August 7, 1969

Concerning groups G having a representation V with SV G-free.

G abelian $\Longrightarrow$ G cyclic. In effect there is a an irreducible subrepresentation $W \subset V$ with $\dim_{\mathbb{R}} W \leq 2$. If $\dim W = 1$, then G acts on W by sign so $G = \mathbb{Z}/2\mathbb{Z}$. If $\dim W \geq 2$, then W has ~~an invol~~ a complex structure and G acts via a faithful homomorphism $G \to \mathbb{C}^*$, hence G is cyclic.

Proposition: ~~If p is odd, then any p group such that every abelian~~ Let p be an odd prime and let G be a p group such that every abelian subgroup of G is cyclic. Then G is cyclic.

Proof: Let G be of minimal order such that the proposition is false; then every proper subgroup of G is cyclic. Hence G is generated by 2 elements and so if F is the Frattini subgroup of G, then $Q = G/F$ is a 2-dimensional $\mathbb{F}_p$-vector space. ~~Let x be a generator of~~ We define a homomorphism

$$\theta : Q \longrightarrow F/pF \cong \mathbb{F}_p$$

as follows. Given $q \in Q$ lift it to $x \in G$ and take x^p. Thus

$$\theta(xF) \equiv x^p \mod pF.$$

I claim that θ is a homomorphism since p is odd. Indeed

If $x, y \in G$ and $xyx^{-1}y^{-1} = z \in F$, then

$$(xy)^p = xyxy \cdots xy \quad p \text{ times}$$
$$= zyx^2y\,(xy)^{p-2}$$
$$= z^{1+2}y^2x^3(xy)^{p-3}$$
$$= z^{1+2+\cdots+(p-1)}\,y^p x^p$$

Here we have used that z is central because $\langle x, z\rangle$ + $\langle y, z\rangle$ are necessarily ~~the~~ proper subgroups hence cyclic. As p is odd

$$z^{\frac{p(p-1)}{2}} = \left(z^{\left(\frac{p-1}{2}\right)}\right)^p \in pF$$

hence $\quad (xy)^p \equiv y^p x^p \equiv x^p y^p \quad \text{mod } pF.$

Now as Q is 2-dimensional over $\mathbb{F}_p$, $\exists\, q \in Q \ni q \neq 0$ and $\theta q = 0$. Thus $\exists\, x \in G \ni x \notin F$ and $x^p \in pF$. Therefore the subgroup $\langle x, F\rangle$ is not cyclic and we have a contradiction.

Example to show p odd is necessary. Take the quaternion group of order 8 with ~~generators~~ elements $\pm 1, \pm i, \pm j, \pm k$. Then every proper subgroup is cyclic.

Conclusion: If G is a finite group acting freely on SV, then every odd order Sylow subgroup of G is cyclic

Example of a group of odd order which acts freely on SV.

Let G be the semidirect product of $\mathbb{Z}/p^n\mathbb{Z}$ and a cyclic group $N = \mathbb{Z}/m\mathbb{Z}$ where $(m, p) = 1$. $\mathbb{Z}/p^n\mathbb{Z}$ acts on N by an auto. θ of order p. Let A be the subgroup $\mathbb{Z}/m\mathbb{Z} \times p\mathbb{Z}/p^n\mathbb{Z} \subset G$. Then A is normal abelian cyclic of index p. Let $\chi : A \hookrightarrow S^1 \subset \mathbb{C}^*$ be a faithful character and let V be the induced representation. Now θ acts non-trivially on χ hence $\theta^i\chi$, $0 \le i < p$ are all distinct and V is irreducible. I claim that G acts freely on the sphere SV. To see this recall that V is endowed with a system of imprimitivity $V = \bigoplus_{0 \le i < p} L_i$ where A acts on L_i with character $\theta^i(\chi)$. Thus a vector in V is of the form $v = \sum v_i$, $v_i \in L_i$. Clearly A acts freely since if $a \in A$ and $av = v$, then (for some i $v_i \neq 0$) $av_i = (\theta^i\chi)(a)\, v_i = v_i$ so $(\theta^i\chi)(a) = 1$ and $a = 1$. If $x \in G$ is not in A and $xv = v$, $v \neq 0$, then ~~if x must permute the~~ $x^p v = v$, and $x^p \in A$, so $x^p = 1$. Now look at what happens to x under the homomorphism $\pi : G \to G/N = \mathbb{Z}/p^n\mathbb{Z}$ and one sees that if $n \geq 2$, then $\pi x \in p\mathbb{Z}/p^n\mathbb{Z}$, hence $x \in A$. This is a contradiction, hence G acts freely on SV.

