(finished May 3, 1967)

§2. Formal categories over a field

One knows that using pseudocompact modules over a field it is possible to develop a theory of formal groups parallel to the theory of affine group and moreover related to the latter by adjoint functors called "formalization" + "algebraization". In this section I outline the generalization ~~of these results~~ to formal categories. It is likely that one can further generalize to the case where the base is a pseudo-compact ring.

2.1. Let A be a sfield and let Modc(A) be the category of pseudocompact A-modules. Then we have an equivalence

$$\mathrm{Modc}(A)^{\circ} \sim \mathrm{Mod}\, A^{\circ}$$

(2.1.1)
$$\mathrm{Hom}_{A^{\circ}}(M, A) \longleftarrow M$$

suppose given an additive functor $h: \mathrm{Modc}(A) \longrightarrow \mathrm{Modc}(A)$ which is compatible with inverse limits. By duality $h^{\circ}: \mathrm{Mod}\, A^{\circ} \longrightarrow \mathrm{Mod}\, A^{\circ}$ is compatible with $\varinjlim$'s so

(2.1.2)
$$h^{\circ}(M) \leftarrow h^{\circ}(A^{\circ}) \otimes_{A^{\circ}} M$$

where $h^{\circ}(A^{\circ}) = Q$ is an A°, A° ~~bimodules~~. ~~Not liking these~~ °'s,

~~Let A be regarded as a A°-module (that is, a right A-module) in the obvious way and let $Q = h^{\circ}(A)$ be regarded as an A, A module with right structure coming from the fact that $Q \in \mathrm{Ob}\, \mathrm{Mod}\, A^{\circ}$ and with left structure coming from the endos.~~

Not liking all these $^{\circ}$'s, I think of ~~A~~ A as an A°-module in the obvious way and $Q = h^{\circ}(A)$ as an A, A module with right structure coming from the fact that Q is an object of $\mathrm{Mod}\, A^{\circ}$ and with left structure coming from the endos. of A as ~~a~~ a right A-module produced by left multiplication. Thus 2.1.2 may be rewritten

$$M \otimes_A Q \simeq h^{\circ}(M) \tag{2.1.3}$$

so by duality

$$\begin{aligned} h(M') = h^{\circ}(M)' &= \mathrm{Hom}_{A^{\circ}}(M \otimes_A Q; A) \\ &= \mathrm{Hom}_{A^{\circ}}(M, \mathrm{Hom}_{A^{\circ}}(Q, A)) \end{aligned} \tag{2.1.4}$$

Let $P = \mathrm{Hom}_{A^{\circ}}(Q, A)$. Then P is a left pseudo-compact A-module with ~~[illegible]~~ $(a.f)(q) = af(q)$. Moreover P is a right A-module with $(f.a)(q) = f(qa)$. The right ~~[illegible]~~ multiplication by each element of A is continuous but P is not necessarily pseudocompact as a right A-module. P is therefore a left pseudo compact A, A-module in the following sense.

<u>Definition 2.1.5</u>: A <u>left-pseudo-compact A, A-module</u> is a ~~[illegible]~~ pseudo-compact A-module F endowed with a homomorphism

$$A^{\circ} \longrightarrow \mathrm{End}_{\mathrm{Modc}(A)}(F).$$

If F is a left pseudocompact A,A-module and if N is a pseudocompact A~~A~~-module, set

$$(2.1.6) \qquad F\hat{\otimes} N = \varprojlim_i F\otimes N_i$$

where $N = \varprojlim N_i$ and the $N_i \in \mathrm{Ob}\ \mathrm{Modf}(A)$. Then

$$N \longmapsto F\hat{\otimes} N$$

is a endo-functor of $\mathrm{Modc}(A)$. Moreover by 2.1.4

$$\begin{aligned} h(M') &= \mathrm{Hom}_{A^\circ}(M, \mathrm{Hom}_{A^\circ}(Q,A)) \\ &= \varprojlim_i \mathrm{Hom}_{A^\circ}(Q,A)\otimes_A \mathrm{Hom}_{A^\circ}(M_i, A) \\ &= P\hat{\otimes} M', \end{aligned}$$

so we find

<u>Proposition 2.1.7</u>: Any endo h of $\mathrm{Modc}(\mathbf{A})$ compatible with $\varprojlim$'s is of the form $h(N) = P\hat{\otimes} N$ where P is the left pseudocompact A,A-module $h(A)$.

<u>Remark</u>: Obvious generalization $\mathrm{Modc}(A) \longrightarrow \mathrm{Modc}(B)$. In fact you should probably prove the proposition directly without passing to duals.

2.2. Let $h : \mathcal{a} \longrightarrow \mathrm{Modc}(A)$ be a functor and consider the functor $F \longmapsto \mathrm{Hom}(h, h_F)$ where

$h_F(X) = F \hat{\otimes} h(X)$. If represented by P, then P is a left pseudocompact A,A-cogebra, i.e. a left p.c. A,A-module with

$$P \xrightarrow{\varepsilon} A$$
$$P \xrightarrow{\Delta} P \hat{\otimes} P.$$

Moreover the canonical map

$$hX \longrightarrow P \hat{\otimes} hX$$

gives a functor

$$\tilde{h} : \mathcal{A} \longrightarrow \mathrm{Comc}(P) \quad \text{(pseudocompact } P\text{-comodules)}$$

Again the basic descent argument gives the following result.

<u>Theorem 2.21</u>: Let A be a field, let $\mathcal{A}$ be an abelian category and let

$$\mathcal{A} \underset{g}{\overset{h}{\rightleftarrows}} \mathrm{Modc}(A) \quad = \text{pseudocompact } A\text{-modules}$$

be adjoint functors such that h commutes with $\varprojlim$'s. Then $\mathcal{A}, h$ is equivalent to the category of pseudocompact P-modules ~~when~~ and forgetful functor, where P is the left-pseudocompact
(2.15) A,A-cogebra given by

(2.22) $$\mathrm{Hom}(h, h_F) = \mathrm{Hom}_{\text{left p.c. } A,A \text{ mods}}(P, F).$$

Conversely given such a P, P can be recovered from $\mathrm{Comc}(P)$ via (2.2.2).

Remark 2.2.3: To see that 2.2.1 holds we can use duality (although the argument of §1 is probably cleaner). Thus suppose given adjoint functors (I drop off the °'s.)

$$\mathcal{A} \underset{h}{\overset{g}{\rightleftarrows}} Mod(A)$$

where $\mathcal{A}$ abelian, h faithful and $\varinjlim$ compatible. Then

$$hgM = (hgA)\otimes M$$

where $Q = hgA$ is an A,A module endowed with maps

$$\eta: A \longrightarrow Q$$

$$\psi: Q\otimes_A Q \longrightarrow Q$$

of A,A-modules satisfying usual identities. I checked that this means simply that Q is a ring and $\eta: A \longrightarrow Q$ is a ring homomorphism (but Q is not nec. an A-alg.). Then as h is faithful, exact and $\mathcal{A}$ is abelian we have by descent argument that

$$\mathcal{A} \sim Mod(Q)$$

(2.2.4) $$gM = Q\otimes_A M \quad , \quad hX = X_{[\eta]}$$

Finally one can show that for any A-module

(2.2.5) $$Hom(F\otimes h, h) = Hom_{A,A\text{-mod}}(F, Q).$$

Note that everything above holds for A any ring not necessarily commutative.

<u>Remark 2.2.6:</u> It's interesting to combine the above remark with §1. Thus suppose

$$\begin{tikzcd} \mathcal{A} \arrow[r, "h"] & \mathrm{Mod}\, A \arrow[l, bend right, "k"'] \arrow[l, bend left, "g"] \end{tikzcd}$$

~~to~~ with $\mathcal{A}$ abelian, h exact, g compatible with $\varinjlim$'s. Without making the last assumption we know that

$$\mathcal{A} = \mathrm{Mod}\, Q$$

$$kM = Q \otimes_A M$$

hence $$gM = \mathrm{Hom}_A(Q, M)$$

The last assumption implies that

$$\mathrm{Hom}_A(Q, M) = P \otimes_A M$$

and hence that ~~[illegible]~~ Q is necessarily projective of finite type as a left A-module; ~~[illegible]~~ moreover

$$P = \mathrm{Hom}_A(Q, A) \qquad \cancel{\mathrm{Hom}_A(P, A)}$$

is projective of finite type as a right R-module and ~~[illegible]~~ we have

$$Q = \mathrm{Hom}_{A^\circ}(P, A).$$

This Q is not to be confused with the Q of 2.1.3 which is $\mathrm{Hom}_A(P, A)$ see page 2.12

Denote by $\langle q, p \rangle$ the basic pairing of ~~[illegible]~~ Q and P so that

$$(2.2.7) \quad \begin{cases} \langle aq, p \rangle = a \langle q, p \rangle \\ \langle qa, p \rangle = \langle q, ap \rangle \\ \langle q, pa \rangle = \langle q, p \rangle a \end{cases}$$

Then the product of Q and diagonal of P are related by

$$\langle q_1 \otimes q_2, \Delta p \rangle = \langle q_1 q_2, p \rangle$$

where the LHS is defined to be

(2.2.8) $$\Sigma < q_1, < q_2, p_{(1)} > p_{(2)} >$$

where I use Sweedler's notation $\Delta p = \Sigma\, p_{(1)} \otimes p_{(2)}$.

