September 1, 1969

clean intersections:

A cartesian diagram of manifolds

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

is called <u>clean</u> if $\forall\, x' \in X'$ the diagram of tangent spaces

$$\begin{tikzcd} T_{x'}(X') \arrow[r] \arrow[d] & T_x(X) \arrow[d] \\ T_{y'}(Y') \arrow[r] & T_y(Y) \end{tikzcd}$$

$x = g'x'$
$y' = f'x'$
$y = fx = gy'$

is cartesian, or equiv. if

$$0 \longrightarrow T_{x'}(X') \longrightarrow T_x(X) \oplus T_{y'}(Y') \longrightarrow T_y(Y)$$

is exact. The cokernel of the last map as x' varies over X' gives a vector bundle F on X'.

~~Formula: If f proper and f~~

~~$g^* f_*$~~

Conj Formula: $\nu_{f'} + F = \nu_f$

hence if two of $\nu_{f'}$, F, ~~ν_f~~ are oriented so is the third.

Conj. formula: If f proper + f, f' oriented, then

$$g^* f_* x = f'_*(e(F) g'^* x)$$

where e denotes Euler class.

~~Basic situation~~

Examples: ① Suppose E a v. bundle oriented

$$\begin{array}{ccc} X & \xrightarrow{id} & X \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \cancel{\text{zero sect.}}\ i} \\ X & \xrightarrow[\cancel{\text{sect}}\ i]{\cancel{\text{zero}}} & E \end{array}$$

where $i =$ zero section. Above formula says

$$i^* i_* x = e(E) \cdot x$$

② $$\begin{array}{ccc} \tilde{Y} & \xrightarrow{j} & \tilde{X} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \overset{i}{\hookrightarrow} & X \end{array}$$ blow-up situation

~~Here the formula says that~~ Locally $X =$ normal bundle of i
E

$$\begin{array}{ccc} PE & \overset{j}{\hookrightarrow} & \mathcal{O}(-1) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \overset{i}{\hookrightarrow} & E \end{array}$$

$$0 \to \mathcal{O}(-1) \to g^*E \to F \to 0$$
$$\begin{array}{ccc} \| & & \| \\ \nu_j & \hookrightarrow & g^*\nu_i \end{array}$$

thus the excess of the clean square is F and ~~we have~~ the formula gives

$$i^* f_* z = g_*(e(F) j^* z)$$

which is known to be true

Apply this to the equivariant case.

$$\begin{array}{ccc} Z^G & \xrightarrow{i'} & Z \\ \downarrow f^G & & \downarrow f \\ X^G & \xrightarrow[i]{} & X \end{array}$$

If f embedding, this is clean

and the excess is cokernel of

$$\nu_{f^G} \to i'^* \nu_f$$

which is μ_f. Thus for an oriented embedding f we get

$$i^* f_* z = (f^G)_* (e(\mu_f)\, i'^* z).$$

Let's see how this formula behaves under tom Dieck localization!

$$f_* x = (f^G)_* (\gamma(\mu_f)\, x)$$

where γ is the multiplication char class given by

$$\gamma(V \otimes E) = e(V)^{rg\, E} \sum{}' (a_V)^\alpha c_\alpha(E)$$

V irred repn of G not trivial.

In the theory $U(X^G)[\cancel{\ }\ e(V), e(V)^{-1}, a_{V,n}\ n \geq 1]_{V \in \hat{G} - 0}$ what is the Euler class of $V \otimes E$?

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ X \arrow[r] & V \otimes E \end{tikzcd}$$

V non-trivial irred

$\Rightarrow f_G = id \quad \nu_f = V \otimes E$

$\therefore \quad f_* 1 = \cancel{e(V)}\ \gamma(V \otimes E).$

<u>Conclusion</u>: γ is the mult. characteristic class extending the Euler class.

Basic conclusion is that when the diagram

$$\begin{tikzcd} Z^G \arrow[r, hook] \arrow[d, "f^G"] & Z \arrow[d, "f"] \\ X^G \arrow[r, hook] & X \end{tikzcd}$$

is clean, then μ_f is an honest bundle and

$f_* 1 = (f^G)_* e(\mu_f)$. is integral for tom Dieck loc.

~~Criterion: / Set~~

Conclusions: Applying this to the bundle $V \otimes E$ where E is the canonical bundle over the Grassmannian we have that

$$c(V \otimes E) = e(V)^{rg E} \sum_{\alpha}{}' a_V^{\alpha} c_{\alpha}(E) \quad \text{is integral}$$

hence

$$e(V)\, a_V^{\alpha} \in \operatorname{Im}\{U_G(pt) \to S^{-1} U_G(pt)\}$$

~~for all α. Otherwise put~~

$$\cancel{e(V)(a_{V,n})^j \text{ integral for all } j, n}$$

~~Contrast this with our calculation for $G = \mathbb{Z}_2$ where we showed that a_n was integral for $n \geq 2$ and that $a_1 - \frac{1}{w}$ is integral.~~

$$\cancel{w\left(a_1 - \frac{1}{w}\right)^2 = w a_1^2 - 2a_1 + \frac{1}{w}}$$

$$\cancel{= w a_1^2 + \left(\frac{1}{w} - a_1\right) - a_1}$$

~~implies that a_1 hence w^{-1} is integral which is false? PUZZLE~~

Addition to stuff on the clean intersection formula:

If $i: Z \to X$ is an embedding with normal bundle ν_i, then the clean intersection formula gives

$$\begin{tikzcd} Z \arrow[r, "id"] \arrow[d, "id"'] & Z \arrow[d, "i"] \\ Z \arrow[r, "i"] & X \end{tikzcd} \qquad i^* i_* z = e(\nu_i) z$$

This is what Grothendieck calls the fundamental result of intersection theory. Another special case is:

$$\begin{tikzcd} Z \arrow[r, "\Delta"] \arrow[d, "i"'] & Z \times Z \arrow[d, "i \times i"] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd} \qquad i_* z_1 \cdot i_* z_2 = i_* (e(\nu_i) z_1 z_2)$$

~~Problems with the clean formula in equivariant theory.~~

Let X^2 be a $U^{(ev)}$-oriented compact surface. Then $\exists$ a diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{C}^2 = Z \arrow[d, "p"] \\ & pt \end{tikzcd}$$

where i is an embedding, in fact SO-oriented surfaces embed in $\mathbb{R}^3$. Now as f and p are U-oriented, so is i. Assuming that the clean formula holds in $U_{\mathbb{Z}_2}$ for

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "i"] & X \times X \arrow[d, "i^2"] \\ Z \arrow[r, "\Delta"] & Z \times Z \end{tikzcd}$$

we get hence $\quad Q(i_* 1) = \Delta_Z^*(i_*^2) 1 = i_* e(\eta \otimes \nu_i)$ and

$$w^2 Q(f_* 1) = w^2 Q(p_* i_* 1) = p_* Q(i_* 1) = f_* e(\eta \otimes \nu_i)$$

which I now know is false, since forgetting $\mathbb{Z}_2$ action it gives $f_* e(\nu_i) = f_* c_1(\nu_f) = 0.$ This paradox results from the mistaken identification of

$e(\nu_i) =$ Euler class of the honest normal bundle of i with its orientation

$c_1(\nu_i) =$ first Chern class of the stable normal bundle of i

In this example $e(\nu_i) = 0$ since in fact X embeds in $\mathbb{R}^3$ and there is a trivial non-zero section, whereas

$$f_* c_1(\nu_i)\, P_1$$

is the cobordism class of X.

Critical case of clean formula

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f''"'] \arrow[dr, bend left, "f"] & \\
Y' \arrow[r, "g''"] \arrow[rr, bend right, "g"'] & Z \arrow[r, "h"] & Y
\end{tikzcd}$$

square transversal cartesian
h embedding.
The excess bundle is $\nu_h | X'$.
Suppose h, f hence also f'' and f' are oriented.
Then

$$g^* f_* x = g''^* h^* h_* f''_* x = g''^*(e(\nu_h) f''_* x) = g''^* e(\nu_h)\, g''^* f''_* x$$
$$= e(g''^* \nu_h)\, f'_* g'^* x = f'_*(e(\nu_h | X')\, g'^* x).$$

which is the clean formula. The general case restricts to this one by ~~using supports~~ first restricting to embeddings and then to a neighborhood of X' in Y using supports proper along Y'.

September 5, 1969:

We wanted to be able define the dotted map Φ

$$\begin{tikzcd} \mathcal{U}_G(X) \arrow[r, dashed, "\Phi"] \arrow[d] & R(G)\otimes(\mathbb{Z}\otimes_L \mathcal{U}(X^G)) \arrow[d, "\int"] \\ K_G(X) \arrow[r] & K_G(X^G) \end{tikzcd}$$

directly without using the periodicity theorem. The method would be to define Φ using a characteristic class φ by a formula

$$\Phi(f_* x) = f^G_*(\varphi(\nu_f)\Phi x)$$

To determine what φ must be we compute what it does to the Euler class of a bundle E over X. Let $f: X \to E$ be an embedding. Then

$$\begin{array}{c} e(E) = f^* f_* 1 \\ \downarrow \\ \lambda_{-1}(E|_{X^G}^{\vee}) \end{array}$$

so
$$\lambda_{-1}(E|_{X^G}^{\vee}) = \Phi f^* f_* 1 = f^{G*} f^G_* \varphi(\nu_f)$$
$$= \varphi(\nu_f)\cdot\lambda_{-1}(\nu_f^{\vee G}) = \varphi(E)\lambda_{-1}(\nu_f^{\vee G})$$

Thus
$$\varphi(E) = \lambda_{-1}(\mu_f^{\vee}) = \text{Euler class in } K \text{ theory of } \mu_f$$

is the obvious candidate for φ. Unfortunately ~~the~~ this Euler class can not be defined for virtual bundles without first inverting $\lambda_{-1}(\check{V})$ for any irred. non-trivial representation of G. Thus we see that the fact that Φ exists is a non-stable consequence of the periodicity theorem. The situation is similar to the existence of a multiplicative operation

$$U(X) \longrightarrow \mathbb{Z} \otimes_{L} U(X) \qquad L \to \mathbb{Z} \text{ given by } X+Y-XY$$

sending $\quad c_1(L) \longmapsto c_1(L^k)$.

Representations of the symmetric group:

The Steenrod power method defines a map

$$K(X) \xrightarrow{\ Q\ } \Big(\mathbb{Z} + \prod_{n\geq 1} R(\Sigma_n) \otimes K(X)\Big)^*$$

$$E \longmapsto \sum_{n\geq 0} E^{\otimes n}$$

If we define $\quad R(\Sigma_p) \otimes R(\Sigma_q) \longrightarrow R(\Sigma_{p+q})$ to be the induction from $\Sigma_p \times \Sigma_q$ to Σ_{p+q}, then

$$\mathbb{Z} + \prod_{n\geq 1} R(\Sigma_n) \otimes K(X)$$

becomes a complete graded ring and Q is a homomorphism for the additive structures of K(X) into the units of this graded ring. This follows from the formula

$$(E+F)^{\otimes n} = \sum_{i+j=n} \left\{ \Sigma_n \times_{(\Sigma_i \times \Sigma_j)} (E^{\otimes i} \otimes E^{\otimes j}) \right\}.$$

Next note that if π_ω is the representation of $\Sigma_{|\omega|}$ induced from the trivial representation of $\Sigma_\omega = \Sigma_{\omega_1} \times \cdots \times \Sigma_{\omega_k}$, then

(i) The π_ω form a basis of $R(\Sigma_n)$ where ω runs over the partitions of n (see Atiyah's paper).

(ii) $\pi_\omega = e_{\omega_1} e_{\omega_2} \cdots e_{\omega_k}$ if $\omega = (\omega_1, \dots \omega_k)$ for the product in $\mathbb{Z} \oplus \bigoplus_{n\geq 1} R(\Sigma_n)$ defined by induction, where $e_k \in R(\Sigma_k)$ is the class of the trivial representation.

Therefore

$$\mathbb{Z}[e_1, e_2, \dots] \xrightarrow{\sim} \mathbb{Z} + \bigoplus_{n\geq 1} R(\Sigma_n)$$

and so up to isomorphism

$$Q: K(X) \longrightarrow (\mathbb{Z} + K(X)[[e_1, e_2, \dots]]^+)^\times$$

is the homomorphism with

$$L \longmapsto \sum_{n\geq 0} e_n L^n \qquad \text{where } e_0 = 1.$$

September 8, 1969

Problem: Find a formula for $Q: U^{(ev)}(X) \longrightarrow U^{(ev)}(B\mathbb{Z}_2 \times X)$.

Suppose that $f: X \longrightarrow Y$ is a $U^{(ev)}$-oriented map and let f have a factorization

(1)
$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & Y \times \mathbb{C}^n = Z \arrow[d, "pr_1 = p"] \\ & Y \end{tikzcd}$$

where i is an embedding. Orienting pr_1 in the standard way we get an orièntation of i. As i is an embedding the square

$$\begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[d, "i"'] & X \times X \arrow[d, "i^2"] \\ Z \arrow[r, "\Delta_Z"] & Z \times Z \end{tikzcd}$$

is cartesian and clean, so we have the formula

$$\boxed{Q(i_* 1) = \Delta_Z^* (i^2)_* 1 = (i)_* e(\eta \otimes \nu_i)}$$

in $U_{\mathbb{Z}_2}(Z)$. (These things are almost certainly true if the actual normal bundle of i has a complex ~~structure~~ and probably true in general when ν_i is stably complex).

If $j: Y \longrightarrow Z$ is a section, then

$$j_* : U(Y) \xrightarrow{\sim} U_{proper/Y}(Y \times \mathbb{C}^n)$$

is multiplication with $j_* 1$, ie.

$$j_*(y) = (p^*y)\cdot j_* 1$$

so

$$Q(j_* y) = (p^*Q(y))\,Q(j_* 1)$$
$$= p^*Q(y)\cdot \omega^n\, j_* 1 \qquad \omega = e(\eta)$$
$$= \omega^n\, j_*(Qy).$$

So if p_* is integration over the fibres we have that

$$Q(z) = Q(j_* p_* z) = \omega^n j_*\, Q(p_* z)$$

$$\boxed{p_* Q(z) = \omega^n Q(p_* z) \qquad z \in U_{prop/Y}(Y\times\mathbb{C}^n)}$$

Combining the two boxed formulas

$$\omega^n Q(f_* 1) = \omega^n Q(p_* \iota_* 1)$$
$$= p_* Q(\iota_* 1) = p_* \iota_*(e(\eta\otimes\nu_i) 1)$$

or

(*) $$\boxed{\omega^n Q(f_* x) = f_*(e(\eta\otimes\nu_i)\,Qx) \quad \text{when } f \text{ may be factored as in (1)}}$$

(formula holds with x replacing 1, see page 3)

<u>Remark</u>: This is an unstable formula which may be specialized under the maps

$$U_{\mathbb{Z}_2}(X) \longrightarrow U(\mathbb{RP}^n\times X).$$

Of course after inverting ω the formula (*) is that of tom Dieck

$$Q(f_* x) = f_*\left(e(\eta \otimes \nu_f)x\right) \text{ in } \quad U_{\mathbb{Z}_2}(X)[w^{-1}].$$

Next recall the Gysin sequence

$$U^q_{\mathbb{Z}_2}(X) \xrightarrow{w^k} U^{q+2k}_{\mathbb{Z}_2}(X) \xrightarrow{pr_2^*} U^{q+2k}_{\mathbb{Z}_2}(S^{2k-1} \times X) \longrightarrow \cdots$$

$$\cong$$

$$U^{q+2k}(\mathbb{R}P^{2k-1} \times X) = 0 \text{ if }$$

$q + 2k > 2k-1 + \dim X$

Thus

$$U^q_{\mathbb{Z}_2}(X) \xrightarrow{w} U^{q+2}_{\mathbb{Z}_2}(X)$$

is an isomorphism for $q > \dim X$ and surjective for $q = \dim X$. Thus the equation $(*)$ is a "stable" formula when $\dim(w^r Q(f_* 1)) > \dim Y$ or $2r + 2(\dim Y - \dim X) > \dim Y$ in the sense that going from r to $r+1$ doesn't lose any information.

Question: How does Q commute with Gysin for an embedding?

$$\boxed{Q(i_* x) = i_* e(\eta \otimes \nu_i) Q x}$$

since if
$$\begin{array}{ccccc} W & \xrightarrow{g} & X & \xrightarrow{i} & Z \\ \downarrow & & \downarrow & & \downarrow \\ W^2 & \xrightarrow{g^2} & X^2 & \xrightarrow{i^2} & Z^2 \end{array},$$
$x = g_* 1$ then $Qx = \Delta_X^* g^2_* 1$

$$Q(i_* x) = Q(g_* i_* 1) = \Delta_2^* i^2_* g^2_* 1$$
$$= i_*\{e(\eta \otimes \nu_i) \cdot Qx\}.$$

From now on we take $X = pt$. Given $f: X \to pt$ proper oriented of real dimension $2n$, the stable formula is

$$w^{-2n+1} Q(f_* 1) = f_* e(\eta \otimes (\nu_f + 2n+1))$$

~~Here to make sure the notation makes sense we must look now in $U(BZ_2 \times X)$, i.e. $U(RP^n \times X)$.~~ Note $\nu_f + 2n$ is a well defined up to ~~homotopy~~ isomorphism complex bundle over X of ~~dim~~ complex dimension n.* Now the formula

$$(2) \qquad w^{-2n} Q(f_* 1) = f_* e(\eta \otimes (\nu_f + 2n))$$

is in general false. In effect restrict from $U_{Z_2}(pt) \to U(pt)$ so that $\eta \mapsto 1$, $w \mapsto 0$. Then we get

This is OKAY but you have to be careful to make precise that $\nu_f + 2n$ is a definite cx. bundle and so its e is defined.

$$\begin{aligned} 0 &= f_* e(\nu_f + 2n) \\ &= f_* c_n(\nu_f) \in U^0(pt) = Z \end{aligned}$$

which is not generally the case (e.g. P_1).

