$; the bad set near Y is a union of submanifolds finitely many in number, hence the good points are open and dense near y. So I can replace X by Y and assume $X^{(H)}$ is dense in X. Clearly $\pi_0(X^{(H)}) \to \pi_0(Y)$ is surjective. To prove injectivity I can suppose that Y is connected. If $X^{(H)}$ is disconnected, then the bad set contains submanifolds of codimension 1, necessarily of the form Y^A where $A = \mathbb{Z}_2$, since at a generic point of Y^A, A must act faithfully on the normal space. Now I can remove the bad submanifolds of codimension ≥ 2 from Y if I want to and suppose the bad set is a submanifold of codimension 1. Now the thing to note is that two points reflecting each other through Y^A are conjugate under A

and therefore represent the same element in $\pi_0(X^{(H)})_N$. Therefore modulo the N action, Y^A and its conjugates don't disconnect Y.
q.e.d.

~~$(X - X^{A_0})^A \quad X^A - X^{A_0} \cap X^A = X^A$~~

~~Since $X^{A_0} \cap X^A \subset X^{A_0 A}$ and~~

If $A \in \mathrm{Iso}(X)$ and $A \not\subset A_0$, then $(X - X^{A_0})^A = X^A - X^{A_0} \cap X^A = X^A$ since $X^{A_0} \cap X^A \subset X^{A_0 A}$ and $A_0 A > A_0$. On the other hand if $A \subset A_0$, then $(X - X^{A_0})^A = X^A - X^{A_0}$. ~~and~~ Since X^{A_0} might contain whole components of X^A, we must be careful.

October 25, 1969:

We are trying to prove that $J_G(X) \xrightarrow{\sim} I_G(X)$ and we will now check the reduction to the case where G is a torus. Let F be the flag manifold of a faithful complex representation of G. Let's assume, as we may, that G is a unitary group so that $F = G/T$. One knows that

$$H_G(X) \longrightarrow H_G(X \times F)$$

is a free finite map and hence that the induced map on spectra is surjective and universally open. Moreover we know that

$$H_G(X \times F \times F) \simeq H_G(X \times F) \otimes_{H_G(X)} H_G(X \times F)$$

and that for two rings R, S over a third k that the map

$$\operatorname{Spec}(R \otimes_k S) \longrightarrow (\operatorname{Spec} R) \times_{\operatorname{Spec} k} (\operatorname{Spec} S)$$

of topological spaces is surjective. Therefore we get a diagram

$$\begin{tikzcd}
\operatorname{Spec}(R \otimes_k R) \arrow[dr, "\text{surjective}"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & \\
\operatorname{Spec} R \arrow[d, "f"', "\text{open surj}"] & (\operatorname{Spec} R \times_{\operatorname{Spec} k} \operatorname{Spec} R) \arrow[l, shift right] \arrow[l, shift left] \\
\operatorname{Spec} k &
\end{tikzcd}$$

where $k = H_G(X)$, $R = H_G(X \times F)$. It follows that $\operatorname{Spec} k$ is the topological space quotient of Spec R by the pre-equivalence relation defined by p_1 and p_2. Now we know already that as the maps f, p_1, p_2 are finite, the inverse image of an invariant prime consists of invariant primes. Thus we get a diagram

$$
\begin{tikzcd}
I_G(X\times F\times F) \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \arrow[dr, "\text{surjective}"] & \\
I_G(X\times F) \arrow[d, "\text{open surjective}"] & I_G(X\times F)\times_{I_G(X)} I_G(X\times F) \arrow[l, Rightarrow] \\
I_G(X) &
\end{tikzcd}
$$

which proves that $I_G(X)$ is the topological space quotient of $I_G(X\times F)$ by the couple (p_1, p_2). As the topology and order are equivalent, it follows that $I_G(X)$ is the quotient as ordered sets of $I_G(X\times F)$ by (p_1, p_2).

