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New proof of the decomposition theorem.

1. Let $A$ be a category with an internal product operation $\otimes$, supposed to be unitary, associative, and commutative, and let

$$h : A \rightarrow \text{Mod } A$$

be a tensor functor, i.e. there is given a natural transformation

$$\Delta : hX \otimes hY \rightarrow h(x \otimes y)$$

the unit, the compatible with associativity, and the commutativity isomorphism in $\text{Mod } A$.

Let $P$ be a quasi-$\text{coalgebra}$ over $A$ which acts on $h$ in a fashion compatible with products. This means that there is given a natural transformation

$$\Delta : hX \rightarrow P \otimes hX$$

such that (i) $hX$ is a $P$-comodule,

(ii) $hX \otimes hY \xrightarrow{\Delta \otimes \Delta} (P \otimes hX) \otimes (P \otimes hY) \xrightarrow{(\mu \otimes \text{id}) \otimes \text{id}} (P \otimes P) \otimes hX \otimes hY$

$$\xrightarrow{\text{id} \otimes \text{id}} (P \otimes P) \otimes h(X \otimes Y) \xrightarrow{\mu \otimes \text{id}} P \otimes h(X \otimes Y) \xrightarrow{\text{id} \otimes \text{id}} P \otimes h(X \otimes Y)$$

is commutative for all $X, Y$.

A better way of expressing this is to say that we are given a tensor functor $h' : A \rightarrow \text{Com } P$ such that
is commutative.

Given a ring homomorphism \( u: A \to \mathbb{R} \) we get an \( \mathbb{R} \)-functor

\[
h_u : A \to \text{Mod } \mathbb{R}
\]

\[
X \mapsto R_{\mathbb{R}[A]} hX.
\]

Also given \( \Theta : P \to \mathbb{R} \) a ring homomorphism with \( \Theta s = u \) and \( \Theta t = v \) we obtain a transf.

\[
\overline{\Theta} : h_u \to h_v
\]

defined to be the composition

\[
R_{\mathbb{R}[A]} h(X) \xrightarrow{id \otimes \Delta} R_{\mathbb{R}[A]} P \otimes_A hX \xrightarrow{(id, \Theta) \otimes id} R_{\mathbb{R}[V]} \otimes_A hX.
\]

Here \((id, \Theta)\) is the ring homomorphism from \( R_{\mathbb{R}[A]} P \) to \( R_{\mathbb{R}[V]} \)
given by \( id \) on \( \mathbb{R} \) and \( \Theta \) on \( P \). (In formulas \((id, \Theta)(r \otimes p) = r \Theta p. \)
\( ru(a) \cdot \Theta p = r \cdot u(a) \Theta p = r \cdot \Theta(s(a)p)\)

Notice that if \( \overline{\Theta} \) is a morphism in the category \((A,R)P(R)\), is an isomorphism, then \( \overline{\Theta} \) is an isomorphism.

In particular if \((A,P)\) is an affine groupoid, then \( \overline{\Theta} \) is an isomorphism.
2. Suppose that

\[(2.1) \quad (A, P) \rightarrow (A', P')\]

is a morphism of affine categories; it gives rise to a functor

\[(2.2) \quad (A(R), P(R)) \rightarrow (A'(R), P'(R))\]

in the opposite direction for each ring R. Then there is a commutative square of \(\oplus\)-functors

\[
\begin{array}{ccc}
\text{Com}(P) & \xrightarrow{\oplus h} & \text{Com} P' \\
\text{forget} & & \text{forget} \\
\text{Mod} A & \xrightarrow{\oplus h} & \text{Mod} A'
\end{array}
\]

Consequently, if \(h : A \rightarrow \text{Mod} A\) is a tensor functor on which \(\oplus h : A \rightarrow \text{Mod} A'\) is a tensor functor on which \(P'\) acts.

We shall say that the morphism is fully faithful (resp. an equivalence) if for each \(R\) the functor (2.2) is fully faithful (resp. an equivalence). Fully faithful means that

\[(2.3) \quad P' \leftarrow A' \otimes_A P \otimes_A A'\]

One sees easily that if \(P\) represents the functor \(\text{End} \otimes h\) then \(P'\) given by (2.3) represents the functor \(\text{End} \otimes h'\), where \(h' = A' \otimes_A h\).
Lemma 2. Let \((f, f') : (A, P) \rightarrow (A', P')\) be an equivalence. Then there exists a morphism in the other direction \((g, g') : (A', P') \rightarrow (A, P)\) which is quasi-inverse to \((f, f')\) in the sense that there are isomorphisms of the two composites with the identities.

Proof: We know that for any ring \(R\), the functor
\[
(f^*, f') : (A(R), P(R)) \rightarrow (A(R), P(R))
\]
is an equivalence of categories, and we must show that we can find a quasi-inverse which is independent of \(R\). Take \(R = A\); then \(\text{id}_A \in A(A)\) is an object of \((A(A), P(A))\) hence is isomorphic to \(f^*(g)\) where \(g : A' \rightarrow A\). The isomorphism is given by \(\Theta : P\) such that \(\Theta s = \text{id}_A\), \(\Theta t = gf\).

For any \(u \in A(R)\), \(f^*(g^*u)\) is isomorphic to \(u\) via \(\Theta\), hence \(f^*(g^*u)\) is an equivalence. Thus there exists a morphism \(f^*u \in P(R)\) with target \(g^*u\), which is provided by \(f^*\) onto the given isomorphism of \(u\) to \(f^*(g^*u)\).

Now, \(g^*\) maps objects of \((A, P)(R)\) to the objects of \((A', P')(R)\) and after composition with \(f^*\) there is an isomorphism of \(u \in A(R)\) with \(f^*g^*u = ugf\) given by \(\Phi : A \rightarrow R\). Thus we know \(g^*\) extends to a functor quasi-inverse to \((f^*, f')\) in a canonical way which is functorial in \(R\). Hence we obtain \(g_1 : P' \rightarrow P\), etc.
Definition: Let \((A, P)\) be an affine category and let \(u : A \rightarrow A'\) and \(u' : A' \rightarrow A''\) be ring morphisms. Then to \(u\) we have associated an affine category \((A', P')\) and a fully faithful faithful functor \((A, P) \rightarrow (A', P')\). Same for \(u'\). We say that \(u\) and \(u'\) are equivalent if there exists an equivalence \(g\)

\[(A', P') \xrightarrow{\sim} (A'', P'') \]

\[\alpha \quad \beta \]

\[(A, P)\]

such that \(gu' \simeq u''\).

In terms of categories we have a diagram

\[C' \xrightarrow{F'} C \xleftarrow{F''} C''\]

where \(F'\) and \(F''\) are fully faithful. Then \(F'\) and \(F''\) are equivalent if they have the same essential image.

In concrete terms it means that there exist maps

\[C' \xrightarrow{G_1} C'' \xleftarrow{G_2}\]

a set of functions.
together with isomorphisms

\[ \Theta' : F' \longrightarrow F'' G_1 \]
\[ \Theta'' : F'' \longrightarrow F' G_2 G_1. \]

From these one deduces an isomorphism

\[ F' \longrightarrow F'' G_1 \longrightarrow F' G_2 G_1, \]

hence as \( F' \) is fully faithful an isomorphism

\[ \text{id} \longrightarrow G_2 G_1. \]

Similarly one gets an isomorphism

\[ \text{id} \longrightarrow G_1 G_2 \]

and we checked these two isomorphisms are compatible.