Note that we have from the Gysin sequence

$$\begin{tikzcd} U^1_{Z_2}(S^1) \arrow[r, "\delta"] \arrow[d, equal] & U^0_{Z_2}(pt) \arrow[r, "w"] & U^2_{Z_2}(pt) \arrow[r] & U^2_{Z_2}(S^1) \\ U^1(S^1/Z_2) \simeq U^0(pt) \arrow[ur, "\cdot [Z_2 \to pt]"'] & & & \end{tikzcd}$$

(this should be checked later)

---

* This follows from obstruction theory applied to

$$\begin{tikzcd} & X^{2n} \arrow[d, "\nu_f + 2n"'] \arrow[dr, "\nu_f + 2n+1"] & \\ S^{2n+1} \arrow[r] & BU_n \arrow[r] & BU_{n+1} \end{tikzcd}$$

Thus the two sides of (2) differ by an integral multiple of $[\mathbb{Z}_2 \to pt]$ and this integer may be determined by restriction to trivial group. Thus the correct version of (2) is

(3) $$\boxed{\omega^{-2n} Q(f_* 1) = f_* e(\eta \otimes (\nu_f + 2n)) - \left(\tfrac{1}{2} f_* c_n(\nu_f)\right)[\mathbb{Z}_2 \to pt]}$$

Consider (3) in the case $n=1$, so that X is a surface. Then $\nu_f + 2$ is a well defined line bundle L on X. Consider (3) in $U^0(B\mathbb{Z}_2)$ where there is the group law

$$e(\eta \otimes L) = F(\omega, c_1(L))$$
$$= \omega + F_2(\omega, 0)\, c_1(L) \qquad \text{since } c_1(L)^2 = 0$$
$$[\mathbb{Z}_2 \to pt] = \frac{F(\omega, \omega)}{\omega}$$

Then (3) becomes

(4) $$\omega^2 Q(f_* 1) = (f_* 1)\omega + \left(\tfrac{1}{2} f_* c_1(L)\right)\left[2F_2(\omega, 0) - \frac{F(\omega,\omega)}{\omega}\right]$$

(4′) $$\left\{ \begin{aligned} 2F_2(\omega,0) - \frac{F(\omega,\omega)}{\omega} &= 2(1+\omega G(\omega,0)) - (2 + \omega G(\omega,\omega)) \\ &= \omega\{2G(\omega,0) - G(\omega,\omega)\} \end{aligned} \right.$$

Consider the restriction $U^0(B\mathbb{Z}_2) \longrightarrow U^0(\mathbb{R}P^2)$.

$$U^0(\mathbb{R}P^2) = U^0(pt) + U^{-2}(pt, \mathbb{Z}_2)\cdot \omega$$

where now $\omega^2 = 0$, $2\omega = 0$.

$$G(0,0) = -P_1$$

Thus (3) under the restriction to $\mathbb{R}P^2$ shows that

$$(5) \qquad w\left\{f_* 1 - \left(\tfrac{1}{2} f_* c_1(L)\right) P_1\right\} = 0$$

and since we have

$$U^{-2}(pt) \xrightarrow{2} U^{-2}(pt) \xrightarrow{\cdot w} U^{-2}(pt, \mathbb{Z}_2) \longrightarrow U^{-1}(pt) \xrightarrow{2} U^{-1}(pt)$$

it follows that

$$(6) \qquad \boxed{f_* 1 - \left(\tfrac{1}{2} f_* c_1(L)\right) P_1 \in 2U^{-2}(pt).}$$

Now use the fact that $\operatorname{Ker} w^2 = \operatorname{Ker} w$, which follows after we know $U(pt)$ is torsion free, to obtain

$$(5) \qquad w\, Q(f_* 1) = f_* 1 + \left(\tfrac{1}{2} f_* c_1 L\right)\left[2G(w,0) - G(w,w)\right]$$

Now restrict via $B\mathbb{Z}_2 \to pt$ and this becomes

$$f_* 1 = \left(\tfrac{1}{2} f_* c_1 L\right) P_1$$

which is what we want, except for a discrepancy in sign. Thus for P_1 we have $c_1 L = -c_1 \tau = -2H$ and $f_* c_1 L = -2$?

~~Thus~~ ~~Now let us use the fact which we shall have to eventually prove directly that~~ $\cancel{\operatorname{Ker} w = U(pt)[\mathbb{Z}_2 \to pt]}$. ~~Using (4) and (4') we find that~~

(6) has the following consequences for $U^{-2}(pt)$.

Let

$$T(X) = -\tfrac{1}{2} f_* c_1(L) \in U^0(pt) = \mathbb{Z}$$

Then for $X = \mathbb{C}P^1$, we have

$$\tau_{\mathbb{C}P^1} = \frac{2\theta(1)}{\theta} \qquad c_t(\tau) = (1+tH)^2 = 1+2tH$$

$$c_1(\nu) = -c_1(\tau) = -2H$$

$$T(\mathbb{C}P^1) = -\tfrac{1}{2}(-2) = 1$$

T is the Todd genus of X. Then we have an exact sequence

$$0 \longrightarrow K \longrightarrow U^{-2}(pt) \xrightarrow{\ T\ } \mathbb{Z} \longrightarrow 0$$

From (6) we have

$$\alpha + T(\alpha)P_1 \in 2U^{-2}(pt)$$

Thus $T(\alpha) = 0 \implies \alpha = 2\beta$. $2T(\beta) = 0 \implies T(\beta) = 0 \implies \beta = 2\gamma$ etc, and so K is a 2-divisible abelian group. Therefore if we knew that $U^{-2}(pt)$ were finitely generated beforehand we could conclude that $U^{-2}(pt)$ has no 2-torsion. We conclude also that

$$[X] - T(X)P_1 \in K.$$

September 10, 1969

Induction and restriction formulas for equivariant cobordism.

Let G be a finite group, ~~and~~ let H be a subgroup, and let $j : K \longrightarrow G$ be a homomorphism. Let X be an H-manifold. Then we have maps

$$U_H(X) \xrightarrow[\text{ind}]{\cong} U_G(G \times_H X) \xrightarrow[\text{res}]{} U_K(G \times_H X)$$

whose composition we want to calculate à la Mackey. ~~We have a K orbit decomposition~~

$$\cancel{G/H = \coprod_{KgH} KgH/H \simeq \coprod_{KgH} K/K \cap gHg^{-1}}$$

~~where the sum is taken over the [illegible] elements of $K\backslash G/H$. Denote $i : H \to G$ the inclusion and for double cosets KgH~~

Let S be a system of representatives for the left K- right H cosets of G, so that we have a K-orbit decomposition

$$G/H = \coprod_{g \in S} KgH/H \simeq \coprod_{g \in S} K/K \cap gHg^{-1}.$$

For an element $g \in G$ define maps

$$j_g : K \cap gHg^{-1} \longrightarrow H \quad \text{(induced by } j\text{)}$$

$$i_g : K \cap gHg^{-1} \longrightarrow K \quad \text{(inclusion)}$$

here $K \cap gHg^{-1} = j^{-1}(gHg^{-1})$

There is a corresponding decomposition

$$
\begin{tikzcd}
G\times_H X \arrow[d] & \coprod_{g\in S} K \times_{K\cap gHg^{-1}} X \arrow[l, "\simeq"'] \arrow[d] \\
G/H & \coprod_{g\in S} K/K\cap gHg^{-1} \arrow[l, "\simeq"']
\end{tikzcd}
$$

(i_g, j_g)

Claim $\exists$ commutative diagram

$$
\text{(i)}\quad
\begin{tikzcd}
U_H(X) \arrow[r, "\sim", "\mathrm{ind}\ i_*"'] \arrow[d, "(j_g^*)_{g\in S}"'] & U_G(G\times_H X) \arrow[r, "j^*"] & U_K(G\times_H X) \arrow[d, "\cong"] \\
\prod_{g\in S} U_{K\cap gHg^{-1}}(X) \arrow[rr, "(i_{g*})_{g\in S}", "\cong"'] & & \prod_{g\in S} U_K(K\times_{K\cap gHg^{-1}} X)
\end{tikzcd}
$$

The proof is clear by the universal property of U_H.

Suppose now that X is a G-manifold. Then we have an isomorphism of G-manifolds

$$G\times_H X \xrightarrow{\cong} G/H\times X$$
$$(g,\ x) \longmapsto (gH,\ gx)$$

which enables us to define a transfer or induction homomorphism

$$U_H(X) \longrightarrow U_G(G\times_H X) \cong U_G(G/H\times X) \xrightarrow{(pr_2)_*} U_G(X),$$

which we will denote by i_* (at the expense of confusion of notation.) We have various commutative diagrams

$$
\begin{tikzcd}
U_G(G\times_H X) \arrow[r, "j^*"] \arrow[d, phantom, "\cong" sloped] & U_K(G\times_H X) \arrow[d, phantom, "\cong" sloped] \\
U_G(G/H\times X) \arrow[r, "j^*"] \arrow[d, "pr_{2*}"] & U_K(G/H\times X) \arrow[d, "pr_{2*}"] \\
U_G(X) \arrow[r, "j^*"] \arrow[d, "j_g"] & U_K(X) \arrow[d, "\cong"] \\
\prod_{g\in S} U_{K\cap gHg^{-1}}(X) \arrow[r, "\prod j^*"] \arrow[ru, bend right, "\prod i_{g*}"'] & \prod_{g\in S} U_K(K\times_{K\cap gHg^{-1}} X) \arrow[u, "\prod j"']
\end{tikzcd}
$$

which yields the Mackey formula

$$\boxed{j^* i_* = \sum i_{g*} j_g^*}$$

September 12, 1969.

Total Steenrod operation.

Recall that if $i: H \to G$ is an injection of finite groups and if X is a G-manifold, there is the induction or transfer map

$$i_*: U_H(X) \longrightarrow U_G(X)$$

defined as the composition

$$U_H(X) \xrightarrow{\sim} U_G(G\times_H X) \xrightarrow{f_*} U_G(X)$$

where f is the map

$$G\times_H X \xrightarrow{\sim} G/H \times X \xrightarrow{pr_2} X$$

$$(g, x) \longmapsto (gH, gx).$$

f is oriented because it is a covering with fibre G/H.

If $f: K \to G$ is a homomorphism of finite groups we have a restriction (or inflation) map defined for G manifolds

$$f^*: U_G(X) \longrightarrow U_K(X).$$

We have the basic formula for an injection $i: H \to G$

$$i_* i^* x = i_* 1 \cdot x$$

where $i_* 1 = [G/H \longrightarrow pt] \in U^0_G(pt).$

Mackey formula:

$$\begin{tikzcd} & H \arrow[d, "i"] \\ K \arrow[r, "j"] & G \end{tikzcd}$$

Let S be a system of representatives for the cosets $\{KxH\}$ so that

$$G/H = \coprod_{g \in S} KgH/H \cong \coprod_{g \in S} K/j^{-1}(gHg^{-1})$$

and let

$$j_g : j^{-1}(gHg^{-1}) \longrightarrow H \qquad \text{be given by}$$
$$k \longmapsto g^{-1} j(k) g$$

and let $i_g : j^{-1}(gHg^{-1}) \longrightarrow K$ be the inclusion.

Then

$$\boxed{j^* i_* = \sum_{g \in S} (i_g)_* (j_g)^*}$$

Let

$$\tilde{F}(X) = \dot{\prod_{n \geq 1}} U^{n*}_{\Sigma_n}(X^n) \qquad * \text{ even}$$

denotes the subset of the direct product consisting of sequences $(\alpha_n)_{n \geq 1}$ such that for $n = i+j$ $i, j > 0$

$$\operatorname{res}^{\Sigma_n}_{\Sigma_i \times \Sigma_j} \alpha_n = \alpha_i \otimes \alpha_j ,$$

Here $\alpha_n \in U^{nq}_{\Sigma_n}(X^n)$ and $\alpha_i \otimes \alpha_j$ denotes the image of $\alpha_i \times \alpha_j$ under the map

$$U^{iq}_{\Sigma_i}(X^i) \times U^{jq}_{\Sigma_j}(X^j) \longrightarrow U^{nq}_{\Sigma_i \times \Sigma_j}(X^i \times X^j).$$

Define operations of addition and multiplication in $\tilde{F}(X)$ by

$$(\alpha_n) + (\beta_n) = \Big(n \mapsto \sum_{\substack{i+j=n \\ i,j \geq 0}} \operatorname{ind}^{\Sigma_n}_{\Sigma_i \times \Sigma_j} \alpha_i \otimes \beta_j\Big)$$

$$(\alpha_n)(\beta_n) = (\alpha_n \beta_n)$$

Here we adopt the convention that $\alpha_0 = 1$ and that

$$\operatorname{ind}^{\Sigma_n}_{\Sigma_0 \times \Sigma_n} 1 \otimes \beta_n = \beta_n$$

<u>Proposition:</u> $\tilde{F}(X)$ is a ring.

<u>Proof:</u> First we show that addition and multiplication are well-defined. Given $n = a+b$ $a, b > 0$

$$\operatorname{res}^{\Sigma_n}_{\Sigma_a \times \Sigma_b} \sum_{i+j=n} \operatorname{ind}^{\Sigma_n}_{\Sigma_i \times \Sigma_j} \alpha_i \otimes \beta_j = ?$$

So we need to use the Mackey formula. Recall

$$\Sigma_n / \Sigma_i \times \Sigma_j \cong i,j \text{ shuffles of } \{1, \dots, n\}$$

hence an orbit of $\Sigma_a \times \Sigma_b$ on this contains a unique representative of the form

$$\underbrace{\overset{a'}{\rule{1cm}{0.4pt}}\ \overset{a''}{\rule{1cm}{0.4pt}}}_{a}\ \underbrace{\overset{b'}{\rule{1cm}{0.4pt}}\ \overset{b''}{\rule{1cm}{0.4pt}}}_{b} \qquad \begin{aligned} i &= a'+b' \\ j &= a''+b'' \end{aligned}$$

Let $g_{a',a'',b',b''}$ be the (i,j)-shuffle permutation sending

$$\overset{a'}{\rule{1cm}{0.4pt}}\ \overset{b'}{\rule{1cm}{0.4pt}}\ \overset{a''}{\rule{1cm}{0.4pt}}\ \overset{b''}{\rule{1cm}{0.4pt}} \longmapsto \overset{a'}{\rule{1cm}{0.4pt}}\ \overset{a''}{\rule{1cm}{0.4pt}}\ \overset{b'}{\rule{1cm}{0.4pt}}\ \overset{b''}{\rule{1cm}{0.4pt}}.$$

~~This is an i,j-shuffle~~ where $i = a'+b'$ and $j = a''+b''$. ~~Observe that as a', a'', b', b'' run over integers ≥ 0 with $a = a'+a''$, $b = b'+b''$ we get each double coset Σ_a~~ Fix the double coset $(\Sigma_a \times \Sigma_b)\, g_{a'a''b'b''} (\Sigma_i \times \Sigma_j)$ and calculate the contribution $(\iota_g)_* (\gamma_g)^* \alpha_i \otimes \beta_j$ to the Mackey formula

$$H = \Sigma_i \times \Sigma_j \quad , \quad K = \Sigma_a \times \Sigma_b$$

$$K \cap gHg^{-1} = \Sigma_{a'} \times \Sigma_{a''} \times \Sigma_{b'} \times \Sigma_{b''}$$

ι_g : $K \cap gHg^{-1} \longrightarrow K$ is product of inclusions $\Sigma_{a'} \times \Sigma_{a''} \longrightarrow \Sigma_a$
$\Sigma_{b'} \times \Sigma_{b''} \longrightarrow \Sigma_b$

$j_g : K \cap gHg^{-1} \to H$ is the map

$$\Sigma_{a'} \times \Sigma_{a''} \times \Sigma_{b'} \times \Sigma_{b''} \longrightarrow \Sigma_i \times \Sigma_j$$

Thus

$$j_g^*(\alpha_i \otimes \beta_j) = \alpha_{a'} \otimes \beta_{a''} \otimes \alpha_{b'} \otimes \beta_{b''}$$

$$(\iota_g)_* j_g^*(\alpha_i \otimes \beta_j) = \left(\operatorname{ind}_{\Sigma_{a'} \times \Sigma_{a''}}^{\Sigma_a} (\alpha_{a'} \otimes \beta_{a''})\right) \otimes \left(\operatorname{ind}_{\Sigma_{b'} \times \Sigma_{b''}}^{\Sigma_b} (\alpha_{b'} \otimes \beta_{b''})\right)$$

Note that i, j are determined by a', b', a'', b'' so we have

$$\operatorname{res}^{\Sigma_n}_{\Sigma_a \times \Sigma_b} \sum_{i+j=n} \operatorname{ind}^{\Sigma_n}_{\Sigma_i \times \Sigma_j} \alpha_i \otimes \beta_j = \sum_{\substack{a'+a''=a \\ b'+b''=b}} \operatorname{ind}^{\Sigma_a}_{\Sigma_{a'} \times \Sigma_{a''}} (\alpha_{a'} \otimes \beta_{a''}) \otimes \left(\operatorname{ind}^{\Sigma_b}_{\Sigma_{b'} \times \Sigma_{b''}} \alpha_{b'} \otimes \beta_{b''}\right)$$

which proves that $(\alpha_n) + (\beta_n)$ indeed is again an element of $\tilde{F}(X)$.