2.3. Tensor product.

Suppose A commutative and that

$$\begin{tikzcd} \mathcal{A} \arrow[dr, "h"] & \\ & \mathrm{Modc}(A) \\ \mathcal{A}' \arrow[ur, "h'"'] & \end{tikzcd}$$

are two functors and let

(2.3.1) $$\begin{aligned} h \otimes h' : \mathcal{A} \times \mathcal{A}' &\longrightarrow \mathrm{Modc}(A) \\ X, Y &\longmapsto hX \otimes h'Y \end{aligned}$$

where the tensor product is the completed product of pseudocompact A-modules. Suppose End h and End h' represented by P, P' in the sense of 2.2.2 (strengthened as below). Then (sauf erreur) End $h \otimes h'$ is represented by $P * P'$ defined to be the quotient of the completed tensor product $P \otimes P'$ of left p.c. A-modules by the relations

(2.3.2) $$pa \otimes p' = p \otimes p'a \qquad \text{for all } a \in A.$$

In effect given $\theta : h \otimes h' \longrightarrow F \hat{\otimes} (h \otimes h')$ one can (when at least one has the adjoint functors g, g') apply it to $gA, g'A$ getting

$$P \otimes P' \longrightarrow F \hat{\otimes} (P \otimes P') \longrightarrow F \hat{\otimes} (A \otimes A) \cong F,$$

and the two right A module structure merge on F. I didn't check everything, but everything (seems to) works smoothly provided we assume the following stronger version of (2.2.2)

$$(2.3.4) \qquad \boxed{\mathrm{Hom}(h_G, h_F) = \mathrm{Hom}(G \hat{\otimes} P, F)}.$$

So now if $\mathcal{a}$ is a category with a unitary associative commutative tensor operation, if h is provided with a compatibility with respect to this structure, and if P is a left p.c. A,A cogebra representing End h, then we obtain maps

$$P * P \longrightarrow P$$

$$A \longrightarrow P$$

of p.c. A,A cogebras which make P into a left pseudocompact A,A-algebra. Thus we abut on the definition of a formal category over A namely a left pseudocompact A,A-algebra.

One can describe the situation dually: Instead of P we consider $Q = \mathrm{Hom}_A(P, A)$ which is a ring under A and provided with

$$Q \xrightarrow{\varepsilon} A$$

$$Q \xrightarrow{\Delta} Q. \otimes Q.$$

where Δ is subject to (among others) the relation

$$(2.3.5) \qquad \Delta L_a = L_a \otimes id = id \otimes L_a \qquad L_a = \text{left mult of } a \text{ on } Q$$

implying that $\mathrm{Im}\,\Delta \subset \{ z \in Q.\otimes Q. \mid (a \otimes 1) z = (1 \otimes a) z \}$.

The category $\mathcal{A}$ is thus dual to the category of right Q modules, h° being the underlying A° module. The tensor product of Q° modules M, N being defined to be $M \otimes_A N$ with Q°-action given by

$$(m \otimes n)\, q = m q_{(1)} \otimes n q_{(2)} ,$$

the condition 2.3.5 assuring this is well-defined.

2.4. Some examples of formal categories.

2.4.1. (In this example A is not ~~a~~ field but a (topologically) pseudocompact semi-simple ring k^S where S is a set (and k is a field)). Let C be a (small) category with objects set C_{ob} and morphism set C_{fl}. Let

$$A = k^{C_{ob}}$$
$$P = k^{C_{fl}}$$

whence the composition map $C_{fl} \times_{C_{ob}} C_{fl} \longrightarrow C_{fl}$ $(\theta_1, \theta_2) \mapsto \theta_2 \circ \theta_1$ gives a map

$$P \longrightarrow P \hat{\otimes}_A P$$
$$\delta_f \longmapsto \sum_{f_1, f_2 \ni\ f_2 \circ f_1 = f} \delta_{f_1} \otimes \delta_{f_2}$$

$$\prod_{x \to y} k \longrightarrow \prod_{x \to y} k \otimes \prod \ \left(\prod_y k\right)_{y \to z}$$

A pseudocompact A module is a product $\prod_{x \in C_{ob}} M_x$ of pseudo compact k-module. Then

$$P \hat{\otimes}_A M \cong \prod_{x \to y} M_y$$

and to give a comodule structure on M is to give

$$\prod_x M_x \rightrightarrows \prod_{x \to y} M_y$$

that is, to make $x \mapsto M_x$ a <u>covariant</u> functor from C to $\mathrm{Mod}(k)$.

~~3.4.2. Let $k \to K$ be a finite field extension, let $A = K$, $P = K \otimes_k K$. Then~~

$$\cancel{Q = \mathrm{Hom}_{K^\circ}(K \otimes_k K, K) = \mathrm{Hom}_k(K,K)}$$

~~the pairing~~

$$\cancel{\langle \theta, x \otimes y \rangle}$$

2.4.2. Let $k \to A$ be ~~faithfully flat~~ a morphism of rings ~~which is finite locally free, and~~ such that A is a faithful projective k module of finite type. Apply the considerations of 2.2.6 to the situation

$$\begin{tikzcd} \mathrm{Mod}\, k = \mathcal{A} \arrow[r, "h"] & \mathrm{Mod}\, A \arrow[l, bend right, "k"'] \arrow[l, bend left, "g"] \end{tikzcd}$$

where
$$h = A \otimes_k (?)$$
$$g = \text{underlying } k \text{ module}$$
$$kM = A' \otimes_A M \qquad\qquad A' = \mathrm{Hom}_{k^\circ}(A, k).$$

Thus we can think of $\mathcal{A}$ as either the category of P-comodules where

$$P = A \otimes_k A$$

(that is, A-modules with descent data) or as the category of Q modules where

$$Q = A \otimes_k A'$$

and the algebra structure is given by

$$(a \otimes \lambda)\cdot(b \otimes \mu) = a \otimes \langle \lambda, b\rangle \mu .$$

Therefore $Q = \mathrm{Hom}_{k^\circ}(A,A)$ with product $\theta_1 \cdot \theta_2 = \theta_1 \circ \theta_2$. When A is commutative, then the diagonal of Q is the ugly map

$$\mathrm{Hom}_k(A,A) \xrightarrow{\mu_A^*} \mathrm{Hom}_k(A\otimes_k A, A) \cong \underbrace{\mathrm{Hom}_k(A,A) \otimes \mathrm{Hom}_k(A,A)}_{A}$$

$$\Delta\theta = \sum_i \theta_i' \otimes \theta_i'' \iff \theta(ab) = \sum_i \theta_i'(a)\,\theta_i''(b) .$$

Suppose A hence k is a field. If L is an intermediate field, then we have a quotient A,A cogebra

$$A \otimes_k A \twoheadrightarrow A \otimes_L A$$

or the subring $\mathrm{End}_{L^\circ}(A)$ of $\mathrm{End}_{k^\circ}(A)$ (containing A) determined by the map

$$\begin{tikzcd}
\mathrm{Mod}\, k \arrow[rd] \arrow[d] & \\
\mathrm{Mod}\, L \arrow[r] & \mathrm{Mod}\, A
\end{tikzcd}$$

The Jacobson–Bourbaki theorem asserts that one thus obtains a 1-1 correspondence between intermediate L and subrings of $\mathrm{End}_k A$ containing A.

Interesting further cases are

(a) A/k Galois extension of fields $\Rightarrow$ $A \otimes_k A \cong A \otimes_k k^G$.
and ~~[illegible]~~ $\mathrm{End}_k(A) = A \otimes_k k[G]$ ~~[illegible]~~ with twisted multiplication.

(b) A/k purely inseparable of height 1 $\Longrightarrow$ $\mathrm{End}_k(A)$ = mixed universal envelopping algebra of $\mathrm{Der}_k(A)$ as a Lie ring + A-module + actions on A.