<u>Special case</u>: $\underset{y}{(\mathbb{Z}/7\mathbb{Z})} \times_\theta \underset{x}{(\mathbb{Z}/9\mathbb{Z})} = G$

$$\begin{cases} xyx^{-1} = y^2 \\ x^9 = y^7 = 1 \end{cases}$$

Relation with periodic cohomology: If G acts freely on a homotopy sphere (manifold) S of dimension $n-1$ (preserving orientation), then one has the Gysin sequence for the sphere fibration

$$S \longrightarrow P_G \times_G S \longrightarrow B_G$$

which is

$$\longrightarrow H^{q-n}(B_G, \mathbb{Z}) \xrightarrow{\cup e} H^q(B_G, \mathbb{Z}) \longrightarrow H^q(P_G \times_G S, \mathbb{Z}) \longrightarrow \cdots$$

Now $P_G \times_G S \sim S/G$ is a CW complex of dimension $n-1$ so one sees that

$$\cup e : \; H^q_G(pt, \mathbb{Z}) \xrightarrow{\cong} H^{q+n}_G(pt, \mathbb{Z}) \qquad q>0$$

and that

$$\begin{tikzcd} H^{n-1}(S/G, \mathbb{Z}) \arrow[r, "\text{int over fiber}"'] \arrow[d, phantom, "\cong"] & H^0_G(pt, \mathbb{Z}) \arrow[r, "\cup e"] \arrow[d, phantom, "\cong"] & H^n_G(pt, \mathbb{Z}) \arrow[r] & 0 \\ \mathbb{Z} \arrow[r, "g = |G|"] & \mathbb{Z} \end{tikzcd}$$

Thus $H^n_G(pt, \mathbb{Z})$ is $\cong \mathbb{Z}/g\mathbb{Z}$ with generator e. Therefore the cohomology of G is periodic with period n.

Conversely one knows (SWAN) that if $H^*_G(pt, \mathbb{Z})$ is periodic of period n, then G acts freely on a finite CW complex of the homotopy type of S^{dn-1} for some d. These results have been made more precise by Wall.

Cohomology of a semi-direct product:

$$G = N \times_\sigma C \qquad |N|,\ |G| \text{ rel. prime}$$

The Hochschild–Serre spec. seq. degenerates yielding split exact sequences

$$H^0(G,\mathbb{Z}) = \mathbb{Z}$$

$$0 \longrightarrow H^q(C,\mathbb{Z}) \longrightarrow H^q(G,\mathbb{Z}) \longrightarrow H^q(N,\mathbb{Z})^C \longrightarrow 0. \quad q>0$$

Suppose that N and C are cyclic of orders n, c resp. and ~~coprime~~ Then

$$H^*(N,\mathbb{Z}) = \mathbb{Z}[\eta]/(n\eta)$$

where $\quad \eta \in H^2(N,\mathbb{Z}) \cong \mathrm{Hom}(N,\mathbb{Q}/\mathbb{Z}) \cong \hat{N}$

is the character with $\eta(u) = 1/n \mod \mathbb{Z}$ where u is the generator of N. Thus

$$c\cdot\eta = \eta^q \qquad q \in (\mathbb{Z}/n\mathbb{Z})^*$$

and so $\quad \eta^{\varphi(n)}$ is invariant under C. Thus

$$H^{2\varphi(n)}(N,\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$$

$$H^{2\varphi(n)}(C,\mathbb{Z}) \cong \mathbb{Z}/c\mathbb{Z}$$

and so $\quad H^{2\varphi(n)}(G,\mathbb{Z}) \cong \mathbb{Z}/nc\mathbb{Z} \quad$ since $(n,c)=1$.

Therefore one knows that $H^*(G,\mathbb{Z})$ is periodic of period $\varphi(n)$.

<u>Example</u> of a group with periodic cohomology which does not act freely on SV for any ~~representation~~ V. Take primes p,q with $p \mid q-1$ and form the (non-trivial) semi-direct product

$$G = \mathbb{Z}/q\mathbb{Z} \times_\theta \mathbb{Z}/p\mathbb{Z}$$

By the above the cohomology of G is periodic of period $2p$. G can't act freely on SV, V irreducible complex (say p, q are odd) because either V is 1-dimensional non-faithful or else V is induced from a non-trivial character of $\mathbb{Z}/q\mathbb{Z}$, whence there is a non-zero $\mathbb{Z}/p\mathbb{Z}$-invariant.

I don't yet know the kind of 2 groups that can act freely on the sphere of a representation. But if G is nilpotent and G acts freely on SV, V irred complex, then V is the tensor product of such representations for each Sylow subgroup. In effect if G_1 acts freely on $V_1 - 0$ and G_2 acts freely on $V_2 - 0$ and if $|G_1|, |G_2|$ are rel. prime, then $G_1 \times G_2$ acts freely on $V_1 \otimes V_2 - 0$

August 12, 1969

General facts about G manifolds, G finite odd order

X G-manifold connected. The principal orbit submanifold is where the isotropy representations are trivial. As G is of odd order, the non-principal part is of codim ≥ 2 (†) so $X_{princ.}$ is connected. The map $X_{princ.} \longrightarrow$ subgps of G given by $x \longmapsto G_x$ is locally constant, hence constant, hence $G_x = N$ a normal subgp of G for all $x \in X_{princ.}$. G/N acts on X.

Conclusion: If G acts faithfully on X, then X_{princ} is the open subset where G acts freely.

~~[illegible]~~

(†) We recall that the strata of X are all of even codimension, since G being of odd order, all irreducible real representations of G have complex structures.