Now we check the descent for J_G. Recall that if $A_1, \ldots, A_N$ are representatives for the conjugacy classes of elementary p-abelian subgroups of G and if N_i is the normalizer of A_i, then as a set

$$J_G(X) = \coprod_{i=1}^{N} \pi_0(X^{A_i})_{N_i}$$

To show that

$$(*) \qquad J_G(X\times F\times F) \rightrightarrows J_G(X\times F) \longrightarrow J_G(X)$$

is exact in the category of sets, it clearly suffices to consider the fiber coming from A_i and to show that

$$\pi_0\left(\left((X\times F\times F)^{A_i}\right)_{N_i}\right) \rightrightarrows \pi_0\left(\left((X\times F)^{A_i}\right)_{N_i}\right) \longrightarrow \pi_0\left(X^{A_i}\right)_{N_i}$$

is exact. As dividing ~~f~~ by a group is a left adjoint it suffices to show that without the N_i is exact. By $X \mapsto X^A$ and $X \longmapsto \pi_0(X)$ commute with products hence we must see that

$$\pi_0(X^A) \times \pi_0(F^{A_i}) \times \pi_0(F^{A_i}) \rightrightarrows \pi_0(X^{A_i}) \times \pi_0(F^{A_i}) \longrightarrow \pi_0(X^{A_i})$$

is exact. This is true provided $F^{A_i} \neq \phi$ which is clear since A_i is an abelian group and hence leaves some flag invariant.

It remains to check that the diagram $*$ is exact in the category of ordered sets, or equivalently that if $\varphi : J_G(X\times F) \longrightarrow S$ is a map of partially ordered sets factoring through $J_G(X\times F) \xrightarrow{f} J_G(X)$, then $\bar{\varphi} : J_G(X) \longrightarrow S$ is compatible with the ordering. It will suffice to show that if $(A,\lambda) \geq (B,\mu)$ in $J_G(X)$ then each can be lifted to $J_G(X\times F)$ preserving the order. We may suppose that $B \supset A$ and that

$$\begin{tikzcd} G/A \arrow[d, "\cdot 1"] \arrow[dr, "\lambda"] & \\ G/B \arrow[r, "\mu"'] & X \end{tikzcd}$$

is homotopy commutative or equivalently that $\mu \mapsto \lambda$ under the map $\pi_0(X^B) \longrightarrow \pi_0(X^A)$. But choose a point $P \in \pi_0(F^B)$ and ~~that there is a correspond~~ let Q be the corresponding point in $\pi_0(F^A)$ whence

$$\begin{tikzcd} G/A \arrow[drr, "\lambda\times Q"] \arrow[d, ".1"'] & & \\ & & X\times F \\ G/B \arrow[urr, "\mu\times P"'] & & \end{tikzcd}$$

is homotopy commutative. Thus we have lifted $(A,\lambda) \geq (B,\mu)$ to $(A, \lambda\times Q) \geq (B, \mu\times P)$ and are finished.

Since we can assume $G = U$, $F = G/T$ ~~we can~~ and since

$$J_G(X\times G/T) = J_T(X)$$
$$I_G(X\times G/T) = I_T(X)$$

we reach the following.

<u>Conclusion:</u> To prove $J_G \xrightarrow{\sim} I_G$ it suffices to consider the case where G is a torus.

October 27, 1969

Proof that $J_G(X) \xrightarrow{\sim} I_G(X)$ when G is a torus.

Proposition 1: Let A be an elementary p-abelian subgroup of G. Then there is ~~[illegible]~~ an isomorphism

$$H_G(X)_{\mathfrak{p}_A} \xrightarrow{\sim} H_G(X^A)_{\mathfrak{p}_A}$$

Proof: We have to show that $H_G(X,X^A)_{\mathfrak{p}_A} = 0$; as ~~the~~ ~~[illegible]~~ $H_G(X,X^A)$ is a $H_G(X - X^A)$ module, it suffices to prove that $H_G(X)_{\mathfrak{p}_A} = 0$ if $X^A = \emptyset$. By use of the spectral sequence

$$E_2^{pq} = H^p(X/G, \mathcal{O} \longmapsto H_G^q(\mathcal{O})_{\mathfrak{p}_A}) \Longrightarrow H_G^{p+q}(X)_{\mathfrak{p}_A}$$

we must show that if ~~B~~ $(G/B)^A = \emptyset \Rightarrow H_G(G/B)_{\mathfrak{p}_A} = 0$.

