Concerning groups $G$ having a representation $V$ with $SV \ G$-free.

$G$ abelian $\rightarrow G$ cyclic. In effect, there is an irreducible subrepresentation $W \subset V$ with dim$_R W \leq 2$.

If dim $W = 1$, then $G$ acts on $W$ by sign so $G = \mathbb{Z}/2\mathbb{Z}$.
If dim $W = 2$, then $W$ has a complex structure and $G$ acts via a homomorphism $G \rightarrow \mathbb{C}^*$, hence $G$ is cyclic.

**Proposition:** Let $p$ be an odd prime and let $G$ be a $p$-group such that every abelian subgroup of $G$ is cyclic. Then $G$ is cyclic.

**Proof:** Let $G$ be of minimal order such that the proposition is false; then every subgroup of $G$ is cyclic. Hence $G$ is generated by $2p^k$ elements and so if $F$ is the Frattini subgroup of $G$, then $Q = G/F$ is a $2$-dimensional $F_p$-vector space.

We define a homomorphism

$$\theta : Q \rightarrow F/pF \cong \mathbb{F}_p$$

as follows. Given $q \in Q$, lift it to $x \in G$ and take $x^p$. Thus

$$\theta (x^p) \equiv x^p \ mod \ pF.$$

I claim that $\theta$ is a homomorphism since $p$ is odd. Indeed
If \( x, y \in G \) and \( xy^{-1}x^{-1}y = z \in F \), then

\[
(xy)^p = xyxy^{-1}x^{-1}y = z^p y^x y^{-p} x^{-1} y^{-1} x^{-1}
\]

\[
= z \cdot y^x y^{-p} z^{-1} = z y^x y^{-p} z^{-1}
\]

\[
= z^{1+2 \cdot (p-1)} y^p x^p
\]

Here we have used that \( z \) is central because \( \langle x, z \rangle \neq \langle y, z \rangle \) are necessarily proper subgroups hence cyclic. As \( p \) is odd

\[
z^{p(\frac{p-1}{2})} \equiv (z^{\frac{p-1}{2}})^p \equiv pF
\]

hence

\[
(xy)^p \equiv x^p y^p x^p \equiv x^p y^p \mod pF.
\]

Now as \( Q \) is 2-dimensional over \( F_p \), \( \exists \gamma \in Q \neq \Theta \neq 0 \) and \( \Theta \gamma = 0 \). Thus \( \exists x \in G \neq x \in F \) and \( x^p \in pF \)

Therefore the subgroup \( \langle x, F \rangle \) is not cyclic and we have a contradiction.

Example to show \( p \) odd is necessary. Take the quaternion group of order 8 with elements \( \pm 1, i, j, k \).

Then every subgroup is cyclic.

Conclusion: If \( G \) is a finite group acting freely on \( SV \), then every odd order subgroup of \( G \) is cyclic.
Example of a group of odd order which acts freely on $S^V$. Let $G$ be the semidirect product of $\mathbb{Z}/p^n\mathbb{Z}$ and a cyclic group $N = \mathbb{Z}/m\mathbb{Z}$ where $(m, p) = 1$. The $\mathbb{Z}/p^n\mathbb{Z}$ acts on $N$ by an automorphism $\theta$ of order $p$. Let $A$ be the subgroup $\mathbb{Z}/m\mathbb{Z} \times p\mathbb{Z}/p^n\mathbb{Z} \leq G$. Then $A$ is normal abelian cyclic of index $p$. Let $\chi: A \rightarrow \mathbb{C}^*$ be a faithful character and let $V$ be the induced representation. Now $\theta$ acts non-trivially on $X$ hence $\theta^i X$, $0 < i < p$ are all distinct and $V$ is irreducible. I claim that $G$ acts freely on the sphere $S^V$. To see this, recall that $V$ is endowed with a system of imprimitivity $V = \bigoplus_{x \in G} L_x$ where $A$ acts on $L_x$ with character $\theta^i(x)$. Thus a vector in $V$ is of the form $v = \sum v_i$, $v_i \in L_i$. Clearly $A$ acts freely since if $aeA$ and $\alpha v = v$, then $\left(\theta^i(x)(a) v_i = v_i \right.$ for some $x \in A$) and $\theta^i(x)(a) = 1$ and $a = 1$. If $x \in G$ is not in $A$ and $xv = v$, then $x$ must permute the $i$. This contradiction, hence $G$ acts freely on $S^V$.