~~[illegible] near an orbit [illegible]~~

A neighborhood of an orbit Gx is of the form $G \times_{G_x} T_x$ where T_x is the tangent space to X at x. Note this result even holds complex-analytically if X is a complex manifold since if m_x is the maximal ideal of $\mathcal{O}_x$ the sequence

$$0 \longrightarrow m_x^2 \longrightarrow m_x \longrightarrow m_x/m_x^2 \longrightarrow 0$$

splits as a sequence of G-modules, and so ~~[illegible]~~

~~[illegible]~~ we obtain local coordinates at x giving an étale map $U_x \longrightarrow T_x$ near x.

So to complete local study we replace X by a representation V of G which we suppose is endowed with a complex structure. Assume G faithful and let $Z \subset V$ be the singular cone

$$Z = \{v \in V \mid \exists g \neq 1 \ni gv = v\}.$$

~~Suppose~~ Z is ~~a complete analytic~~ the union of a finite set of subspaces of V, namely the ~~[illegible]~~ fixed point submanifold V^g for each $g \neq 1$.

Suppose that Z is a divisor, i.e. $Z = \bigcup_{i=1}^{n} W_i$ where the W_i are distinct hyperplanes contained in Z. Let H_i be the subgroup of G fixing W_i elementwise, so that $H_i \neq 1$, as $W_i = V^g$ for some $g \neq 1$. Let H_{ij} be the subgroup fixing $W_i \cap W_j$ elementwise. Then H_{ij} acts faithfully on $W/W_i \cap W_j$. As this space is 2 dimensional and H_{ij} is of odd order, H_{ij} must be abelian (degree of an irred. repn. divides ~~that of~~ the order of the group.) H_{ij} commutes with H_i and H_j, it normalizes $W_i / W_i \cap W_j$ and $W_j / W_i \cap W_j$, and hence these are the two eigenspaces of the representation of H_{ij}. Note that the resulting 1-diml repns of H_{ij} are distinct since they are distinct when restricted to $H_i \times H_j$. Assuming V endowed with a unitary structure invariant under G, it follows that $W_i^{\perp}$ and $W_j^{\perp}$ are perpendicular lines for $i \neq j$. Thus V contains an invariant subspace

$$V' = L_1 + \cdots + L_n \qquad L_i = W_i^{\perp}$$

with complement

$$(V')^{\perp} = \cap W_i.$$

As $(V')^{\perp}$ is totally singular for G it is a non-faithful representation of G. Moreover G must act faithfully on V' since otherwise there would be a singular element of the form $\sum_{i=1}^{n} v_i \neq 0 \quad v_i \in L_i - 0$ not contained in Z. Similarly the singular subset of G acting on V' is the union of the hyperplanes $W_i \cap V'$.

<u>Conclusion</u>: If G acts on V and V_{sing} is a divisor, then $V = V' \oplus V''$ where V'' is a non-faithful representation of G and where V' has a system of imprimitivity

$$V' = L_1 + \cdots + L_n \qquad \dim L_i = 1.$$

The singular set of V' is the union of $(L_i)^{\perp}$ and G permutes the lines giving an exact sequence

$$1 \longrightarrow G \cap T_n \longrightarrow G \longrightarrow \Sigma_n$$

Hence G is a subgroup of the normalizer of a torus.

Note that G odd is essential since consider Σ_3 acting on $V = \{(x,y,z) \in \mathbb{C}^3 \mid x+y+z=0\}$. Then V_{sing} is union of the three hyperplanes

$$x = y$$
$$y = z$$
$$x = z$$

which are not mutually perpendicular.

~~Thm~~ If G acts faithfully on a complex vector space V, then V_{sing} is a union of its irreducible components

$$V_{sing} = \bigcup_{i=1}^{n} W_i$$

and each W_i is a subspace. If $H_i = \{g \mid g|W_i = id\}$, then H_i acts freely on $W_i^{\perp} - \{0\}$.

Remarks (added Oct. 7, 1969) In Cartan-Eilenberg one finds the result that a group has periodic cohomology iff all abelian subgroups cyclic iff its Sylow subgroups are cyclic for odd primes and generalized quaternion for $p=2$. Consult Milnor's Amer. J. paper on free actions on spheres for a reference to old paper of Zassenhaus where among other things (odd) groups which act freely on SV are classified. See Zassenhaus's book for classification of groups all of whose Sylow subgroups are cyclic.

August 19, 1969

Equivariant cobordism revisited:

(analysis of the group law in the equivariant case

Let G be an abelian compact Lie group with character group $\hat{G}$ and let us consider a Chern theory Q on the category of G-manifolds (Gysin for proper U-oriented maps) such that $c_1(\chi)$ is a unit in $Q(pt)$ for ~~two~~ ~~distinct element~~ $\chi \in \hat{G}$, $\chi \neq 1$.