It is clear that ~~addition~~ multiplication is well-defined on $\tilde{F}(X)$ and is associative with unit $(1)_{n \geq 1}$. Addition is pretty clearly associative with zero $(0)_{n \geq 1}$ and inverse. It remains to prove distributivity:

$$[(\alpha_n) + (\beta_n)](\gamma_n) = \left(n \longmapsto \sum_{i+j=n} (\operatorname{ind} \alpha_i \otimes \beta_j)\, \gamma_n\right)$$

$$(\alpha_n)(\gamma_n) + (\beta_n)(\gamma_n) = \left(n \longmapsto \sum_{i+j=n} \operatorname{ind} \alpha_i \gamma_i \otimes \beta_j \gamma_j\right)$$

These are equal because

$$\mathrm{ind}(\alpha_i\gamma_i \otimes \beta_j\gamma_j) = \mathrm{ind}\left((\alpha_i \otimes \beta_i)\ \mathrm{res}\ \gamma_n\right)$$
$$= \left(\mathrm{ind}\,(\alpha_i \otimes \beta_i)\right)\cdot\gamma_n \quad .$$

Thus it is for the distributive law that we must restrict to the subset $\tilde{F}(X)$ of $\prod_{n\geq 1} \mathcal{U}_{\Sigma_n}(X^n)$.

Clearly $\tilde{F}(X)$ is a contravariant functor of X with values in rings. If $f: X \to Y$ is proper and $\mathcal{U}^{(ev)}$ oriented then we can define

$$f_* : \tilde{F}(X) \longrightarrow \tilde{F}(Y) \qquad \text{by}$$

$$f_*(\alpha_n) = (f_*^n \alpha_n)$$

It is immediate that f_* is an additive homomorphism and that the homotopy + ~~[illegible]~~ transversal cartesian axioms hold for $\tilde{F}(X)$. Finally one notes that

$$\tilde{F}(X \amalg Y) \simeq \tilde{F}(X) \oplus \tilde{F}(Y).$$

In effect as a Σ_n manifold

$$(X \amalg Y)^n = \coprod_{i+j=n} \Sigma_n \times_{(\Sigma_i \times \Sigma_j)} (X^i \times Y^j).$$

Thus $\tilde{F}(X)$ ~~[illegible]~~ satisfies the ~~any~~ axioms so there is a unique natural ring homomorphism

$$\tilde{Q} : \mathcal{U}^{(ev)}(X) \longrightarrow \tilde{F}(X)$$

compatible with Gysin homomorphism. The nth component is

$$\tilde{Q}_n : U^{2q}(X) \longrightarrow U^{2qn}_{\Sigma_n}(X^n).$$

These are the <u>external</u> Steenrod operations. Next restrict to diagonal:

Set

$$F(X) = \dot{\prod_{n\geq 1}} U^{n*}_{\Sigma_n}(X) \qquad *\ \text{even}$$

be the subset of the direct product defined in same way as $\tilde{F}(X)$ and let the same formulas be used to define a ring structure on $F(X)$. Then the diagonal gives a map

$$\Delta : \tilde{F}(X) \longrightarrow F(X)$$

which is a natural ring homomorphism, <u>not</u> compatible with Gysin. However if $i: Z \to X$ is an embedding, then the clean intersection formula gives that

$$\Delta^*_{X^n}(i^n)_* z = f_*(e(\eta_n \otimes \nu_i)\, \Delta^*_{Z^n} z)$$

where η_n is the augmentation ideal of $R\Sigma_n$. Set

$$Q = \Delta\tilde{Q} : U^{(ev)}(X) \longrightarrow F(X)$$

with nth component

$$Q_n : U^{2q}(X) \longrightarrow U^{2qn}_{\Sigma_n}(X).$$

These are the <u>internal</u> Steenrod operations. For an oriented embedding

$i: Z \to X$, we have

$$Q_n(i_* z) = i_*\big(e(\eta_n \otimes \nu_i)\, Q_n z\big)$$

where ν_i denotes the honest normal bundle of i. For a general proper $U^{(ev)}$-oriented map $f: X \to Y$ we have

$$e(\eta_n)^r\, Q_n(f_* x) = f_*\big(e(\eta_n \otimes (\nu_f + r))\, Q_n x\big)$$

for r large (i.e. $r > \dim X$), so that $\nu_f + r$ is a bundle on X well-defined up to isomorphism.

This is all wrong – mistake on page 10:
It can be rectified if one works with $U^*[\frac{1}{2}]$, but then is less interesting

<u>Exactness for $\tilde{F}$:</u>

Let $\tilde{F}_k(X) = \dot{\prod}_{1\le n\le k} U^{n*}_{\Sigma_n}(X^n)$. If U is an open subset of X, set

$$\tilde{F}_k(X,U) = \dot{\prod}_{1\le n\le k} U_{\Sigma_n}\big((X,U)^n\big)$$

where

$$(X,U)^n = \Big(X^n, \bigcup_{i=0}^{n-1} X^{n-i-1}\times U\times X^i\Big)$$

I want to determine if the sequence

$$\tilde{F}_k(X,U) \longrightarrow F_k(X) \longrightarrow F_k(U)$$

is exact, at least for nice U, i.e. complement of a submanifold.

We first consider $k=2$.

$$\tilde{F}_2(X) = \{(\alpha,\beta)\in U^{2*}_{\mathbb{Z}_2}(X^2)\times U^*(X) \mid \operatorname{res}\alpha = \beta\otimes\beta\}.$$

Let $j: U\longrightarrow X$ be the inclusion and suppose that

$$(j^2)^*\alpha = 0\,,\quad j^*\beta = 0 \qquad (\alpha,\beta)\in\tilde{F}_2(X).$$

~~Since we have read we~~ Let $\beta'\mapsto\beta$ under the map from $U(X,U) \xrightarrow{\ \ } U(X)$. There are exact sequences

$$(*)\qquad \begin{tikzcd} \arrow[r, "\delta"] & \mathcal{U}_{\mathbb{Z}_2}(X^2, \mathcal{U}^2) \arrow[r] \arrow[d, "res"] & \mathcal{U}_{\mathbb{Z}_2}(X^2)^{\alpha} \arrow[r] \arrow[d, "res"] & \mathcal{U}_{\mathbb{Z}_2}(\mathcal{U}^2) \arrow[r, "\delta"] \arrow[d, "res"] & \cdots \\ \arrow[r, "\delta"] & \mathcal{U}(X^2, \mathcal{U}^2)^{\mathrm{im}(\beta'\otimes\beta')} \arrow[r] & \mathcal{U}(X^2)^{\beta\otimes\beta} \arrow[r] & \mathcal{U}(\mathcal{U}^2) \arrow[r, "\delta"] & \cdots \end{tikzcd}$$

NO only for G-trivial spaces is inflation defined

and the bottom sequence is a retract of the top using the "inflation" map $\mathcal{U}(X) \longrightarrow \mathcal{U}_G(X)$ corresponding to the group homomorphism $G \longrightarrow e$. Here $\mathrm{im}(\beta'\otimes\beta')$ ~~is an abuse of notation for the product~~ denotes the image of $\beta'\otimes\beta'$ under the map

$$\mathcal{U}(X^2, X\times\mathcal{U} \cup \mathcal{U}\times X) \longrightarrow \mathcal{U}(X^2, \mathcal{U}^2).$$

By diagram chasing $\overset{\mathrm{in}\,(*)}{\text{one}}$ sees that $\exists\, \alpha' \in \mathcal{U}_{\mathbb{Z}_2}(X^2, \mathcal{U}^2)$ restricting to $\mathrm{im}\,(\beta'\otimes\beta')$ and which gives α on forgetting supports.

Next we consider the commutative diagram

$$(**)\qquad \begin{tikzcd} \alpha' \quad \mathcal{U}_{\mathbb{Z}_2}(X^2, \mathcal{U}^2) \arrow[r] \arrow[d, "res"] & \overset{(\alpha'|V)}{\mathcal{U}_{\mathbb{Z}_2}(\cancel{(X\times\mathcal{U})\cup(\mathcal{U}\times X)}\,V, \mathcal{U}^2)} \arrow[d, "res"] & \mathcal{U}(\cancel{X\times\mathcal{U}\cup\mathcal{U}\times X}\,V, X\times\mathcal{U}) \arrow[l, "ind"', "\sim"] \arrow[d, "{(id,\tau)}"] \\ \beta'\otimes\beta' \quad \mathcal{U}(X^2, \mathcal{U}^2) \arrow[r, "res"] & \mathcal{U}(V, \mathcal{U}^2) & \mathcal{U}(V, X\times\mathcal{U}) \oplus \mathcal{U}(V, \mathcal{U}\times X) \arrow[l, "\sim"] \end{tikzcd}$$

Here $V = (X\times\mathcal{U})\cup(\mathcal{U}\times X)$. We must know ~~that~~ for a closed subset Y of a $\mathbb{Z}_2$-manifold V such that $Y \cap \tau Y = 0$ that

$$\mathrm{ind}: \ \mathcal{U}_Y(V) \xrightarrow{\ \sim\ } \mathcal{U}_{\mathbb{Z}_2,\, Y\cup\tau Y}(V)$$

~~Now It seems necessary to require that~~ No hypotheses $\overset{\text{on } Y}{\text{are}}$ needed since V is normal and so $\exists$ an open $W \supset Y$ with $W \cap \tau W = 0$.

The diagram $(**)$ shows that $(\alpha'|V) \in U_{\mathbb{Z}_2}(V, U^2)$ is zero. So we consider the map of exact sequences

$$\begin{tikzcd} U_{\mathbb{Z}_2}(X^2; V) \arrow[r] \arrow[d] & U_{\mathbb{Z}_2}(X^2; U^2) \arrow[r] \arrow[d] & U_{\mathbb{Z}_2}(V, U^2) \arrow[d] \\ U(X^2; V) \arrow[r, "\beta'\otimes\beta'"] & U(X^2; U^2) \arrow[r, "\mathrm{im}(\beta'\otimes\beta')"] & U(V, U^2) \end{tikzcd} \qquad \alpha' \mapsto 0$$

of which the bottom is a retract of the top and diagram chasing shows that $\exists\, \alpha''$ hitting α' and $\beta'\otimes\beta'$. Then $(\alpha'', \beta') \in \tilde{F}(X, U)$ hits (α, β) and so the sequence

$$\tilde{F}_2(X, U) \longrightarrow \tilde{F}_2(X) \longrightarrow \tilde{F}_2(U)$$

is exact.

Now let us suppose that ~~[illegible]~~ we have

$$Y \xrightarrow{i} X \xleftarrow{j} U$$

where i is ~~an~~ $U^{(\omega)}$ oriented closed embedding (of dim $2r$) with complement j. Then I wish to check that the Thom-Gysin isomorphism

$$\hat{i}_* : \quad \tilde{F}^q(Y) \xrightarrow{\sim} \tilde{F}^{q+2r}(X, U)$$

~~[illegible]~~ given by the product of the Thom isomorphisms

$$(i^n)_* : \quad U^{nq}_{\Sigma_n}(Y^n) \xrightarrow{\sim} U^{n(q+2r)}_{\Sigma_n}((X, U)^n)$$

is a well-defined $\tilde{F}(X)$-module homomorphism. But all this ~~everything~~

follows ~~[illegible]~~ formally from the properties of $\tilde{F}$ such as its ring structure, Gysin homomorphism, etc., which we have already checked.

September 15, 1969: Trace and ~~[illegible]~~ norm.

Let $f: X \longrightarrow Y$ be a finite covering (etale) of degree d. Then we have the ~~[illegible]~~ Gysin map

$$f_* : U^q(X) \longrightarrow U^q(Y)$$

which is a kind of trace map from X to Y. Now I want to define a norm map

$$\mathrm{Norm}\, f : U^{2q}(X) \longrightarrow U^{2qd}(Y).$$

The reason for this is that one needs a formula

$$e(f_* E) = \mathrm{Norm}_f\, e(E)$$

if E is ~~an~~ $U^{(ev)}$-oriented bundle over X.

~~[illegible]~~

First suppose that f is a principal G-bundle. Then we have

$$U^{2q}(X) \xrightarrow{Q_d} U^{2qd}_{\Sigma_d}(X^d) \xrightarrow{\mathrm{res}^{\Sigma_d}_G} U^{2qd}_G(X^d).$$

It is more clear to think of $X^d = \mathrm{Map}(G, X)$ where G acts on itself on the right. Thus this map sends $f_* 1$, $f: Z \to X$ into

$$\mathrm{Map}(G,f) : \mathrm{Map}(G,Z) \longrightarrow \mathrm{Map}(G,X).$$

Since G acts on X there is a canonical equivariant map

$$X \longrightarrow \mathrm{Map}(G,X)$$
$$x \longmapsto (g \longmapsto gx)$$

and so after pull-back we get a map

$$U^{2q}(X) \longrightarrow U^{2qd}_G(X) \simeq U^{2qd}(Y)$$

which we defined to be Norm f. (Assume Y connected)

In the general case there is a ~~normal~~ Galois covering $\tilde{Y} \longrightarrow Y$ with group G and a G-set S such that

$$X = S \times_G \tilde{Y} \qquad \text{(left action)}$$

Then we ~~assume that~~ define ~~the~~ Norm f as the composition

$$U^{2q}(X) \cong U^{2q}_G(S \times \tilde{Y}) \longrightarrow U^{2qd}_G(\tilde{Y}) \cong U^{2qd}(Y)$$

where the middle map takes $W \xrightarrow{f} S \times \tilde{Y}$ into the element represented by:

$$\begin{array}{ccccc}
 & & \prod_{s\in S} W_s & = & \mathrm{Map}_{/S}(S, W) \\
 & & \downarrow {\scriptstyle \prod f_s} & & \downarrow \\
\tilde{Y} & \xrightarrow{\ \Delta\ } & \prod_S \tilde{Y} & = & \mathrm{Map}_{/S}(S, S \times \tilde{Y})
\end{array}$$

From this definition one sees that the key case is the norm with respect to a map $pr_2 : S \times X \longrightarrow X$ and the general case is reduced to this one by descent. Hence verifications of transitivity +

$$e(f_* E) = \mathrm{Norm}_f \, e(E)$$

are trivial.

Analogues of the other elementary symmetric functions:

Let $f: X \longrightarrow Y$ be proper etale of degree d and let $1 \leq j \leq d$. Assuming Y connected we can find a Galois covering $\tilde{Y} \longrightarrow Y$ with group G and a set S on which G acts such that $X = \tilde{Y} \times_G S$. Then define

$$\sigma_j(f) : \mathcal{U}^{2g}(X) \longrightarrow \mathcal{U}^{2jg}(Y)$$

as the composition

$$\mathcal{U}^{2g}(X) \simeq \mathcal{U}^{2g}_G(\tilde{Y} \times S) \xrightarrow{\ \sigma_j\ } \mathcal{U}^{2jg}_G(\tilde{Y}) \cong \mathcal{U}^{2jg}(Y)$$

where σ_j is defined for a trivial map $pr_2 : \tilde{Y} \times S \longrightarrow \tilde{Y}$ as follows. ~~Suppose~~

Change notation to $pr_2 : X \times S \longrightarrow X$ and suppose $\alpha \in \mathcal{U}^{2g}(X \times S)$ represented by $Z \longrightarrow X \times S$ where

$$Z = \coprod_{s \in S} Z_s$$

Let I runs over the subsets of S with j elements and consider the maps

$$\begin{tikzcd}
 & \coprod_I \prod_{s \in I} Z_s \arrow[d] \\
\coprod_I X \arrow[r, "\amalg \Delta"] \arrow[d] & \coprod_I \prod_{s \in I} X \\
X &
\end{tikzcd}$$

This represents an element of $U^{2jq}(X)$ which we denote by $\sigma_j(\alpha)$. Note that everything is perfectly natural for a group acting on X and S.

There is a (curious?) relation between the norm and external Steenrod-operations. Namely if $X \to Y$ is a principal G bundle, then $\mathrm{Norm}_f [Z \to X]$ is represented by

$$\begin{array}{ccl} & & \prod_{g\in G} Z \\ & & \big\downarrow \prod_g f \quad \text{represents } Q_G(f_*1). \\ X & \longrightarrow & \prod_{g \in G} X \\ x & \longmapsto & (gx) \end{array}$$

in $U_G(X) \simeq U(Y)$.

Alternative definition of $\sigma_k(f) : U^{2q}(X \times S) \longrightarrow U^{2qk}(X)$.

$$U^{2q}(X\times S) \xrightarrow{Q_k} U^{2qk}_{\Sigma_k}((X\times S)^k) = U^{2qk}_{\Sigma_k}(X^k \times S^k) -$$

$$\xrightarrow{\Delta_X^*} U^{2qk}_{\Sigma_k}(X \times S^k) \longrightarrow U^{2qk}_{\Sigma_k}(X \times (S^k)_{reg}) \xrightarrow{\beta} U^{2qk}(X)$$

Here $(S^k)_{reg} \subset S^k$ is the subset of ~~dist~~ k-tuples all of whose terms are distinct and β is the composition

$$U^{2qk}_{\Sigma_k}(X \times (S^k)_{reg}) \longrightarrow U^{2qk}((X \times S^k_{reg})/\Sigma_k) = U^{2qk}(X \times (S^k_{reg})/\Sigma_k)$$

$$\xrightarrow{(pr_1)_*} U^{2qk}(X)$$

This suggests that perhaps we can analyze what happens to the ring structure. Thus there should be ring homomorphisms

$$U(X\times S) \longrightarrow \dot{\prod_{k\geq 1}} U_{\Sigma_k}(X^k\times S^k) \xrightarrow{\Delta_X^*} \dot{\prod_{k\geq 1}} U_{\Sigma_k}(X\times S^k)$$

$$\longrightarrow \dot{\prod_{k\geq 1}} U_{\Sigma_k}(X\times S^k_{reg}) = \dot{\prod_{k\geq 1}} U(X\times(S^k_{reg}/\Sigma_k))$$

and the problem is how to handle the sum maps $pr_{1*}: U(X\times(S^k_{reg}/\Sigma_k)) \longrightarrow U(X)$ with the ring structure.