2.4.3. Let G be a monoid ~~of~~ acting as endomorphisms of a field A. Let $Q = A[G]$ with twisted product and let Δ be given by

$$\begin{array}{ccc} \Delta: Q & \longrightarrow & \underbrace{Q \otimes Q}_{A} \\ \| & & \wr\| \\ A[G] & \longrightarrow & A[G \times G] \\ g & \longmapsto & g \otimes g \end{array}$$

Then we obtain a formal category.

At this point we've got the notation terribly screwed up, the problem being that the Q of 2.2.6 is ~~[illegible]~~ the same as the ~~[illegible]~~ transposed ring of the Q used in 2.1.3 to describe the dual of $\mathcal{A}$. The way to resolve things is to forget Q and work directly with the left-p.c. A,A-cogebra ~~[illegible]~~ P.

Let $P = A^G$ with
$t: A \longrightarrow A^G$ given by $t(a)(g) = a^g$

Using the notation ~~[scribbled out]~~ $\sum f(g)\,\delta_g$ for a function $f: G \to A$ we have that

$$P \hat{\otimes}_A P$$

has the basis $\delta_x \otimes \delta_y \quad x, y \in G$ as a left A-module.

Define

$$\Delta : P \longrightarrow P \hat{\otimes} P \qquad\qquad \varepsilon : P \longrightarrow A$$

by

$$\delta_g \longmapsto \sum_{xy=g} \delta_x \otimes \delta_y \qquad\qquad \delta_g \longmapsto 1$$

$$s : A \longrightarrow P$$

$$s(a) = \sum_g a\,\delta_g$$

$$t(a) = \sum_g a^g\,\delta_g$$

(Check

$$\Delta s(a) = s(a) \otimes 1$$

$$\Delta t(a) = \sum_{xy} \underbrace{a^{xy}\,\delta_x}_{(a^y)^x \delta_x = \delta_x a^y} \otimes \delta_y = \sum_{x,y} \delta_x \otimes \overset{a^y}{\delta_y} = 1 \otimes t(a).)$$

A pseudocompact P module M is given by a map

$$M \xrightarrow{\Delta} P \hat{\otimes} M$$

Dually $M' = \mathrm{Hom}_A(M, A)$ is a right P' module. Extrapolating the formulas 2.2.7 + 2.2.8 to the present situation (where $Q = \mathrm{Hom}_A(P, A)$) we find that P' is a free right A module with base $\hat{x} \quad x \in G$ given by

$$\langle f, \hat{x} \rangle = f(x)$$

with multiplication given by

$$\hat{x}\hat{y} = \widehat{yx}$$
$$a\hat{x} = \hat{x}a^x$$

In other words P' is the ~~transpose~~ opposite of the ring $\mathring{A}[G]$, so to given a right P'-module is the same as giving a left $\mathring{A}[G]$-module. <u>Conclusion</u>: The category of ~~free~~ pseudocompact P-modules is dual to the category of $\mathring{A}[G]$-modules.

2.5. Let P be a left pseudocompact A,A-bigebra. Then P is a pseudocompact A-algebra for the left A-structure and so is a product of local pseudocompact A-algebras

$$P \cong \prod_{\mathfrak{m} \in \operatorname{Max} P} P_{\mathfrak{m}}$$

Assume P split, i.e. the residue field of each $P_{\mathfrak{m}}$ is rational over A; also assume P reduced. Then

$$P = \prod_G A$$

where G is the set $\operatorname{Hom}_{A\text{-alg}}(P, A)$. ~~It is quickly~~ It seems clear that one is in the situation of the example 2.4.3. Therefore a reduced P over an algebraically closed field always comes from a monoid of endomorphisms of the field.

2.6. If P is local over a field of characteristic zero, then we can define $\Omega = I/I^2$ ~~[illegible]~~ and the de Rham complex of P. Ω is a pseudocompact A-module, ~~[illegible]~~ and it would seem true that the old arguments of yours could be carried through to this case to show that p.c. de Rham complexes over a field of char 0 are same as a local formal categories over that field. Of course one ~~uses~~ prefers to $L = \mathrm{Hom}_A(\Omega, A)$ and its mixed Lie algebra structure over A.

2.7. If P is local over a field A of char. p ~~and if [illegible]~~ ~~[illegible]~~ and if P is of height 1 ~~[illegible]~~ in the sense ~~of~~ that $x^p = 0$ for all $x \in I$, then one expects the Jacobson theory to generalize and so P should be equivalent to a mixed ~~[illegible]~~ restricted Lie algebra ~~[illegible]~~ over A.

I'm tired, May 5, 1969

A first attempt which failed about April 28, 1969

~~[illegible]~~ Formal category schemes

By an *adic ring* we shall mean an inverse system of rings P_n $n \in \mathbb{N}$ such that for each n there is a commutative ~~[illegible]~~ triangle

$$\begin{array}{ccc} P_n & \longrightarrow & P_{n-1} \\ & \searrow & \nearrow\!\!\wr \\ & P_n/I_n^{\,n} & \end{array}$$

where $I_n = \operatorname{Ker}\{P_n \to P_0\}$. ~~[illegible]~~ ~~[illegible]~~ Equivalently it may be defined as a ring P endowed with a filtration F_nP such that

$$F_0P = P, \qquad F_pP \cdot F_qP \subset F_{p+q}P$$

$$\operatorname{gr} P = \oplus\, F_pP/F_{p+1}P \quad \text{is generated by } \operatorname{gr}_0 P \text{ and } \operatorname{gr}_1 P$$

$$P \simeq \varprojlim P/F_pP$$

Given two adic rings P and Q one defines their tensor product as direct sum in the category and checks that

$$(P\otimes Q)_n \cong P_i \otimes P_j \Big/ ~~[illegible]~~ \left(\operatorname{Im}\{I_i\otimes P_j \oplus P_i \otimes I_j \to P_i\otimes P_j\}\right)^{n+1}$$

if $i,j \geq n$.

A *formal category* with object ring A is defined ~~[illegible]~~ to be an adic A,A algebra P together with maps

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A *formal category* with object ring A will be defined as an inverse system of A,A algebras P

(1) $$\cdots \longrightarrow P_1 \xrightarrow{\pi} P_0$$

which is an adic ring, and which is endowed with an isom.

(2) $$\varepsilon_0 : P_0 \xrightarrow{\cong} A$$

of A,A algebras and ~~maps~~ homomorphisms of A,A-algs.

(3) $$\Delta_{k,l} : P_{k+l} \longrightarrow (P_k)_t \otimes_{A\,s} (P_l) \qquad k,l \in \mathbb{N}$$

This data is subject to a number of obvious conditions, namely compatibility with the π maps (1), with the isom (2), and coassociativity.

Thus regarding A as an adic ring in the obvious way we have in the category of adic ring a co-category object

$$A \overset{s}{\underset{t}{\rightrightarrows}} P \xrightarrow{\varepsilon} A \qquad \Delta : P \longrightarrow P_t \otimes_A P .$$

with ε an isomorphism in degree 0.

Equivalent definition: Consider the ~~category~~ Th of thickenings of A, i.e. whose objects are surjective ring homomorphisms $R \to A$ with nilpotent kernel. Let Th_n be the full subcategory of object with $(\text{kernel})^{n+1} = 0$. Then a formal category with object ring A may be identified with a functor $C : Th \longrightarrow Cat$ ~~with~~ such that $Ob\, C$ is represented by A and such that $Fl\, C$ restricted to Th_n is representable for each n.

(I now wish to recover the functor

$$R \longmapsto \mathrm{Hom}(A,R)$$

$$R \longmapsto \varinjlim_n \mathrm{Hom}(P_n, R)$$

from rings (enough to do for $R \in$ ~~Q~~. Th ~~A~~) to categories as the endos. of a forgetful functor. The first thing one tries is P-stratified modules, i.e. A-modules M endowed with

$$\Delta_k : M \longrightarrow (P_k)_t \otimes M \qquad k \in \mathbb{N}$$

satisfying the counit + coassociativity identities. However this doesn't seem to work since P itself is not a P-stratified module, rather it is a kind of P-stratified inverse system. Hence the following considerations)

Suppose ~~[illegible]~~ P is an inverse system of A,A algebras endowed with (2) and (3). We do not suppose that ~~[illegible]~~ P is adic. Let ~~[illegible]~~ $\mathrm{Inv}(A)$ be the category of inverse systems of A-modules indexed by $\mathbb{N}$ and let ~~[illegible]~~ ~~[illegible]~~ $\mathrm{Stinv}(P)$ be the category of P-stratified inverse systems of A-modules, i.e. inverse systems $M = \{M_n\}$ endowed with

$$\Delta_{k,\ell} : M_{k+\ell} \longrightarrow (P_k)_t \otimes M_\ell \qquad k,\ell \in \mathbb{N}$$

such that

(i) $\Delta_{k,\ell}(a\,m) = (s(a) \otimes 1)\, \Delta_{k,\ell}(m)$

(ii) ~~[illegible]~~ $(\pi \otimes id)\, \Delta_{k,\ell} = \Delta_{k-1,\ell}\, \pi$

$$(id \otimes \pi)\, \Delta_{k,\ell} = \Delta_{k,\ell-1}\, \pi$$

where π refers to the structural maps $P_k \to P_{k-1}$ or $M_\ell \to M_{\ell-1}$.