Special case: $(\mathbb{Z}/7\mathbb{Z}) \times (\mathbb{Z}/9\mathbb{Z}) = G$

$$\begin{cases} xyx^{-1} = y^2 \\ x^7 = y^7 = 1 \end{cases}$$
Relation with periodic cohomology: If $G$ acts freely on a homotopy sphere $S$ of dimension $n-1$, then one has the Lyalin sequence for the sphere fibration

$$S \rightarrow P_G \times_G S \rightarrow B_G$$

which is

$$\rightarrow H^{g-n}(B_G, \mathbb{Z}) \xrightarrow{\text{ve}} H^{g}(B_G, \mathbb{Z}) \rightarrow H^{g}(P_G \times_G S, \mathbb{Z}) \rightarrow \cdots$$

Now $P_G \times_G S \sim S/G$ is a CW complex of dimension $n-1$ so one sees that

$$\text{ve}: H^g_G(\ast, \mathbb{Z}) \xrightarrow{\cong} H^{g+n}_G(\ast, \mathbb{Z}) \quad g > 0$$

and that

$$H^{-n}(S/G, \mathbb{Z}) \xrightarrow{\text{int}_\ast} H^0_G(\ast, \mathbb{Z}) \xrightarrow{\text{ve}} H^n_G(\ast, \mathbb{Z}) \rightarrow 0$$

Thus $H^n_G(\ast, \mathbb{Z}) \cong \mathbb{Z}/g\mathbb{Z}$ with generator $e$. Therefore the cohomology of $G$ is periodic with period $g$.

Conversely one knows that if $H^*_G(\ast, \mathbb{Z})$ is periodic of period $n$, then $G$ acts freely on a finite CW complex of the homotopy type of $S^{dn-1}$ for some $d$. These results have been made more precise by Wall.

Cohomology of a semi-direct product:

$$G = N \rtimes_C G \quad |N|, |G| \text{ rel. prime}$$
The Hochschild–Serre spec. seq. degenerates yielding split exact sequences

\[ H^0(G, \mathbb{Z}) = \mathbb{Z}, \]

\[ 0 \rightarrow H^1(G, \mathbb{Z}) \rightarrow H^1(G, \mathbb{Z}) \rightarrow H^1(N, \mathbb{Z})^C \rightarrow 0. \]

Suppose that \( N \) and \( C \) are cyclic of orders \( n, c \) respectively. Then

\[ H^*(N, \mathbb{Z}) = \mathbb{Z}[\eta]/(\eta^n) \]

where \( \eta \in H^2(N, \mathbb{Z}) \cong \text{Hom}(N, \mathbb{Q}/\mathbb{Z}) \cong \hat{N} \)

is the character with \( \eta(u) = 1/n \mod \mathbb{Z} \) where \( u \) is the generator of \( N \). Thus

\[ c \cdot \eta = \eta^b \quad \text{for } b \in (\mathbb{Z}/n\mathbb{Z})^* \]

and so \( \eta^{\varphi(n)} \) is invariant under \( C \). Thus

\[ H^2(\varphi(n))(N, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \]

\[ H^2(\varphi(n))(C, \mathbb{Z}) \cong \mathbb{Z}/c\mathbb{Z} \]

and so \( H^2(\varphi(n))(G, \mathbb{Z}) \cong \mathbb{Z}/nc\mathbb{Z} \) since \( (n,c) = 1 \).

Therefore one knows that \( H^*(G, \mathbb{Z}) \) is periodic of period \( \varphi(n) \).

Example of a group with periodic cohomology which does not act freely on \( SV \) for any representation \( V \). Take primes \( p, q \) with \( p \nmid q-1 \) and form the semi-direct product

\[ G = \mathbb{Z}/p\mathbb{Z} \ltimes \mathbb{Z}/q\mathbb{Z} \]
By the above the cohomology of $G$ is periodic of period 2p: $G$ can't act freely on $SV$, $V$ irreducible complex (say $p,q$ are odd) because either $V$ is 1-dimensional non-faithful or else $V$ is induced from a non-trivial character of $\mathbb{Z}/p\mathbb{Z}$, whence there is a non-zero $\mathbb{Z}/p\mathbb{Z}$-invariant.

I don't yet know the kind of 2 groups that can act freely on the sphere of a representation. But if $G$ is nilpotent and $G_i$ acts freely on $SV_i$, $V_i$ irreducible complex, then $V$ is the tensor product of such representations for each Sylow subgroup. In effect if $G_i$ acts freely on $V_i - 0$ and $G_i$ acts freely on $V_i - 0$ and if $|G_1|, |G_2|$ are rel. prime, then $G_1 \times G_2$ acts freely on $V_1 \odot V_2 - 0$.
General facts about $G$ manifolds, $G$ finite odd order

$X$, $G$-manifold connected. The principal orbit set manifold is where the isotropy representations are trivial. As $G$ is of odd order, the non-principal part is of codim $\geq 2$ so $X_{\text{prin}}$ in connected. The map $X_{\text{prin}} \rightarrow \text{subgps of } G$ given by $x \mapsto G_x$ is locally constant, hence constant, hence $G_x = N$ a normal subgp of $G$ for all $x \in X_{\text{prin}}$. $G/N$ acts on $X$.