Observe that induction has a multiplicative analogue

$$U^{2q}(X) \simeq U^{2q}_G(G\times X) \xrightarrow{\text{Norm}} U^{2qn}_G(X)$$

and that we have the other symmetric functions. In particular for $G = \mathbb{Z}_n$ the diagram

$$\begin{tikzcd}
U^{2q}(X) \arrow[d, "Q_n"] \arrow[r, phantom, "\simeq"] & U^{2q}_{\mathbb{Z}_n}(\mathbb{Z}_n\times X) \arrow[r, "\text{Norm}"] & U^{2qn}_{\mathbb{Z}_n}(X) \\
U^{2qn}_{\Sigma_n}(X) \arrow[urr, "\text{rest}^{\Sigma_n}_{\mathbb{Z}_n}"'] & &
\end{tikzcd}$$

commutes. Maybe one can fit the σ_j coherently as $n\to\infty$?

September 16, 1969. Roots of orientations.

Let F be a generalized cohomology theory with products. Let E be a vector bundle over a finite CW cx X, let d be an integer ≥ 1. Assume that $F^q(pt)$ is uniquely d-divisible for $q<0$ and that $d(E) = E+\cdots+E$ (d times) is endowed with an orientation for F. Then I want to show how to take the "dth root" of this orientation and get one for E.

Recall that an F-orientation for E is (by defn.) a Thom class $U \in F^q(E, E-X)$ where $q = \dim E$ such that

$$F^*(X) \xrightarrow{\sim} F^{*+q}(E, E-X)$$
$$\alpha \longmapsto \alpha . U$$

and more generally that

$$F^*(Y) \xrightarrow{\sim} F^{*+q}(E|Y, E|Y-Y)$$

for all subspaces Y of X. It's enough by the Atiyah-Hirz. spectral sequence to require that U gives an isomorphism over each point $x \in X$. Thus F-orientation = class $U \in F^q(E, E-X)$ such that U is a generator on each fiber.

If U_1 and U_2 are two F-orientations of E, then $\exists!$ unit $\lambda \in U^0(X)$ such that $U_2 = \lambda \cdot U_1$.

If E and E' are bundles with F-orientations U_E and $U_{E'}$, then $E+E'$ is endowed with the F-orientation

$U_E \cdot U_{E'}$, where this denotes the image of $U_E \otimes U_{E'}$ under the map

$$\begin{array}{c} F^q(E, E-X) \otimes F^{q'}(E', E'-X) \longrightarrow F^{q+q'}(E\times E', E\times(E'-X) \cup (E-X)\times E') \\ \downarrow \\ F^{q+q'}(E \times_X E', E\times_X E' - X) \end{array}$$

Note that under the isomorphism $E\times_X E' \simeq E'\times_X E$ the Thom class $U_E \cdot U_{E'}$ corresponds to $(-1)^{qq'} U_{E'} \cdot U_E$.

So now suppose that U_E is an F-orientation for E. Then $U_E \cdot \ldots \cdot U_E$ d times is an orientation for dE. Conversely given an F-orientation $U_{dE} \in F^{dq}(dE, dE-X)$, I want to know whether it comes from a U_E. Note that if $\lambda \in U^0(X)^*$, then

$$(\lambda U_E)\cdots(\lambda U_E) = \lambda^d (U_E \cdots U_E)$$

which shows that U_E is unique up to a dth root of 1 if it exists. So for simplicity suppose that X is connected ~~and that~~ endowed with a basepoint x_0. Then any orientation U'_E such that $U'_E|x = U_E|x$ and $U'^d_E = U^d_E$ is of the form λU_E where $\lambda^d = 1$ and $\lambda = 1 + a$ $a \in \tilde{F}^0(X)$. Now a is nilpotent:

<u>Lemma:</u> Let R be a ring and let I be an ideal in R every element of which is nilpotent. If I is d-uniquely divisible, then so is $(1+I)$ under multiplication.

Proof: The universal example is the ring $\mathbb{Z}+I$ where I is the augmentation ideal of $\mathbb{Z}[\frac{1}{d}][X]/(X^N)$, N large. Here I is nilpotent and

$$gr\,(1+I)^{\times} \simeq gr\,I^{+}$$

so the result is clear.

This lemma proves that $\lambda^d = 1 \Rightarrow \lambda = 1$ and so guarantees the uniqueness of u_E provided $F(pt)$ is a $\mathbb{Z}[\frac{1}{d}]$ algebra.

Induction over the skeleton uses only the weaker hypothesis that $F^q(pt)$ is a $\mathbb{Z}[\frac{1}{d}]$ module for $q<0$. In effect suppose that we have shown that $\lambda = 1$ on Y and that $X = Y \cup e^l$ where we may assume $l \geq 1$ since X is connected. Then we have an exact sequence of abelian groups

$$\tilde{F}^{-1}(Y) \longrightarrow F^0(X,Y) \longrightarrow \tilde{F}^0(X) \longrightarrow \tilde{F}^0(Y) \xrightarrow{\delta} F^1(X,Y)$$

where by induction we can assume that $\tilde{F}^q(Y)$ is d-uniquely divisible for $q \leq 0$. Also can assume every element of $\tilde{F}^q(Y)|^{q\leq 0}$ is nilpotent. Now $F^q(X,Y) = F^q(e^l, \dot{e}^l) \simeq F^{q-l}(pt)$. So if $l \geq 2$ this is d-uniquely-divisible. For $l=1$ we have to be careful and we argue as follows. Recall that λ restricts to 1 at each point of X, thus λ has same value 1 at the endpoints of e^1 so $\delta\lambda = 0$. The rest is the 5 lemma which shows that $\tilde{F}^q(X)$ is d-uniquely-divisible for $q \leq 0$.

For the existence of U_E given U_{dE} we proceed ~~as follows~~ in a similar fashion. First we must find U_E over the 1-skeleton. We may assume X has a single 0 cell ~~U_E given over x_0. This means that we choose~~ x_0. It is necessary to suppose U_E given over x_0 otherwise its hopeless. The orientation representation $\pi_1(X,x_0) \longrightarrow \{\pm 1\}\cdot U_E$ of E must be trivial that U_E be extended over the 1-skeleton; this requires either that $2=0$ in $F^0(pt)$ or that E be orientable in the usual sense. Now consider the step of extending from Y to $X = Y \cup e^l$ where $l \geq 2$. We are given U_E over Y with $U_E{}^d = U_{dE}|Y$. We consider

$$\delta U_E \in F^{n+1}(X^E, Y^E) \simeq F^{n+1}(e^E, \dot{e}^E) \qquad n = \dim E$$

and want to show it is zero. Let V_E be the standard orientation of $E|e$. Then $U_E|\dot{e} = \lambda V_E|\dot{e}$ where $\lambda \in 1+\tilde{F}^0(\dot{e})$. Moreover

$$\delta U_E = \delta\lambda \cdot V_E \quad \text{in } F^{n+1}(e^E, \dot{e}^E).$$

Now the important thing is that as $l \geq 2$, $1+\tilde{F}^0(\dot{e}) = 1 + F^{-l+1}(pt)\cdot\varepsilon^{l-1}$ with $(\varepsilon^{l-1})^2 = 0$ is uniquely d-divisible. Thus as $U_E^d|\dot{e} = U_{dE}|\dot{e}$ extends over the disk,

$$\delta(\lambda^d) = \delta(1+da) = d\,\delta a = 0$$

so $\delta\lambda = \delta a = 0$. Thus U_E extends over the disk and by our previous uniqueness argument can be modified to agree with U_{dE}.

Conclusion: Suppose that $F^q(pt)$ is a $\mathbb{Z}[\frac{1}{d}]$-module for $q<0$ and that E is a oriented vector bundle in the usual sense. Then any F-orientation U_{dE} on dE gives rise to a d-th root F-orientation on E provided this can be done over the zero skeleton. This dth root is uniquely determined by its restriction to the zero skeleton.

Remarks: I tried without success to produce a direct construction of U_E given U_{dE} and $U_E|X^{(1)}$. One idea would have been to find a Galois covering $\pi: \tilde{X} \to X$ with group G such that $\pi^* E$ is fiber homotopically trivial. Then one has $U_{\pi^* E}$ by existence of dth roots and it's invariant under G so ought to descend. However not every element of $J(X)$ can be killed by a finite covering e.g. X simply-connected. For X= torus this is true and might be useful.

September 17, 1969. On $SO^*(X)[\frac{1}{2}]$.

Preliminaries on formal groups over $\mathbb{Z}[\frac{1}{2}]$-algebras:

Let $F(X,Y)$ be a formal group law over a $\mathbb{Z}[\frac{1}{2}]$-algebra R. Then if γ_0 is the coordinate curve

$$(F_2\gamma_0)(X) = (X^{1/2}) +^F (-X^{1/2}) = F(X^{1/2}, -X^{1/2})$$

and ~~this is zero iff~~ clearly

$$F_2\gamma_0 = 0 \iff F(X,-X) = 0$$

$$\iff I(X) = -X.$$

Call such an F *symmetric*.
Since R is a $\mathbb{Z}[\frac{1}{2}]$-algebra, the group of curves of F is 2-uniquely-divisible and we may produce a change of variable rendering the law symmetric. Thus let

$$\gamma(X) = \left(\gamma_0 - \tfrac{1}{2} V_2 F_2 \gamma_0\right)(X)$$

$$= X -^F \left(\tfrac{1}{2}\right)_F F(X,-X)$$

or

$$\boxed{F(\gamma X, \gamma X) = F(X, I(-X))}$$

Then

$$\gamma(-X) = I\gamma X$$

so if we set

$$F_s(X,Y) = \gamma^{-1}(F(\gamma X, \gamma Y))$$

the law F_s is symmetric since $\quad F(\gamma X, \gamma(-X)) = \gamma X +^F I\gamma X = 0$

~~Let F_{univ} be a universal symmetric law over a $\mathbb{Z}[\frac{1}{2}]$-algebra.~~

Example: Over $\mathbb{Z}[\frac{1}{2}]$ consider the law $X+Y+XY$

Then

$$(1+\gamma(X))^2 = \frac{1+X}{1-X}$$

The old logarithm is

$$\log(1+X) = \sum_{n\geq 1} (-1)^{n-1} \frac{X^n}{n}$$

and the new one is

$$\frac{1}{2} \log \frac{1+X}{1-X} = \sum_{n \text{ odd}} (-1)^{n-1} \frac{X^n}{n}$$

Let F_{univ} be a universal symmetric law over a $\mathbb{Z}[\frac{1}{2}]$-algebra. Let $\gamma_{F_{univ}}$ be the symmetrizer of F_{univ} over $L[\frac{1}{2}]$ ie so that

$$\gamma^{-1}_{F_{univ}} * F_{univ} = F_{univ,s}.$$

Recall in general that if $\alpha, \beta : L \longrightarrow A$ give laws F_α, F_β and $\theta^{-1} * F_\alpha = F_\beta$ $(\Leftrightarrow l_\alpha \circ \theta = l_\beta)$ then we have an isomorphism multiplicative

$$\hat{\theta} : A_\alpha \otimes_L U(X) \xrightarrow{\simeq} A_\beta \otimes_L U(X)$$

$$c_1(L) \longmapsto \theta(c_1 L)$$

Applying this in the case where $A = L[\frac{1}{2}]$ and where $F_\alpha = F_{univ}$, $F_\beta = F_{univ,s}$, $\theta = \gamma_{F_{univ}}$ we get an isomorphism

$$U(X)[\tfrac{1}{2}] = L[\tfrac{1}{2}]_\alpha \otimes_L U(X) \xrightarrow[\sim]{\hat{\theta}_{F_{univ}}} L[\tfrac{1}{2}]_\beta \otimes_L U(X)$$

$$c_1 L \longmapsto \gamma_{F_{univ}}(c_1 L) \, .$$

Here
$$\ell_\alpha(Z) = \sum p_n \frac{Z^n}{n+1}$$
$$\ell_\beta(Z) = \sum_{n \text{ even}} p_n \frac{Z^n}{n+1}$$

and
$$\ell_\alpha \circ \gamma_{F_{univ}} = \ell_\beta \, .$$

Combining the above with the standard isomorphism

$$L[\tfrac{1}{2}]_\beta \otimes_L U(X) \simeq L[\tfrac{1}{2}] \otimes_{L_s} (L_s \otimes_L U(X))$$

where $L_s \to L[\frac{1}{2}]$ sends F_{univ} to $F_{univ,s}$ and $L \to L_s$ sends F_{univ} to $F_{univ,s}$, we get an isomorphism

$$\boxed{\begin{aligned} U(X)[\tfrac{1}{2}] &\xrightarrow{\sim} L[\tfrac{1}{2}] \otimes_{L_s} (L_s \otimes_L U(X)) \\ c_1(L) &\longmapsto \gamma(c_1^u L) \\ \text{where } \gamma &= \Big(\sum p_n \frac{Z^{n+1}}{n+1}\Big)^{-1} \circ \Big(\sum_{n \text{ even}} p_n \frac{Z^{n+1}}{n+1}\Big) \in L[\tfrac{1}{2}][[Z]] \end{aligned}}$$

The map in the opposite direction is easier to describe: start with

$$U(X) \longrightarrow U(X)[\tfrac{1}{2}]$$

$$c_1(L) \longmapsto \Big(\sum_{n\,\mathrm{even}}' P_n \frac{z^{n+1}}{n+1}\Big)^{-1}\Big(\sum_{n\geq 0}' P_n \frac{(c_1 L)^{n+1}}{n+1}\Big) = \gamma^{-1}(c_1 L)$$

and observe that $c_1(L)$ and $-c_1(L^\vee)$ have the same image so that F^U becomes symmetric. Hence it induces a map

$$L_s \otimes_L U(X) \longrightarrow U(X)[\tfrac{1}{2}]$$

which in turn extends to an L-linear map

$$L[\tfrac{1}{2}] \otimes_{L_s} (L_s \otimes_L U(X)) \longrightarrow U(X)[\tfrac{1}{2}]$$

which is the inverse isomorphism to the one on the preceding page.

~~Remark / We know that~~

<u>Proposition</u>: $L[\frac{1}{2}]$ is a polynomial ring over L_s with one generator of every odd degree. L_s is a poly. ring over $\mathbb{Z}[\frac{1}{2}]$ with one generator of every even degree.

<u>Proof</u>: We know that $L[\frac{1}{2}]$ is a graded ring which is a polynomial ring with one generator of each degree. L_s is a retract of L. Consider the projection operator $\varepsilon: L_s \to L[\frac{1}{2}] \to L_s$

acting on the indecomposable subspace of $L[\frac{1}{2}]$ which is $\mathbb{Z}[\frac{1}{2}]$ in each degree. We know that

$$\varepsilon P_n = \begin{cases} 0 & n \text{ odd} \\ P_n & n \text{ even} \end{cases}$$

so $\varepsilon = (-1)^n$ on $Q_n(L[\frac{1}{2}])$ and the proposition follows immediately.

Theorem: ~~Proposition~~ $$L_\rho \otimes_L U(X) \xrightarrow{\sim} SO(X)[\tfrac{1}{2}].$$

or equivalently $SO[\frac{1}{2}]$ is a universal gen. coh. theory with products and U-classes such that the group law F^{SO} is symmetric.

Proof: ~~Let H be a mult. gen. coh. theory with U-classes such that $H(pt)$ is a $\mathbb{Z}[\frac{1}{2}]$ algebra and is~~

Step 1: F^{SO} is symmetric: Recall how the negative is calculated in a cobordism theory Γ. Let $\alpha \in \Gamma(X)$ be represented as $\alpha = f_* 1$ where $f: Z \to X$ is proper and oriented for the theory, which means that $\nu_f : Z \to BO$ is pulled back to a map $Z \xrightarrow{o_f} A$. More precisely we have

$$\begin{tikzcd} & A \arrow[d, "\pi \text{ given}"] \\ Z \arrow[ur, "o_f"] \arrow[r, "\nu_f"'] \arrow[ur, phantom, bend right=10, "h"] & BO \end{tikzcd}$$

where h is a homotopy from πo_f to ν_f. Now

$$\nu_f = \nu_i - n \simeq (\nu_i + 1) - (n+1) \overset{\theta}{\simeq} (\nu_i+1) - (n+1) \simeq \nu_f,$$

where θ is the isomorphism that is identity on ν_i and $(n+1)$ but (-1) on ~~the~~ the trivial line bundle added to ν_i, gives a self homotopy of ν_f to itself which may be added to h to get a new orientation of f which we denote $-\sigma_f$. Of course $-f_* 1$ is represented by $(f, -\sigma_f)$.