(iii) $\Delta_{0,\ell} m = 1 \otimes m$, i.e.

$$M_\ell \xrightarrow{\Delta_{0,\ell}} P_0 \otimes M_\ell \xrightarrow{\varepsilon \otimes id} A \otimes M_\ell \simeq M_\ell$$

is the identity.

(iv)
$$\begin{tikzcd} M_{k+\ell+m} \arrow[r, "\Delta_{k+\ell,m}"] \arrow[d, "\Delta_{k,\ell+m}"] & P_{k+\ell} \otimes M_m \arrow[d, "\Delta_{k,\ell} \otimes id"] \\ P_k \otimes M_{\ell+m} \arrow[r, "id \otimes \Delta_{\ell,m}"] & P_k \otimes P_\ell \otimes M_m \end{tikzcd}$$
commutes.

Let

$$h: \mathcal{S}tinv(P) \longrightarrow \mathcal{I}nv(A)$$

be the forgetful functor.

If $M \in Ob\, \mathcal{I}nv(A)$ let

$$M(n)_k = M_{n+k}$$

and let $T^n : M(n) \longrightarrow M$ be the map in $\mathcal{I}nv(A)$ obtained from $\pi^n : M_{n+k} \to M_k$. Then the category of Artin-Rees pro-objects in $Mod(A)$ is obtained by inverting all the maps T^* for all M. Denote this category by $AR(A)$. Observe that there are also maps $T^n : M(n) \to M$ for M in $\mathcal{S}tinv(P)$ defined the same way and we can also form the associated Artin-Rees category $StAR(P)$. Then we have a commutative diagram of categories + functors

$$\begin{tikzcd}
\mathrm{Stinv}(P) \arrow[r, "h"] \arrow[d, "Q"] & \mathrm{Inv}(A) \arrow[d, "Q"] \\
\mathrm{StAR}(P) \arrow[r, "\bar{h}"] & AR(A)
\end{tikzcd}$$

where Q are the canonical functors and $\bar{h}$ is the functor induced by h.

<u>Proposition:</u> { Assume $P_n \to P_{n-1}$ surjective and P_n right A-flat $\forall n$ } If F is an A_-^A module, then

$$\mathrm{Hom}^{*}(\bar{h}, \bar{h}_F) \cong \mathrm{Hom}_{AR(A,A)}(P_{*}, F) \quad \left(= \varinjlim_n \mathrm{Hom}_{A,A}((P_n)_{*}, F)\right)$$

(FALSE)

<u>Proof.</u> Here $h_F(M) = F \otimes M$. Given $\theta : \bar{h} \to \bar{h}_F$ define (an Artin-Rees map) $\theta^\# : P_t \to F$ by

$$\bar{h}(P) \xrightarrow{\theta(P)} F \otimes \bar{h}(P)$$
$$\bar{h}(P) = P$$

<u>Proof</u>: Given $\theta : \bar{h} \to \bar{h}_F$ (an additive transformation not necessarily A-linear) one obtains an AR map

$$\theta(P) : \bar{h}(P) \longrightarrow \bar{h}_F(P)$$

which may be represented by a map of inverse systems

$$P_{m+\cdot} \longrightarrow F \otimes P_{\cdot} \qquad P(m) \longrightarrow F \otimes P$$

for some m. Composing with $\varepsilon_\cdot : P_\cdot \longrightarrow A$ one obtains a AR map

$$\theta^\# : P \longrightarrow F$$

May 6, 1969

Dear Serre,

If Bourbaki should happen to have an extra copy of his redaction on gèbres, I would very much appreciate him sending it to me. I have been working recently on a slight generalization of cogebras and bigebras that might perhaps make good exercises in case Bourbaki plans to write a [redacted] chapter on gebras.

Let A be a ring, unitary but not necessarily commutative. By an A,A-cogebra I shall mean an A-bimodule P (that is an $A\otimes_{\mathbb{Z}} A^{\circ}$-module) endowed with morphisms

$$P \xrightarrow{\varepsilon_P} A$$
$$P \xrightarrow{\Delta_P} P\otimes_A P$$

of A-bimodules satisfying the counit and coassociativity identities. If P is an A,A-cogebra, then by a P-comodule I mean an A-module M endowed with a morphism

$$M \xrightarrow{\Delta_M} P\otimes_A M$$

of A-modules which is compatible with ε_P and Δ_P in the sense that

$$(\varepsilon\otimes id)\Delta_M = 1\otimes m$$
$$(\Delta_P\otimes id)\Delta_M = (id\otimes\Delta_P)\Delta_M$$

The additive category of P-comodules will be denoted $Com(P)$; it is abelian if P is flat as a right A-module.

Example: Suppose that

$$\mathcal{A} \underset{g}{\overset{h}{\rightleftarrows}} \mathrm{Mod}\, A$$

are adjoint functors with h left adjoint to g. If the functor hg preserves inductive limits one knows that

$$hg(A) \otimes_A M \xrightarrow{\sim} hg(M)$$

where $P = hg(A)$ is an A-bimodule, the right structure coming from the endomorphisms of A produced by right multiplications. The canonical adjunction morphisms

$$hg \longrightarrow id \qquad gh \longrightarrow id$$

furnish us with morphisms

$$P = hg(A) \longrightarrow A$$

$$P \otimes_A P \cong hghg(A) \longrightarrow hg(A) = P.$$

It is easy to see that P is thereby an A,A-cogebra and that h induces a functor

$$h' : \mathcal{A} \longrightarrow \mathrm{Com}(P)$$

The faithfully flat descent argument of Grothendieck may be used to prove the following result.

<u>Theorem 1</u> <u>Let</u> $\mathcal{A}$ <u>be an abelian category</u> and let

$$h : \mathcal{A} \longrightarrow \mathrm{Mod}\, A$$

be a faithful exact functor which possesses a right adjoint preserving inductive limits. Then

$$h': \mathcal{A} \longrightarrow \mathrm{Com}(P)$$

is an equivalence of categories, where P is the A,A-cogebra hgA.

(For some purposes it is useful to have a definition of the A,A-cogebra P not using the adjoint g. For any A-bimodule F, let $h_F(X) = F \otimes_A hX$. Then one can prove

$$(*) \qquad \mathrm{Hom}(h, h_F) \cong \mathrm{Hom}_{A\text{-bimod}}(P, F)$$

where on the left one has morphisms of functors.)

Suppose now that A is a skew field and that

$$h: \mathcal{A} \longrightarrow \mathrm{Modf}(A)$$

is a faithful exact functor where the right side denotes the finite dimensional A-modules. Then every object of $\mathcal{A}$ is of finite length and $\mathcal{A}$ is noetherian. h extends to a functor

$$h_e: \mathrm{Ind}\,\mathcal{A} \longrightarrow \mathrm{Mod}\,A$$

where $\mathrm{Ind}\,\mathcal{A}$ is the locally noetherian category associated to $\mathcal{A}$ by Gabriel. One shows easily that h_e satisfies the hypotheses of Theorem 1 and concludes the following.

Theorem 2. Let

$$h: \mathcal{A} \longrightarrow \mathrm{Modf}(A)$$

be a faithful exact functor where A is a skew field. Then $\mathcal{A}$

is equivalent to the category of P-comodules which are finite dimensional A-modules, where P is the A,A-cogebra given by (A).

From now on all rings are commutative. By an A,A-bigebra I mean an A,A-cogebra P endowed with a ring structure such that the maps

$$A \underset{t}{\overset{s}{\rightrightarrows}} P \xrightarrow{\varepsilon_P} A \qquad\qquad P \xrightarrow{\Delta_P} P \otimes_A P$$

$$s(a) = a.1$$
$$t(a) = 1.a$$

are ring homomorphisms. Such an animal gives rise to a covariant functor from the category of rings to the category of (small) categories by associating to R the category $C(R)$ with

$$Ob\, C(R) = Hom_{rings}(A, R)$$
$$Fl\, C(R) = Hom_{rings}(P, R),$$

and with $s, t, \varepsilon_P, \Delta_P$ yielding the source, target, identity section, and composition ~~[illegible]~~ defining the category structure. For this reason Grothendieck suggests calling such a P an <u>affine category</u>, the bigebra of an affine group scheme over A being a special case.