Replacing X by $X \times G/{}_pG$ we could have assumed that all isotropy groups of X ~~[illegible]~~ are contained in ${}_pG$. Thus we may suppose B is elementary abelian and $(G/B)^A = \emptyset \Longrightarrow A \not\subset B \Longrightarrow B \cap A < A$. Now by the formula for the cohomology of an elementary p-group, the cartesian square

$$\begin{tikzcd} & B \arrow[dr] & \\ A \cap B \arrow[ur] \arrow[dr] & & {}_pG \\ & A \arrow[ur] & \end{tikzcd}$$

gives rise to a cocartesian square ~~[illegible]~~

But $(G/B)^A = \phi \iff A \not\subset B \iff B \cap A < A$. Let X be a non-trivial character of A which is trivial on $B \cap A$ and extend X to a character of G/B. Then $e(X) \in H^2_G(pt)$ vanishes in $H^2_B(pt)$ and restricts to a generator of $H^*_A(pt)/n_A$. Thus $e(X) \notin p_A$, so $H_G(G/B)_{p_A} = 0$.

Proposition 2: If A is an elementary abelian subgroup of a torus G, then the map

$$\rho : H^*_G(X) \otimes_{H^*_G(pt)} \left(H^*_G(pt)/p_A\right) \longrightarrow H^*_A(X)$$

induces a homeomorphism of spectra.

Proof: We will show that the kernel is nilpotent and that every element on the ~~left~~ right comes from the ~~right~~ left after being raised to a sufficiently high power of p. ~~We may replace $H_A(pt)$ by $H_G(pt)/p_A$. First we show that Ker ρ consists of nilpotent elements.~~ Writing $X = U_1 \cup \cdots \cup U_n$ where the U_i are tubes around orbits we reduce to checking this when $X = G/H$ and that it's true for $X = U \cup V$ if it is true for U, V and $U \cap V$. For an abelian compact Lie group H we have that $H^*_H(pt)/n_H = S(\hat{H}/p\hat{H})$. So:

$$H^*_G(G/H) \otimes_{H^*_G(pt)} \left(H^*_G(pt)/p_A\right) / \text{nilpotents} = S(\hat{H}/p\hat{H}) \otimes_{S(\hat{G})} S(\hat{A})$$

$$= S(\hat{H}/p\hat{H} +_{\hat{G}} \hat{A}) = S(\widehat{H \cap A})$$

and it suffices to check that

$$H^*_{H\cap A}(pt) \longrightarrow H^*_A(G/H)$$

induces a homeomorphism of spectra. But $A/A\cap H$ acts freely on G/H so

$$\mathrm{Spec}\{H_A(G/H)\} = \mathrm{Spec}\{H^0(G/AH, AxH \mapsto H_A(AxH)\}$$

$$= \mathrm{Spec}\{H^0(G/AH)\otimes H^*_{A\cap H}(pt)\}$$

$$= \mathrm{Spec}\{H^*_{A\cap H}(pt)\}$$

since G/AH is connected. Therefore we see that ρ is a homeomorphism if $X = G/H$.

More generally suppose that G/H acts freely on X. Then

$$\mathrm{Spec}\{H^*_G(X)\} = \mathrm{Spec}\{H^0(X/G, Gx \mapsto H^*_G(Gx)\}$$

$$= \mathrm{Spec}\{H^0(X/G)\otimes H^*_H(pt)\}$$

Thus

$$\mathrm{Spec}\{H^*_G(X)\otimes_{(H_G(pt))}(H_G(pt)/p_A)\} = \text{part of Spec } H^*_G(X) \text{ over } \overline{\{p_A\}}$$

$$= \text{part of } H^0(X/G)\otimes H^*_H(pt) \text{ over } \overline{\{p_A\}}$$

$$= \pi_0(X/G)\times \mathrm{Spec}\, H^*_H(pt)\otimes_{H_G(pt)}(H_G(pt)/p_A)$$

$$= \pi_0(X/G)\times \mathrm{Spec}\, H^*_{H\cap A}(pt)$$

$$= \mathrm{Spec}\, H^0(X/G)\otimes H^*_{H\cap A}(pt) = \mathrm{Spec}\{H^0(X/A, Ax \mapsto H_A(Ax)\}$$

$$= \operatorname{Spec}\{H_A^*(X)\}.$$

Thus we have checked the proposition if there is a single orbit type.