Let

$$Q(\mathbb{P}^\infty) = \varprojlim_V Q(\mathbb{P}V)$$

$$\cong \varprojlim_{\underline{n}} Q(pt)[X] \Big/ \prod_\chi (X - c_1(\chi))^{n_\chi}$$

$$\overset{\text{defn}}{=} Q(pt)\{X\}$$

false

where $\underline{n}$ runs over functions from $\hat{G}$ to integers ≥ 0 almost everywhere 0. $Q(\mathbb{P}^\infty)$ is natural transformation from Pic_G to Q. Similarly set $Q(\mathbb{P}^\infty \times \mathbb{P}^\infty)$ = natural transformations from $\mathrm{Pic}_G \times \mathrm{Pic}_G$ to ~~the~~ Q $\overset{\text{defn}}{=} Q(pt)\{X_1, X_2\}$. Then tensor product gives us a homomorphism

$$Q(\mathbb{P}^\infty) \longrightarrow Q(\mathbb{P}^\infty \times \mathbb{P}^\infty)$$

$$X \longmapsto F(X_1, X_2)$$

and $F(X_1, X_2)$ is a kind of "formal group law". I like to think of F as defining an abelian group structure on the functor $A \mapsto D(A)$ where A runs over the category of $Q(pt)$-algebras and where

$$D(A) = \{a \in A \mid \exists n_\chi \text{ with } \prod_\chi (a - c_1(\chi))^{n_\chi} = 0\}.$$

Lemma: $c_1(\chi) \in Q(pt)^*$ for all $\chi \neq 1 \Rightarrow [c_1(\chi) - c_1(\chi')] \in Q(pt)^*$ for $\chi \neq \chi'$.

Proof: If $c_1(\chi) - c_1(\chi')$ is not a unit then there is a non-zero $Q(pt)$-algebra A in which $c_1(\chi) = c_1(\chi')$, whence using the group-structure of $D(A)$, we have $c_1(\chi \otimes \chi'^{-1}) = c_1(\chi) \overset{D}{-} c_1(\chi') = c_1(\chi) \overset{D}{-} c_1(\chi) = 0$. This contradicts fact that $c_1(\chi \otimes \chi'^{-1}) \in A^*$.

By the Chinese remainder theorem

$$Q(pt)\{X\} \xrightarrow{\sim} \prod_\chi Q(pt)[[X - c_1(\chi)]].$$

Another way of putting this is to say there exists idempotents $\delta_\chi(X)$ in $Q(pt)\{X\}$ such that

$$\delta_\chi(c_1(\chi')) = \begin{cases} 0 & \chi' \neq \chi \\ 1 & \chi' = \chi \end{cases}$$

and such that

$$1 = \sum_\chi \delta_\chi$$

as a topological sum in case G isn't finite. (Also $X\delta_\chi - c_1(\chi)$ is top. nilp. Thus given $a \in D(A)$ there exists a decomposition of 1 as a sum of orthogonal idempotents

$$1 = \sum_\chi \delta_\chi(a) \qquad \text{finite sum}$$

such that

$$a\,\delta_\chi(a) - c_1(\chi)$$

is nilpotent for each χ. Such a decomposition of 1 may be identified with a point of $\hat{G}$ with values in Spec A. There are maps

$$\begin{aligned} \hat{G}(A) &\longrightarrow D(A) \longrightarrow \hat{G}(A) \\ 1=\textstyle\sum e_\chi &\longmapsto \textstyle\sum e_\chi \cdot c_1(\chi) \\ & \qquad\quad a \longmapsto 1=\textstyle\sum \delta_\chi(a) \end{aligned}$$

whose composition is the identity. These are homomorphisms as one sees by local calculations on Spec A. One thus has an isomorphism of group-valued functors

$$D(A) \xleftarrow{\ \sim\ } \hat{G}(A) \times D_0(A)$$

where $D_0(A) =$ nilpotent elements in A.

Next consider what we need to describe the group law on the functor D. Locally on Spec A every element of $D(A)$ is uniquely expressible in the form $c_1(\chi) + x$ where $\chi \in \hat{G}$ and x is nilpotent. The group structure is given by formulas

$$x \overset{D}{+} y = F(x,y) \qquad F \text{ ordinary formal group law}$$

$$c_1(\chi) \overset{D}{+} x = c_1(\chi) + \varphi_\chi(x)$$

where $\varphi_\chi(X)$ is a power series with coefficients in $Q(pt)$ with leading term $a_1 X$, $a_1 \in Q(pt)^*$ (otherwise $x \longmapsto c_1(\chi) \overset{D}{+} x$ wouldn't be an isomorphism).

August 25, 1969:

Analysis of Gysin sequence

Attempt to get integral φ^* on U
Counterexample to $U_G(PE)$ result $G = \mathbb{Z}_2$

$$\longrightarrow U^{q}_{\mathbb{Z}_p}(X) \xrightarrow{\cdot w} U^{q+2}_{\mathbb{Z}_p}(X) \xrightarrow{\pi^*} U^{q+2}_{\mathbb{Z}_p}(S^1\times X) \xrightarrow{\delta} U^{q+1}_{\mathbb{Z}_p}(X) \cdots$$

π^* sends a class represented by

$$\begin{tikzcd} & Z \arrow[d, "\text{cod } q+2"] \\ X \arrow[r] & X\times V \end{tikzcd} \qquad \text{~~into~~} \qquad \begin{tikzcd} & & Z \arrow[d] \\ S^1\times X \arrow[r] & X \arrow[r] & X\times V \end{tikzcd}$$

whereas δ sends the class represented by

$$\begin{tikzcd} Z \arrow[d] \\ S^1\times X \end{tikzcd} \qquad \text{into} \qquad \begin{tikzcd} Z \arrow[d] \\ S^1\times X \arrow[d] \\ X \end{tikzcd}.$$