Therefore \( u' : A \longrightarrow A' \) and \( u'' : A \longrightarrow A'' \) are equivalent if there is a diagram

\[
\begin{array}{ccc}
A' & \xleftarrow{g_2} & A'' \\
\downarrow{u'} & \nearrow & \downarrow{u''} \\
A & \xrightarrow{g_1} & A'
\end{array}
\]

where \( \Theta' \) and \( \Theta'' \) are isomorphisms. From such a diagram one gets isomorphisms

\[
A'' \otimes_A h_u = A'' \otimes_A h_{u'} \Theta' \]
\[
A' \otimes_A h_u = A' \otimes_A h_{u''} \Theta''.
\]
Example: Recall that if \( \theta: \Omega \to Q \) is a multiplicative transformation, where \( Q \) is a Chern theory, such that
\[
\theta(x^*) = x \cdot x^*
\]
with \( x \in Q(pt)^* \), where \( \iota: pt \to \mathbb{P}^1 \), then by RR there is a unique multiplicative transformation \( \hat{\theta}: K \to Q^{**} \) such that \( \theta = \hat{\theta} : \Omega \to Q \). Here the notation is as follows: \( \hat{\theta} \) denotes a power series in \( (Q(pt)[[X]]^*) \), \( \hat{\theta} \) is the unique multiplicative transf \( K \to Q^{**} \) \( \hat{\theta}(L) = \hat{p}(c^\theta(L)) \)

\[ p(x) = x p(x) \] and
\[
\hat{\theta}: \Omega \to Q
\]
is the unique multiplicative transf.
\[
\hat{\theta}(f \cdot x) = f_x(\hat{\theta} x, \hat{\theta}(y_x)).
\]

As equivalently by RR the unique transf
\[
\hat{\theta}(c^\theta(L)) = \hat{p}(c^\theta(L)).
\]

Let \( A \) be the Lawvere ring with universal law \( Fun_{A} \) and let \( A \to O(pt) \) be given by \( Fun_{A} \to F^\Omega \). Let \((A,P)\) be the affine category associating to each ring its category of formal group laws:

Objects = formal group laws \( R \in \text{Hom}(A,R) \)

\[
\text{Hom}(F,F') = \left\{ \text{power series } p(x) = \sum_{n=0}^\infty a_n x^{n+1}, a_n \in \mathbb{R}^* \mid p*F = F' \right\}
\]
Composition is defined as follows: Given \( u : F \to F' \) and \( v : F' \to F'' \), say \( u = (F, p) \) \( v = (F', q) \), then \( v \circ u = (F, qp) \).

Make \((A, p)\) act on \( \Omega \) as follows. Given a formal group law \( F \) over \( R \), let

\[ \Omega_F = R \otimes_A \Omega \]

where \( u : A \to R \) sends \( F_{univ} \) to \( F \). Given \( p(x) \in R[[X]]^* \), let

\[ p^{-1} : \Omega \to R \otimes_A \Omega \]

be the transformation (multiplicative) such that

\[ p^{-1} c_1(L) = p^{-1}(c_1(L)). \]

Then

\[ p : F^{\Omega} = p^{-1} \circ F^{\Omega} \]

where \( u : A \to R \) sends \( F_{univ} \) to \( F \). Note that

\[ \text{in}_1 : R \xrightarrow{\sim} \Omega_F(pt), \]

hence given \( q(x) \in R[[X]]^* \) there is a unique multiplicative transformation

\[ \hat{q}^*: \Omega \to \Omega_F \]

such that

\[ \hat{q}^* f_* x = f_* (\hat{q}^* x \cdot \hat{q}^*(v_f)) \]

where

\[ \hat{q}^* : K \to \Omega_F^* \]

is the multiplicative characteristic class (genus) given by
\[ \hat{\vartheta}(L) = (\text{in}_1 \vartheta)(\text{in}_2 \vartheta) c_1 L \]

From now on we identify \( \vartheta_F(pt) \) with \( R \) via \( \text{in}_1 \).

Note that then \( \hat{\vartheta} c_1(L) \).

\[ \hat{\vartheta}_F = \vartheta \]

Note that
\[ \hat{\vartheta} c_1(L) = \vartheta(c_1(L)) \]
hence
\[ \hat{\vartheta}_F = \vartheta(\vartheta c_1(L)) = \vartheta F \]
so if \( \varphi : A \to R \) sends \( \vartheta \) to \( \vartheta F \), then the diagram
\[
\begin{array}{c}
A \rightarrow \Omega(pt) \\
\downarrow \varphi \downarrow \hat{\vartheta} \\
R \rightarrow \Omega_F(pt)
\end{array}
\]
is commutative, so in fact \( \hat{\vartheta} \) induces a transformation
\[ \hat{\vartheta} : \Omega \vartheta F \to \Omega_F \]
which is \( R \)-linear and satisfies
\[ \hat{\vartheta} c_1(L) = \vartheta(c_1(L)) \]

Therefore if \((F, p)\) is a morphism in \((A(R), P(R))\) from \( F \)
to \( F' = p \times F \), we have a multiplicative transformation
\[ \hat{p}^{-1} : \Omega_F \to \Omega_{F'} \]
Given \( p, q \in \mathbb{R}[[X]]^* \).

\[
\begin{align*}
\Omega_F & \xrightarrow{\overset{\sim}{p}} \Omega_{p^*F} \xrightarrow{\overset{\sim}{q}} \Omega_{q^*p^*F}. \\
\overset{\sim}{q}^{-1} p^{-1} (c_{iL}) & = \overset{\sim}{\delta}^{-1} (p^{-1}(c_{iL})) \\
& = p^{-1}(q^{-1}(c_{iL})) \\
& = (q^{-1}p)^{-1} c_{iL}.
\end{align*}
\]

Thus we have defined an action of \( \otimes (A, \hat{\mathbb{R}}) \) on \( \Omega \).

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**Examples 1:** Suppose \( h = \Omega \otimes \hat{\mathbb{Z}}(p) \) and \((A, \hat{\mathbb{R}})\) is the affine category associating to each \( \hat{\mathbb{Z}}(p) \) algebra its category of formal group laws. Let \((A', \hat{\mathbb{R}}')\) be the full subcategory associating to a \( \hat{\mathbb{Z}}(p) \) algebra its full subcategory of typical group laws, and let \( f : (A, \hat{\mathbb{R}}) \to (A', \hat{\mathbb{R}}') \) be the inclusion functor. Then this functor is an equivalence of categories; indeed, Cartier's recoordination defines a map

\[
A' \xrightarrow{\sim} A
\]

such that \( g f = \text{id}_A \) and such that \( f g \) is isomorphic to \( \text{id}_A \).

Hence if we define

\[
\Omega' = A'[\hat{g}] \otimes_{A} \Omega(p)
\]

then there is an isomorphism

\[
A \otimes_{A'} \Omega' \xrightarrow{\sim} \Omega(p).
\]
This isomorphism is the composition
\[ A \otimes A \otimes A \cong A \otimes (A \otimes A) \cong \Omega \otimes A \cong A \otimes \Omega = 2 \Omega \]

Example 2: Suppose \( h = \Omega \otimes \Omega \), \( (A',p') = \) formal group laws over \( R \) (an alg. over \( Q \)), \( (A(R),p(R)) \) full subcategory consisting of the law - \( X+Y \). Then
\[ \Omega' = A' \otimes \Omega 
\]
is a universal Chern theory with values in \( \Omega \) algebras with law \( X+Y \).
By \( R \)-\( R \) this theory is \( \otimes X \to H^*(X,\Omega) \). Thus
\[ \Omega = A \otimes H^*(X,\Omega). \]
(more detail required here - K-theory relation with Conner- Floyd theme)

Example 3: \( h = \mathcal{N} \) unoriented cobordism \( \mathcal{N} \) formal laws of height \( n \) over the \( F_2 \)-alg. \( R \)
(\( \mathcal{N}(R) \) = full subcategory consisting of the law - \( X+Y \). Then
\[ \mathcal{N}(X) \cong \mathcal{N}'(X) \]

By formal group law theory is a polynomial ring over \( F_2 \) with generators in dimensions \( \geq n-1 \). Thus using Thom's theorem
\[ \mathcal{N}(pt) = \mathbb{Z}_2 \] by uniqueness theorem in generalized cohomology theory
\[ \mathcal{N}(X) = H^*(X,\mathbb{Z}_2). \]

(more detail. One sees that \( F^1 \) has a unique log of form
\[ L(X) = \sum \alpha_n x^n \] with \( \int_a \alpha_n = 0, i \geq 0 \) \( a_0 = 1 \)
We now wish to consider the case of laws which are locally isomorphic for the \( \text{fpqc} \) topology. Thus in the situation on page 6, we are given group laws \( F' \) over \( A' \) and \( F'' \) on \( A'' \) and morphisms \( g_2: A' \to A' \) and \( g_1: A'' \to A'' \) together with isomorphisms \( g_2(F') \cong F' \) and \( g_1(F'') \cong F'' \). Now we wish to understand what happens in the case that we are given.