Next we want to use induction on the number of orbit types, so suppose $X = U \cup V$, and consider the square

$$(*)\quad \begin{array}{ccc} H_G(X)/p_A & \longrightarrow & \{H_G(U)/p_A\} \times_{\{H_G(U\cap V)/p_A\}} \{H_G(V)/p_A\} \\ \downarrow & & \downarrow \\ H_A(X) & \longrightarrow & H_A(U) \times_{H_A(U\cap V)} H_A(V) \end{array}$$

We claim that the horizontal arrows ~~induce~~ homeomorphisms of spectra. This is clear for the bottom by Mayer-Vietoris. For the top, first let $x \in H_G(X)$ and suppose that $x|U \equiv x|V \equiv 0$ mod p_A. Let K and I be the kernel and ~~cokernel of~~ image of the map

$$H_G(X) \longrightarrow H_G(U) + H_G(V).$$

By Artin-Rees, $\exists\, k$ such that $I \cap p_A^{n+k}(H_G(U)+H_G(V)) \subset I p_A^n$. Thus ~~then~~ for $p^a > k$ we have that $(x^{p^a}/U, x^{p^a}/V) \subset p_A I$. Since

$$K/p_A^n \longrightarrow H_G(X)/p_A^n \longrightarrow I/p_A^n \longrightarrow 0$$

is exact if follows that $\exists\, y \in K$ with $x^{p^a} - y \in p_A H_G(X)$. Thus $x^{p^a N} = y^N \equiv 0$ mod p_A and so the kernel of the upper row of $(*)$ is nilpotent.

Now suppose give $u \in H_G(U)$ and $v \in H_G(V)$ such that $\quad u|(U\cap V) \equiv v|(U \cap V) \quad \bmod \mathfrak{p}_A$. Consider the exact sequences

$$H_G(X) \longrightarrow H_G(U) + H_G(V) \longrightarrow C \longrightarrow 0$$

$$C \hookrightarrow H_G(U\cap V)$$

By Artin-Rees

$$C \cap \mathfrak{p}_A^{n+k} H_G(U\cap V) \subset \mathfrak{p}^n C.$$

Thus $\quad (u^{p^a}|_{U\cap V} - v^{p^a}|_{U\cap V}) = (u|U\cap V - v|U\cap V)^{p^a} \subset \mathfrak{p}C \quad$ if $a \gg 0$ and so from the exactness of

$$H_G(X)/\mathfrak{p} \longrightarrow H_G(U)/\mathfrak{p} + H_G(V)/\mathfrak{p} \longrightarrow C/\mathfrak{p} \longrightarrow 0$$

we see $\exists\, x \in H_G(X) \ni$

$$\begin{aligned} x|U &\equiv u^{p^a} \\ x|V &\equiv v^{p^a} \end{aligned} \quad \bmod \mathfrak{p}.$$

~~and~~ Therefore the cokernel of the top row of $(*)$ is killed by a power of Frobenius.

By assumption for U, V and $U\cap V$ the right hand vertical arrow has its kernel + cokernel killed by a power of Frobenius. By diagram chasing it must be so for the first vertical arrow, which concludes the proof of proposition 2.

<u>Remark</u>: In view of the fact that I don't yet understand decomposition of a G-manifold, it is worth remarking that in the above argument 1) to get nilpotency ~~one does~~ of the kernel one

doesn't need to know anything about $U \cap V$ 2) to get surjectivity (modulo action of Frobenius) one needs only the nilpotency assertion for $U \cap V$.