The ring structure of P permits one to define a P-comodule structure on $M \otimes_A N$, if M, N are two P-comodules, and hence Com(P) is a tensor category. There is a generalization of the representation theorem of Grothendieck ~~and is~~ for tensor categories which reads as follows.

Theorem 3: Let $\mathcal{A}$ be an abelian category and let $h: \mathcal{A} \longrightarrow \mathrm{Modf}\, A$ be a faithful exact functor where A is a field. Suppose that $\mathcal{A}$ is endowed with a ~~bilinear functor $X, Y \mapsto X \otimes Y$ and that h is~~ coherently associative commutative and unitary tensor product operation $X, Y \longmapsto X \otimes Y$ and that h is compatible with this structure. (As h is faithful this means that there is given natural isomorphisms

(*) $$h(X \otimes Y) \simeq (hX) \otimes (hY) \qquad X, Y \in \mathrm{Ob}\, \mathcal{A}$$

and an object 1 of $\mathcal{A}$ with an isomorphism

(**) $$h1 \simeq A$$

such that the standard isomorphisms

$$[(hX) \otimes (hY)] \otimes hZ \simeq hX \otimes [hY \otimes hZ]$$

$$hX \otimes hY \simeq hY \otimes hX$$

$$A \otimes hX \simeq hX \simeq hX \otimes A$$

come via (*) and (**) from isomorphisms in $\mathcal{A}$.

$$[X \otimes Y] \otimes Z \simeq X \otimes [Y \otimes Z]$$

$$X \otimes Y \simeq Y \otimes X$$

$$1 \otimes X \simeq X \simeq X \otimes 1 .)$$

Then $\mathcal{A}$ is equivalent with all the structure to the category of P-comodules with its natural tensor product where P is the

A, A - bigebra representing the functor C from rings to categories given by

$$\mathrm{Ob}\, C(R) = \mathrm{Hom}_{\text{rings}}(A, R)$$

$$\mathrm{Hom}_{C(R)}(u, v) = \mathrm{Hom}^{\otimes}(h_u, h_v),$$

where ~~[illegible]~~ if $u: A \longrightarrow R$ is a morphism of rings then $h_u: \mathcal{A} \longrightarrow \mathrm{Mod}\, R$ is the functor $X \longmapsto R_{[u]} \otimes_A hX$, and where ~~[illegible]~~ $\mathrm{Hom}^{\otimes}$ denotes natural transformations compatible with tensor product.

The above is a small sample of statements about affine group schemes which can be generalized to affine categories or at least to affine groupoids. Roughly speaking about any general assertion about algebraic groups not involving commuting elements seems to have a generalization of some kind. (This doesn't make much theory, of course!) For example there is a replacement for the Lie algebra and a generalization of Cartier's theorem that in characteristic zero the Lie algebra determines the formal completion at the identity; ~~this theorem is presented~~ this theorem is presently being exposed by Illusie in the Berthelot seminar on crystalline cohomology.

Sincerely yours

Daniel Quillen

May 8, 1969:

Generalities on category objects:

Let $\mathcal{C}$ be a category, with fibre products for simplicity. By a category in $\mathcal{C}$ I mean a collection of maps

$$\begin{tikzcd} C_{ob} \arrow[r, bend left, "\varepsilon"] & C_{fl} \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \end{tikzcd} \qquad C_{fl} \times_{(s,t)} C_{fl} \xrightarrow{\Delta} C_{fl}$$

subject to the category axioms. Let $p: \mathcal{E} \longrightarrow \mathcal{C}$ be a fibrant functor. By a covariant functor on C with values in $\mathcal{E}$ I mean an object F of $\mathcal{E}$ over (wrt p) C_{ob} endowed with a map

$$\theta : s^*F \longrightarrow t^*F$$

subject to the functor axioms, i.e.

$$\begin{tikzcd} & F \arrow[dl, "\wr"'] \arrow[dr, "\sim"] & \\ \varepsilon^* s^* F \arrow[rr, "\varepsilon^*(\theta)"] & & \varepsilon^* t^* F \end{tikzcd}$$

commutes and

$$pr_2^*\theta \circ pr_1^*\theta \cong \Delta^*(\theta) .$$

A more intrinsic way of putting this is to think of C as being a functor $T \longmapsto C(T)$ from $\mathcal{C}$ to Cat and to think of F as associating to an object S of $\mathcal{E}$ a functor from $C(pS)$ to Sets. Indeed given $\xi : S \longrightarrow F$ in $\mathcal{E}$ write $\xi \in F(x)$ where $x = p(\xi) : pS \longrightarrow C_{ob}$. Then given $\xi \in F(x)$ and $u : x \to y$ in $C(pS)$ we have maps

$$S \xrightarrow{\ \xi\ } F$$

$$\begin{tikzcd} pS \arrow[r, "u"] \arrow[rr, bend left, "x"] & C_{f\ell} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & C_0 \end{tikzcd}$$

which furnish~~es~~ us with a map $S \longrightarrow s^*F$; ~~hence~~ ~~[illegible]~~ composing with $s^*F \longrightarrow t^*F$ we get a new map $\eta: S \longrightarrow F$ which we write ~~[illegible]~~

$$\eta = F(u)\,\xi \in F(y).$$

system?

Suppose now that $C.$ is a simplicial object in $\mathcal{C}$. By a cosimplicial ~~object~~ system $F^\cdot$ of $\mathcal{E}$ over $C.$, I mean a rule associating to each integer $p \geq 0$ an object F^p of $\mathcal{E}$ over C_p and to each $\varphi: [p] \longrightarrow [q]$ a map

$$F_\varphi : C_\varphi^* F^p \longrightarrow F^q \qquad (\text{recall } C_\varphi : C_q \longrightarrow C_p),$$

~~[illegible]~~ all this subject to the evident transitivity conditions. A simplicial ~~object~~ system $E.$ of $\mathcal{E}$ over $C.$ consists of F_p over C_p and

$$F_\varphi : F_q \longrightarrow C_\varphi^* F_p .$$

Cosimplicial ~~objects~~ systems are natural from the point of view of cohomology. In effect given an object S of $\mathcal{E}$ over T in $\mathcal{C}$

~~we know a functor F(T) on the category C(T) and hence can form the associated~~ we obtain a cosimplicial system $F(\mathfrak{S})$ over the ~~co~~simplicial set $C(T)$ and so we can form the cosimplicial set

$$\Gamma(C_0(T), F^0(S)) \rightrightarrows \Gamma(C_1(T), F^1(S)) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \cdots\cdots$$

which depends contravariantly on S.

Given a category C in $\mathcal{C}$ let Sing C (or simply $C_\cdot$) be the simplicial object with

$$(\mathrm{Sing}\, C)_q = \overbrace{C_{fl} \underset{C_{ob}}{\times} \cdots \underset{C_{ob}}{\times} C_{fl}}^{q}$$

Then a functor from C with values in $\mathcal{E}$ gives rise to a simplicial system ~~object~~ $F_\cdot$ over $C_\cdot$ with

$$F_q = F \underset{C_{ob}}{\times} \overbrace{C_{fl} \underset{C_{ob}}{\times} \cdots C_{fl}}^{q},$$

more precisely

$$F_q = (C_{f_q})^* F,$$

where $f_q : [0] \longrightarrow [q]$ is the first vertex. It also gives rise to a cosimplicial system ~~object~~ with

$$F^q = \overbrace{C_{fl} \underset{C_{ob}}{\times} \cdots\cdots \underset{C_{ob}}{\times} C_{fl}}^{q} \underset{C_{ob}}{\times} F,$$

more precisely

$$F^q = (C_{l_q})^* F,$$

where $\ell_q : [0] \to [q]$ is the last vertex.