In the case of SO one sees that changing the orientation of a bundle E, that is, the isom $\Lambda^n E \simeq 1$ is the same for stable bundles as the operation just described. ~~If~~ Let L is a complex line bundle over a manifold X and $L^\vee$ the dual bundle and $\theta : L \simeq L^\vee$ the conjugate linear isomorphism given by a hermitian structure. Then

$$\begin{tikzcd} & X \arrow[dl, "i"'] \arrow[dr, "j"] & \\ L \arrow[rr, "\theta", "\simeq"'] & & L^\vee \end{tikzcd}$$

i, j zero section

is commutative but the orientations of L and $L^\vee$ are reversed by θ. Thus $\theta(i_* 1) = -j_* 1$ ~~and so~~ in $SO(L^\vee, L^\vee - X)$ so

$$\boxed{c_1^{SO}(L) = -\, c_1^{SO}(L^\vee)}$$

proving that F^{SO} is symmetric. By the splitting principle one sees that for an arbitrary complex bundle

$$\boxed{c_n^{SO}(\check{E}) = (-1)^n c_n^{SO}(E)}$$

~~Part~~ Step 2: $U[\frac{1}{2}]$ orientation classes for SO-bundles: We want to produce a multiplicative stable natural transformation

$$SO(X) \longrightarrow U(X)[\tfrac{1}{2}]$$

$$c_1^{SO}(L) \longmapsto \gamma^{-1}(c_1^{U}L)$$

where γ is determined by

$$F^{U}(\gamma X, \gamma X) = F^{U}(X, I^{U}(-X))$$

so that $\cancel{\gamma(\ldots)}$ $\gamma(-X) = I(\gamma X)$ or better

$$\boxed{\gamma^{-1}(IX) = -\gamma^{-1}X}$$

(Thus we think of γ^{-1} as an approximation to a logarithm and

$$\gamma^{-1} \circ \sum_n P_n \frac{X^{n+1}}{n+1} = \sum_{n \text{ even}} P_n \frac{X^{n+1}}{n+1}.)$$

So let

$$\varphi(X) = \frac{\gamma^{-1}(X)}{X} \in 1 + U(pt)[\tfrac{1}{2}][[X]]^{+}$$

and extend it to a mult. characteristic class from complex bundles to $U(\cdot)[\frac{1}{2}]$. I claim that as $U(pt)[\frac{1}{2}]$ is a $\mathbb{Z}[\frac{1}{2}]$-algebra it is possible to apply our work on roots of orientations to orient SO bundles for $U[\frac{1}{2}]$ as follows:

Given E an $SO(n)$-bundle, let $\lambda_{E_{\mathbb{C}}} \in \tilde{U}^{2n}(X^{E_{\mathbb{C}}})$ be the standard Thom class. Use the isomorphism

$$E + E \simeq E + iE \simeq E_{\mathbb{C}}$$

to identify X^{E+E} with $X^{E_{\mathbb{C}}}$. Over any point $x \in X$ we have a Thom class $\mu_E \in \widetilde{SO}^n(X^E)$ coming from the orientation of E and I claim that

$$\mu_E^2 = (-1)^{\frac{n(n-1)}{2}} \lambda_{E_{\mathbb{C}}}|x$$

In effect if $e_1, \dots, e_n$ is a frame for $E|x$ with the correct orientation μ_E, then $(e_1,0), \dots, (e_n,0), (0,e_1), \dots, (0,e_n)$ is the frame for $(E+E)|x$ with orientation μ_E^2, while $(e_1,0),(0,e_1), \dots (e_n,0),(0,e_n)$ is the frame with orientation $\lambda_{E_{\mathbb{C}}}|x$. Conclude that there is a unique Thom class $\mu_E \in \tilde{U}^n(X^E)[\frac{1}{2}]$ such that

$$\boxed{\mu_E^2 = (-1)^{\frac{n(n-1)}{2}} \varphi(E_{\mathbb{C}}) \cdot \lambda_{E_{\mathbb{C}}}}$$

and such that μ_E has the correct orientation at each point $x \in X$. By the uniqueness it is clear that

$$\mu_E \mu_F = \mu_{E+F}$$

$$\left((\mu_E\mu_F)^2 = (-1)^{ef} \mu_E^2 \mu_F^2 = (-1)^{ef + \frac{e(e-1)}{2} + \frac{f(f-1)}{2}} \varphi((E+F)_{\mathbb{C}}) \lambda_E \lambda_F = (-1)^{\frac{(e+f)(e+f-1)}{2}} \varphi((E+F)_{\mathbb{C}}) \lambda_{E+F} \right),$$

hence we obtain a multiplicative stable $(\mu_{1_{\mathbb{R}}}^2 = \lambda_{1_{\mathbb{C}}})$ transformation

$$SO^*(X) \longrightarrow U^*(X)[\tfrac{1}{2}]$$

such that

$$\boxed{e(E)^2 = (-1)^{\frac{n(n-1)}{2}} c_n(E_{\mathbb{C}}) \varphi(E_{\mathbb{C}})}$$

if E is an $SO(n)$ bundle. Thus for a complex line bundle L, we have

$$e(L)^2 = -c_1(L)c_1(L^\vee)\,\varphi(L)\varphi(L^\vee)$$
$$= -\gamma^{-1}(c_1L)\,\gamma^{-1}(c_1L^\vee)$$
$$= (\gamma^{-1}(c_1L))^2 \qquad \text{by } \square \text{ page 12,}$$

so
$$\boxed{e(L) = \gamma^{-1}(c_1L)}\,.$$

(twisted by γ) Now observe that we could have replaced $U[\frac{1}{2}]$ by any mult. gen. coh. th. V with U-classes and symmetric group law. In this case $\gamma = id$ so there is a transf. mult + stable

$$\Phi : SO(X) \longrightarrow V(X)$$

such that
$$c_1^{SO}(L) \longmapsto c_1^V(L).$$

By Riemann-Roch Φ is compatible with Thom classes of complex bundles, hence if $\mu_E \in \tilde{SO}^{2n}(X^E)$, then

$$\Phi(\mu_E^{SO})^2 = \Phi(\mu_E^2) = (-1)^{\frac{n(n-1)}{2}}\Phi(\lambda^{SO}_{E_{\mathbb{C}}})$$
$$= (-1)^{\frac{n(n-1)}{2}}\lambda^V_{E_{\mathbb{C}}} = (\mu_E^V)^2$$

and so by uniquess of roots $\Phi(\mu_E^{SO}) = \mu_E^V$. Therefore Φ is compatible with all SO-oriented Gysin homomorphisms and so is unique by the universal property of SO^*.

This shows that $SO[\frac{1}{2}]$ is a universal symmetric Chern theory and proves the theorem.

As customary we may define (following Bott) Pontryagin classes in SO as Chern classes of the complexification

$$p_t(E) = \sum t^n p_n(E) = c_t^{SO}(E_{\mathbb{C}})$$

As $E_{\mathbb{C}}$ is self dual the odd Chern classes are of order 2 ~~vanish~~, so only $p_{2n}(E) \in SO^{4n}[\tfrac{1}{2}]$ are possibly non-zero. If $E = L_1 + \cdots + L_n$, L_i complex line bundle ($SO(1)$-bundle), then

$$p_t(E) = c_t(E+E^\vee) = \prod_{i=1}^{n} (1-t^2 x_i^2).$$

September 18, 1969. Symplectic theory.

Symplectic group:

Let V be a complex vector space of dimension $2n$ endowed with a non-degenerate anti-symmetric quadratic form ω and let G be the autos. of (V, ω). Let K be a maximal compact subgroup of G, and let (x, y) be a hermitian inner product on V invariant under K. Then there is a unique anti-linear operator J such that

$$\omega(x, y) = (x, Jy)$$

Since

$$\begin{aligned}(x, J^2y) &= \omega(x, Jy) = -\omega(Jy, x) = -(Jy, Jx) \\ &= -\overline{(Jx, Jy)} = -\overline{\omega(Jx, y)} = \overline{\omega(y, Jx)} \\ &= \overline{(y, J^2x)} = (J^2x, y)\end{aligned}$$

and

$$(x, J^2x) = -(Jx, Jx) < 0$$

J^2 is a negative self-adjoint operator and so K must leave the eigenspaces of J^2 invariant. As K is maximal $J^2 = -aI$ where a real >0. By changing $(\,,\,)$ we can suppose $J^2 = -I$, whence K is the group of unitary matrices of V leaving J fixed.

Conclusion: $Sp(n)$ is the group of autos of $\mathbb{H}^n$, $\mathbb{H}$ = quaternions, preserving the scalar product $\sum_{k=1}^{n} x_k x_k^*$ or equivalently the subgroup of unitary autos. of $\mathbb{H}^n$ which commute with j.

Remarks:

$$Sp(1) \xrightarrow{\sim} SU(2) = Spin(4) = S^3 \quad \text{analogous to} \quad U(1) \cong SO(2).$$

$Sp(n)$ is simply-connected, hence first non-zero homotopy group is $\pi_3\, Sp(n) = \mathbb{Z}$.

One has

$$Sp(n)/Sp(n-1) \xrightarrow{\sim} S^{4n-1}$$

which gives rise to a Wang sequences

$$\longrightarrow H^{q-4n+1}(Sp(n-1)) \longrightarrow H^q(Sp(n)) \xrightarrow{i^*} H^q(Sp(n-1)) \xrightarrow{d} H^{q-4n+2}(Sp(n-1))$$

By induction one knows $H^*(Sp(n-1))$ is an exterior algebra with generators of degree $4k-1$, $k=1,2,\cdots,n-1$ and this are killed by d for dimensional reasons. Thus i^* is onto and so $H^*(Sp(n))$ is an exterior algebra with generators of degrees $4k-1$, $k=1,\ldots,n$.

If V is a mult. gen. coh. theory with Thom class for $Sp(1) = S^3$ bundles, then the projective bundle theorem holds for quaternionic bundles so one can define quaternionic chern classes

$$cq_i^V(E) \in V^{4i}(X)$$

by the usual procedures and

$$V^*(BSp(1)) = V^*(pt)[cq_1^V]$$

$$V^*(BSp(n)) = V^*(pt)[cq_1^V, \ldots, cq_n^V].$$

Now we have maps

$$Sp(n) \longrightarrow U(2n) \longrightarrow SO(4n) \longrightarrow O(4n)$$

$$MSp^*(X) \longrightarrow MU^*(X) \longrightarrow MSO^*(X) \longrightarrow MO^*(X)$$

and I would like to know what happens to the quaternionic Cherns classes.

The following geometric considerations are important. Let V be a quat. vector space of dimension n and let $H = \mathcal{O}(-1)$ be the quat. (sub-) line bundle. Then we have

$$\begin{tikzcd}
P_{\mathbb{R}}(H) \arrow[r] \arrow[d, equal] & P_{\mathbb{C}}(H) \arrow[r] \arrow[d, equal] & \mathbb{H}P(V) \\
\mathbb{R}P(V) \arrow[r] & \mathbb{C}P(V) \arrow[ur] &
\end{tikzcd}$$

In this limit this gives

$$B(\mathbb{R}^*) \longrightarrow B(\mathbb{C}^*) \longrightarrow B(\mathbb{H}^*)$$

and each is a projective bundle over the following space. The important thing about the map

$$BU(1) \longrightarrow BSp(1)$$

described above is that it is on one hand the splitting space for H as a complex bundle and on the other hand it represents the ~~map~~ base extension map $Pic_{\mathbb{C}}(X) \longrightarrow Pic_{\mathbb{H}}(X)$, $L \longmapsto \mathbb{H}\otimes_{\mathbb{C}} L$. This means that in calculating U classes of H I can assume $H = \mathbb{H}\otimes_{\mathbb{C}} L$, so

$$c_1^{u}(H) = c_2^{u}(\mathbb{H}\otimes_{\mathbb{C}} L) \qquad \text{both Euler classes.}$$

$$= c_2^{u}(L + L^{\vee}) \qquad (jL = L^{\vee})$$
$$= c_1^{u}(L)\, c_1^{u}(L^{\vee})$$

Thus
$$cq_t^{u}(E) = \prod_{i=1}^{n} (1 + t\, x_i\, I x_i)$$

if E is a quaternionic bundle $= H_1 + \cdots + H_n$ where $H_i = \mathbb{H} \otimes_{\mathbb{C}} L_i$ & $x_i = c_1(L_i)$.

Now passing from U to a symmetric theory such as SO this becomes

$$\boxed{cq_1^{SO}(\mathbb{H} \otimes_{\mathbb{C}} L) = -c_1^{SO}(L)^2}$$

(It is necessary to be more systematic. Thus given a quat. bundle F I want to know its Chern and Pontryagin classes – forgetting $\mathbb{H}$ action. Also want formulas for $\mathbb{H} \otimes_{\mathbb{C}} E$ and $\mathbb{H} \otimes_{\mathbb{R}} E$:)

<u>Forgetting structure</u>:

1) If E complex vector bundle of dimension n, then

$$P_{t^2}^{SO}(E) = c_t^{SO}(E)\, c_t^{SO}(\check{E}) \qquad \text{mod 2 torsion}$$
$$= \prod_{i=1}^{n} (1 - t^2 x_i^2)$$

if $c_t^{SO}(E) = \prod (1 + t x_i)$

2) If H is a quaternionic line bundle, then

$$cq^u_{t^2}(\mathbb{H}) = 1 + t^2 c_2^u(\mathbb{H})$$

Note that $c_1^u(\mathbb{H} \otimes_{\mathbb{C}} L) = c_1^u(L + L^\vee) = c_1^u(L) + I(c_1^u L) \neq 0$ so $cq^u_{t^2}(E)$ is a bit of a mess. In fact as remarked before, the splitting principle allows us for calculation to assume that $H = \mathbb{H} \otimes_{\mathbb{C}} L = L + L^\vee$ as a complex bundle. Then

$$E \ (\mathbb{C}x) \longmapsto cq_t(\mathbb{H} \otimes_{\mathbb{C}} E)$$

is the multiplicative characteristic class with

$$L \longmapsto 1 + t\, c_1 L \cdot I(c_1 L)$$

~~[illegible]~~ So the desired formula for expressing ~~[illegible]~~ $c_t^u(F \text{(over } \mathbb{H}))$ in terms of $cq_t^u(F)$ is found by writing ~~[illegible]~~

$$\prod_i (1 + tX_i)(1 + tIX_i)$$

~~[illegible]~~ a function of the elementary symmetric functions of the $X_i I(X_i)$. This is possible ~~since~~ if

$$(1 + tX)(1 + tIX) = 1 + t(X + IX) + t^2(X \cdot I(X))$$

$$= 1 + t\,\varphi(X \cdot IX) + t^2(X \cdot I(X))$$

Thus it seems that $\exists$ power series φ with

$$X + I(X) = \varphi(X I(X))$$

which is true, in fact the general statement which one can prove is that if $G(X,Y) = G(Y,X)$, then $G(X,IX)$ is a function of $XI(X)$.

Thus the conclusion is that if φ is chosen so that

$$(1+tX)(1+tIX) = 1 + t\varphi(X\cdot IX) + t^2(X\cdot IX)$$

then

$$c_t^u(F) = \prod_{i=1}^{n}\left(1+t\varphi(y_i) + t^2 y_i\right)$$

where

$$cq_t^u(F) = \prod_{i=1}^{n}(1+t y_i).$$

For a symmetric theory this simplifies to

$$c_t^{SO}(F) = cq_{t^2}^{SO}(F).$$

3) Combining the above

$$p_{t^2}^{SO}(F) = c_t^{SO}(F)\, c_t^{SO}(F^\vee) \qquad F^\vee = F \text{ via } j$$

$$= \left(c_t^{SO}(F)\right)^2 = \left(cq_{t^2}^{SO}(F)\right)^2$$

$$\boxed{p_t^{SO}(F) = \left\{cq_t^{SO}(F)\right\}^2}$$

One concludes that ~~for~~ the map

$$BSp \longrightarrow BSO$$

has for induced map

$$\begin{aligned} H^*(BSO) &\longrightarrow H^*(BSp) \\ p_n^H &\longmapsto \sum_{i+j=n} cq_i^H \cdot cq_j^H \end{aligned}$$

and more generally for any symmetric theory. Now

$$\begin{aligned} H^*(BSO, \mathbb{Z}[\tfrac{1}{2}]) &= \mathbb{Z}[\tfrac{1}{2}, p_1^H, p_2^H, \dots] \\ H^*(BSp, \mathbb{Z}) &= \mathbb{Z}[cq_1, cq_2, \dots] \end{aligned}$$

and

$$p_n^H \longmapsto 2cq_n^H$$

modulo decomposables so one concludes that

$$\begin{aligned} BSp &\longrightarrow BSO \\ MSp &\longrightarrow MSO \end{aligned}$$

are homotopy equivalences off 2 and hence

$$MSp^*(X)[\tfrac{1}{2}] \xrightarrow{\sim} MSO^*(X)[\tfrac{1}{2}].$$

I want eventually to find a proof of this avoiding homotopy theory.

also SU off 2

September 21, 1969. Symplectic and oriented cobordism off 2

In the following everything will be over $\mathbb{Z}[\frac{1}{2}]$ without there being any special notation.

Let $L \longrightarrow L_s$ be the map symmetrizing the universal group law. We shall prove that the natural maps

$$Sp \longrightarrow U \longrightarrow SO$$

induce isomorphisms

Thm: $\boxed{Sp^*(X) \xrightarrow{\cong} L_s \otimes_L U^*(X) \xrightarrow{\cong} SO^*(X)}$

or that $Sp(X)$ and $SO(X)$ are universal symmetric theories.