Recall the situation in the bottom of page 6, where we said that \( u: A \to A' \) and \( u': A \to A'' \) are equivalent if

\[
A' \xrightarrow{g_2} A'' \xrightarrow{g_1} A'
\]

and in this case

\[
\begin{align*}
A'' \otimes h_{A'} & \cong h_{A''} \\
A' \otimes h_{A''} & \cong h_{A'}
\end{align*}
\]

Now suppose that \( u, u' \) are not equivalent but instead that after a faithfully flat map \( A' \to B' \) the map \( A'' \to B'' \) is equivalent to \( A \to A'' \). Then what we have is an exact sequence

\[
h_{A'} \to h_{B'} \to h_{B' \otimes_{A'B'} A''}
\]
We now wish to consider the case of \textit{albility} law which are locally isomorphic for the \textit{fpgc} topology.

Thus in the situation on page 6, we are given a group laws $F'$ over $A'$ and $F''$ on $A''$ and morphisms $g_1: A' \to A''$ together with isomorphism $g_2(F') \cong F$, $g_2(F'') \cong F$. Now we wish to understand what happens in the case that

Recall the situation in the bottom of page 6, where we said that $u: A \to A'$ and $u'': A \to A''$ are equivalent if

\[
\begin{array}{ccc}
A' & \xrightarrow{g_2} & A'' \\
\uparrow & & \uparrow \\
A & \looparrowleft & A'
\end{array}
\]

and in this case

\[
\begin{align*}
A'' \otimes h_{A'} & \cong h_{A''} \\
A' \otimes h_{A''} & \cong h_{A'}
\end{align*}
\]

Now suppose that $u$, $u''$ are not equivalent but instead that after a faithfully flat map $A' \to B'$ the map $A'' \to A'$ is equivalent to $A'' \to A''$. Then what we have is an exact sequence

\[
h_{A'} \longrightarrow h_{B'} \longrightarrow h_{B' \otimes_{A'B'}}
\]
\begin{align*}
\hat{h}_{B'} & \cong B' \otimes_{A''} \hat{h}_{A''} \\
\hat{h}_{A''} & \cong A'' \otimes_{B'} \hat{h}_{B'}.
\end{align*}

I now wish to apply these considerations to K-theory and \( \Omega \). First the formal group picture:
Let $Q$ be a theory with products and with complex vector bundles. Then there is a ring homomorphism (called characteristic numbers with values in $Q$)

$$Q.(pt) \longrightarrow Q.(MU) \cong Q.(BU)$$

which I would like to describe using formal group laws.

Adams' description of $Q.(BU)$: $Q.(BU) \cong \bigoplus_{i=0}^{\infty} Q.(pt) b_i$, where $\{b_i\}$ is dual base to $\{c_i\}$. Then $Q.(BU) \longrightarrow Q.(BU)$ carries the $b_i$ into generators for $Q.(BU)$ where $b_0 = 1$. Then

$$Q.(BU) = Q.(pt)[b_1, b_2, \ldots] \quad \text{deg } b_i = 2i$$

My description:

$$\text{Hom}_{Q.(pt)-\text{alg}}(Q.(BU), R) \cong \{\text{Mult. char classes } R \rightarrow Q \otimes_{Q.(pt)} R\}$$

$$\cong \{\text{Power series } \sum_{n=0} a_n x^n, a_n \in R, a_0 = 1\}$$

$$\cong \text{Hom}_{Q.(pt)-\text{alg}}(Q.(pt)[b_1, \ldots], R)$$

Classical description of $Q.(pt) \longrightarrow Q.(BU)$:

Given $f: M^n \longrightarrow pt$ compact almost complex manifold, let

$$S^\infty(M^n) = f_! C_\infty(V_f)$$
be the $x$-th characteristic number of $M^n$. Then

$$ \Phi [M^n] = \sum b^x \Phi_x (M^n) $$

My description: Let $c_b : \mathbb{R} \to \mathbb{Q} [b_0, b_2, ...]$ be the universal characteristic class given on line bundles by

$$ c_b (L) = \sum_{i=0}^{\infty} b^i c_i (L)^i $$

$b_0 = 1$.

Then there is a unique multiplicative homomorphism

$$ s_b : \hat{\Omega} (\cdot) \to \mathbb{Q} [\cdot] $$

given by

$$ \hat{c}_b (f \cdot x) = f (x \cdot c_b (v_f)) $$

and

$$ s_b : \Omega (\text{pt}) \to \mathbb{Q} (\text{pt}) [b] $$

is the characteristic numbers map $\Phi$.

One knows in general that if $Q'$ is a Chern theory, and if $q : \mathbb{R} \to (Q')^*$ is a mult. char. class, and if $\hat{q} : \hat{\Omega} \to Q'$ is the induced operation, then

$$ \hat{q} c_i^Q (L) = c_i^{Q'} (L) \cdot \hat{q} (L) = p (c_i^{Q'} L) $$

where $p (X) = \sum_{n \geq 0} a_n X^{n+1} \in \mathbb{Q} (\text{pt}) [\mathbb{Q}]$, $p (c_i L) = \Phi (L)$. Hence

$$ \hat{q} (F \Omega) = p* F^{Q'} $$
and therefore for

\[ s_b = \hat{e}_b : \Omega(pt) \rightarrow Q(pt)[b] = Q(\mathbb{B}U) \]

we have

\[ s_b(F^\Omega) = \prod_b \times F^Q \]

where

\[ p_b(x) = \sum_{x \geq 0} b_x x^{x+1} \quad b_0 = 1 \]

Examples.

1. \[ \pi_*(\mathbb{M}U) \rightarrow H_*(\mathbb{M}U) \cong H_*(\mathbb{B}U) \cong \mathbb{Z}[b] \]

\[ F^\Omega \mapsto (\sum_{n \geq 0} b_n x^{x+1}) \times (x+y) \]

2. \[ \pi_*(\mathbb{M}U) \rightarrow \mathbb{K}_*(\mathbb{B}U) \cong \mathbb{Z}[b^2, b_1, b_2, \ldots] \]

\[ F^\Omega \mapsto (\sum_{n \geq 0} b_n x^{x+1}) \times (x+y - \beta xy) \]

3. \[ \pi_*(\mathbb{M}O) \rightarrow H_*(\mathbb{B}O, \mathbb{F}_2) \cong \mathbb{F}_2[b_1, b_2, \ldots] \]

\[ F^n \mapsto (\sum_{n \geq 0} b_n x^{x+1}) \times (x+y) \]
Let $K^*$ be the periodic generalized cohomology theory of Atiyah–Hirzebruch. Let us review how this is defined. If $X$ is a compact space (resp. pointed space), let

$$K(X) = \text{groth group of v.b. over } X$$

(resp. $K^*(X) = \text{Ker}\{ K(X) \to K(\text{basepoint}) \}$.)

By the periodicity theorem of Bott there is a canonical element $\beta \in K(S^2)$ such that

$$\beta : K(X) \to \tilde{K}(S^2 \wedge X)$$

is an isomorphism. This said we define for a pointed space $X$

$$K^{2n}(X) = \hat{K}(X)$$

$$K^{2n+1}(X) = \hat{K}(S^1 \wedge X)$$

and we define the suspension isomorphism:

$$K^{2n+1}(X) \to K^{2n+2}(S^1 \wedge X)$$

$\text{id}$ def. $\to$ def. $\tilde{K}(S^1 \wedge X) \to \hat{K}(S^1 \wedge X)$
\[ K^{2n}(X) \xrightarrow{\beta} K^{2n+1}(S^1X) \]

The associated spectrum \( BU \) : 

For a compact space \( X \), let \( K(X) \) be the kernel of \( K(X) \to H^*(X, \mathbb{Z}) \).