Conclusion: If $G$ acts faithfully on $X$, then $X_{\text{prin}}$ is the open subset where $G$ acts freely.

Remark: we recall that the strata of $X$ are all of even codimension, since $G$ being of odd order, all irreducible real representations of $G$ have complex structures.

A neighborhood of an orbit $G_x$ is of the form $G_x \cdot T_x$ where $T_x$ is the tangent space to $X$ at $x$. Note this result even holds complex-analytically if $X$ is a complex manifold since if $m_x$ is the maximal ideal of $O_x$ the sequence

$$0 \rightarrow m_x^2 \rightarrow m_x \rightarrow m_x/m_x^2 \rightarrow 0$$

splits as a sequence of $G$-modules, and so
we obtain local coordinates at \( x \), giving an étale map \( U_x \to T_x \) near \( x \).

So to complete local study we replace \( X \) by a representation \( V \) of \( G \) which we suppose is endowed with a complex structure. Assume \( G \) faithful and let \( Z \subset V \) be the singular cone

\[
Z = \{ v \in V | \exists g \neq 1 \Rightarrow gv = v \}.
\]

\( Z \) is the union of a finite set of subspaces of \( V \), namely the fixed point submanifold \( V_g^G \) for each \( g \neq 1 \).

Suppose that \( Z \) is a divisor, i.e. \( Z = \bigcup_{i=1}^{m} W_i \) where the \( W_i \) are distinct hyperplanes contained in \( Z \). Let \( H_i \) be the subgroup of \( G \) fixing \( W_i \), so that \( H_i \neq 1 \), and \( W_i = V_g^G \) for some \( g \neq 1 \). Let \( H_{ij} \) be the subgroup fixing \( W_i \cap W_j \). Then \( H_{ij} \) acts faithfully on \( V/W_i \cap W_j \). As this space is 2-dimensional and \( H_{ij} \) is of odd order, \( H_{ij} \) must be abelian (degree of an irreducible representation divides the order of the group.) \( H_{ij} \) commutes with \( H_i \) and \( H_j \), it normalizes \( W_i/W_i \cap W_j \) and \( W_j/W_i \cap W_j \), and hence these are the two eigenspaces of the representation of \( H_{ij} \). Note that the resulting irreps of \( H_{ij} \) are distinct since they are distinct when restricted to \( H_i \times H_j \).

Assume \( V \) endowed with a unitary structure invariant under \( G \). It follows that \( W_i^G \) and \( W_j^G \) are perpendicular lines for \( i \neq j \). Thus \( V \) contains an invariant subspace.
\[ V' = L_1 + \ldots + L_n \]

with complement \((V')^\perp = \bigwedge W_i\).

As \((V')^\perp\) is totally singular for G it is a non-faithful representation of G. Moreover G must act faithfully on \(\mathcal{V}\) since otherwise there would be a singular element of the form \(\sum v_i = 0\), \(v_i \in L_i\), not contained in \(\mathcal{Z}\).

Similarly the singular subset of G acting on \(\mathcal{V}\) is the union of the hyperplanes \(W_i \cap \mathcal{V}\).

Conclusion: If G acts on \(\mathcal{V}\) and \(V_{\text{sing}}\) is a divisor, then \(\mathcal{V} = V' \oplus V''\) where \(V''\) is a non-faithful representation of G and where \(V'\) has a system of imprimitivity \(V' = L_1 + \ldots + L_n\) \(\dim L_i = 1\).

The singular set of \(V'\) is the union of \((L_i)^\perp\) and G permutes the lines giving an exact sequence

\[ 1 \rightarrow G_0 \rightarrow G \rightarrow \Sigma_n \rightarrow 1 \]

Hence G is a subgroup of the normalizer of a torus.

Note that G odd is essential since consider \(\Sigma_3\) acting on \(V = \{(x, y, z) \in \mathbb{C}^3 \mid x + y + z = 0\}\). Then \(V_{\text{sing}}\) is union of the hyperplanes

\[ x = y \]
\[ y = z \]
\[ x = z \]

which are not mutually perpendicular.
If $G$ acts faithfully on a complex vector space $V$, then $V_{sing}$ is a union of its irreducible components

$$V_{sing} = \bigcup_{i=1}^{\infty} W_i$$

and each $W_i$ is a subspace. If $H_i = \{ g \in G | W_i = \text{id}_F \}$, then $H_i$ acts freely on $W_i - \{0\}$.