Examples: 1.] Suppose $\mathbf{C}$ = Sets and $\mathcal{E}_T$ = Sets / T. Then C is a category and F is a functor from C to Sets in the usual sense. The simplicial and cosimplicial systems are

$$\cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \coprod_{x_0 \to x_1} F(x_0) \rightrightarrows \coprod_{x \in C_{ob}} F(x_0)$$

$$\prod_{x \in C_{ob}} F(x_0) \rightrightarrows \prod_{x_0 \to x_1} F(x_1) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \cdots$$

2.] Let G be a group and define a category with $C_{ob} = e$, $C_{fl} = G$ and with composition $C_{fl} \times C_{fl} \longrightarrow C_{fl}$, $x, y \longmapsto y \circ x$ given by $y \circ x = yx$. Then a covariant functor from C to sets is a (left) G-set M and the cosimplicial system is

$$M \rightrightarrows \mathrm{Map}(G, M) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathrm{Map}(G \times G, M) \cdots\cdots$$

3.] Suppose C = affine schemes, $\mathcal{C}$ = quasi-coherent modules. Then a category C is Spec applied to

$$A \underset{t}{\overset{s}{\rightrightarrows}} P \xrightarrow{\varepsilon} A \qquad P \xrightarrow{\Delta} P \otimes_A P$$

whereas a functor with values in $\mathcal{E}$ is an A-module M endowed with a map of P-modules

$$_s P \otimes_A M \longrightarrow P \otimes_A M$$

or equivalently a map of A-modules

$$M \longrightarrow P \otimes_A M.$$

compatible with composition, i.e. a P-comodule.

Additions to preceding: If F functor on C with values in $\mathcal{E}$, then $\forall S \in \mathcal{E}$ we have $F_S : C(pS) \longrightarrow$ sets and for every map $S' \xrightarrow{v} S$ in $\mathcal{E}$ a map

$$\begin{tikzcd}
C(pS) \arrow[r, "F_S"] \arrow[d, "C(pv)"'] & \text{sets} \\
C(pS') \arrow[ur, "F_{S'}"'] \arrow[u, phantom, shift left=6ex, "F_v \Downarrow"] &
\end{tikzcd}$$

Cohomology.

Suppose that ~~[struck-out text]~~ $p: \mathcal{E} \to \mathcal{C}$ is an additive fibered category over $\mathcal{C}$. This means that the fibers of p are additive categories and that the base change u^* is an additive functor. In particular for any map $u: X \to Y$ in $\mathcal{C}$ and objects F, G over X, Y resp.,

$$\mathrm{Hom}_{/u}(F, G) = \mathrm{Hom}_{/X}(F, u^*G)$$

is an abelian group.

Suppose now that C is a category in $\mathcal{C}$ and that F is a functor on C with values in $\mathcal{E}$. More generally suppose $C_\bullet$ is a simplicial object of $\mathcal{C}$ and that $F^\cdot$ is a cosimplicial system over $C_\cdot$ in $\mathcal{E}$. Then for any object S of $\mathcal{E}$ we have a cosimplicial abelian group

$$\Gamma(C_0(pS), F^0(S)) \rightrightarrows \Gamma(C_1(pS), F^1(S)) \; \substack{\rightarrow\\ \rightarrow\\ \rightarrow}$$

$$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$$

$$\prod_{u: pS \to C_0} \mathrm{Hom}_{/u}(S, F^0) \rightrightarrows \prod_{v: pS \to C_1} \mathrm{Hom}_{/v}(S, F^1) \; \substack{\rightarrow\\ \rightarrow\\ \rightarrow}$$

and it is natural to interpret the cohomology of this cosimplicial abelian group as derived functors of some kind. With this end in mind we consider some examples.

Main example: Let $\mathcal{C}$ be a topos and let $\mathcal{E}_T = \mathcal{C}/T$. Then the cosimplicial systems over $C_\cdot$ form a topos; in fact it is a special case of the fiber topos of Verdier's last exposé in SGAA as follows. Each of the fibers $\mathcal{C}/T$, $T \in Ob\,\mathcal{C}$ has a topology so the fibers of $\mathcal{E}$ all have topology and the base changes are continuous. Thus $\mathcal{E}$ becomes a fibered site and one sees that a sheaf is a functor $\mathcal{E}^\circ \longrightarrow$ (sets) which is representable on each fiber. This means that a sheaf is a rule associating to each T in $\mathcal{C}$ an object F_T of $\mathcal{C}/T$ and to each map $f: T' \longrightarrow T$ a map $F_f : f^*F_T \longrightarrow F_{T'}$, all this subject to transivity conditions.

Let

$$\pi : \mathrm{Cosys}(\mathcal{C}_\cdot) \longrightarrow \mathrm{Cosets}$$

be the morphism sending $F^\cdot$ to $\nu \mapsto \Gamma(C_\nu, F^\nu)$. Then the Leray spectral sequence for π is

$$E_2^{pq} = \check{H}^p(\nu \longmapsto H^q(C_\nu, F^\nu)) \Longrightarrow H^{p+q}(C, F).$$

One uses the fact that the $\check{H}^*$ are the derived functors of $\check{H}^\circ$ on Cosab.

Example 2: Suppose given a fibered site $\mathcal{E} \longrightarrow \mathcal{C}$ and a simplicial object in $\mathcal{C}$, we can do the same as above. For example if

$$\nu \rightsquigarrow X_\nu$$

is a simplicial topological space, $\mathcal{C}$ = top spaces, $\mathcal{E}_X$ = étale spaces/X

then we can consider the topos of cosimplicial systems of sheaves over $X_\cdot$, i.e. $F^\nu_\bullet$ over X_ν with $F^\varphi : (X_\varphi)^* F^p \longrightarrow F^q$ for each $\varphi : [p] \to [q]$.
One gets the spectral sequence

$$E_2^{pq} = \check{H}^p\big(\nu \longmapsto H^q(X_\nu, F^\nu)\big) \Longrightarrow H^*(X_\cdot, F^\cdot)$$

<u>Example 2a</u>: If $X_\cdot$ is a simplicial set considered as a discrete simplicial topology space, we find for any simplicial system F over $X_\cdot$, that

$$H^*(X,F) = \check{H}^p\big(\nu \longmapsto C^\nu(X,F)\big)$$

$$C^\nu(X,F) \overset{\text{defn}}{=} \prod_{x \in X_\nu} F(x)$$

<u>Example 2b</u>: If $X_\cdot = \overline{W}(G)$ where G is a topological group

$$\cdots\ G \times G \rightrightarrows G \rightrightarrows e,$$

then we get for any abelian group A a spectral sequence

$$E_2^{pq} = \check{H}^p\big(\nu \longmapsto H^q(G^\nu, A)\big) \Longrightarrow H^{p+q}(BG, A)$$

first signalled by Milnor.

<u>Remark</u>: It seems that the cosimplicial systems of sheaves over the singular complex of a Cat in top spaces always give the good cohomology. One might ask when it's enough to consider only functors on the category with values in

the fiber category $\mathcal{E}$ with $\mathcal{E}_X$ = Et spaces/X. Here's an example to show that this doesn't work when G is not discrete. Take a connected group G. Then a functor is an étale space F over a pt endowed with a map

$$\begin{array}{ccc} \Theta : G\times F & \longrightarrow & G\times F \\ (g, f) & \longmapsto & (g, gf) \end{array}$$

of étale spaces over G. Thus $G\times F \to F$ is a ~~continuous action~~ of G on F; as G is connected the action is trivial. So there are not enough functors.

To remedy this defect one (Giraud?) introduces the gross topos of sheaves having the advantage that ~~the image ... is a sheaf if a representable functor is~~ the map from spaces over X to sheaves over X is fully faithful.

Example 3: Suppose C is a category scheme and that F is a functor with values in the fiber of quasi-coherent sheaves. We work in the gross Zariski, étale, fppf or fpqc topology on schemes and associate to a quasi-coh. sheaf F (Serre sense) on X the gross sheaf

$$\begin{array}{c} U \\ \downarrow f \\ X \end{array} \longmapsto \Gamma(U, f^*F)$$

Now the functor F gives rise to a cosimplicial system of quasi-coherent sheaves on the simplicial object Sing C. Hence

$$E_2^{pq} = \check{H}^p(\nu \mapsto H^q(C_\nu, F^\nu)) \Rightarrow H^*(C., F),$$

where $H^*(C_\nu, F^\nu)$ is the Zariski cohomology of F^ν in the case of any of the above topologies. So if C is affine with $A = \Gamma(C_{ob}, \mathcal{O}_{C_{ob}})$, $P = \Gamma(C_{fl}, \mathcal{O}_{C_{fl}})$ and F is given by the P-comodule $\Gamma(C_{ob}, F) = M$, then the spectral sequence degenerates showing that

$$(A) \qquad H^n(C_\cdot, F^\cdot) \cong \check{H}^n\big(\nu \longmapsto \overbrace{P \otimes_A \cdots \otimes_A P \otimes_A M}^{\nu}\big).$$

If P is right flat over A, then the P-comodules form an abelian category and the left side is a sequence of cohomological functors which is effaceable, hence the derived functors of $\check{H}^0$. Thus

$$H^n(C_\cdot, F^\cdot) = \mathrm{Cotor}^n_P(A, M)$$

Thus we have shown

<u>Proposition</u>: Let A, P etc. be an affine category and let M be a P-comodule. To the affine category we can associate a simplicial object $C_\cdot$ in the category of schemes, and to M we can associate a cosimplicial system $F^\cdot$ in the fibre category of quasi-coherent sheaves. Then in the (fibre) topos of (small) Zariski sheaves over schemes (= gross Zariski sheaves) we obtain a cosimplicial system over $C_\cdot$ and hence cohomology groups of this system $H^*(C_\cdot, F^\cdot)$ are defined. Conclusion:

$$H^*(C_\cdot, F^\cdot) = \check{H}^*\big(\nu \longmapsto \overbrace{P \otimes \cdots \otimes P \otimes M}^{\nu}\big)$$

$$\big(= \mathrm{Cotor}^*_P(A, M) \quad \text{when } P \text{ is right } A\text{-flat}\big)$$

Remarks 1. You have not yet understood when the cohomology in the cosimplicial sense equals that in the category of functors. This uses on an effacability argument.