First define $U^* \longrightarrow Sp^*$ by introducing the Sp orientation on complex bundles such that

$$e^{Sp}(E)^2 = (-1)^{\dim E} e^{Sp}(\mathbb{H} \otimes_{\mathbb{C}} E).$$

More precisely if E is a $U(n)$-bundle, let $\mu_E \in Sp^{2n}(E, E-X)$ be defined by

$$\mu_E^2 = (-1)^n \lambda_{\mathbb{H} \otimes_{\mathbb{C}} E}$$

where λ is the given Thom class for Sp. Then for a complex line bundle L we have

$$e_1^{Sp}(L)^2 = - e^{Sp}(\mathbb{H} \otimes_{\mathbb{C}} L).$$

Now for a complex bundle E we have

$$\mathbb{H} \otimes_{\mathbb{C}} E \simeq \mathbb{H} \otimes_{\mathbb{C}} \bar{E}$$

$$x \otimes v \longmapsto xj \otimes v$$

thus

$$c_1(L)^2 = c_1(L^\vee)^2$$

and so as the orientations of $L, L^\vee$ differ by -1

$$\boxed{-c_1^{Sp}(L) = c_1^{Sp}(L^\vee)}$$

Thus we have made Sp^* into a symmetric Chern theory.

Suppose V is a symmetric Chern theory and consider the composition

$$\Phi : \quad Sp \longrightarrow U \longrightarrow V.$$

This is clearly compatible with Euler classes of symplectic bundles. So if L is a complex line bundle

$$\begin{aligned} \Phi\left\{c_1^{Sp}(L)\right\}^2 &= \Phi\left\{-e^{Sp}(\mathbb{H}\otimes_{\mathbb{C}} L)\right\} \\ &= -e^V(\mathbb{H}\otimes_{\mathbb{C}} L) \\ &= -c_1^V(L)\,c_1^V(L^\vee) \\ &= c_1^V(L)^2 \end{aligned}$$

as V is symmetric.

~~So by Now by universal Now by universal considerations~~

$\cancel{\Phi c_1^{Sp}(L) = \text{power}}$

So as square roots of these classes are unique $\Phi\, c_1^{Sp}(L) = c_1^V(L)$

and so Φ is compatible with Gysin homomorphisms. Thus Sp is a universal symmetric Chern theory.

Now use the same argument for SO. Let V be a symmetric Chern theory. If E is an $SO(n)$ bundle define a Thom class λ^V_E by

$$\left(\lambda^V_E\right)^2 = (-1)^{\frac{n(n-1)}{2}}\left(\mu^V_{\mathbb{C}\otimes_{\mathbb{R}} E}\right)$$

where μ^V denotes the Thom class for complex bundles. Note that if E is already a $U(m)$ bundles, then

$$\left(\lambda^V_E\right)^2 = (-1)^{\frac{2m(2m-1)}{2}}\left(\mu^V_{E\oplus E^*}\right)$$

(here use that V is symmetric)

$$= \mu^V_E \cdot \left((-1)^m \mu^V_{E^*}\right) \cong \left(\mu^V_E\right)^2.$$

and so $\lambda^V_E = \mu^V_E$. Thus we see that for a symmetric theory that the U orientation extends uniquely to an SO-orientation. Hence there is a unique map $SO \longrightarrow V$ compatible with complex Chern classes and so SO is a universal symmetric Chern theory.

The composition

$$SO(X) \xleftarrow{\sim} Sp(X) \longrightarrow U(X)$$

sends $\quad c_1^{SO}(L) \longmapsto \sqrt{-e^{Sp}(\mathbb{H}\otimes_{\mathbb{C}} L)} \longmapsto \sqrt{-c_1^u(L)c_1^u(L^*)}$
and hence is not the map constructed on Sept. 17. There we symmetrized F^u by γ^{-1} given by

$$F(\gamma x, \gamma x) = F(x, I(-x)) \quad \Rightarrow \gamma^{-1}(-x) = I\gamma^{-1}(x)$$

Here we symmetrize F^u by β given by

$$\boxed{\beta(x) = \sqrt{-xI(x)} = x\sqrt{-\frac{I(x)}{x}}}$$

Note that

$$\beta(Ix) = Ix\sqrt{-\frac{x}{Ix}} = -x\sqrt{-\frac{Ix}{x}} = -\beta(x).$$

Therefore

$$\boxed{SO \longrightarrow U \quad \text{sends} \quad c_1^{SO}(L) \longmapsto \beta\{c_1^u(L)\}}.$$

Remark: In the preceding we perhaps should be more precise and introduce the isomorphism

$$\sigma: V(E, E-X) \simeq V(E^*, E^*-X)$$

and write $\mu_E \cdot \sigma(\mu_E)$ instead of μ_E^2. The whole argument could be cleaned up in this respect

September 22, 1969 SU-cobordism off 2

Recall that the Sp Thom classes of symplectic bundles extend uniquely to ~~[illegible]~~ Thom classes for complex bundles which are symmetric, i.e.

$$\sigma^*(\lambda_{E^*}) = (-1)^n \lambda_E$$

where E is a $U(n)$-bundle and $\sigma: E \to E^*$ is the anti-linear map given by the hermitian structure. Let V be a theory over SU with Thom class μ_E^V for an $SU(n)$-bundle. Then V is also over Sp so we have the symmetric Thom class λ_E^V. Let $\varphi^V(E) \in V(X)$ be such that

$$\boxed{\mu_E^V = \varphi^V(E)\,\lambda_E^V}$$

Then

$$\varphi^V(E+F) = \varphi^V(E)\cdot\varphi^V(F)$$

$$\varphi^V(E) = 1 \quad \text{if} \quad E \text{ is an } Sp\text{-bundle.}$$

Thus

$$(*)\quad \begin{cases} \varphi^V \in G\{V(BSU)\} \\ \varphi^V \mid BSp = 1. \end{cases}$$

Conversely given a theory V over Sp and such a multiplicative class $\varphi^{\#}$ one obtains an SU-structure on V extending the Sp-structure. So we see that

(In all this we are off 2 and theory = mult. gen. coh. theory)

$$(**)\qquad (\text{theories}/SU) \simeq (\text{theories}/Sp \text{ endowed with a } \varphi)$$

is an equivalence of categories. We are going to determine the initial object of the right-hand category, but before getting into the specific calculations we prove an easy result.

<u>Proposition:</u> $SU(pt)$ is a flat $Sp(pt)$-algebra and there is a Conner-Floyd type isomorphism

$$(\text{off } 2)\qquad SU(pt) \otimes_{Sp(pt)} Sp(X) \xrightarrow{\sim} SU(X).$$

<u>Proof:</u> We are going to show that the functor Γ associating to an $Sp(pt)$-algebra R the group

$$\Gamma(R) = \{\varphi \in \mathcal{G}(Sp_R(BSU)) \mid \varphi | BSp = 1\}$$

is represented by a flat $Sp(pt)$-algebra Γ. Hence if V is a theory over Sp endowed with a $\varphi \in \Gamma(V(pt))$, then there is a unique homomorphism

$$\Gamma \otimes_{Sp(pt)} Sp(X) \longrightarrow V(X)$$

of theories over Sp endowed with φ. Note that by flatness the left-side is a theory. Thus the left side must by formula $(**)$ above be $SU(X)$, so setting $X = pt$ we find that $\Gamma = SU(pt)$ and the proposition follows.

Now let V be a theory over U and recall that

$$R^\times = (1+\bar{R}) \text{ with multiplication}$$

for any $V(pt)$-algebra R we have

$$\mathrm{Hom}_{ab}(\tilde{K}, V_R^\times) = \mathcal{G}V_R(BU) = \mathrm{Hom}_{V(pt)\text{-alg}}(V_*(BU), R)$$

Now we have an exact sequence of functors

$$0 \longrightarrow S\tilde{K}(X) \longrightarrow \tilde{K}(X) \xrightarrow{\det} \mathrm{Pic}(X) \longrightarrow 0$$

and there is a canonical $\overset{\text{set-theoretic}}{}$ section of the det map. This gives a set isomorphism

$$\tilde{K}(X) = S\tilde{K}(X) \times \mathrm{Pic}(X)$$

or a homotopy equivalence

$$BU = BSU \times BU_1 .$$

Hence
$$V_*(BU) = V_*(BSU \times BU_1)$$
$$= V_*(BSU) \otimes_{V(pt)} V_*(BU_1)$$

since $V_*(BU_1)$ is a free $V(pt)$-module. It follows that $V_*(BSU)$ is a projective $V(pt)$-module (even free since $U^0(pt) = \mathbb{Z}$) and so for a $V(pt)$-module R

$$V_R^*(BSU) = \mathrm{Hom}_{V(pt)\text{-modules}}(V_*(BSU), R).$$

Also $V_*(BSU)$ is a Hopf algebra over $V(pt)$ and

$$\mathcal{G}V_R^*(BSU) = \mathrm{Hom}_{V(pt)\text{-alg}}(V_*(BSU), R) = \mathrm{Hom}_{ab}(S\tilde{K}, V_R^\times).$$

Similarly $V_*(BSp)$ is a ~~free~~ Hopf algebra free as a module over $V(pt)$ and

$$\mathrm{Hom}_{V(pt)\text{-alg}}(V_*(BSp), R) = \mathrm{Hom}(\tilde{K}Sp, V_R^\times)$$

Consequently

$$\underline{\Gamma}^V(R) = \mathrm{Ker}\left\{ \mathcal{G}V_R(BSU) \longrightarrow \mathcal{G}V_R(BSp) \right\}$$

$$= \mathrm{Hom}_{V(pt)\text{-alg}}(\Gamma, R)$$

where

$$\Gamma = V(pt) \otimes_{V_*(BSp)} V_*(BSU).$$

It remains to show that Γ is flat over $V(pt)$. Recall that the composition

$$\begin{tikzcd} KSp \arrow[r] \arrow[rr, bend left, "2"] & KU \arrow[r] \arrow[rr, bend right, "E \mapsto E+E^*"'] & KSp \arrow[r] & KU \end{tikzcd}$$

is multiplication by 2 since for a quaternionic bundle E

$$\mathbb{H} \otimes_{\mathbb{C}} E \cong E \oplus E$$

$$q \otimes v \longmapsto qv + qiv$$

Thus there is a decomposition

$$KU = KSp \oplus KA$$

(A stands for anti-conjugate)

$$KSK = KSp \oplus KSA$$

giving rise to an H-space isomorphism

$$BSU \simeq BSp \times BSA$$

Now clearly

$$\Gamma = V_*(BSA)$$

is a retract of $V_*(BSU)$ and hence is projective as a $V(pt)$ module. This completes the proof of the proposition.

Corollary: $SU^*(pt)$ has no odd-torsion.

Proof: If p is an odd prime $0 \to Sp(pt) \xrightarrow{p} Sp(pt)$ is exact so ~~by~~ flatness so is $0 \to SU(pt) \xrightarrow{p} SU(pt)$.

Remarks: Recall that an $SU(n)$ bundle is a $U(n)$-bundle E endowed with ~~an element~~ a section $\omega \in \Gamma(\Lambda^n E)$ of norm 1. If E is a $U(n)$-bundle, then $E + E^*$ is canonically an $SU(2n)$ bundle where $\omega|(\text{open set}) = e_1 \wedge \cdots \wedge e_n \wedge e_1^* \wedge \cdots \wedge e_n^*$ for any orth. basis $\{e_i\}$ of E over (open set), where $\{e_i^*\}$ is the dual basis for E^*.

An $SU(n)$ bundle (E, ω) has a dual (E^*, ω^*) where if $\omega = e_1 \wedge \cdots \wedge e_n$, then $\omega^* = e_1^* \wedge \cdots \wedge e_n^*$. Observe that the two $SU(2n)$-~~structures~~ on $E + E^*$ given by $\omega \wedge \omega^*$ and by the above always coincide, so

$$(*) \qquad \mu_E \cdot \mu_{E^*} = \mu_{E+E^*}$$

for any theory V over SU. Let $\sigma : E \to E^*$ be the conjugate-linear isomorphism given for the hermitian form and let $\psi(E)$ be defined by

$$(**) \qquad \boxed{(-1)^n \sigma^* \mu_{E^*} = \psi(E) \cdot \mu_E}$$

Then

$$\psi \in \mathcal{G}\{V(BSU)\} = \mathrm{Hom}_{ab}(SK, V^\times).$$

Now recall from page 1 that

$$\mu_E = \varphi(E) \lambda_E$$
$$\sigma^* \lambda_{E^*} = (-1)^n \lambda_E ,$$

so

$$\psi(E) \cdot \mu_E = (-1)^n \sigma^* (\varphi(E^*) \lambda_{E^*}) = \varphi(E^*) \lambda_E \qquad \text{or}$$

$$\boxed{\psi(E) = \frac{\varphi(E^*)}{\varphi(E)}}$$

Combining the above formulas $(*), (**)$ we have

$$(***) \qquad \boxed{(-1)^n (id + \sigma)^* \mu_{E+E^*} = \psi(E) \cdot \mu_E^2}$$

which is valid even without the assumption "off 2". Now if the theory V happens to be over U in some way not necessary compatible with the given map ~~from~~ $SU \to V$, then the Gysin sequence for the sphere fibration

$$S(\Lambda^n E) \longrightarrow BSU_n \longrightarrow BU_n$$

gives

$$V(BU_n) \xrightarrow{e(\det E_n)} V(BU_n) \longrightarrow V(BSU_n) \xrightarrow{\delta} \cdots$$

Now we know that

$$V(BU_n) = V(pt)[[c_1, c_2, \cdots, c_n]]$$

and ~~so~~ by splitting principle, we have

$$c_1(\det E) = c_1(L_1 \otimes \cdots \otimes L_n)$$

$$\equiv c_1(E) \text{ mod deg } 2.$$

Thus $c_1(\det E)$ is a non-zero divisor in $V(BU_n)$ and we see δ in the Gysin sequence is zero. Thus

$$V(BU_n) \longrightarrow V(BSU_n)$$

is surjective, in fact

$$V(BSU_n) = V(BU_n)/(e(\det E)).$$

This means that any char. class ρ for SU-bundles in V ~~[illegible]~~ is the restriction of a class for U-bundles, in other words the value $\rho(E)$ is independent of the trivialization of ~~the~~ $\det E$. Therefore (***) shows that μ_E^2 is independent of the trivialization of $\det E$ and hence if we are off 2, that also μ_E

(all off 2)

is independent of the trivialization. So we have proved

Proposition: If V is a theory over SU and off 2, then the Thom class μ_E in V of an SU_n bundle is independent of the trivialization of $\det E$, and hence depends only on the underlying U_n-bundle.

We might try to apply this to determine the image of SU in U. As $BU_1 = \mathbb{C}P^\infty$ has the Künneth property for U, $U_*(BU_1)$ is the bicommutative Hopf algebra over $U(pt)$ such that for any $U(pt)$-algebra R

$$\mathrm{Hom}_{U(pt)\text{-alg}}(U_*(BU_1), R) = \mathrm{Hom}_{ab}(\mathrm{Pic}, U_R^\times)$$

$$= \left\{ \sum r_n X^n \in R[[X]] \;\middle|\; \begin{array}{l} r_0 = 1 \text{ and} \\ \sum r_n F(X,Y)^n = \sum r_n X^n \cdot \sum r_n Y^n \end{array} \right\}$$

Thus $U_*(BU_1)$ is the affine coordinate ring of the group scheme

$$\underline{\mathrm{Hom}}(G, \hat{\mathbb{G}}_m)$$

where G is the formal group over $U(pt)$ defined by the law F. A point of this group scheme over R is a character ~~from~~ of G over R.

$$U_*(BU_1) = \bigoplus_{n \geq 0} U_*(pt)\, e_n \qquad \text{where}$$

$$e_m \cdot e_n = \text{coefficient of } X^m Y^n \text{ in } \sum e_k F(X,Y)^k$$

Define

$$T : \mathcal{U}(X) \longrightarrow \mathcal{U}_*(BU_1) \otimes_{\mathcal{U}(pt)} \mathcal{U}(X)$$

by

$$T(f_* x) = f_*\left(\tilde{T}(\nu_f) \cdot Tx\right)$$

where $\tilde{T}$ is the multiplicative characteristic class with

$$\tilde{T}(L) = \sum_{n \geq 0} e_n \otimes c_1(L)^n$$

Note that if $E = L_1 + \cdots + L_k$

$$\tilde{T}(E) = \prod_{j=1}^{k} \sum_n e_n \otimes c_1(L_j)^n = \sum_n e_n \otimes c_1(L_1 \otimes \cdots \otimes L_k)^n$$

$$\boxed{\tilde{T}(E) = \tilde{T}(\det E)}$$

$$\boxed{T(f_* x) = f_*\left(\tilde{T}(\det \nu_f) \cdot Tx\right)}$$

Thus the maps

$$(*) \qquad S\mathcal{U}^*(X) \xrightarrow{\alpha} \mathcal{U}^*(X) \underset{T}{\overset{1 \otimes id}{\rightrightarrows}} \mathcal{U}_*(BU_1) \otimes_{\mathcal{U}(pt)} \mathcal{U}^*(X)$$

satisfy

$$T(\alpha x) = 1 \otimes \alpha x.$$

<u>Conjecture</u>: $(*)$ is exact off 2.

Some evidence for this conjecture is that if $T(f_* 1) = 1 \otimes f_* 1$, then

$$f_* c_1(\det \nu_f)^n = 0 \qquad \text{all } n \geq 1.$$

Calculation of $V_*(BSU)$ for a theory V over U:

(done better on page 15)

(In the following the "off 2" hypothesis will not be assumed unless mentioned.)