Remarks (added Oct. 7, 1969) In Cartan-Eilenberg one finds the result that a group has periodic cohomology iff all its Sylow subgroups are cyclic for odd primes and generalized quaternion for $p=2$. Consult Milnor's Amer. J. paper on free actions on spheres for a reference to old paper of Zassenhaus where among other things groups which act freely on $S^7$ are classified. See Zassenhaus's book for classification of groups all of whose Sylow subgroups are cyclic.
Equivariant cobordism revisited:

Let $G$ be an abelian compact Lie group with character group $\hat{G}$ and let us consider a Chern theory $Q$ on the category of $G$-manifolds (again for proper $U$-oriented maps) such that $c_1(x) \in \hat{G}$ is a unit in $Q(pt)$ for $x \neq 1$.

Let

$$Q(P^n) = \lim_{\leftarrow V} Q(P^V)$$

$$= \lim_{n} Q(pt)[X]/\prod_{x}(x - c_1(x))^n$$

$$= Q(pt)[X]$$

where $n$ runs over functions from $\hat{G}$ to integers $\geq 0$ almost everywhere.

$Q(P^n)$ is a natural transformation from $\text{Pic}_0$ to $Q$.

Similarly set $Q(P^n \times P^n) = \text{natural transformation from } \text{Pic}_0 \times \text{Pic}_0$ to $Q$ defined $Q = Q(pt)\{X_1, X_2\}$. Then tensor product gives a homomorphism

$$Q(P^n) \longrightarrow Q(P^n \times P^n)$$

$$X \longmapsto F(X_1, X_2)$$

and $F(X_1, X_2)$ is a kind of "formal group law". I like to think of $F$ as defining an abelian group structure on the functor $A \mapsto OA$ where $A$ runs over the category of $Q(pt)$-algebras and where
\[ D(A) = \{ a \in A \mid \exists x \text{ with } \prod_{x} (a - c_{i}(x))^{n_{i}} = 0 \} \]

**Lemma:** \( c_{i}(x) \in \mathbb{Q}(\mathfrak{p}t)^{\times} \) for all \( x \neq 1 \) \( \Rightarrow \) \( [c_{i}(x) - c_{i}(x')] \in \mathbb{Q}(\mathfrak{p}t)^{\times} \) for \( x \neq x' \).

**Proof:** If \( c_{i}(X) - c_{i}(X') \) is not a unit then there is a non-zero \( \mathbb{Q}(\mathfrak{p}t) \)-algebra \( A \) in which \( c_{i}(X) = c_{i}(X') \), whence using the group-structure of \( D(A) \), we have \( c_{i}(X \circ X'^{-1}) = c_{i}(X) \odot c_{i}(X') = c_{i}(X) \odot c_{i}(X) = c_{i}(X) - c_{i}(X) = 0 \). This contradicts fact that \( c_{i}(X \circ X'^{-1}) \in A^{\times} \).

---

By the Chinese remainder theorem,

\[ \mathbb{Q}(\mathfrak{p}t)\{X\} \xrightarrow{\sim} \prod_{X} \mathbb{Q}(\mathfrak{p}t) \left[ [X - c_{i}(X)] \right] . \]

Another way of putting this is to say there exists idempotents \( \delta_{x}(X) \) in \( \mathbb{Q}(\mathfrak{p}t)\{X\} \) such that \( \delta_{x}(c_{i}(X')) = \begin{cases} 0 & x' \neq x \\ 1 & x' = x \end{cases} \)

and such that

\[ 1 = \sum_{X} \delta_{x} \]

as a topological sum in case \( G \) isn't finite. Thus given \( a \in D(A) \) there exists a decomposition of \( 1 \) as a sum of orthogonal idempotents

\[ 1 = \sum_{X} \delta_{x}(a) \quad \text{finite sum} \]

such that

\[ a \delta_{x}(a) = c_{i}(X) \]
is nilpotent for each $x$. Such an decomposition of $1$ may be identified with a point of $\hat{G}$ with values in $\text{Spec } A$. There are maps

$$\hat{G}(A) \xrightarrow{\varphi} D(A) \xrightarrow{\pi} \hat{G}(A)$$

$$1 = \sum x \mapsto \sum x^c_1(x)$$

$$a \mapsto 1 = \sum \delta_x(a)$$

whose composition is the identity. These are homomorphisms as one sees by local calculations on $\text{Spec } A$. One thus has an isomorphism of group-valued functors

$$D(A) \xleftarrow{\sim} \hat{G}(A) \times D_0(A)$$

where $D_0(A) =$ nilpotent elements in $A$.