~~Letter to Cartier — outline of paper~~

So I let $F: C^\circ \longrightarrow \mathcal{A}b$ be a functor. The point is to show that

$$\gamma^*: \quad R^q \varprojlim_{C} F \xrightarrow{\sim} R^q \varprojlim_{\Delta/\mathrm{Sing}\, C} F\gamma$$

$$\gamma: \Delta/\mathrm{Sing}\, C \longrightarrow C \qquad \text{last vertex.}$$

Method (classical) consists of

(i) check directly for $q=0$

(ii) show the 2nd functor vanishes for the induced F.

In fact (for (ii)) you take functor

$$g: \Delta/\mathrm{Sing}\, C \longrightarrow \Delta$$

$$g_*: \mathrm{Hom}((\Delta/\mathrm{Sing}\, C)^\circ, \mathcal{A}b) \longrightarrow \mathrm{Hom}(\Delta^\circ, \mathcal{A}b).$$

is exact, moreover gives s.s. of Leray

$$R^q \varprojlim_{\Delta/\mathrm{Sing}\, C} F\gamma = R^q \varprojlim_{\Delta} g_*(F\gamma) = \check{H}^q \underline{g_*(F\gamma)}$$

$$g_*(F\gamma) = \Big(\nu \rightsquigarrow \prod_{X_0 \to \cdots \to X_q} F(X_q)\Big)$$

$$= C^\cdot(C, F).$$

Thus $$R^q \varprojlim_{\Delta/\mathrm{Sing}\, C} F\gamma = \check{H}^q(C, F).$$

this is effaceable by the usual argument

May 9, 1969. Pseudogroups versus formal groupoids.

Example: G Lie group acting on a manifold M. Then one gets a Lie category (category object in the cat. of manifolds). More generally let a Lie alg $\mathfrak{g}$ act as derivations of the sheaf of functions $\mathcal{O}_M$ on a manifold M. The associated de Rham complex is

$$\mathcal{O}_M \longrightarrow \mathcal{O}_M \otimes_k \mathfrak{g}' \longrightarrow \mathcal{O}_M \otimes_k \Lambda^2 \mathfrak{g}' \longrightarrow \cdots$$

where $k = \mathbb{R}$, and where

$$d : \mathcal{O}_M \longrightarrow \mathfrak{g}' \otimes \mathcal{O}_M$$

is given by

$$d = \sum_{i=1}^{n} e(\omega_i)\, \theta(X_i)$$

where ω_i and X_i are dual bases of $\mathfrak{g}'$ and $\mathfrak{g}$, resp.

On the other hand we obtain a pseudogroup on M as follows. Say that a germ of vector field X is in the pseudogroup if it comes from $\mathfrak{g}$, i.e. if $\exists\ u \in \mathfrak{g}$ such that $X = i(u)d : \mathcal{O}_M \to \mathcal{O}_M$. Of course we lose structure this way unless $\mathfrak{g}$ acts faithfully on the manifold. Given a point $m \in M$, we obtain a filtration on $\mathfrak{g}$ induced by the maps

$$\mathfrak{g}_k = \operatorname{Ker}\{\mathfrak{g} \longrightarrow J_k(T)(x)\}$$

In the tradition of pseudogroups one makes regularity assumption and assumes that the function

$$x \longmapsto \dim \mathfrak{g}_k(x)$$

is locally constant. So one gets a ~~[illegible]~~ Lie algebra bundle

$$L = \mathcal{O}_X \otimes_k \mathfrak{g}$$

endowed with a filtration

$$L_q(x) = \mathrm{Ker}\left\{ \mathfrak{g} \longrightarrow J_q(T)(x) \right\}.$$

Although L is the base extension of $\mathfrak{g}$ from k to $\mathcal{O}_X$, the L_q are usually non-constant.

$$L = \mathrm{Hom}_k(\mathfrak{g}', \mathcal{O}_M) = \mathrm{Hom}_{\mathcal{O}_M}(\underbrace{\mathfrak{g}' \otimes \mathcal{O}_M}_{\Omega}, \mathcal{O}_M).$$

Suppose now given a de Rham complex

$$\mathcal{O}_M \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \longrightarrow \cdots$$

or equivalently a formal category. Assume Ω^1 flat. ~~[illegible]~~ Set

$$L = \underline{\mathrm{Hom}}_{\mathcal{O}_M}(\Omega, \mathcal{O}_M)$$

so L is a mixed Lie ring. ~~[illegible]~~ Sections of L ~~[illegible]~~ ~~[illegible] vector fields~~ give rise to vector fields on M, ~~[illegible] one can define a filtration of L by subbundles~~ so define

~~[illegible]~~

$$X \in \Gamma(L_0) \iff Xf = 0 \quad \text{all } f$$

Note that L_0 is a Lie algebra over $\mathcal{O}_M$, since

$$[X, fY] = \cancel{} + f[X,Y].$$

Now proceed as before

$$X \in \Gamma(L_q) \iff [X, L] \subset L_{q-1} \text{ and } X \in \Gamma(L_{q-1}).$$

$$\left(\underline{\text{Check}}: [fX, L] \subset -(Lf)\cdot X + f[X,L] \subset L_{q-1}. \right)$$

One checks that

$$[L_p, L_q] \subset L_{p+q} \qquad p,q \geq -1$$

~~[crossed out]~~ First observe true if p or $q = -1$. Hence can assume $p, q \geq 0$ and use induction on $p+q$, the case $p=q=0$ being evident from the above. Now

$$[L_p, L_q] \subset [L_{p-1}, L_q] \subset L_{p+q-1} \quad \text{by ind}$$

$$[[L_p, L_q], L] \subset \underline{[[L_p, L], L_q] + [L_p, [L_q, L]]} \subset L_{p+q-1} \quad \text{using inducti}$$

So we now have a filtered Lie algebra L_0 over $\mathcal{O}_M$.
In the example above the bracket on $L = \mathcal{O}_X \otimes \mathfrak{g}$ differs from its bracket as vector fields, e.g. let X_i be a basis for $\mathfrak{g}$, then

$$[f\otimes X, g\otimes Y] = fg[X,Y] + f(Xg)Y - g(Yf)X$$

Is there a canonical way of reorganizing the bracket?

For a pseudogroup there appears to be a basic map of bundles

(★) $$L \longrightarrow J_\infty(T_M)$$

of Lie algebras which doesn't seem to exist in general. For example take a principal bundle P over M with group G and consider the formal category associated to the complex of G-invariant forms on P. Then one has $L =$ G-invariant vector fields on P and the ~~basic~~ exact sequence

$$0 \longrightarrow P \times_G \mathfrak{g} \longrightarrow L \longrightarrow T_M \longrightarrow 0$$

of bundles on M. Thus $L_0 = P \times_G \mathfrak{g}$ = invariant fields on P tangent to the fibers. Now locally $P = M \times G$ so an invariant vector field is the sum of an element of $\mathfrak{g}$ and ~~an element~~ a vector field on M, these two commuting of course. Thus $[L_0, L] \subset L_0$ and so $L_0 = L_1 = L_2 = L_3 = \cdots$, which makes one hope that ★ can be defined by induction viz:

$$L/L_n \hookrightarrow J_n(T_M).$$

Unfortunately I can't seem to define such a map. I seem also to need the map

(1) $$D: L \longrightarrow \Omega \otimes L$$

whose solutions give the actual ~~[illegible]~~ Lie algebra $\mathfrak{g}$ of vector fields. From such a D I can construct

(2) $$L \longrightarrow J_n(T_M)$$

by induction using the ~~[illegible]~~ formula

$$J_n T \simeq J_{n-1}T \underset{\Omega\otimes J_{n-2}T}{\times} \mathrm{Ker}\{D : \Omega^1\otimes J_{n-1}T \to \Omega^2 \otimes J_{n-2}T\}.$$

Conversely given (2) I obtain D, assuming the L_n filtration is complete.