We start with the exact sequence of functors

$$0 \longrightarrow S\tilde{K} \longrightarrow \tilde{K} \xrightarrow{\det} Pic \longrightarrow 0$$

to Ab, and the isomorphism of functor to sets

$$S\tilde{K} \times Pic \simeq \tilde{K}.$$

This gives rise to a homotopy equivalence

$$BSU \times BU_1 \sim BU$$

and shows that BSU is a retract of BU, hence $V_*(BSU)$ is a projective $V(pt)$ module. In fact we have a coalgebra isomorphism

$$V_*(BSU) \otimes_{V(pt)} V_*(BU_1) \simeq V_*(BU)$$

so that the sequence of Hopf algebras

$$(*) \qquad 0 \longrightarrow V_*(BSU) \longrightarrow V_*(BU) \xrightarrow{\det} V_*(BU_1) \longrightarrow 0$$

is exact and canonically cosplit. If R is a $V(pt)$-algebra, then taking the points with values in R, we get an exact sequence of abelian groups

$$(**) \qquad 0 \longrightarrow \mathrm{Hom}(Pic, V_R^\times) \longrightarrow \mathrm{Hom}(\tilde{K}, V_R^\times) \longrightarrow \mathrm{Hom}(S\tilde{K}, V_R^\times).$$

Recall that if G is a formal group over a ring Λ given by a formal group law F, so that G is endowed with a coordinate curve, then there is an exact sequence of formal groups (not all of dimension 1) over Λ

$$0 \longrightarrow K \longrightarrow \hat{W}_\Lambda \xrightarrow{\pi} G \longrightarrow 0$$

where π is the map given by universal property of $\hat{W}$ (represents curves). This gives rise via duality to ~~the~~ an exact sequence of affine groups

$$0 \longrightarrow \underline{\mathrm{Hom}}(G, \hat{G}_{m,\Lambda}) \longrightarrow W_\Lambda \longrightarrow \underline{\mathrm{Hom}}(K, \hat{G}_{m,\Lambda}) \longrightarrow 0.$$

We shall now check that this sequence gives rise by passing to coordinate rings to the sequence $(*)$ on page 10, in the case of the group law $F^\vee$ over $V(pt)$.

Recall that the map $\pi : \hat{W} \longrightarrow G$ is defined by

$$\pi(1 + a_1 t + \cdots + a_n t^n) = \sum_{i=1}^{n} \gamma_0(\lambda_i)$$

where $\gamma_0 : D \longrightarrow G$ is the coordinate curve and where the λ_i are the dummy variables with

$$1 + a_1 t + \cdots + a_n t^n = \prod (1 + \lambda_i t)$$

Now take (G, γ_0) to be defined by $F^\vee(X,Y)$ over $V(pt)$ so that the coordinate ring of G is

$$A(G) = V(pt)[[X]] \qquad \Delta X = F(X\hat{\otimes}1, 1\hat{\otimes}X)$$

The coordinate ring of $\hat{W}_{V(pt)}$ is

$$A(\hat{W}) = V(pt)[[C_1, C_2, \cdots]] \qquad \Delta C_n = \sum_{i+j=n} C_i \hat{\otimes} C_j$$

and the map

$$\pi^* : A(G) \longrightarrow A(\hat{W})$$

is given by

$$X \longmapsto f(C_1, \cdots, C_n, \cdots)$$

where f is determined by

$$f(\sigma_1(\underline{Z}), \cdots, \sigma_n(\underline{Z}), 0, \cdots) = \sum_{i=1}^{n} {}^F Z_i \qquad \underline{Z} = (Z_1, \cdots, Z_n)$$

for each n. I claim that

$$(***) \qquad \begin{array}{ccc} A(\hat{W}) = V(pt)[[\underline{C}]] & \xrightarrow[\cong]{C_i \mapsto c_i} & V^*(BU) \\ \uparrow \pi^* & & \uparrow \det^* \\ A(G) = V(pt)[[X]] & \xrightarrow[X \mapsto c_1]{} & V^*(BU_1) \end{array}$$

is commutative. To check this it is enough to show that

$$c_1(\det E) = f(c_1 E, c_2 E, \cdots)$$

where is a vector bundle over any space X. But then by splitting principle I can suppose $E = L_1 + \cdots + L_n$, so

$$f(c_1 E, c_2 E, \cdots) = f(\sigma_1(\underline{\lambda}), \cdots, \sigma_n(\underline{\lambda}), 0, \cdots) = \sum_{i=1}^{n} {}^F \lambda_i = c_1(\det E)$$

where $\lambda_i = c_1(L_i)$. Thus the square $(***)$ is commutative and so by duality

$$\begin{tikzcd} A(W) \arrow[d, "\pi_*"] & \simeq & V_*(BU) \arrow[d, "\det_*"] \\ A(\underline{\mathrm{Hom}}(G,\hat{\mathbb{G}}_m)) & \simeq & V_*(BU_1) \end{tikzcd}$$

commutes. Therefore we get a commutative diagram of exact sequences of Hopf algebras

$$\begin{tikzcd} 0 \arrow[r] & A(\underline{\mathrm{Hom}}(K,\hat{\mathbb{G}}_m)) \arrow[r] \arrow[d, phantom, "\wr|"] & A(W) \arrow[r] \arrow[d, phantom, "\wr|"] & A(\underline{\mathrm{Hom}}(G,\hat{\mathbb{G}}_m)) \arrow[r] \arrow[d, phantom, "\wr|"] & 0 \\ 0 \arrow[r] & V_*(BSU) \arrow[r] & V_*(BU) \arrow[r] & V_*(BU_1) \arrow[r] & 0 \end{tikzcd}$$

which was to be proved.

The next thing to do is to note that as functors there is an ~~exact~~ isomorphism of exact sequences of flat sheaves

$$\begin{tikzcd} 0 \arrow[r] & \underline{\mathrm{Hom}}(G,\hat{\mathbb{G}}_m) \arrow[r] \arrow[d, phantom, "\wr|"] & \underline{\mathrm{Hom}}(W,\hat{\mathbb{G}}_m) \arrow[r] \arrow[d, phantom, "\wr|"] & \underline{\mathrm{Hom}}(K,\hat{\mathbb{G}}_m) \arrow[r] \arrow[d, phantom, "\wr|"] & 0 \\ 0 \arrow[r] & \underline{\mathrm{Hom}}(G,\hat{\mathbb{G}}_m) \arrow[r] & \underline{\mathrm{Map}}(G,\hat{\mathbb{G}}_m) \arrow[r] & \underline{Z}^2_s(G,\hat{\mathbb{G}}_m) \arrow[r] & 0 \end{tikzcd} \quad (*)$$

since any extension splits locally.

where $\Gamma(R, \underline{Z}^2_s(G,\hat{\mathbb{G}}_m)) = \{ f(X,Y) \in 1 + R[[X,Y]]^+ \mid f \text{ symmetric 2-cocycle of } G \text{ values in } \hat{\mathbb{G}}_m \}$

i.e.
$$f(X,Y) = f(Y,X)$$
$$\frac{f(Y,Z)\ f(X,F(Y,Z))}{f(F(X,Y),Z)\ f(X,Y)} = 1$$

(note by setting $Y=0$ we have $f(0,Z) = f(X,0) = f(0,0) = 1$ so the cocycle is normalized.)

<u>Conclusion:</u>
$$\boxed{\mathrm{Hom}_{V(pt)\text{-alg}}(V_*(BSU), R) = Z^2_s(G, \hat{G}_m)(R)}$$

Taking points with values in R in $*$ we get the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(G,\hat{G}_m) \arrow[r] \arrow[d, equal] & W(R) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_R(K,\hat{G}_m) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_R(G,\hat{G}_m) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(G,\hat{G}_m) \arrow[r] \arrow[d, equal] & W(R) \arrow[r] \arrow[d, equal] & Z^2_s(G,\hat{G}_m)(R) \arrow[r] \arrow[d, equal] & H^2_{s,R}(G,\hat{G}_m) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{ab}(\mathrm{Pic}, V_R^\times) \arrow[r] & \mathrm{Hom}_{ab}(\tilde{K}, V_R^\times) \arrow[r] & \mathrm{Hom}_{ab}(S\tilde{K}, V_R^\times) \arrow[ru, "\delta"] & &
\end{tikzcd}$$

<u>Remarks:</u> The above formula for $V_*(BSU)$ permits one to describe it as a quotient of a polynomial ring.

$V_*(BSU)$ as a sub-Hopf algebra of $V_*(BU)$ has to be a polynomial ring (? I think, because $W \to \mathrm{Hom}(K,\hat{G}_m)$ is surjective $\Rightarrow$ all frobenius operators on $\mathrm{Hom}(K, G_m)$ are surjective $\Rightarrow$ smooth).

<u>September 24, 1969</u>:

I want to make more explicit the isomorphism between multiplicative characteristic classes for SU-bundles with values in V_R and cocycles. So let $\varphi \in G\{V_R^*(BSU)\}$ where V is a theory over U (not assuming "off 2") and R is a $V(pt)$ algebra. Extend φ to a char. class for U-bundles by setting

$$\tilde{\varphi}(E) = \varphi(E - \det E)$$

Then

$$\begin{aligned} \tilde{\varphi}(E+F) &= \varphi(E+F-\det E \cdot \det F) \\ &= \varphi\big((E-\det E)+(F-\det F)+(\det E+\det F-\det E\det F)\big) \\ &= \tilde{\varphi}(E)\tilde{\varphi}(F)\, c(\det E, \det F) \end{aligned}$$

where for any two line bundles

$$c(L_1, L_2) = \varphi(L_1 + L_2 - L_1L_2).$$

Note that

$$c : \mathrm{Pic} \times \mathrm{Pic} \longrightarrow V_R^{\times}$$

~~is~~ represented by

$$c(L_1, L_2) = f(c_1L_1, c_1L_2)$$

where $f(X,Y) \in R[[X,Y]]$ satisfies the identities

$$f(0,0) = 1 \qquad f(X,Y) = f(Y,X).$$

$$f(Y,Z)\, f(X, F(Y,Z)) = f(F(X,Y), Z)\, f(X,Y)$$

where the cocycle identity comes from the calculation

$$\varphi(L_1+L_2+L_3-L_1L_2L_3) = \varphi(L_1+L_2-L_1L_2 + L_1\overbrace{L_2}^{+L_3}-L_1L_2L_3)$$

$$= \varphi(L_1+L_2-L_1L_2)\,\varphi(L_1L_2+L_3-L_1L_2L_3)$$

$$= f(X_1,X_2)\,f(F(X_1,X_2),X_3) \qquad X_i = c_1L_i.$$

and the calculation the other way ~~away~~ round. Thus we have constructed a map

$$\boxed{\operatorname{Hom}_{V(pt)\text{-alg.}}(V_*(BSU),R) \xrightarrow{\ \sim\ } \underline{Z}^2_s(G,\hat{G}_m)(R)}.$$

This map is in fact an isomorphism since given $f(X,Y)$ we can define $\tilde{\varphi}$ on $U(n)$ bundles by induction on n using the splitting principle and the formula

$$\tilde{\varphi}(E+L) = \tilde{\varphi}(E)\,c(\det E, L).$$

Suppose now that we work "off 2" whence we have decompositions

$$BU \simeq BSp \times BA$$

$$BSU \simeq BSp \times BSA$$

of H-spaces. Suppose now that V is a symmetric theory over U, i.e. a theory over Sp. Then on the formal group level we have an exact sequence

$$0 \longrightarrow K^a \longrightarrow W^a \longrightarrow G \longrightarrow 0$$

where W^a represents anti-symmetric curves $(\gamma(-x) = -\gamma(x))$.

Then we have a diagram analogous to that on page 14

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Hom}_R(G,\hat{\mathbb{G}}_m) & \longrightarrow & \mathrm{Hom}_R(W^a_*,\hat{\mathbb{G}}_m) & \longrightarrow & \mathrm{Hom}_R(K^a_*,\hat{\mathbb{G}}_m) & \longrightarrow & \mathrm{Ext}^1_R(G,\hat{\mathbb{G}}_m) & \to 0 \\
& \| & & \| & & \| & & \| & \\
0 \to & \mathrm{Hom}_R(G,\hat{\mathbb{G}}_m) & \longrightarrow & \mathrm{Map}^a_R(G,\hat{\mathbb{G}}_m) & \longrightarrow & \underline{Z}^{2,a}_s(G,\hat{\mathbb{G}}_m)(R) & \longrightarrow & H^2_{s,R}(G,\hat{\mathbb{G}}_m) & \to 0 \\
& \| & & \| & & \| & \nearrow_{\delta} & & \\
0 \to & \mathrm{Hom}_{ab}(\mathrm{Pic},V_R^\times) & \longrightarrow & \mathrm{Hom}_{ab}(KA,V_R^\times) & \longrightarrow & \mathrm{Hom}_{ab}(SKA,V_R^\times) & & &
\end{array}$$

where

$$\underline{Z}^{2,a}_s(G,\hat{\mathbb{G}}_m)(R) = \left\{ f(X,Y)\in R[[X,Y]] \;\middle|\; \begin{array}{l} f(X,Y)=f(Y,X) \\ f(Y,Z)f(X,F(Y,Z)) = f(F(X,Y),Z)f(X,Y) \\ f(-X,-Y) = f(X,Y)^{-1} \end{array} \right\}$$

More explicitly the isomorphism

$$\boxed{\mathrm{Hom}_{ab}(SKA, V_R^\times) \xrightarrow{\sim} \underline{Z}^{2,a}_s(G,\hat{\mathbb{G}}_m)(R)}$$

$$\varphi \longmapsto f$$

is ~~[illegible]~~ given by the same formula as before

$$f(c_1L_1, c_1L_2) = \varphi(L_1+L_2-L_1L_2)$$

but since $\varphi(E^*) = \varphi(E)^{-1}$, it follows that

$$\begin{aligned} f(-c_1L_1, -c_1L_2) &= f(c_1(L_1^*), c_1(L_2^*)) \qquad \text{since } V \text{ symmetric} \\ &= \varphi((L_1+L_2-L_1L_2)^*) \\ &= f(c_1L_1, c_1L_2)^{-1}. \end{aligned}$$

We shall now prove the conjecture on page 9. We recall the ~~composition~~ sequence of H-space maps

$$BSA \xrightarrow{i} BA \xrightarrow{d} BU_1$$

and the section $s: BU_1 \longrightarrow BA$ of d. These give rise to a homotopy equivalence

$$\begin{aligned} BU_1 \times BSA &\longrightarrow BA \\ (x\ ,\ y) &\longmapsto s(x)+i(y) \end{aligned}$$

Now consider the maps of H-spaces

$$(*) \qquad BSA \xrightarrow{i} BA \underset{(0,id)}{\overset{(d,id)}{\rightrightarrows}} BU_1 \times BA$$

as well as the non H-space map $\pi: BA \longrightarrow BSA$ given by $\pi x = i^{-1}(x - sdx)$. We are going to show that (*) remains exact, in fact split exact, after any functor from spaces to another category is applied. The idea is that the maps of spaces

$$\partial_0 = (d, id)$$
$$\partial_1 = (0, id)$$
$$\sigma_0 = pr_2$$
$$\sigma_{-1}(x,y) = \cancel{sdy +} \ sy + i\pi x$$

satisfy

$$\left\{ \begin{array}{ll} \pi i = id & \\ \sigma_0 \partial_0 = \sigma_0 \partial_1 = id & \\ \sigma_{-1}\partial_0 = id & \sigma_{-1}\partial_1 = i\pi \end{array} \right.$$

Now if a functor is applied these identities still hold so that i is injective and also

$$\partial_0 x = \partial_1 x \implies x = \sigma_{-1}\partial_0 x = \sigma_{-1}\partial_1 x = i\pi x.$$

We conclude that the sequence of $V(pt)$-modules

$$(*) \qquad 0 \longrightarrow V_*(BSA) \longrightarrow V_*(BA) \rightrightarrows V_*(BU_1) \otimes V_*(BA)$$

is split exact. ~~Taking~~

Recall that $(*)$ is a sequence of maps of Hopf algebras over $V(pt)$; taking ring homomorphisms into a $V(pt)$-algebra R transforms $(*)$ into

$$\mathrm{Hom}_{ab}(SKA, V_R^\times) \longleftarrow \mathrm{Hom}_{ab}(KA, V_R^\times) \leftleftarrows \mathrm{Hom}_{ab}(\mathrm{Pic} \times KA, V_R^\times)$$

where the two last maps are induced by

$$KA \rightrightarrows \mathrm{Pic} \times KA$$

$$x \longmapsto (\det x, x)$$

$$\longmapsto (0, x)$$

Now suppose that $V = Sp_*$. Tensoring $(*)$ with $Sp^*(X)$ gives an exact sequence

$$(**) \quad 0 \longrightarrow Sp_*(BSA) \otimes_{Sp(pt)} Sp^*(X) \xrightarrow{\lambda_*} Sp_*(BA) \otimes_{Sp(pt)} Sp^*(X) \rightrightarrows Sp_*(BU_1) \otimes_{Sp(pt)} Sp(BA) \otimes_{Sp(pt)} Sp^*(X)$$

Recall that the first theory is isomorphic to $SU(X)$ and that the

~~[illegible]~~

second is $U(X)$; it is also clear that i_* is isomorphic to the natural map $SU(X) \longrightarrow U(X)$. The last theory is the universal theory over Sp endowed with a multiplicative character class for U-bundles vanishing on Sp-bundles and with a characteristic class on line bundles sending products to products. The last theory is therefore $U_*(BU_1) \otimes_{U(pt)} U^*(X)$ and the two maps

$$U^*(X) \rightrightarrows U_*(BU_1) \otimes_{U(pt)} U^*(X)$$

send μ_E to μ_E and $\tilde{T}(\det E)\mu_E$ respectively. Thus we see that (**) on page 19 is isomorphic to (*) on page 9 and we conclude

<u>Theorem:</u> (off 2) the sequence

$$0 \longrightarrow SU^*(X) \longrightarrow U^*(X) \underset{T}{\overset{1\otimes}{\rightrightarrows}} U_*(BU_1) \otimes_{U(pt)} U^*(X)$$

is exact.