Next consider what we need to describe the group law on $D$, the functor $D$. Locally on $\text{Spec } A$ every element of $D(A)$ is uniquely expressible in the form $c_1(x) + x$ where $X \in \hat{G}$ and $x$ is nilpotent. The group structure is given by formulas

$$x + y = F(x, y) \quad \text{F ordinary formal group law}$$

$$c_1(x) \cdot x = c_1(x) + q_x(x)$$

where $q_x(x)$ is a power series with coefficients in $Q(pt)$ with leading term $a_1 X$, $a_1 \in Q(pt)^*$ (otherwise $x \mapsto q_1(x)x$ wouldn't be an isomorphism).
August 25, 1969:

Analysis of Eysin sequence

\[ \rightarrow U^i_{Z_p}(X) \xrightarrow{\cdot \omega} U^{g+2}_{Z_p}(X) \xrightarrow{\pi^*} U^{g+2}_{Z_p}(S^1 \times X) \xrightarrow{\delta^*} U^{g+1}_{Z_p}(X). \]

\[ \pi^* \text{ sends a class represented by } \frac{Z}{\text{card } g+2} \]
\[ X \xrightarrow{\text{into}} S^1 \times X \xrightarrow{\text{into}} X \times X \]

whereas \( S^1 \times X \)
\[ \frac{Z}{\text{card } g+2} \]
\[ X \xrightarrow{\text{into}} S^1 \times X \xrightarrow{\text{into}} X \]

The reason \( \delta \pi^* \) is zero is because the map \( S^1 \times X \xrightarrow{pr_2} X \) factors into \( S^1 \times X \xrightarrow{\text{inid}} D^2 \times X \xrightarrow{pr_1} X \) and because integrating a class on \( S^1 \) which comes from \( D^2 \) must give \( 0 \). Now note that there is an exact sequence:

\[ \rightarrow U^i_{Z_p}(Z_p \times X) \xrightarrow{\cdot \omega} U^i_{Z_p}(S^1 \times X) \xrightarrow{\delta^*} U^i_{Z_p}( (S^1 - Z_p) \times X ) \rightarrow \]

\[ \delta^* \]
\[ U^{i-1}_{Z_p}(X) \xrightarrow{\cdot \omega} U^i_{Z_p}(S^1 \times X) \xrightarrow{\delta^*} U^i_{Z_p}( (S^1 - Z_p) \times X ) \rightarrow \]

\[ \delta^* \]
which is a Wang type exact sequence with differential
\[ U^0(X) \xrightarrow{\delta} U^0(X) \]
probably given by \( 1 - t^* \) where \( t: X \to X \) is the action of the generator of \( \mathbb{Z}_p \) on \( X \). So suppose for simplicity that \( X \) is \( \mathbb{Z}_p \)-trivial whence we obtain isomorphisms
\[ U^0_{\mathbb{Z}_p}(S^1 \times X) = \frac{U^0_{\mathbb{Z}_p}(S^1 \times \mathbb{Z}_p \times X)}{U^0_{\mathbb{Z}_p}(S^1 \times \mathbb{Z}_p)} \cong U^0(X) \cdot 1 \oplus U^0(X) \cdot t \cdot 1. \]
The map diagram
\[ \begin{array}{ccc}
U^0(X) & \xrightarrow{\cdot 1} & U^0(X) \\
\downarrow \text{incl} & & \downarrow \text{incl} \\
U^0(X) & \xrightarrow{\cdot t} & U^0(X) \\
\end{array} \]
commutes.

\[ U^0_{\mathbb{Z}_p}(S^1 \times X) \xrightarrow{\cdot t} U^0_{\mathbb{Z}_p}(S^1 \times X) \]
commutes and so the map \( \delta \) which is the top composition \( x \mapsto t^* x \) is the induction map \( U^0(X) \to U^0_{\mathbb{Z}_p}(X) \). The map
\[ \begin{array}{ccc}
U^0(X) & \xrightarrow{\cdot 1} & U^0_{\mathbb{Z}_p}(S^1 \times X) \\
\downarrow \text{incl} & & \downarrow \text{incl} \\
S^1 \times \mathbb{Z}_p & \xrightarrow{\cdot t} & S^1 \times \mathbb{Z}_p \\
\end{array} \]
is zero since \( S^1 \times 2 \xrightarrow{fp_{S^1}} X \) is the boundary of \( D^2 \times 2 \xrightarrow{fp_{D^2}} X \).
Thus the Gysin sequence under consideration appears to be
\[ \rightarrow U^0(X) \xrightarrow{\text{ind}} U^0_{Z_p}(X) \xrightarrow{\omega} \tilde{U}^{8+2}_{Z_p}(X) \xrightarrow{\beta} U^{8+1}(X) \rightarrow \ldots \]

where \( \beta \) does as follows:

\[
\begin{array}{ccc}
\text{given} & \text{form} \\
\downarrow & \downarrow \\
X & X \times V \\
\end{array}
\]

and you get

\[
\begin{array}{ccc}
W & \rightarrow & Z \\
\downarrow g & \downarrow \\
S^1 \times X & \rightarrow & X \rightarrow X \times V \\
\downarrow p_2 & \\
S^1 \times Z_p & \rightarrow & X \\
\downarrow p_2 & \\
X & & \\
\end{array}
\]

Therefore \( \beta \) is capping with the homology element in \( U_1^1(BZ_p) \) represented by the free \( Z_p \) manifold \( S^1 \). Note that the image of \( \beta \) is of order \( p \) since

\[ U^0(X) \xrightarrow{\text{ind}} U^0_{Z_p}(X) \xrightarrow{\text{rest}} U^0(X) \]

is multiplication by \( p \).