The reason $\delta\pi^*$ is zero is because the map $S^1\times X \xrightarrow{pr_2} X$ factors into $S^1\times X \xrightarrow{i\times id} D^2\times X \xrightarrow{pr_1} X$ and because integrating a class on S^1 which comes from D^2 must give 0. Now note that there is an exact sequence

~~where $i : pt \to S^1$ is the inclusion. Now δ must be a $U^*_{\mathbb{Z}_p}(X)$ module map by general nonsense and~~

$$\begin{array}{ccccccc} \longrightarrow U^{q-1}_{\mathbb{Z}_p}(\mathbb{Z}_p\times X) & \xrightarrow{i_*} & U^{q}_{\mathbb{Z}_p}(S^1\times X) & \xrightarrow{j^*} & U^{q}_{\mathbb{Z}_p}((S^1-\mathbb{Z}_p)\times X) & \longrightarrow \cdots \\ \wr\!\!\mid & & & & \wr\!\!\mid & \\ U^{q-1}(X) & \longrightarrow & U^{q}((S^1\times X)/\mathbb{Z}_p) & \longrightarrow & U^{q}(X) & \xrightarrow{\delta} \end{array}$$

which is a Wang type exact sequence with differential

$$U^q(X) \xrightarrow{\ \delta\ } U^q(X)$$

probably given by $1-t^*$ where $t: X \to X$ is the action of the generator of $\mathbb{Z}_p$ on X. So suppose for simplicity that X is $\mathbb{Z}_p$-trivial whence we obtain isomorphisms

$$U^q_{\mathbb{Z}_p}(S^1 \times X) = U^q_{\mathbb{Z}_p}(S^1/\mathbb{Z}_p \times X) \simeq U^q(X)\cdot 1 \oplus U^{q-1}(X)\cdot \iota_* 1 .$$

The ~~map~~ diagram

$$\begin{tikzcd}
U^{q+1}(X) \arrow[r, "\cdot \iota_* 1"] \arrow[d, phantom, "\cong"] & U^{q+2}_{\mathbb{Z}_p}(S^1/\mathbb{Z}_p \times X) = U^{q+2}_{\mathbb{Z}_p}(S^1 \times X) \arrow[r, "\delta"] & U^{q+1}_{\mathbb{Z}_p}(X) \\
U^{q+1}_{\mathbb{Z}_p}(\mathbb{Z}_p \times X) \arrow[r, "\iota_*"] \arrow[rru, bend right, "\text{int over } \mathbb{Z}_p \to pt"'] & U^{q+2}_{\mathbb{Z}_p}(S^1 \times X) \arrow[ru, "\delta = \text{integration over } S^1"'] &
\end{tikzcd}$$

commutes

commutes and so the map ~~is~~ which is the top composition, $x \mapsto \delta(\iota_* 1 \cdot x)$ is the induction map $U^q(X) \longrightarrow U^q_{\mathbb{Z}_p}(X)$. The map

$$U^{q+1}(X) \xrightarrow{\ \cdot 1\ } U^{q+2}_{\mathbb{Z}_p}(S^1/\mathbb{Z}_p \times X) = U^{q+2}_{\mathbb{Z}_p}(S^1 \times X) \xrightarrow{\ \delta\ } U^{q+1}_{\mathbb{Z}_p}(X)$$

$$\begin{array}{ccccccc}
Z & & S^1/\mathbb{Z}_p \times Z & & S^1 \times Z & & S^1 \times Z \\
\downarrow f & \longmapsto & \downarrow id \times f & \longmapsto & \downarrow & \longmapsto & \downarrow \\
X & & S^1/\mathbb{Z}_p \times X & & S^1 \times X & & S^1 \times X \\
 & & & & & & \downarrow \\
 & & & & & & X
\end{array}$$

~~and this~~ is zero since $S^1 \times Z \xrightarrow{f pr_2} X$ is the boundary of $D^2 \times Z \xrightarrow{f pr_2} X$.
Thus the Gysin sequence under consideration appears to be

$$\longrightarrow \mathcal{U}^q(X) \xrightarrow{\text{ind}} \mathcal{U}^q_{\mathbb{Z}_p}(X) \xrightarrow{\cdot w} \tilde{\mathcal{U}}^{q+2}_{\mathbb{Z}_p}(X) \xrightarrow{\beta} \mathcal{U}^{q+1}(X) \longrightarrow \cdots$$

where β ~~does~~ does as follows:

$$\left(\text{given } \begin{array}{c} Z \\ \downarrow \\ X \longrightarrow X \times V \end{array} \quad \text{form} \quad \begin{tikzcd} W \arrow[rrr, dashed] \arrow[d, "g"] & & & Z \arrow[d] \\ S^1 \times X \arrow[r, "pr_2"] & X \arrow[rr] & & X \times V \end{tikzcd} \right.$$

$$\text{and you get} \quad \left. \begin{tikzcd} W/\mathbb{Z}_p \arrow[d, "g/\mathbb{Z}_p"] \\ S^1/\mathbb{Z}_p \times X \arrow[d, "pr_2"] \\ X \end{tikzcd} \right).$$

Therefore β is capping with the homology element in $\mathcal{U}_1(B\mathbb{Z}_p)$ represented by the free $\mathbb{Z}_p$ manifold S^1. Note that the image of β is of order p since

$$\mathcal{U}^q(X) \xrightarrow{\text{ind}} \mathcal{U}^q_{\mathbb{Z}_p}(X) \xrightarrow{\text{rest}} \mathcal{U}^q(X)$$

is multiplication by p.