Proof that $J_G(X) \xrightarrow{\sim} I_G(X)$ as sets, when G is a torus. We calculate the fiber over $p_A \subset H^*_G(pt)$. To show that

$$\pi_0(X^A) \xrightarrow{\sim} \mathrm{Spec}\; H_G(X) \otimes_{H_G(pt)} k(p_A).$$

But

$$H_G(X) \otimes_{H_G(pt)} k(p_A)$$

$$\cong H_G(X^A) \otimes_{H_G(pt)} k(p_A) \qquad \text{by prop 1.}$$

$$\cong \Big(H_G(X^A) \otimes_{H_G(pt)} \{H_G(pt)/p_A\} \Big) \otimes_{H_A(pt)} \{H_A(pt)_{p_A}\}$$

so by proposition 2 this has the same spectrum as

$$H_A(X^A)_{m_A} .$$

But as A acts trivially on X^A there is a single orbit type so

$$\operatorname{Spec} H_A(X^A) = \pi_0(X^A) \times \operatorname{Spec} H_A(pt).$$

Thus

$$\operatorname{Spec}\{H_G(X) \otimes_{H_G(pt)} k(p_A)\} = \pi_0(X^A).$$

as claimed.

It remains to check the ordering. Suppose p, p' are invariant primes in $H_G(X)$ coming from $\lambda \in \pi_0(X^A)$ and $\lambda' \in \pi_0(A')$, respectively and that $p < p'$. Then $p_A \subset p_{A'}$ in $H_G(pt)$ and so we know that $A \supset A'$ and that $X^A \subset X^{A'}$. We must show that $\lambda \mapsto \lambda'$. We may replace X by $X^{A'}$ without affecting the spectrum containing $p_{A'}$ since

$$H_G(X)_{p_{A'}} = H_G(X^{A'})_{p_{A'}} \quad \text{by prop. 1.}$$

But now we can decompose $X^{A'}$ into its connected components affecting a corresponding decomposition of the spectrum of $H_G(X^{A'})$. It is then clear that if λ were not in the component of λ', then we could not have $p \subset p'$.

<u>Localization theorem</u>: Let G be a compact Lie group acting on a nice G-space X and let A be an elementary p-abelian subgroup of G with associated prime ideal $\mathfrak{p}_A$ in $H_G^*(pt)$. Then

$$H_G^*(X)_{\mathfrak{p}_A} \xrightarrow{\cong} H_G^*(GX^A)_{\mathfrak{p}_A}$$

<u>Proof</u>: It suffices to show that if $X^A = \phi$ then $H_G^*(X)_{\mathfrak{p}_A} = 0$, by the Atiyah–Segal argument. It suffices to show that there is no invariant prime ideal in $H_G(X)$ whose inverse image in $H_G(pt)$ is contained in $\mathfrak{p}_A$. In effect $H_G(X)$ is a finite $H_G(pt)$ module so if $H_G(X)_{\mathfrak{p}_A} \neq 0$ we know by Nakyama that there is a prime $\mathfrak{p}$ in $H_G(X)$, necessarily invariant, whose inverse image in $H_G(pt)$ is $\mathfrak{p}_A$. By our theorem $\mathfrak{p}$ is represented by a homotopy class φ

$$G/A' \xrightarrow{\varphi} X$$

and the fact that the inverse image of $\mathfrak{p}$ in $H_G(pt)$ is contained in $\mathfrak{p}_A$ means that $\exists$ a diagram

$$\begin{tikzcd} G/A' \arrow[r] & X \arrow[d] \\ G/A \arrow[u] \arrow[r] & pt \end{tikzcd}$$

so $X^A \neq \phi$, a contradiction.

<u>Basic special case</u>: Invariant prime ideals in $H^*_G(pt)$ correspond to conjugacy classes of elementary p-abelian subgroups of G.