If we ignore \( p \)-torsion, then the map \( \beta \) is zero and \( \frac{1}{p} \text{rest} \) is the left inverse to \( \text{ind} \). Thus we obtain have that \( \omega \) is an isomorphism when restricted to its image and we have a ring isomorphism:

\[
\begin{align*}
U^0_{Z_p}(X)[\frac{1}{p}] & \cong (U^0_{Z_p}(X)[\frac{1}{p}] / \ker \omega) \times U^0(X)[\frac{1}{p}] \\
\end{align*}
\]

\[ U^0_{Z_p}(X)[\frac{1}{p}, \omega] \]
Hence it seems that we do not get the unstable operations we need.
August 27, 1969

I know that for each integer $k \geq 0$, there is a unique multiplicative operation

$$\psi^k : U^w(X) \rightarrow U^w(X)[\frac{1}{k}]$$

degree $0$

given by

$$\psi^k c_i(L) = c_i(L \otimes^k).$$

It's clear that $\psi^k$ induces the identity on $U^*(pt)$, hence $\psi^k x$ is without denominators whenever $x$ is in the subring $\cup_{i \geq 0} U^i(X)$ generated by $U^*(pt)$ and the Chern classes. According to Novikov, $\psi^k : U^\otimes(X) \rightarrow U^\otimes(X)$ for $k \geq 0$. (This is because $U^\otimes(X) = [X, E_8]$ where $E_8$ is a space without torsion and only even dimensional cells. One uses the Hurewicz spectral sequence to show $U^\otimes(M)$ has no torsion for $k < r$ and concludes that $\psi^k$ is integral for $E_8$ by Steenrod-Hatcher.)

It follows that the stable Adams operations $\psi^k : U^w \rightarrow U^{w-k}$ are integral in degrees $\leq 0$.

Question: Can $\psi^k : U^w \rightarrow U^{w-k}$ be defined by some Steenrod method.

Try $k=2$:

$$U^w(X) \xrightarrow{Q} U^w_{\mathbb{Z}_2}(X) \rightarrow U^w_{\mathbb{Z}_2}[\omega^k] \xrightarrow{\gamma_0} U(X)[\omega^k, a_1, \ldots]$$
Recall that tom Dieck's localization theorem says that there is an isomorphism
\[ U_G(X)[\mathcal{E}] \cong U(X)[e_1^*, e_2^*, a_2^*, a_3^*, \ldots] \text{ in } \hat{\mathcal{E}}. \]

The map from \( U_G(X) \) to the right side comes by regarding the right side as an equivariant theory with equivariant homomorphism \( f_* \) defined for a proper oriented \( G \)-map \( f : X \to Y \) by
\[ f_*(x) = f^G_* \{ \varphi(\mu_f) \} \]
where \( \mu_f \) and \( \varphi \) are as follows. Note that without trivial components
\[ \mu_f|X^G = \mu_f^G + \mu_f \]
where \( \mu_f \) is a \( G \)-bundle on \( X^G \), hence
\[ \mu_f = \bigoplus_{i \in \hat{G}-0} V_i \otimes \mathbb{Q} \mu_f^i. \]

Then
\[ \varphi(\mu_f) = \prod_{i \in \hat{G}-0} e_i \text{ rank } \mu_f^i \sum \lambda^a \left[ \sum_{a} a^i \cdot c_a(\mu_f^i) \right] \]
or equivalently \( \varphi \) is the multiplicative characteristic class which is given by
\[ \varphi(V_i \otimes L) = e_i \sum_{n \geq 0} a^i c_1(L)^n, \quad a^i = 1. \]

Here \( V_i \) is the \( i \)-th irred. rep of \( G \) and \( L \) is a \( G \)-trivial line bundle.