If we ignore p-torsion, then the map β is zero and $\frac{1}{p}$rest is ~~an~~ left inverse to ind. Thus we ~~obtain~~ have that w is an isomorphism when restricted to its image and we have a ring isom.

$$\mathcal{U}^q_{\mathbb{Z}_p}(X)[\tfrac{1}{p}] \xrightarrow[\sim]{(\text{can}, \text{rest})} \left(\mathcal{U}^q_{\mathbb{Z}_p}(X)[\tfrac{1}{p}]/\operatorname{Ker} w\right) \times \mathcal{U}^q(X)[\tfrac{1}{p}]$$

$$\left(\mathcal{U}^q_{\mathbb{Z}_p}(X)[\tfrac{1}{p}]/\operatorname{Ker} w\right) \cong \mathcal{U}^q_{\mathbb{Z}_p}(X)[\tfrac{1}{p}, \tfrac{1}{w}].$$

Hence it seems that we do not get the unstable operations we need.

August 27, 1969:

I know that for each integer $k \neq 0$, there is a unique multiplicative operation

$$\psi^k : U^{ev}(X) \longrightarrow U^{ev}(X)[\tfrac{1}{k}] \qquad \text{of degree } 0$$

given by

$$\psi^k c_1(L) = c_1(L^{\otimes k}).$$

It's clear that ψ^k induces the identity on $U^*(pt)$, hence $\psi^k x$ is without denominators whenever x is in the subring of $U^{ev}(X)$ generated by $U^*(pt)$ and the chern classes (of elements of $K(X)$). According to Novikov $\psi^k : U^{2q}(X) \longrightarrow U^{2q}(X)$ for $q \geq 0$. (This is because $U^{2q}(X) = [X, E_q]$ where E_q is a space without torsion and only even dimensional cells. One ~~~~ uses ~~~~ the Serre spectral sequence to show $\Omega^k MU_n$ has no torsion for $k < n$ and concludes that ψ^k is integral for E_q by Stong-Hattori). It follows that the stable Adams operations $\psi^k x \cdot k^{-\frac{\deg x}{2}}$ are integral in degrees ≤ 0.

Question: ~~~~ Can $\psi^k : U^{ev} \to U^{ev}$ be defined by some Steenrod method.

Try $k=2$:

$$\begin{array}{ccc}
U^{ev}(X) & \xrightarrow{Q} & U^{ev}_{\mathbb{Z}_2}(X) \\
 & & \downarrow \\
 & & U^{ev}_{\mathbb{Z}_2}(X)[w^{-1}] \underset{+\Omega}{\cong} U(X)[w^{-1}, w, a_1, \cdots]
\end{array}$$

Recall that tom Dieck's localization theorem says that ~~[illegible]~~ there ~~[illegible]~~ is an isomorphism

$$U_G(X)[e_i^{-1}]_{i\in\hat{G}-0} \longrightarrow U(X)[e_i^{-1}, e_i, a_1^i, a_2^i, \dots]_{i\in\hat{G}-0}$$

~~[illegible]~~ The map from $U_G(X)$ to the ~~[illegible]~~ right side comes by regarding the ~~left~~ right side as an equivariant theory with Gysin homomorphism f_* defined for a proper oriented G-map $f: X \longrightarrow Y$ by

$$f_*(x) = f_*^G\{x\gamma(\mu_f)\}$$

where ~~[illegible]~~ μ_f and γ are as follows. Note that $\nu_f | X^G = \nu_{f^G} + \mu_f$ where μ_f is a G-bundle without G-trivial components on X^G, hence

$$\mu_f = \bigoplus_{i\in\hat{G}-0} V_i \otimes \mu_f^i$$

Then

$$\gamma(\mu_f) = \prod_{i\in\hat{G}-0} e_i^{\operatorname{rank}\mu_f^i} \sum_\alpha (a^i)^\alpha c_\alpha(\mu_f^i)$$

or equivalently γ is the multiplicative characteristic class which is given by

$$\gamma(V_i\otimes L) = e_i \sum_{n\geq 0} a_n^i\, c_1(L)^n, \qquad a_0^i = 1.$$

Here V_i is the ith irred. rep. of G and L is a G-trivial line bundle.