Then there is a canonical isomorphism

\[ [X, BU] \cong K(X), \]

hence if \( X \) is a pointed connected finite complex, there is an isomorphism

\[ K(X) \cong K(X) = [X, BU], \]

Therefore for a pointed finite complex

\[ K^{2n}(X) = [S^1 \wedge X, BU], \]

\[ K^{2n}(X) = [X, Z \times BU]. \]

From the suspension isomorphism we have

\[ [X, U] \xrightarrow{\sim} [S^1 \wedge X, Z \times BU], \]

\[ [X, Z \times BU] \xrightarrow{\sim} [S^1 \wedge X, U]. \]

Therefore we have homotopy equivalences

\[ U \xrightarrow{\sim} \Omega(Z \times BU), \]

\[ Z \times BU \xrightarrow{\sim} \Omega U. \]
and so we have a spectrum \( BU \) given by

\[
\begin{align*}
BU_{2n} &= \mathbb{Z} \times BU \\
BU_{2n+1} &= U.
\end{align*}
\]

Now I wish to check the Atiyah trick condition, i.e. that \( \text{K}^0 \) as a functor on the suspension category is expressed by Künneth complexes. So given \( X \) in \( A \) and \( u \in \text{K}^0(X) \) we have \( X = \bigoplus_{i=2n}^\infty Y \) with \( Y \) a finite pointed complex for some \( n \geq 0 \) and

\[ \text{K}^0(X) \cong \text{K}^{2n}(Y) = [Y, \mathbb{Z} \times BU]. \]

Now \( BU = \lim_{\rightarrow} \text{Grass}_{\infty} \). \( K(\text{Grass}_{\infty}) \) is finitely generated free \( K^*(pt) = \mathbb{Z}[\beta, \sigma^{-1}] \) module. Clearly \( u \) comes from a map \( Y \rightarrow [-1, r, X] \times G_{\infty} = E \) for some \( r, m \) where \( E \) is a Künneth complex. Thus \( u \) comes from \( X = \bigoplus_{i=2n}^\infty Y \rightarrow \bigoplus_{i=2n}^\infty E \) where \( \bigoplus_{i=2n}^\infty E \) is of Künneth type.

Since this condition holds I know that the functor

\[ R \mapsto (\text{End}^0 \text{K}^*)(R) \]

is represented by the flat affine groupoid

\[ K^*(BU) = \lim_{\rightarrow} K_{*+2n}(\mathbb{Z} \times BU) \]

over \( K(\text{pt}) \).

To calculate the functor \( \text{End}^0 \text{K}^* \) suppose given a graded \( K(\text{pt}) \) algebra \( R \) and a multiplicative stable transf

\[ \gamma \]

of

\[ K^* \rightarrow R \odot K(\text{pt}) K^* \]
composing \( \gamma \) with the Todd map

\[
\Omega^* \xrightarrow{\gamma} K^*
\]

we obtain a stable multiplicative transf.

\[
\Omega^* \xrightarrow{\gamma} R \otimes_{K^*(pt)} K^*
\]

which, as we know, is given by a power series

\[
\varphi(x) = \sum_{n=0} a_n x^{n+1}, \quad a_0 = 1, \quad a_n \in (R \otimes_{K^*(pt)} K^*(pt))^n \cong R^n.
\]

However \( \varphi \) cannot be arbitrary; one knows that

\[
(\gamma \Phi)(c_i^L) = \varphi(c_i^L)
\]

and hence that

\[
(\gamma \Phi)(F^\gamma) = \varphi \ast F^\gamma.
\]

\[
\gamma(F^\gamma)
\]

Now \( F^\gamma(x, y) = x + y - \beta xy \), therefore

\[
\varphi \ast F^\gamma = x + y - \delta(\beta)xy.
\]

Conversely by the Conner-Floyd theorem

\[
\Omega^*(X) \otimes_{\mathbb{Q}(pt)} K^*(pt) \xrightarrow{\mathbb{Q}} K^*(X)
\]

we know that if \( \varphi \ast F^\gamma = x + y - \pi xy \) for some \( \pi \in R^* \), then \( \gamma \) induces a transf from \( K^* \) to \( K_R^* \), so we conclude.
Proposition: \( \text{Hom}_{\mathcal{K}}(\mathcal{B}(\mathcal{U}), R) \cong \{(\sigma, \tau, \varphi)\} \)

\( \sigma, \tau \in R^* \times R^2 \), \( \varphi(x) = \sum_{n=0} a_n x^{n+1} \), \( a_n \in R_{\geq 0} \), \( a_0 = 1 \) and

\[ X + Y - \tau XY = \varphi \ast (X + Y - \sigma XY). \]

Interchange \( \tau \) and \( \sigma \). Should get \( \tau = \text{t(s)} \) and \( \sigma = \text{s(t)} \).

Using this, we shall calculate the structure of \( \mathcal{K}(\mathcal{B}(\mathcal{U})) \).

First observe that above implies

\[ \varphi(x) + \varphi(y) - \tau \varphi(x) \varphi(y) = \varphi(X + Y - \sigma XY) \]

\[ X + a_1 x^2 + y + a_1 y^2 - \tau XY = X + Y - \sigma XY + a_1 (X + Y)^2 \quad \text{mod} \]

\[ \therefore \quad -\tau = -\sigma + 2a_1 \]

\[ \therefore \quad \tau = \sigma - 2a_1 \]

Thus

\[ (-\sigma^{-1}X) \ast (X + Y + XY) = \varphi \ast (-\sigma^{-1}X) \ast (X + Y + XY) \]

\[ (-\sigma^{-1}X) \ast \varphi \ast (-\sigma^{-1}X) \]

is an automorphism of the law \( X + Y + XY \). But one knows that the latter are of the form

\[ \sum_{n \geq 1} b_n x^n \quad \text{where} \ b_n \in R^* \]

where there is a map \( \bigoplus_{n \geq 0} \mathbb{Z}(\mathbb{T}_n) \to R \) such that \( \mathbb{T}_n \to b_n \).
Therefore

\[ K_0(BU) \cong \mathbb{Z}[T, T^{-1}] \otimes_{\mathbb{Z}[T]} \bigoplus_{n \geq 0} \mathbb{Z}[\sigma^r \sigma^{-r}] \]

where \( s, t \) from \( K_0(\mathbb{U}) \cong \mathbb{Z}[\beta, \beta^{-1}] \) to \( K_0(BU) \) are given by

\[ t_0(\beta) = \sigma \]
\[ s_0(\beta) = T \sigma \]

I can use the same method to calculate \( H_0(\mathbb{U}) \).

Thus a ring homomorphism from \( H_0(\mathbb{U}) \) to \( R \) is the same thing as a multiplicative stable operation \( K^* \rightarrow H^* \otimes \mathbb{Z} R \), that is, a unit \( \sigma \) in \( R \) and a power series \( \varphi(X) = \sum_{n \geq 0} a_n X^n + 1, a_0 = 1, a_n \in R^{-2n} \) such that

\[ X + Y - \sigma XY = \varphi \ast (X + Y) \]
\[ (-\sigma^{-1}X) \ast (X + Y + XY) = \varphi \ast (X + Y) \]
\[ X + Y + XY = (X \sigma X) \varphi \ast (X + Y) \]

In other words, \( 1 + (-\sigma X) \varphi \) is an exponential function. One knows these are of the form

\[ \sum_{n \geq 0} b_n X^n \]

where there is a ring homomorphism

\[ \bigoplus_{n \geq 0} \mathbb{Z} \frac{T^n}{n!} \rightarrow R \]

sending \( \frac{T^n}{n!} \rightarrow b_n \). Therefore as \( b_1 = -\sigma \).
\[ H_*(BU) \cong \bigoplus_{n \geq 0} \mathbb{Z} \mathfrak{q}^{\frac{1}{n}} \otimes \mathbb{Z}[\sigma, \sigma^{-1}] \cong \mathbb{Q}(\sigma, \sigma^{-1}). \]

This agrees well with what Kan told me!

The calculation of \( K_*(BU) \) may be simplified as follows. Let \( R \) be a graded \( K(\mathbb{P}^1) \) algebra where \( \beta \mapsto \sigma \in R_2 \otimes (R)^* \). A multiplicative stable operation

\[ \gamma : K^* \longrightarrow R \otimes K^* \]

is determined by its restrictor to \( K^0 \); \( K^0 \) is the additive extension of the restriction to line bundles.

\[ \gamma(L) = \sum_{n \geq 0} b_n (L - 1)^n \]

where \( b_n \in R_0 \). As \( \gamma(L L') = \gamma(L) \gamma(L') \), one knows that

\[ b_n = \binom{b_1}{n} \]

\[ \gamma(L) = L^{b_1} \]

and precisely that there is a ring homomorphism

\[ \bigoplus_{n \geq 0} \mathbb{Z} \big( \frac{T}{n} \big) \longrightarrow R \]

\[ \big( \frac{T}{n} \big) \longmapsto b_n \]