<u>Remarks:</u> 1.) $\operatorname{Hom}_{\mathbb{Z}[\frac{1}{2}]\text{-algs.}}\left(SU^*(pt), R\right) = \{(F,c) \mid F \text{ symmetric group law over } R$

and where $c(X,Y)$ is a symmetric cocycle for F with values in $\hat{\mathbb{G}}_m$

$$\frac{c(Y,Z)\,c(X,F(Y,Z))}{c(F(X,Y),Z)\;c(X,Y)} = 1$$

$$c(X,Y) = c(Y,X)$$

which also satisfies

$$c(-X,-Y) = c(X,Y)^{-1}.$$

2.) It is not true that $SU(pt)[P_1] = U(pt)$

3.) $U_*(BU_1)$ is an affine category over $U(pt)$ and T gives an action on $U^*(X)$. This category structure is different from the Hopf algebra structure. In effect the map

$$U(X) \xrightarrow{T} U_*(BU_1) \underset{U(pt)}{\otimes} U(X) \xrightarrow{id \otimes T = T'} U_*(BU_1) \underset{U(pt)}{\otimes} U_*(BU_1) \underset{U(pt)}{\otimes} U(X)$$

sends

$$f_* 1 \longmapsto f_*(\tilde{T}(\det \nu_f)) \longmapsto f_*(\tilde{T}'(\det \nu_f) \cdot T' \tilde{T}(\det \nu_f))$$

and $L \longmapsto \tilde{T}'(L) \cdot T'(\tilde{T}(L))$ is still a homomorphism from Pic to U^*. Of course all this appears Hopf-algebra-ish over Sp since the operation T leaves Sp alone.

<u>September 25, 1969</u> <u>On the localization theorem of tom Dieck</u>

Let G be a finite group and let $\mathcal{C}$ be a crible on the category of transitive G-sets so that if $X \in \mathcal{C}$ and if $\exists$ G-map $Y \longrightarrow X$ then $Y \in \mathcal{C}$. Extend $\mathcal{C}$ to a crible $\tilde{\mathcal{C}}$ on the category of G-manifolds by saying $X \in \tilde{\mathcal{C}}$ iff all orbits of X belong to $\mathcal{C}$. Given an arbitrary G-manifold X the subset $X_{\mathcal{C}}$ of orbits in $\mathcal{C}$ is open since if $x \in X_{\mathcal{C}}$ has stabilizer H, then the stabilizers of nearby points are subgroups of H. Thus we have

$$X_{\mathcal{C}} \xrightarrow{j} X \xleftarrow{i} X_{\bar{\mathcal{C}}}$$

where j is an open embedding and where $X_{\bar{\mathcal{C}}}$ is the complement of $X_{\mathcal{C}}$.

Let F be an equivariant multiplicative cohomology theory on nice G-spaces ~~manifolds~~. Then F gives a contravariant functor from the category of transitive G-spaces to rings. If $X \in \mathcal{C}$ and $Y \notin \mathcal{C}$ then as there is no map from Y to X it is possible for $F(X) = 0$ and $F(Y) \neq 0$. <u>Problem:</u> Does there exist a theory (exact) F such that

$$F(X) = 0 \iff X \in \mathcal{C}$$

for all transitive G-spaces X. Note that for any nice G-space $X \in \tilde{\mathcal{C}}$, then $F(X) = 0$ since by considering the spectral sequence

$$E_2^{pq} = H^p(X/G, x \longmapsto F^q(Gx)) \Longrightarrow F^{p+q}(X)$$

one sees that as 1_X dies on each orbit it is nilpotent hence $1_X = 0$.

Consequently for an arbitrary nice G-space X

$$\begin{array}{ccccccc} F(X, X_{\bar{\mathcal{C}}}) & \longrightarrow & F(X) & \xrightarrow{\sim} & F(X_{\bar{\mathcal{C}}}) & \xrightarrow{\delta} & F^{*+1}(X, X_{\bar{\mathcal{C}}}) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

since $F(X, X_{\bar{\mathcal{C}}})$ is a $F(X - X_{\bar{\mathcal{C}}}) = F(X_{\mathcal{C}})$ module.

<u>tom Dieck's method</u>: Let $S \subset \hat{G}$ be a set of irreducible ~~nontrivial~~ complex representations of G and let $\mathcal{C}$ be the crible given by

$$G/H \in \mathcal{C} \iff \exists V \in S \ni V^H \neq 0.$$

Let F be an equivariant mult. coh. theory with Thom isomorphism for complex bundles. Claim that

$$\{ F(G/H) = 0 \quad \text{all} \quad G/H \in \mathcal{C} \}$$

$$\iff \{ e(V) \text{ is a unit in } F(pt) \text{ for all } V \in S \}.$$

($\Leftarrow$): As $G/H \in \mathcal{C}$ $\exists$ $V \in S$ with $V^H \neq 0$, that is, $\exists$ an equivariant map $G/H \longrightarrow SV$. In other words V pulled back to G/H has an invariant section hence $f^*e(V) = e(f^*V) = e(V) 1_{G/H} = 0$ where $f: G/H \longrightarrow pt$. As $e(V)$ is a unit $1_{G/H} = 0$. ($\Rightarrow$). It follows that $F(X) = 0$ if X is a G-space in $\tilde{\mathcal{C}}$. Thus $F(SV) = 0$ for all $V \in S$. So by the Gysin sequence

$$\begin{array}{ccccccccc} F(SV) & \longrightarrow & F(pt) & \xrightarrow{e(V)} & F(pt) & \longrightarrow & F(SV) & \longrightarrow \\ \| & & & & & & \| & \\ 0 & & & & & & 0 & \end{array}$$

$e(V)$ is a unit in $F(pt)$ for all $V \in S$.

The tom Dieck method shows that for cribles obtained from a set S there is a map $F \to S^{-1}F$ universal for killing $1_{G/H}$ for all G/H in the crible. Observe that we have the usual polarity

$$(\text{cribles}) \underset{\mathcal{C}_S \leftarrow\!\shortmid\, S}{\overset{\mathcal{C} \mapsto S_{\mathcal{C}}}{\rightleftarrows}} (\text{subsets of } \hat{G})$$

$$\mathcal{C} \longmapsto \{V \mid SV \in \tilde{\mathcal{C}}\}$$

$$\begin{array}{l}\text{crible generated} \\ \text{by orbit types} \\ \text{of } SV \text{ with} \\ V \text{ in } S\end{array} \longleftarrow\!\shortmid\; S$$

Not every subset of $\hat{G}$ comes from a crible, e.g. take $S = 0 \in \hat{G}$, the trivial representation; then $\mathcal{C}_S$ = all orbit types and $S(\mathcal{C}_S) = \hat{G}$.

Question: Is it true that every crible comes from an S, i.e. is

$$\mathcal{C} = \mathcal{C}(S_{\mathcal{C}}) \; ? \qquad \text{(NO see example 3)}$$

Equivalently if $G/H \notin \mathcal{C}$, does there exist a $V \in \hat{G}$ such that SV is in the crible $\tilde{\mathcal{C}}$ but $V^H = 0$?

Examples: 1. $\mathcal{C}$ = all orbit types G/H, $H \neq G$, and $S = \hat{G} - 0$.

2. Suppose N is a normal subgroup of G and let $S = \{V \in \hat{G} \mid V \text{ does not come from } (G/N)^\wedge\}$. Note that if V is a G-representation so is V^N, hence if V is irred. either $V^N = V$ or $V^N = 0$

and ~~[illegible]~~ so $S = \{V \in \hat{G} \mid V^N = 0\}$. The corresponding crible is

$$C_S = \{ G/H \mid (G/H)^N = \phi \}$$
$$\Updownarrow$$
$$NxH \neq xH \quad \text{all } x$$
$$\Updownarrow$$
$$NH \neq H$$
$$\Updownarrow$$
$$C_S = \{ G/H \mid H \not\supset N \}$$

Clearly $S(C_S)$ = those $V \in \hat{G}$ with SV without points fixed under $N = S$.

3. S = all $V \in \hat{G} \ni G$ acts freely on SV. C ~~[illegible]~~ = principal homogeneous spaces. It's necessary to assume that $S \neq \phi$, which implies that G has very special form, in order that $C = C_S$. This example shows that not every crible comes from an S and therefore if we wish to construct a universal theory for which $F(X) = 0$ for all $X \in C$, we cannot use the tom Dieck method.

4. Let G be a finite abelian group. If V is an irreducible representation then V is 1-dimensional and is given by a character $\chi : G \to S^1$ and $\chi(G) = \mu_n$ is cyclic. Conversely if $H \subset G$ is such that G/H is cyclic then H occurs as the kernel of a character. This shows that the admissible orbit types of form C_S are of the form G/H where $H \subset \operatorname{Ker} \chi$ for some $\chi \in S$. Moreover the corresponding

S are subsets of $\hat{G}$ closed under Galois conjugation, the point being that if $H = \mathrm{Ker}\,\chi$, then also $H = \mathrm{Ker}\,\chi^\sigma$ where σ is an auto. of the cyclotomic field $\mathbb{Q}(\mu_\infty)/\mathbb{Q}$.

We now take up the localization theorem of tom Dieck in a slightly more general form. Let N be a normal subgroup of G supposed finite for the moment. Let F be an equivariant multiplicative coh. theory on G-manifolds. Then if $f: X \to Y$ is an embedding with normal bundle ν_f, the diagram

$$\begin{array}{ccc} X^N & \xrightarrow{r_X} & X \\ \downarrow{\scriptstyle f^N} & & \downarrow{\scriptstyle f} \\ Y^N & \xrightarrow{r_Y} & Y \end{array}$$

is clean cartesian with excess bundle μ_f fitting into an exact sequence

$$0 \longrightarrow \nu_{f^N} \longrightarrow \nu_f | X^N \longrightarrow \mu_f \longrightarrow 0, \qquad \text{and}$$

there is the clean intersection formula

$$f^N_*\left(e(\mu_f)\, r_X^* x\right) = r_Y^* f_* x.$$

The bundle μ_f is a G-bundle on X^N such that $\mu_f^N = 0$.

We see from this that we shall have to be able to classify G-bundles E over a G/N space such that $E^N = 0$, ~~them~~ or at least their Euler classes with values in an equivariant cohomology theory on G/N-manifolds.

For simplicity suppose that G is a semi-direct product of N and Q. Let E be a G bundle over a Q space X. For each $V \in N^\wedge$ let E_V be the subbundle of E transforming like V under N. Then if $\mathcal{O}$ is the orbit of $V \in N^\wedge$ under the action of $G/N = Q$ we have that

$$E_{\mathcal{O}} = \bigoplus_{V \in \mathcal{O}} E_V$$

is a G-subbundle of E. Clearly

$$E = \bigoplus_{\mathcal{O}} E_{\mathcal{O}}$$

as $\mathcal{O}$ ranges over the orbit set $\hat{N}/_Q$. (So far have not used the homomorphism $Q \to G$). Let $Q_V \subset Q$ be the stabilizer of $V \in N^\wedge$. Then Q_V acts on E_V and as $E_{\mathcal{O}}$ has the E_V, $V \in \mathcal{O}$ as a system of imprimitivity, it follows that $E_{\mathcal{O}}$ is the bundle induced from E_V as a $N \cdot Q_V$ bundle under the inclusion $NQ_V \to G$.

To simplify even further suppose that N is abelian in which case V is 1-dimensional and E_V is just a Q_V bundle over X tensored with V. Therefore it appears that a G bundle over a $G/N = Q$ space X consists of giving for each V in a system of representatives for the orbits of Q on $N^\wedge$, a Q_V-bundle over X. So we obtain the standard recipe of the physicists

$$K_G(X) = \bigoplus_{\hat{N} = \coprod_{V \in S} Q \cdot V} K_{Q_V}(X).$$

$\left\{ \begin{array}{l} G = N \rtimes Q \\ N \text{ abelian} \end{array} \right.$

This is a mess.

September 26, 1969. Oriented cobordism

We try to see what can be proved for SO using the power operations method. Thus consider the square

$$SO^{-2q}(pt) \longrightarrow SO^{-4q}(B\mathbb{Z}_2).$$

Again we have the formula

$$(*) \qquad w^{2q+1}\, Q(f_* 1) = f_*\, e(\eta \otimes (\nu_f + 4q+2))$$

obtained by an embedding

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{R}^{4q+2} \arrow[d] \\ & pt \end{tikzcd}$$

where $w = c_1(\eta \otimes_{\mathbb{R}} \mathbb{C})$. Precise signs are insignificant since w is of order 2 since $\eta \otimes_{\mathbb{R}} \mathbb{C}$ is of order 2

$$w = c_1(\eta \otimes_{\mathbb{R}} \mathbb{C}) = c_1((\eta \otimes_{\mathbb{R}} \mathbb{C})^{-1}) = -c_1(\eta \otimes_{\mathbb{R}} \mathbb{C}) = -w$$

and the group law for SO is symmetric.

Next consider the Gysin sequence

$$\begin{tikzcd} \arrow[r] & \widetilde{SO}^{q+1}(S^1/\mathbb{Z}_2) \arrow[r] & SO^{q}(B\mathbb{Z}_2) \arrow[r, "w"] & \widetilde{SO}^{q+2}(B\mathbb{Z}_2) \arrow[r] & \widetilde{SO}^{q+2}(S^1/\mathbb{Z}_2) \arrow[r, "\delta"] & {} \\ & \widetilde{SO}^{q}(pt) \arrow[u, "\wr"] \arrow[ur, "\text{induction}"'] & & & SO^{q+1}(pt) \arrow[u, equal] & \end{tikzcd}$$

This shows that for $q \geq 1$ that $\widetilde{SO}^{q}(B\mathbb{Z}_2) = w\, SO^{q-2}(B\mathbb{Z}_2)$

~~[illegible]~~ is killed by 2. Recall from preceding calculations that the induction homomorphism is multiplication by

$$\xi = 2 + w\,G(w, w).$$

Now since the group law is symmetric

$$0 = F(X,-X) = X + (-X) + X(-X)\,G(X,-X) = -X^2 G(X,-X)$$

so $\quad G(X,-X) = 0$

so $\quad G(w,w) = G(w,-w) = 0.$

Thus ~~the~~

$$\boxed{\xi = 2}.$$

Next we note Atiyah's isomorphism:

$$SO^q(\mathbb{R}P^{2n}) \xrightarrow{\sim} \mathcal{N}^{q-2n}(pt) \qquad \text{for } q > 0$$

The map associates to a proper $\overset{\text{oriented}}{}$ map $X \longrightarrow \mathbb{R}P^{2n}$ the manifold X, in other words it is the composition

$$SO^q(\mathbb{R}P^{2n}) \longrightarrow \mathcal{N}^q(\mathbb{R}P^{2n}) \xrightarrow{\int_{\mathbb{R}P^{2n}}} \mathcal{N}^{q-2n}(pt)$$

The map is surjective because given a compact manifold X of dimension $2n-q$ there is a map $f : X \longrightarrow \mathbb{R}P^{2n}$ ~~such that~~ ~~[illegible]~~ and an isomorphism $\det(\tau_X) \simeq f^*\mathcal{O}(1)$. Then

$$\det \nu_f \cong f^*\det(\tau_{\mathbb{R}P^{2n}}) \cdot (\det \tau_X)^{-1} \cong f^*\det\big((2n+1)\mathcal{O}(1) - \theta\big) \cdot (\det \tau_X)^{-1}$$

$$\cong f^*\mathcal{O}(1) \cdot (\det \tau_X)^{-1} \simeq 1$$

so f has (a canonical) orientation and so defines an element of $SO^q(\mathbb{R}P^{2n})$. (Surjectivity holds even for $q=0$). The map is injective because if $X=\partial Y$ then the map $f: X \longrightarrow \mathbb{R}P^{2n}$ extends to Y since $\det \tau_X$ extends to $\det \tau_Y$ and since $\overset{q \geq 1 \Rightarrow}{2n \geq \dim Y}$, hence (X,f) is bordant to 0.

The Atiyah isomorphism shows that

$$SO^q(B\mathbb{Z}_2) \hookrightarrow \mathfrak{N}^q(B\mathbb{Z}_2) \qquad q>0$$

and hence that the formula (*) on page 1 has the same information as the corresponding formula in $\mathfrak{N}$. In other words if $f_*1 \in \operatorname{Ker}\{SO^{-2q}(pt) \longrightarrow \mathfrak{N}^{-2q}(pt)\}$ all we can conclude is that

$$w^{-2q+1}\, Q(f_*1) = 0 .$$

If n is least such that $w^{n+1} Q(f_*1)=0$, then

$$w^n Q(f_*1) = 2a \qquad a \in SO(pt)$$

and restricting one sees that $2a=0$ if $n \geq 1$ a contradiction. Hence

$$Q(f_*1) = 2a$$

which isn't much help. However it gives us the formula

$$f_*\left(e(\eta \otimes (\nu_f + 4q+2)\right) = 0 \quad \text{in } SO^2(B\mathbb{Z}_2)$$