So now consider $G = \mathbb{Z}_2$ and let $\omega = c_1(\eta)$ where η is the non-trivial character of $\mathbb{Z}_2$. Then the tom Dieck map associates to the $\mathbb{Z}_2$ map $f^2: Z^2 \longrightarrow X^2$ the map $f: Z \longrightarrow X$ ~~[illegible]~~

and the bundle $\mu_f = \eta \otimes \nu_f$, so one has a commutative diagram

$$\begin{tikzcd} U^{ev}(X) \arrow[r, "Q"] \arrow[dr, "\alpha"'] & U^{ev}_{\mathbb{Z}_2}(X) \arrow[d, "t.D."] \\ & U^{ev}(X)[w^{-1}, w, a_1, a_2, \cdots] \end{tikzcd}$$

where

$$\alpha(f_* 1) = f_*\left(w^{\operatorname{rank} \nu_f} \sum_\alpha{}' a^\alpha c_\alpha(\nu_f)\right) \qquad \alpha(c_1(L)) = w \sum_{n\geq 0}{}' a_n c_1($$

Thus α is in fact the gigantic characteristic class ~~of~~ maps and so we see that all stabilizable operations in $U^{ev}(X)$ can be obtained from the Steenrod procedures.

The operation $\psi^2 : U^{ev}(X) \longrightarrow U^{ev}(X)[\tfrac{1}{2}]$ is obtained as the composition

$$U^{ev}(X) \xrightarrow{\ \alpha\ } U^{ev}(X)[w^{-1}, w, a_1, \cdots] \xrightarrow{\ \lambda\ } U^{ev}(X)[\tfrac{1}{2}]$$

where λ is the $U^{ev}(X)$ algebra map given by

$$\lambda\left(w \sum_{n\geq 0}{}' a_n X^{n+1}\right) = F^U(X, X),$$

which of course guarantees that

$$\lambda\alpha\, c_1(L) = \lambda\left(w \sum{}' a_n c_1(L)^{n+1}\right) = c_1(L^{\otimes 2}).$$

Observe that λ is uniquely determined by the formula

$$\lambda\alpha = \psi^2$$

since α is in fact the universal stabilizable operation for U^{ev}.

So now our problem to define an integral p^2 reduces to determining whether or not we can define a $\bar{\lambda}$ such that there is a commutative square

$$\begin{tikzcd} U^{ev}_{\mathbb{Z}_2}(X) \arrow[r, "\bar{\lambda}"] \arrow[d, "t.D."] & U^{ev}(X) \arrow[d] \\ U^{ev}(X)[\omega^{-1}, \omega, a_1, \dots] \arrow[r, "\lambda"] & U^{ev}(X)[\tfrac{1}{2}] \end{tikzcd}$$

For spaces without 2 torsion, $\bar{\lambda}$ if it exists, is unique and is a ring homomorphism. Let's examine the situation when $X = pt$ and use the "formal group law" of $U^{ev}_{\mathbb{Z}_2}$.

We recall that as the theory $U^{ev}_{\mathbb{Z}_2}(X)$ satisfies the projective bundle theorem it has a generalized formal group law which for each $U_{\mathbb{Z}_2}(pt)$ (ev understood from now on) -algebra A gives an abelian group structure on

$$D(A) = \{ a \in A \mid (a(a-\omega))^n = 0 \quad \text{some } n \}.$$

This arises as follows. Given $a \in D(A)$ we have a homomorphism

$$U_{\mathbb{Z}_2}(\mathbb{P}(n(1+\eta))) = U_{\mathbb{Z}_2}(pt)[X]/(X(X-\omega))^n \longrightarrow A$$
$$X \longmapsto a$$

for some n. Similarly for a'. Now $\exists$ bundle map

$$\mathbb{P}(n(1+\eta)) \times \mathbb{P}(n'(1+\eta)) \xrightarrow{\mu} \mathbb{P}(n''(1+\eta))$$
$$\mathcal{O}(1) \boxtimes \mathcal{O}(1) \qquad\qquad \mathcal{O}(1)$$

for some n''
whence maps

$$\mathcal{U}_{\mathbb{Z}_2}(pt)[X]/(X(X-\eta))^{n''} \xrightarrow{\mu^*} \mathcal{U}_{\mathbb{Z}_2}(\mathbb{P}(n(1+\eta))\times\mathbb{P}(n'(1+\eta)))$$

$$\cong \mathcal{U}_{\mathbb{Z}_2}(pt)[X,X']/((X(X-\omega))^n,(X'(X'-\omega))^{n'})$$

$$\longrightarrow A \qquad \begin{array}{l} X \mapsto a \\ X' \mapsto a' \end{array}.$$

Then $a * a'$ is the image of X under the composition of the above maps.