We now calculate \( \gamma(\beta) \). Recall stability of \( \gamma \Rightarrow \) commutativity of

\[ \begin{array}{ccc}
\beta & \longrightarrow & K^{-2}(\mathbb{P}^1) \gamma \longrightarrow (R \otimes K(\mathbb{P}^1))^{-2} \gamma \longrightarrow \sigma \gamma \sigma^{-1} \otimes \beta \\
\downarrow & & \downarrow \left( \sigma \gamma \sigma^{-1} \otimes \beta \right)
\end{array} \]

\[ \begin{array}{ccc}
\beta & \longrightarrow & K^0(S^2) \gamma \longrightarrow R \otimes K(\mathbb{P}^1) K(S^2) \\
\downarrow & & \downarrow \left( \sigma \gamma \sigma^{-1} \otimes (1-H) \right)
\end{array} \]
where $H$ is the Hopf bundle $O(1)$ on $S^2 \rightarrow \mathbb{C}P^1$. Now

$$\gamma(1-H^{-1}) = 1 - H^{-b_1} = b_1(-H^{-1}) \quad \text{since} \quad (1-H^{-1})^2 = 0$$

Thus

$$\sigma \tau^{-1} = b_1 \quad \text{i.e.} \quad \sigma = \gamma(b) = b_1 \tau.$$

So we conclude that any stable operator

$$\gamma: K^* \longrightarrow R \otimes_{K^*(kt)} K^* \quad K^*(kt) \overset{\tau}{\longrightarrow} R \quad \beta \longmapsto \tau$$

is of the form

$$\begin{align*}
\gamma(b) &= \tau \tau \\
\gamma(L) &= L^2 \quad \text{i.e.} \quad \sum_{n \geq 0} (\frac{t}{n})(L-1)^n
\end{align*}$$

Thus

$$K^*(BU) \cong \left[\bigoplus_{n \geq 0} Z(T_n) \otimes Z[T^r]\right] \otimes_{K^*(kt)} K^*(kt)$$

where $\text{Spec} \Gamma = \text{Aut} \hat{G}_m$. Moreover, the canonical homomorphism

$$K^*(X) \overset{\Delta}{\longrightarrow} K^*(BU) \otimes_{K^*(kt)} K^*(X) \cong \Gamma \otimes_{\mathbb{Z}} K^*(X)$$

is the unique additive extension given as line bundles by

$$\Delta(L) = "LT" \quad \text{i.e.} \quad \sum_{n \geq 0} (\frac{t}{n})(L-1)^n$$
Stong-Hattori theorem as formulated by Adams says that

\[ 0 \to \Pi_1(\text{MU}) \to K_*(\text{MU}) \xrightarrow{\Delta} K_*(\text{MU}) \otimes K_*(\text{BU}) \]

is exact. I want to make these maps explicit. Recall that

\[ \text{Hom}_{K(\mathbb{F})}(K_*(\text{MU}), R) = \text{Hom}_{K(\mathbb{F})}(\Omega^*, R \otimes K^*) \]

\[ \cong \{ \sum a_n x^{a_n} \mid a_0 = 1, a_n \in R_n \} \]

we call canonical map

\[ \gamma : \Omega^*(X) \to K_*(\text{MU}) \otimes_{K(\mathbb{F})} K^*(X) \]

given by

\[ \gamma(c_i^*(L)) = \sum_{n \geq 0} b_n c_i^*(L)^{n+1} \]

where

\[ K_*(\text{MU}) = \mathbb{Z}[b_1, b_2, \ldots] \otimes_{\mathbb{Z}} K(\mathbb{F}) \]

Also canonical map

\[ \gamma_1 : K^*(X) \to K_*(\text{BU}) \otimes K^*(X) \]

given by

\[ \gamma_1(L) = L^T \]

\[ \gamma_1(\beta) = T \beta \]

where

\[ K_*(\text{BU}) = \bigoplus_{[I_0]} \mathbb{Z} \left[ T \right] \left[ I^{-1} \right] \otimes_{\mathbb{Z}} K(\mathbb{F}) \]

Therefore combining \( \gamma \) and \( \gamma_1 \) we obtain
an operation
\[ \Omega(X) \xrightarrow{\phi} K(MU) \otimes_{K(pt)} K(X) \xrightarrow{id \otimes \gamma_1} K(MU) \otimes_{K(pt)} K'(BY) \otimes_{K(pt)} K(X) \]
hence by the universal property of the tensor product, there exists a unique ring homomorphism
\[ K(MU) \xrightarrow{\mu} K(MU) \otimes_{K(pt)} K'(BY) \]
such that the two compositions below are equal
\[ \Omega(X) \xrightarrow{\mu \otimes id} K(MU) \otimes_{K(pt)} K'(BY) \xrightarrow{id \otimes \gamma_1} K(MU) \otimes_{K(pt)} K'(BY) \otimes_{K(pt)} K'(BY) \]

I now wish to calculate \( \mu \). Now
\[ K(MU) = \mathbb{Z} \left[ b_1, b_2, \ldots \right] \otimes_{\mathbb{Z} \left[ \beta, \beta^{-1} \right]} \mathbb{Z} \left[ \beta, \beta^{-1} \right] \]
\[ K(BY) = \bigoplus_{n \geq 0} \mathbb{Z} \left[ T^n \right] \left[ T^{-1} \right] \otimes \mathbb{Z} \left[ \beta, \beta^{-1} \right] \]

Thus
\[ K(MU) \otimes_{K(pt)} K(BY) = \mathbb{Z} \left[ b_1, \ldots \right] \otimes \bigoplus_{n \geq 0} \mathbb{Z} \left[ T^n \right] \left[ T^{-1} \right] \otimes \mathbb{Z} \left[ \beta, \beta^{-1} \right] \]

Moreover
\[ \left( id \otimes \gamma_1 \right) \otimes c_i \mathcal{L} = \left( id \otimes \gamma_1 \right) \sum b_n (c_i \mathcal{L})^{n+1} \]
\[ = \sum b_n (\gamma_1 c_i \mathcal{L})^{n+1} \]

Now
\[ \gamma_1 c_i \mathcal{L} = \gamma_1 (\beta^{-1} (1-L^{-1})) = (\gamma \beta)^{-1} (1-L^{-1}) \]
\[
\begin{align*}
\sum_{n=1}^\infty \beta n^{-1} (T^{-1} - 1) &= -\beta n^{-1} T^{-1} \sum_{n=1}^\infty \left( \frac{L^{-1} - 1}{(L^{-1})^n} \right) \\
&= T^{-1} \sum_{n=1}^\infty \left( \frac{T}{(\beta n^{-1} L)} \right) \left( 1 - L^{-1} \right)^n \\
V_i c_i^L &= T^{-1} \sum_{n=1}^\infty \left( \frac{T}{(\beta n^{-1} L)} \right) (c_i K^L) \\
\end{align*}
\]

So

\[
(id \otimes \gamma)(\eta c_i^L) = \left( \sum_{n=0}^\infty b_n X^{n+1} \right) \left( T^{-1} \sum_{n=1}^\infty \left( \frac{T}{(\beta n^{-1} L)} \right) X^n \right) (c_i K^L)
\]

Therefore

\[
\sum_{n=0}^\infty (\mu b_n) X^{n+1} = \left( \sum_{n=0}^\infty b_n X^{n+1} \right) \left( T^{-1} \sum_{n=1}^\infty \left( \frac{T}{(\beta n^{-1} L)} \right) X^n \right)
\]

These formulas simplify with the following change of notation. These write

\[K_\beta(M\omega) = Z[\omega_0 L^{-1}, \omega_0, \omega_1, \ldots] \quad t: K_\beta(\mu t) \to K_\beta(\eta \omega_0)
\]

with

\[\gamma: \Omega^*(X) \to K_\beta(M\omega) \otimes K_\beta(\mu t)
\]

given by

\[
\gamma(c_i^L) = \sum_{n=0}^\infty a_n (1 - L^{-1})^{n+1}
\]

Thus

\[a_n = b_n \beta^{-n-1} \in K_\beta(\psi^2(\mu) M\omega) \quad \text{and}
\]

\[
\sum_{n=0}^\infty (\mu a_n) X^{n+1} = \left( \sum_{n=0}^\infty a_n X^{n+1} \right) \left( T^{-1} \sum_{n=1}^\infty \left( \frac{T}{(\beta n^{-1} L)} \right) X^n \right)
\]
Here is a summary of our calculations:

**Proposition 1:** There is an isomorphism

\[ K^*(\mu) \cong \mathbb{Z}\left[ a_0^{-1}, a_0, a_1, \ldots \right] \quad a_i \in K_{-2}(\mu) \]

such that the canonical maps

\[ K^*(pt) \xrightarrow{t} K^*(\mu) \xleftarrow{S} \Omega^*(pt) \]

\[ \gamma : \Omega^*(X) \longrightarrow K^*(\mu) \otimes K^*(pt) K^*(X) \]

are given by

\[ t(\beta) = a_0^{-1} \]

\[ S(F^\alpha) = \left( \sum_{n \geq 0} \alpha_n x^{nt} \right)^\ast (x + y - xy) \]

\[ \gamma (c_i^\alpha(L)) = \sum_{n \geq 0} a_n (1 - L)^{nt} \]