So now consider \( G = \mathbb{Z}_2 \) and let \( \omega = c_1(\eta) \) where \( \eta \) is the non-trivial character of \( \mathbb{Z}_2 \). Then the tom Dieck map associated to the \( \mathbb{Z}_2 \) map \( f^2 : \mathbb{Z}^2 \to \mathbb{X}^2 \) the map \( f : \mathbb{Z} \to \mathbb{X} \) follows.
and the bundle $\mu_f = \eta \otimes V_f$, so one has a commutative diagram

\[
\begin{array}{ccc}
U^0(X) & \xrightarrow{\alpha} & U^0_{\mathbb{Z}_2}(X) \\
\downarrow \lambda & & \downarrow \text{t.o.} \\
U^0(X)[w^{-1}, w, a_1, a_2, \ldots] & & \\
\end{array}
\]

where

\[
\alpha(f, 1) = f \left( w \text{rank } V_f \sum \frac{a}{x} C_x(V_f) \right)
\]

Thus $\alpha$ is in fact the characteristic class map and so we see that all stabilizable operations in $U^0(X)$ can be obtained from the Steenrod procedures.

The operation $\Phi^2 : U^0(X) \to U^0(X)[\frac{1}{2}]$ is obtained as the composition

\[
U^0(X) \xrightarrow{\lambda} U^0(X)[w^{-1}, w, a_1, \ldots] \xrightarrow{\lambda} U^0(X)[\frac{1}{2}]
\]

where $\lambda$ is the $U^0(X)$ algebra map given by

\[
\lambda \left( \sum_{n \geq 0} a_n x^{n+1} \right) = F^U(X, x),
\]

which of course guarantees that

\[
\lambda \times c_1(L) = \lambda \left( \sum a_n c_1(L)^{n+1} \right) = c_1(L^2).
\]

Observe that $\lambda$ is uniquely determined by the formula

\[
\lambda \alpha = \Phi^2
\]

since $\alpha$ is in fact the universal stabilizable operation for $U^0$. 

So now our problem to define an integral $\varphi^2$ reduces to determining whether or not we can define a $\mathcal{I}$ such that there is a commutative square

\[
\begin{array}{ccc}
U^{eo}_\mathbb{Z}_2(X) & \xrightarrow{\mathcal{I}} & U^{eo}(X) \\
\downarrow \text{t.D.} & & \downarrow \\
U^{eo}(X)[\omega^1_{a_{1}, a_{2}, a_{3}}] & \xrightarrow{\lambda} & U^{eo}(X)[\frac{1}{2}]
\end{array}
\]

For spaces without 2 torsion, $\mathcal{I}$ if it exists, is unique, and is a ring homomorphism. Let's examine the situation when $X = pt$ and use the "formal group law" of $U^{eo}_\mathbb{Z}_2$.

We recall that as the theory $U^{eo}_\mathbb{Z}_2(X)$ satisfies the projective bundle theorem, it has a generalized formal group law which for each $U^{eo}_\mathbb{Z}_2(pt)$ (co understood from now on) $\Delta$-algebra $A$ gives an abelian group structure on

\[
D(A) = \{a \in A | (a(a-w))^n = 0 \text{ some } n\}.
\]

This arises as follows. Given $a \in D(A)$ we have a homomorphism

\[
U^{eo}_\mathbb{Z}_2(P(n(1+\eta))) = U^{eo}_\mathbb{Z}_2(pt) [X]/(X(X-w))^n \longrightarrow A
\]

for some $n$. Similarly for $a'$. Now $E$ bundle map

\[
O(i) \boxtimes O(i) \rightarrow O(i)
\]

\[
P(n(1+\eta)) \times P(n'(1+\eta)) \xrightarrow{\mu} P(n^n(1+\eta))
\]

for some $n''$.

Hence we have:
\[ U_2(\text{pt}) [X]/(X(x-y)) \xrightarrow{\mu^*} U_2(P(n+1+\eta) \times P(n'+1+\eta)) \]

\[ \cong U_2(\text{pt}) [X',X]/(X(x-w)) \cong (x'(x'-w)) \]

\[ \xrightarrow{\rightarrow} A \quad X \mapsto a \quad X' \mapsto a' \]

Then \( a \ast a' \) is the image of \( X \) under the composition of the above maps.

I also recall that if \( w^{-1} \) exists in \( A \), then locally on \( \text{Spec} A \) any element \( a \) of \( D(A) \) is either nilpotent or of the form \( w + x \) where \( x \) is nilpotent and the group law on \( D(A) \) is determined by the rules

\[ x \ast y = F^A(x,y) \quad \text{if} \quad x, y \text{ nilpotent} \]

\[ w \ast x = w + \sum_{n \geq 0} b_n x^{n+1} \]

where \( b_n \in U_2(\text{pt})[w^{-1}] \) and \( b_0 \) is a unit. I want now to determine in terms of \( w \) and \( a_n \) what are the \( b_n \).