I also recall that if ω^{-1} exists in A, then locally on Spec A any element a of $D(A)$ is either ~~of the form~~ nilpotent or of the form $\omega + x$ where x is nilpotent and the group law $*$ on $D(A)$ is ~~~~ determined by ~~~~ the rules

$$x * y = F^{\mathcal{U}}(x,y) \qquad \text{if } x, y \text{ nilpotent}$$

$$\omega * x = \omega + \sum_{n\geq 0}{}' b_n x^{n+1}$$

where $b_n \in \mathcal{U}_{\mathbb{Z}_2}(pt)[\omega^{-1}]$ and b_0 is a unit. I want now to determine in terms of ω and a_n what are the b_n.

~~~~ Now if ~~~~ L is a line bundle ~~~~ over X and $\mathbb{Z}_2$ acts trivially on L, then

$$\omega * c_1(L) = c_1(\eta\otimes L)$$

Under the tom Dieck map this element becomes

$$\omega \sum_{n\geq 0} a_n c_1(L)^n$$

$$\begin{array}{c} X \\ \downarrow f \\ X \longrightarrow \eta\otimes L \end{array}$$

$f^{\mathbb{Z}_2} = id$, $\mu_f = \eta L$
~~~~

and therefore we have that

$$b_{n-1} = w a_n \qquad \text{for} \quad n \geq 1.$$

The good formula is

$$\boxed{w * x = w \sum_{n \geq 0} a_n x^n}.$$

(Puzzle: this seems to imply that a_1 is invertible. DISASTER: the projective bundle theorem is false for U_G^* even when G is $\mathbb{Z}_2$. Everything you've done about formal group laws for U_G is completely wrong. The place the old argument breaks down is as follows: suppose $E = L_1 + L_2$ and we want to prove that $1, c_1(\mathcal{O}(1))$ forms a base for $U_G(\mathbb{P}E)$. So we have

$$\mathbb{P}L_1 \xrightarrow{i} \mathbb{P}(L_1+L_2) \longleftarrow \mathbb{P}L_2$$

and we know that

$$1 \text{ and } i_* 1 = c_1(\mathcal{O}(1) \otimes f^* L_2) \qquad\qquad f : \mathbb{P}E \longrightarrow X$$

form a base for $\mathbb{P}E$. Now before you analyzed what happened with $\mathbb{P}^n$ first to construct the formal group law and then used the group law to show that

$$c_1(\mathcal{O}(1)) - c_1(L_2^\vee) = c_1(\mathcal{O}(1) \otimes L_2)\{1 + \text{nilpotent}\}$$

so that necessarily 1 and $c_1(\mathcal{O}(1))$ were a basis. Here no such argument is possible - the best example is to consider the equivariant theory

$$X \longmapsto H^*(X^G, \mathbb{Z})[w, w^{-1}] = Q(X)$$

with Gysin given by $\qquad f_*^G(w^{\dim \nu_f})$.

Then for a line bundle L over X on which G-acts trivially we have

$$c_1(\eta \otimes L) = \omega = c_1(\eta)$$

so we would get a contradiction if we could use a group structure on $D(A)$ and conclude that

$$c_1(L) = c_1(\eta \otimes (\eta \otimes L)) = \omega * \omega = c_1(\eta \otimes \eta) = 0$$

which is false. Observe what happens for $Q(\mathbb{P}(\eta+\eta))$. The generators are 1 and $c_1(\mathcal{O}_{\mathbb{P}(2\eta)}(1) \otimes \eta) = c_1(\mathcal{O}_{\mathbb{P}(2)}(1))$ but

$$c_1(\mathcal{O}_{\mathbb{P}(2\eta)}(1)) = c_1(\eta \otimes \mathcal{O}_{\mathbb{P}(2)}(1)) = \omega$$

which lives in $Q(pt)$.

August 30, 1969

<u>Impossibility of defining an integral ψ^2 on U^{ev} by the Steenrod method:</u>

We have the following diagram of solid arrows

$$\begin{tikzcd}
c_1(L) \arrow[d, mapsto] & U(X) \arrow[d] \arrow[ddr, "\psi^2"] & \\
e(\rho\otimes L) \arrow[d, mapsto] & U_{\mathbb{Z}_2}(X) \arrow[d, "t.D."] \arrow[r, dashed] & U(X) \arrow[d] \\
\omega\sum\limits_{n\geq 0} a_n c_1(L)^{n+1} & U(X)[\omega^{-1}, \omega, a_1, \ldots] \arrow[r, "\lambda"] & U(X)[\tfrac{1}{2}] \\
& \omega\sum a_n X^{n+1} \arrow[r, mapsto] & F^{U}(X,X)
\end{tikzcd}$$

and we would like to know whether there exists a dotted arrow natural in X. We will now show there isn't such a dotted arrow even for $X = pt$. The proof is based on the following:

<u>Proposition:</u> The elements $a_1 - \omega^{-1}$ and a_n for $n \geq 2$ are in the image of the map $U_{\mathbb{Z}_2}(pt) \xrightarrow{t.D.} U(pt)[\omega^{-1}, \omega, a_1, \ldots]$.

<u>Proof:</u> According to tom Dieck's integrality theorem we have a map of exact sequences

$$\begin{tikzcd}
? \arrow[r] \arrow[d, "\cong"] & U_{\mathbb{Z}_2}(X) \arrow[r] \arrow[d] & U_{\mathbb{Z}_2}(X)[\omega^{-1}] \arrow[r] \arrow[d] & ? \arrow[d, "\cong"] \\
? \arrow[r] & U(B\mathbb{Z}_2\times X) \arrow[r] & U(B\mathbb{Z}_2\times X)[\omega^{-1}] \arrow[r] & ?
\end{tikzcd}$$