2) There is an isomorphism

\[ K^*(\mathbb{B}L) \cong \bigoplus_{n \geq 0} \mathbb{Z}[T_n] \otimes \mathbb{Z} \mathbb{Z}[T, T^{-1}] \otimes \mathbb{Z} \frac{1}{T} \]

\[ T \in K_0(\mathbb{B}L) \quad \tau \in K_2(\mathbb{B}L) \]

such that the canonical maps

\[ K^*(pt) \xrightarrow{t} K^*(\mathbb{B}L) \xleftarrow{S} K^*(pt) \]

\[ \gamma_1 : K^*(X) \longrightarrow K^*(\mathbb{B}L) \otimes K^*(pt) K^*(X) \]

are given by
\[ t(\beta) = T \]
\[ s(\beta) = T^T \]
\[ \gamma(L) = L^T \]

3) The unique ring homomorphism

\[ \mu: K(MU) \to K(MU) \otimes_{K(pt)} K(BU) \]

such that the following two compositions are equal

\[ (A) \quad \Omega(X) \to K(MU) \otimes_{K(pt)} K(X) \to K(MU) \otimes_{K(pt)} K(BU) \]

is given by

\[ \sum_{n \geq 0} (\mu a_n) X^{n+1} = \left( \sum_{n \geq 0} a_n X^{n+1} \right) \circ \left( \sum_{n \geq 1} \left( \frac{T}{n} \right) (-1)^{n-1} X^n \right) \]

4) The Stong-Hatieri theorem implies that for a torsion-free finite complex \( X \), the sequence \((A)\) is exact. Equivalently, an algebraic function on invertible power series

\[ P \left( \sum a_n X^{n+1} \right) = P(a_0, a_1, \ldots) \in \mathbb{Z}[[x_0^{-1}, a_0, a_1, \ldots]] \]

can be expressed in the form

\[ P \left( \sum a_n X^{n+1} \right) = Q \left( \left( \sum a_n X^{n+1} \right) \ast (x+y-xy) \right) \]

where \( Q \) is an algebraic function on formal group laws (i.e., \( Q \in \Omega(pt) \)) if and only if
\[ P(\sum a_n x^{n+1}) = P\left( \sum a_n x^{n+1} \circ \left( \sum_{n=1}^{\infty} (-1)^{n-1} x^n \right) \right) \]

**Remark 1:** In terms of schemes, we have that the diagrams

\[
\begin{array}{ccc}
\text{(power series)} \times \text{Aut } \hat{\mathbb{G}}_m & \longrightarrow & \text{(invertible)} \\
\text{(formal gp laws)} & \longrightarrow & f \\
\text{f} & \longrightarrow & f \ast (x + y - xy) \\
f \times g & \longrightarrow & fg
\end{array}
\]

becomes exact after \( \mathbb{G}_a \) is applied.

**Remark 2:** According to Hatton, the map \( \pi_1(MU) \longrightarrow K_0(MU) \) is injective onto a direct summand, i.e., the cokernel of the map is a free \( \mathbb{Z} \)-module. It would be nice to know if the sequence (it isn't seen below)

\[ 0 \rightarrow \pi_1(MU) \rightarrow K_0(MU) \rightarrow K_0(MU) \otimes K_0(\mathbb{Z}) \rightarrow K_0(MU) \otimes K_0(\mathbb{Z}) \otimes K_0(\mathbb{Z}) \]

were also exact as this would yield Hatton's result, as well as the fact that it remains exact after tensoring with any ring \( R \) (any submodule of a flat \( \mathbb{Z} \)-module is flat).

In any case the sequence \( \text{(**)} \) is of the Amitsur form

\[ 0 \rightarrow A \rightarrow B \rightarrow A \otimes A \rightarrow B \otimes A \otimes A \rightarrow \cdots \\
A = \pi_1(MU) \\
B = K_0(MU)
\]

This may be seen by noting that \( A = \pi_1(MU) \) and \( B = K_0(MU) \).
is the same as giving power series $f_1(x), \ldots, f_r(x)$ with coefficients in $R$ such that

$$f_i(x) * (x + y - xy) = f_j(x) * (x + y - xy) \quad \forall i, j,$$

or equivalently series $f_1, u_2, \ldots, u_r$ (with $f_i = f_i u_2 \cdots u_r$) where $u_2$ stabilizes $x + y - xy$ and hence $u_2^r$ gives rise to a map $R \to R$, where $\text{Spec } R = \text{Aut } \hat{G}_m$. Thus

$$B \otimes_A \mathcal{O}B \cong B \otimes \mathcal{O} \mathcal{P} \cong \mathcal{O}_{P, n-1} \cong B \otimes K_0(\mathcal{B}) \otimes \cdots \otimes K_0(\mathcal{B})$$

CLAIM (**), not exact in degree 1.

To see this, simplify notation and write (**) as a complex

$$0 \to A \to Q^0 \to Q^1 \to Q^2 \to \cdots$$

Let $B^1 = \ker Q^1 \to Q^2$ so that we have exact sequences

$$0 \to A \to Q^0 \to B^1 \to 0 \quad \text{(by Stong - Hatcher)}$$

$$0 \to B^1 \to Q^1 \to B^2 \to 0 \quad \text{by hypothesis}$$

Tensoring with $\mathbb{Z}/p\mathbb{Z}$ this remains exact since $B^* \otimes Q^*$ is torsion free. Thus

$$0 \to A \otimes \mathbb{Z}/p\mathbb{Z} \to Q \otimes \mathbb{Z}/p\mathbb{Z} \to Q^0 \otimes \mathbb{Z}/p\mathbb{Z}$$

is exact. But in the case at hand, we know that

$$0 \to (\Omega(\text{pt}) \otimes \mathbb{Z}/p\mathbb{Z})[P_{-1}] \to K_0((\mu)_* \otimes \mathbb{Z}/p\mathbb{Z}) \to K_0((\mu)_* \otimes \mathbb{Z}/p\mathbb{Z}),$$

is exact. (This is because modulo $p^s$ we know that...
\[ \text{Claim: for each n that } \]
\[
0 \longrightarrow \Omega^1(MU) \otimes \mathbb{Z}/p^n \mathbb{Z} \longrightarrow K_1(MU) / p^n \longrightarrow K_1(MU) / p^n \otimes \mathbb{Z} \longrightarrow \cdots
\]
is exact. This is just faithfully flat descent for the morphism
\[ G_{\mathbb{Z}/p^n} \rightarrow G_{\mathbb{Z}/p^n} \otimes \mathbb{Z}/p^n \mathbb{Z} \]
which is a torsor for \((\mathbb{Z}/p^n)\).

**Proof of Steen-Hattori Theorem:** Let \( z \in K_1(MU) \) be a primitive element. Then by rational considerations there exists an integer \( n \) such that \( nz \) comes from \( \Omega^1(MU) \). On the other hand I know that \( z \in \lim_{\rightarrow n} \Omega^1(MU) [P_{-1}] \otimes \mathbb{Z}/p^n \mathbb{Z} \) which means
\[
z = \sum_{g \in \mathbb{N}} (p_{-1})^{g} \omega_g
\]
where the \( \omega_g \) are polynomials in the other generators tending to zero in the \( p \)-topology. For such a thing to be of the form \( \omega / n \) with \( \omega \in \Omega^1(MU) \) it must be that \( \omega_g = 0 \) for \( g < 0 \). Thus \( z \in \Omega^1(MU) \otimes \mathbb{Z}/p^n \) for all \( p \) so \( z \in \Omega^1(MU) \).

Hattori's result that \( \Omega^1(MU) \) is a direct summand of \( K_1(MU) \).
follows easily. In effect, as $K(MU) = \mathbb{Z}[b_0, b_1, \ldots] \otimes \mathbb{Z}[b_1, \ldots]$ is free in each dimension, it suffices to show that if $nz$ is in $\Omega^*(pt)$ so is $Z$. But if $nz$ is primitive so is $Z$ as $K(MU) \otimes \mathbb{F}$ is torsion-free. (This last observation is what Adams must have meant by Hattori's proof showing that $\Omega^*(pt) = \mathbb{F} K(MU)$.)