Now if \( L \) is a line bundle over \( X \) and \( \mathbb{Z}_2 \) acts trivially on \( L \), then

\[ w \ast c_1(L) = c_1(\eta \otimes L) \]

Under the form Dixmey transform element becomes

\[ w \sum_{n \geq 0} a_n c_1(L)^n \]

\[ \eta \otimes L \]

\[ f^2 = \text{id}, \mu = \mathbb{Z} \]
and therefore we have that
\[ b_n = w - a_n \quad \text{for } n \geq 1. \]

The good formula is
\[ w \times x = w \sum_{n \geq 0} a_n x^n. \]

(Puzzle: this seems to imply that \( a_1 \) is invertible. \textbf{Disaster:} the projective bundle theorem is false for \( U^*_G \) even when \( G \) is \( \mathbb{Z}_2 \). Everything you've done about formal group laws for \( U_G \) is completely wrong. The place the argument breaks down is as follows: suppose \( E = L_1 + L_2 \) and we want to prove that \( b_1, c_1(\mathcal{O}(1)) \) forms a base for \( U_G(PE) \). So we have
\[ \mathcal{P}(L_1) \rightarrow \mathcal{P}(L_1+L_2) \leftarrow \mathcal{P}(L_2) \]
and we know that
\[ 1 \text{ and } \chi^*1 = c_1(\mathcal{O}(1) \otimes f^*L_2) \]
form a base for \( PE \). Now before you analyzed what happened with \( P^n \) first to construct the formal group law, and then used the group law to show that
\[ c_1(\mathcal{O}(1)) - c_1(L_2^*) = c_1(\mathcal{O}(1) \otimes L_2 \{ 1 + \text{nilpotent} \}) \]
so that necessarily \( 1 \) and \( c_1(\mathcal{O}(1)) \) were a basis. Here no such argument is possible - the best example is to consider the equivariant theory
\[ X \rightarrow H^*(X^G, \mathbb{Z}) [w, w^{-1}] = Q(x) \]
with \( \text{Yssin given by } f_*^G(\omega \cdot \text{dim } E_\mathfrak{g}). \)
Then for a line bundle \( L \) over \( X \) on which \( G \) acts trivially we have 
\[
c_1(\eta \otimes L) = \omega = c_1(\eta)
\]
so we would get a contradiction if we could use a group structure on \( D(4) \) and conclude that 
\[
c_1(L) = c_1(\eta \otimes (\eta \otimes L)) = \omega \ast \omega = c_1(\eta \otimes \eta) = 0
\]
which is false. Observe what happens for \( Q(\mathbb{P}(\eta + \eta)) \).

The generators are 1 and 
\[
c_1(\mathcal{O}(1) \otimes \eta) = c_1(\mathcal{O}(\mathbb{P}(2), 1)) \text{ but }
\]
\[
c_1(\mathcal{O}(1)) = c_1(\eta \otimes \mathcal{O}(\mathbb{P}(2), 1)) = \omega
\]
which lives in \( Q(\text{pt}) \).
August 30, 1969

Impossibility of defining an integral \( \varphi^2 \) on \( U^w \) by the Steenrod method:

We have the following diagram of solid arrows:

\[
\begin{array}{cccc}
\omega \sum_{n=0}^{\infty} a_n x^{n+1} & \rightarrow & F(X, X) \\
\uparrow & & & \\
\varphi^2 & \rightarrow & U(X) & \rightarrow U(X)[\frac{1}{2}] \\
\downarrow & & & \\
U(X)[w, w^5 a_n, \ldots] & \rightarrow & U(X)[\frac{1}{2}] \\
\downarrow & & & \\
U_2(X) & \rightarrow & U_2(X)[\varphi] \\
\downarrow & & & \\
\varphi^2 & \rightarrow & U(X)[w, w^5 a_n, \ldots] \\
\downarrow & & & \\
U_2(X) & \rightarrow & U_2(X)[\varphi] \\
\end{array}
\]

and we would like to know whether there exists a dotted arrow natural in \( X \). We will now show there isn't such a dotted arrow even for \( X = \text{pt} \). The proof is based on the following:

**Proposition:** The elements \( a_1 = w^{-1} \) and \( a_n \) for \( n \geq 2 \) are in the images of the map \( U_2(\text{pt}) \rightarrow U_2(\text{pt})[w^{-1}, w^5 a_n, \ldots] \).

**Proof:** According to tom Dieck's integrality theorem we have a map of exact sequences:

\[
\begin{array}{cccc}
? & \rightarrow & U_2(X) & \rightarrow U_2(X)[\varphi] & \rightarrow ? \\
\downarrow S & & \downarrow & & \downarrow S \\
? & \rightarrow & U(\mathcal{B}Z_2 \times X) & \rightarrow U(\mathcal{B}Z_2 \times X)[\varphi] & \rightarrow ?
\end{array}
\]