Further questions:

1) do these arguments work in K_G theory

2) structure of the orbits of the $\mathcal{Q}$-action: Thus what is the image of the map

$$H_G(pt) \longrightarrow \left(H_A(pt)/r_A\right)^N$$

<u>Example</u>: Spin (n). Any maximal elementary 2-abelian subgroup A of Spin (n) contains ~~the center~~ Ker π and hence is the inverse image $\pi^{-1}\pi(A)$, where π: Spin (n) $\longrightarrow$ SO(n) is the natural map. Up to conjugation we may suppose that $\pi(A)$ is contained in the diagonal matrices D. $\pi(A)$ is an isotropic subspace of D for the bilinear form Q defining the extension

$$0 \to \mathbb{Z}_2 \longrightarrow \pi^{-1}D \longrightarrow D \to 0$$

Two subgroups of D are conjugate in SO(n) iff they are already conjugate under the Weyl group W so one concludes that <u>conjugacy classes of maximal elementary 2-subgps of Spin(n) are in 1-1 correspondence with W conjugacy classes of maximal isotropic subspaces of D.</u>

Let H be a Sylow p-subgroup of G, so that

$$H^*_G(pt) \longrightarrow H^*_H(pt)$$

is injective. The corresponding map of spectra

$$\mathrm{Spec}\{H^*_H(pt)\}^a \longrightarrow \mathrm{Spec}\{H^*_G(pt)\}^a$$

is surjective, (but not generally bijective) due to the fact that (non-conjugate) elementary abelian subgroups of H may become conjugate in G. fusion?

October 28, 1969

Segal's version of Smith theory leads to the following theorem

Theorem: Let $\mathbb{Z}_p$ act on a compact manifold M everywhere of positive dimension. Assume either that $p=2$ or that p is odd and M is oriented. Then the fixed submanifold can not be a single point.

Proof: Assume that there is at least one fixed point and let $i: pt \longrightarrow M$ be the inclusion. Let $f: M \longrightarrow pt$. By assumption f is equivariantly oriented if p is odd. In any case there is a Gysin homomorphism f_* and hence by transitivity one for i in equivariant cohomology $H_G^*(M) = H^*(EG \times_G M, \mathbb{Z}_p)$. Thus we have

$$(*) \qquad \begin{tikzcd} H_G(pt) \arrow[r, shift left, "f^*"] & H_G(M) \arrow[l, shift left, dashed, "i^*"] \arrow[r, shift left, "f_*"] & H_G(pt) \arrow[l, shift left, dashed, "i_*"] \end{tikzcd}$$

with
$$i^* f^* = id$$
$$f_* i_* = id$$
and
$$f_*(f^* x) = f_* 1 \cdot x = 0$$
since $f_* 1 \in H_G^{-n}(pt) = 0$ for dimensional reasons.

Now consider the triangle

$$\cdots \longrightarrow H_G^*(M, M^G) \longrightarrow H_G^*(M) \longrightarrow H_G^*(M^G) \longrightarrow \cdots$$

and localize with respect to the generator $\omega \in H_G^2(pt)$. We get that

$$H_G^*(M)[w^{-1}] \xrightarrow{\sim} H_G^*(M^G)[w^{-1}]$$

But the left side has rank at least 2 over $H_G^*(pt)[w^{-1}]$ because of the sequences $(*)$. Hence M^G cannot consist of a single point.

Suppose now that the fixed submanifold M^G consists of isolated points and let us apply the localization procedure for restricting to the fixed submanifold.

$$\begin{tikzcd} M^G \arrow[r, "i"] \arrow[d, "f^G"] & M \arrow[d, "f"] \\ pt \arrow[r] & pt \end{tikzcd}$$

$$f_* 1 = f_*^G \, e(\mu_f)$$

$$= \sum_P \nu_P$$

$\nu_{f^G} \to i^* \nu_f \to \mu_f \to 0$

where if the eigenvalues in the normal direction at P are $\zeta^{i_1}, \ldots, \zeta^{i_n}$ $\quad \zeta = e^{2\pi i \frac{1}{p}}$, then

$$\nu_P = \prod_{j=1}^{n} e(\eta^{i_j})^{-1} = \left[\left(\prod_{j=1}^{n} i_j\right) w^{n}\right]^{-1}.$$

~~Remark: In the situation of the theorem, note that the Euler characteristic of $H_G(M)$ as an $H_G(pt)$ module is defined and that~~

$$\cancel{\chi(H_G(M)) = \chi(H(M))}$$

~~using the spectral sequence and the fact that G acts~~