Remark 3: From the fact that (1) on page 16 is exact, one deduces that it is exact for any torsion-free complex. Indeed, after suspensions to kill $\pi$, one can suppose that $X$ is minimal and use skeleton induction. Thus, for a torsion-free complex $X$, $\Omega^*(X)$ can be calculated algebraically from $K(X)$ with its Adams operations: $\Omega^*(X)$ is the invariant elements of $K(MU) \otimes K(X)$.
The general picture about characteristic numbers and Wu relations:

Given a Chern theory $Q$ with formal group law $F^Q$, let $\Gamma$ be the coordinate ring of $Q$ over $F^Q$. Then we have maps (here $Q(MU), \Gamma$ have the universal property of representing skew ring homomorphisms from $Q$ to $\Gamma$).

\[ (**) \quad \Omega(X) \xrightarrow{\gamma} Q(MU) \otimes_{Q(pt)} Q(X) \xrightarrow{\mu \otimes \text{id}} Q(MU) \otimes_{\Gamma} Q(X) \]

where $\mu$ is the unique ring homomorphism such that these two composites are equal. Identifying $Q(MU) \cong Q(pt)[t, \ldots]$ we have that

\[ \gamma = \left( \sum b_n x^{n+1} \right) \]

\[ \sum (\mu b_n) x^{n+1} = \left( \sum b_n x^{n+1} \right) \circ \left( \sum t_n x^{n+1} \right) \]

where $\sum t_n x^{n+1}$, $t_n \in \Gamma$ is the generic auto of $F^Q$.

\[ \gamma \quad \text{is essentially} \quad \text{with values in } \Gamma \]

\[ Q(MU) \cong \text{Hom}^* \left( Q(pt), Q(MU) \right), Q(pt) \cong \text{Hom}^* \left( Q(pt), Q(pt) \right) \cong \text{Hom}^* \left( K, \Gamma \right). \]

The condition

\[ (\mu \otimes \text{id})(z) = (\text{id} \otimes \gamma)(z) \]

on an element $z \in Q(MU) \otimes_{Q(pt)} Q(X)$ is the image of $\gamma$ in the image of $\Gamma$ is called the Wu relations. To say that the Wu relations are complete means that $(**)$ is exact in the middle.
Example: Take $\sigma = H_0^r(X, Z_p)$ denoted $H_0^r(X)$ in the following. Then for $X = \text{pt}$ the maps of $(*)$ correspond to maps of schemes

$$G_{1/p} \times \text{Aut}_{\text{Gr}_a} \xrightarrow{\sigma} G_{1/p} \xrightarrow{\eta} L_{1/p}$$

One knows that the image of $\eta$ is $L_{1/p, \infty} = \text{laws of height } \infty \text{ mod } p$.

Thus

$$\Omega(\text{pt})_{1/p} \xrightarrow{\eta} H_0^r(MU)$$

kills the coefficients of $[p]_{F2}(X)$. Now we know that

$$G_{1/p} / \text{Aut}_{\text{Gr}_a} \cong L_{1/p, \infty}$$

and in fact that there is a section due to Cartier which gives

$$G_{1/p} \cong L_{1/p, \infty} \times \text{Aut}_{\text{Gr}_a} \frac{1}{1/p}$$

From this one sees that

$$\Omega(\text{pt})_{1/p} \xrightarrow{\eta} H_0^r(MU) \xrightarrow{\text{cartier}} H_0^r(MU) \otimes (\alpha/\beta)$$

is exact in the middle, hence the Wu relations are complete.

Strong: Hattori says Wu relations complete for $K$.
June 11, 1969

The cobordism class of a blowup

Let

\[ \tilde{Y} \xrightarrow{f} \tilde{X} \]
\[ Y \xrightarrow{i} X \]

\[ \tilde{Y} = \mathbb{P}E, \quad E = \nu_i \]
\[ \dim E = \nu \]

be a standard blowup diagram. The problem is to calculate \( f_*1 \) in a Chern theory with group law \( F \).

**Theorem:**

\[ f_*1 = 1 + i \left[ \text{res} \frac{\omega (Z)}{I (Z) \prod_{\nu=1}^{\nu} F (Z, \nu)} \right] \]

where \( e_i (E) = \prod_{\nu=1}^{\nu} (1 + t \nu) \).

**Proof:** May assume the Chern theory involved is \( \omega \).

Let \( k : U \rightarrow X \) be the complement of \( U \). As \( f \) is an isomorphism off \( U \), \( f_*1 - 1 \) has a canonical trivialization over \( U \) and so defines an element of \( \Omega Y (X) \). As

\[ l_* : \Omega Y (U) \xrightarrow{\sim} \Omega Y (X) \]

there is a unique \( \xi \) with \( l_* \xi U = f_*1 - 1 \) as elements of \( \Omega Y (X) \).

By excision we may assume that \( X \) is a blowup of \( E \) with \( i \) the zero section, and hence that \( Y \) is the zero section of the line bundle \( \mathcal{O}(1) \) on \( \mathbb{P}E \).
The situation is induced by a map \( \varphi: X \to MU(n) \) transversal to \( BU(n) \) with \( Y \) the inverse image of \( BU(n) \) under \( \varphi \). We can therefore suppose that \( Y = BU(n) \) and \( X \) is the canonical bundle of \( BU(n) \) with \( i \) the zero section. Moreover we can then identify \( j \) with the zero section of \( O(-1) \) over \( PE \). Finally by splitting principle, can assume \( Y = BU(n)^\infty \).

In this case \( c_{\nu}E \) is a non-zero divisor, \( \nu \) and so \( i^*: \Omega^j(X) \to \Omega^j(Y) \) is injective. It suffices therefore to prove that

\[
i^*( f^*1 - 1) = c_{\nu}(E) \text{ res } \frac{\omega(z)}{I(z)^{\nu}F(z, \nu)}
\]

Now to calculate the former use the diagram,

\[
\begin{array}{ccc}
PE & \xrightarrow{i} & O(-1) & \xrightarrow{k} & g^*E \\
\downarrow g & & \downarrow f & & \downarrow h \\
Y & \xrightarrow{i} & E & \xleftarrow{k} &
\end{array}
\]

where \( k \) is the natural inclusion in the canonical sequence

\[
0 \to O(-1) \xrightarrow{k} g^*E \to F \to 0
\]

over \( PE \). Then

\[
i^* f^*1 = i^* h^* k^* 1 = g^* f^* k^* k^* 1 =
\]

\[
g^* f^* c_{\nu-1}(\nu) = g^* c_{\nu-1}(f^* \nu)
\]

\[
g^* (c_{\nu-1}F)
\]
Theorem 2: If $F$ is the canonical quotient bundle on $P^1_E$, then

$$g_x(c_n, F) = 1 + c_n(E) \text{ and } \frac{\omega(Z)}{I(Z) \prod_{\nu=1}^n F(Z, \lambda_{\nu})}$$

Proof: We have with $\lambda = c_1(O(1))$, $\eta = c_1(O(-1)) = I(?)$,

$$c_t(F) = \frac{c_t(E)}{c_t(O(1))} = \frac{c_t(E)}{1+t\eta} = c_t(E)(1-t\eta + t^2\eta^2 - \ldots)$$

so

$$c_{n-1}(F) = c_{n-1}(E) + (-\eta)c_{n-2}(E) + \ldots + (-\eta)^{n-1}$$

Let

$$\alpha(Z) = c_{n-1}(E) + (-I(Z))c_{n-2}(E) + \ldots + (-I(Z))^{n-1}$$

so that

$$\alpha(?) = c_{n-1}(F)$$. 

Then

$$c_n(E) - I(Z)\alpha(Z) = c_n(E) + (-I(Z))c_{n-1}(E) + \ldots + (-I(Z))^n$$

$$= \prod_{\nu=1}^n (\gamma_{\nu} - I(Z))^{\alpha}$$

$$= \prod_{\nu=1}^n \left[F(\gamma_{\nu}, Z) \left\{ 1 + I(Z)G(I(Z), F(\gamma_{\nu}, Z)) \right\} \right]$$

so

$$\frac{c_n(E) - I(Z)\alpha(Z)}{\prod_{\nu=1}^n F(\gamma_{\nu}, Z)} = \prod_{\nu=1}^n \left[ 1 + I(Z)G(I(Z), F(\gamma_{\nu}, Z)) \right] \equiv 1 \mod (Z).$$

Therefore
\[ g_*(c_{n-1}(F)) = g_*(\alpha(\chi)) = \res \frac{\alpha(z) \omega(z)}{\prod_{i \leq n} F(z, x_i)} \]

\[ = \res \frac{I(z) \alpha(z) \omega(z)}{I(z) \prod_{i \leq n} F(z, x_i)} \]

\[ = c_n(E) \res \left( \frac{\omega(z)}{I(z) \prod_{i \leq n} F(z, x_i)} \right) + \res \left[ \frac{c_n(E) - I(z) \alpha(z)}{\prod_{i \leq n} F(x_i, z)} \right] \frac{dz}{I(z)} \]

Since \( I(z) \) has a simple zero at \( z=0 \) and \( I'(0) = -1 \)
the last term using \((\ast)\) is seen to be 1, proving theorem 2.

I now wish to classify those group laws \( F \) for which the\[ \omega/F \] theory, having the\[ f_+ 1 = 1 \] for any blowup. By universal considerations this means that the residue term in theorem 1 is always zero.

**Theorem 3:** Let \( F \) be a formal group law over a ring \( R \) such that

\[ (\ast) \quad \res \frac{\omega(z)}{I(z) \prod_{i \leq n} F(z, x_i)} = 0 \quad \text{for all } n \geq 1 \]

for nilpotent elements \( x_1, \ldots, x_n \) in any \( R \)-algebra. Then

\[ F(x, y) = x + y + \beta x y \]

for some \( \beta \in R \). Conversely any such \( F \) satisfies \((\ast)\).
Proof: We consider the converse statement first. If
\[ F(x, y) = x + y + x y \]
then
\[ \omega(x) = \frac{d x}{1 + bx} \]
and
\[ \omega(x) = \frac{d x}{F_2(x, 0)} = \frac{d x}{1 + bx} \]
so
\[ \text{res} \frac{\omega(z)}{I(2) \prod_{\nu=1}^{n} F(z, w_\nu)} = \text{res} \frac{d z}{-z \left( \prod_{\nu=1}^{n} (z + w_\nu) \right)} \]
which is zero for \( n \geq 1 \) in virtue of the following lemma.

Lemma: Let \( f(z), g(z) \) be polynomials over \( R \) with \( g \)
monic and \( \deg f < \deg g = n \). If \( a \) is the \( (n-1) \)th
coefficient of \( f \), then
\[ \text{res} \frac{f(z)dz}{g(z)} = a \]

Proof: The easiest way to see this is to use that the
sum of the residues is zero and that the residue at \( \infty \) is \(-a\).
But to be precise we first replace \( f(z) \) by \( g(z) \).

\[ \text{res} \frac{g(z)}{g(z)} = a \]
To be precise use induction on the degree of $g$ and check the case $n=1$ from the definition of residue. Next one can easily reduce to the case where $R = k[x_1, \ldots, x_n]$ is a polynomial ring and

$$g(z) = \prod_{\kappa=1}^{n} (z-x_{\kappa}),$$

as $x_i-x_j$ is a non-zero-divisor in $R$ it follows that we can embed $R$ in the ring $R[(x_i-x_j)]_{i\neq j}$ and so suppose that the $x_i-x_j$ are invertible. Now given $f$ we have

$$f(z) = f_1(z)(z-x_n) + f(x_n)$$

so by induction are reduced to showing this to the case where $f=1$. But we have by the division algorithm

$$\prod_{\kappa=1}^{n} (z-x_{\kappa}) = \prod_{\kappa=1}^{n} (z-x_{\kappa}) + g(z)(z-x_n)$$

degree $g \leq n-2$

where the first is a unit and so are done by induction. This proves the lemma and the converse part of thm 3. Now suppose $F$ is a law satisfying (**) . Taking all the $x_i=0$ we see that

$$\text{res} \left( \frac{\omega(z)}{I(z)z^r} \right) = 0 \quad \text{all } r \geq 1.$$  

Now

$$\frac{\omega(z)}{I(z)} = (-1 + q_1 z + q_2 z^2 + \cdots) \frac{dz}{z}$$

as $\omega(z)$ is regular and $I(z) = z (-1 + \text{higher terms})$. Thus

$$\frac{\omega(z)}{I(z)} = - \frac{dz}{z}$$
and we are given that
\[ \text{res } \frac{dz}{Z \prod_{n=1}^{n} F(z, x_n)} = 0 \quad \text{for } n \geq 1. \]

Let \( x \) be a nilpotent element in an \( R \)-algebra. Then
\[ (Z - x) = F(Z, \lambda x) (1 + x G(x, F(\lambda x))) \]
so taking \( x_1 = I(x) \) and all other \( x_n = 0 \) we find
\[ \text{res } \frac{1 + x G(x, F(\lambda x))}{Z - x} \frac{dz}{z^n} = 0 \quad \text{for } n \geq 0 \]

But we can write by the division algorithm
\[ 1 + x G(x, F(\lambda x)) = q(z)(Z - x) + (1 + x G(x, F(\lambda x))) \]
and this residue is
\[ \text{res } \frac{[1 + x G(x, 0)] dz}{(Z - x) Z^n} + \text{res } \left[ \frac{q(z) dz}{z^n} \right] = 0 \quad \text{for } n \geq 1 \]
by lemma

As \( q(z) \) is regular we must have \( q(z) = 0 \) so
\[ (Z - x) = F(Z, \lambda x) (1 + x G(x, 0)) \]
Thus as \( x \) was an arbitrary nilpotent element of any \( R \)-algebra
\[ F(Z, \lambda x) = \frac{Z - x}{1 + x G(x, 0)}. \]
Therefore
\[ F(X, Y) = X + Y + XY G(X, Y) \]

is linear as a function of \( X \), so \( G(X, Y) = G(0, Y) \); by symmetry, \( G(X, Y) = G(X, 0) \), so \( G \) is a constant \( \beta \) and Theorem 3 is proved.

(Remark: See earlier paper for indication that
\[ \text{Res} \left( \frac{\omega(z)}{I(2) T T F(2, x_i)} \right) = \left( \text{Res} \left( \frac{\omega(z)}{I(2) Z^n} \right) + \text{terms of degree } \geq n \text{ in } x_1, \ldots, x_n \right) \]

This appears below.

We observed before that in the stable range \( \iota: Y \to X \),
\[ \text{codim } Y > \dim Y \]
then the cobordism class \( f_* 1 \), where \( f: \tilde{X} \to X \)
depends only on \( i_* 1 \). In fact it seems that

\[ f_* 1 - 1 = a \cdot i_* 1 \]

where \( a \in \Omega^0(pt) \) is the class where \( i \) is \( * \to pt \to e^h \), i.e.
\[ a = \text{Res} \left( \frac{\omega(z)}{I(2) Z^n} \right) \]

* This is because by basic naturality the element \( f_* 1 \) depends only on the map \( X \to MU(h) \) represented by the embedding \( Y \subset X \).
Proof: The element \( f_x^{-1} \) of \( \mathcal{M}_y(X) \) is induced by a map \( X \to MU(n) \) transversal to \( BU(n) \). As \( \dim X \leq 4r-1 \), \( X \) may be deformed into the \( 4r-1 \) skeleton of \( MU(n) \), hence \( f^*(c^\alpha \cdot \text{Thom class}) = 0 \) if \( |x| \geq r \). Now

\[ f_x^{-1} = \left( \prod_{\nu=1}^n \frac{\omega(Z)}{Z_I^{\nu} F(x_{\nu}, I^2)} \right) \]

where \( P = \sum c^\alpha \) is a power series. In the stable range we know that the answer depends only on \( c^1 \). Moreover by the above \( c^\alpha = 0 \) if \( |x| \geq r \).

Thus one sees that \( f_x = 0 \) for \( 0 < |x| < r \), so

\[ f_x^{-1} = f_x(v_0) = v_0 \cdot c^1. \quad (c^1 I^2 = 0). \]

To get \( v_0 \), it suffices to take the case where \( Y \to X \) is \( pt \to C^1 \).

Corollary of the above proof seems to be

\[ \text{Res} \frac{\omega(Z)}{Z_I^{\nu} F(x_{\nu}, I^2)} = \text{Res} \frac{\omega(Z)}{Z_I^{n}} + \text{monomials of degree } \geq n \text{ in the } x_i. \]

since \( l_* : \Omega(BU) \to \Omega(MU) \).