Conclusion: It seems that a (formal) pseudogroup on a **manifold** M is a de Rham complex

$$\mathcal{O}_M \xrightarrow{\delta} \Omega^1 \xrightarrow{\delta} \Omega^2 \longrightarrow \cdots$$

endowed with a **stratification** relative to the map $M \to pt$, that is, a double complex

$$\begin{tikzcd}
\mathcal{O}_M \arrow[r, "\delta"] \arrow[d, "d"] & \Omega^1 \arrow[r, "\delta"] \arrow[d, "d"] & \Omega^2 \arrow[r] \arrow[d] & \cdots \\
\Omega^1_{M/pt} \arrow[r] \arrow[d] & \Omega^1_{M/pt}\otimes\Omega^1 \arrow[r] \arrow[d, dashed] & \Omega^2_{M/pt}\otimes\Omega^2 \arrow[r] \arrow[d, dashed] & \cdots \\
\Omega^2_{M/pt} \arrow[r, dashed] & \cdots \arrow[r, dashed] & \cdots &
\end{tikzcd}$$

For example suppose the pseudogroup is **finite**, e.g. Ω^1 is a vector bundle then taking the sheaf of solutions of d one obtains a complex

$$\underline{\mathbb{R}} \longrightarrow \mathfrak{g}' \longrightarrow \Lambda^2\mathfrak{g}' \longrightarrow \cdots\cdots$$

where $\mathfrak{g}$ is a locally constant sheaf of Lie algebras acting as vector fields on M.

Remarks on DR Chern classes.

A. The true classes live

$$c_p(E) \in H^p(X, \Omega^p \to \Omega^{p+1} \to \cdots)$$

$\to H^p(X, \Omega^p)$ gives ~~Hodge~~ Atiyah classes

$\downarrow$ gives DR classes

$$H^{2p}(X, \Omega^\circ \to \cdots) = H^{2p}_{DR}(X)$$

Example:

$$H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(X, \Omega^1 \to \Omega^2 \to \cdots)$$

$$\begin{array}{ccc} f & \in & \mathcal{O}_X^* \\ \downarrow & & \downarrow \\ \frac{df}{f} & & \Omega^1 \xrightarrow{d} \Omega^2 \to \cdots \end{array}$$

B. The conjectured method is to take the complex of invariant forms on the principal bundle P of E which is kind of "cofibration" of DG cochain algebras, its an extension

(Does $\exists$ an alg. definition of this complex for a perfect complex?)

$$\Omega^\bullet_X \longrightarrow (\pi_* \Omega^\bullet_P)^G \longrightarrow P \times_G \Lambda^\bullet \mathfrak{g}'$$

~~This idea~~ Filtering $A^\bullet(P)^G$ by the ideal gen. by $\Omega^{\geq p}$ one gets a spectral sequence (in the affine case at least, but ~~use~~ can probably Deligne's improvement on Katz-Oda)

$$E_2^{pq} = H^p_{DR}(X, H^q_{DR}(G)) \Longrightarrow H^{p+q}(P)$$

when G reductive. Chara. classes result when one sees why the primitive classes in $H^*(G) = H^*(\Lambda^\bullet \mathfrak{g}')$ are universally transgressive.

C. <u>Problem</u>: Let G be an alg. gp acting on X. What is the significance if any of

$$(\mathbb{R}H^0_G)(\mathbb{R}\Gamma)(\Omega^{\cdot}_X) \;?$$

~~van Est spec~~ If G reductive + X affine (in char 0) then

$$A^{\cdot}(X)^G \longrightarrow A^{\cdot}(X) \quad \text{qis.}$$

Thus it's not the same as the equivariant DR cohomology.

D. van Est spec. ~~sequence~~

G alg. gp, P principal bundle

$$H^p_{gp}(G, H^q_{top}(P)) \Longrightarrow H^{p+q}(A^{\cdot}(P)^G)$$

two cases (i) G nilpotent whence get

$$H^p_{gp}(G, H^q_{top}(P/G)) \Longrightarrow H^{p+q}(A^{\cdot}(P)^G)$$

e.g. $P = G$

$$H^p_{gp}(G) \overset{?}{=} H^p(\mathfrak{g})$$

(ii) G reductive get $H^*_{top}(P) = H^*(A^{\cdot}(P)^G)$.

E. Characterize those formal categories associated to pseudogroups.

Now I must show how to do this trangression universally

General setting of Lie algebra

G ~~Lie~~ aff alg group, over a field K of char. 0, $\mathfrak{g}$ associated Lie algebra

$\boxed{\text{van Est}}$:
$$H^p_{gp}(G, \underline{H^q_{top}(G, K)}) \Longrightarrow H^{p+q}(\mathfrak{g}, K)$$

G reductive $\Downarrow$ $H^*_{gp}(G, V) = 0$? if $\dim V < \infty$

Proof:

~~Look at G as a G-space~~

(i) Consider the algebra of differential forms on G

$$\boxed{A^\cdot(G) = \underset{\sim}{A(G)} \otimes \Lambda \mathfrak{g}'}$$

~~Take its cohomology~~

$$\cancel{A(G) \longrightarrow A(G)\otimes \mathfrak{g}' \longrightarrow A(G)\otimes \Lambda^2 \mathfrak{g}' \longrightarrow}$$

This is a complex of G-modules, hence basic sp. seq

$$E_2^{pq} = H^p_{gp}(G, H^q_{top}(G)) \Longrightarrow H^{p+q}_{gp}(G, A^\cdot(G))$$

$$\int \quad H(\mathfrak{g}, K) \qquad \Uparrow$$

$$E_1^{pq} = H^q_{gp}(A^p(G)) = \begin{cases} 0 & q > 0 \\ \Lambda^p \mathfrak{g}' & q = 0 \end{cases}$$

ΛE

filter by

$$F_p(\Lambda E) = \mathrm{Im}\ \Lambda_p E' \otimes \Lambda E \longrightarrow \Lambda E.$$

Ω

$E = F_0 \qquad F_p \supset F_{p+1} \supset \cdots$

$$A^*(B) \qquad A^*(E) \qquad A^*(F)$$

Consider crude filtration on the base

$$\supset F_p \supset F_{p+1} \supset \cdots$$

$$0 \longrightarrow gr_{p+1} \longrightarrow F_p/F_{p+2} \longrightarrow gr_p \longrightarrow 0$$

$$H^q(gr_p) \longrightarrow H^{q+1}(gr_{p+1}).$$

$$E_1^{pq} = H^q(\Omega^p)$$

$$0 \longrightarrow f^*\Omega_B \longrightarrow \Omega_E \longrightarrow \Omega_f \longrightarrow 0$$

$F_p A^*(E)$ means $\geq p$ terms coming from B.

Then $$F_p A^{p+q}(E) / F_{p+1} A^{p+q}(E) = f^*(\Lambda^p \Omega_B) \otimes \Omega_f^q$$

gives $E_1^{pq} =$

G-sheaf

$$\begin{tikzcd} & F \arrow[d] \\ X\times G \arrow[r, shift left] \arrow[r, shift right] & X \end{tikzcd}$$

?

$$p_1^*F \cong p_2^*F$$

Observe that in the gross ~~[illegible]~~ $\boxed{\text{etale}}$ topos

$$p_1^*F = F\times G$$

$$G \to pt.$$

so that this descent is just $\overbrace{\text{action}}^{\text{a continuous}}$ of G on F.

$$A^*(G) = \underline{\Gamma}(\Omega^*)$$

$$R^+\Gamma(\Omega^*)=0$$

now you calculate

$$H^0_{gp}(G,\Gamma(\Omega^*))$$

e.g. X point

F sets

$$F \to X$$

$$\underline{G}\times F \longrightarrow F$$

Therefore note that there need not exist enough G sheaves on X.

indeed $X\times G$, $X\times G\times G$, etc. belong.

$X \otimes DX \otimes Y \longrightarrow Y$

$X \otimes DX \otimes Y \to Y$

$X \otimes DX$

$\boxed{\cancel{L \subset \mathrm{Der}\, A(X)}}$

$\mathbf{L} = \mathrm{Ham}_{\mathcal{O}_X}(\Omega, \mathcal{O}_X).$

$\mathcal{O}_X \xrightarrow{d} \Omega$

an honest filtered Lie alg.

$\mathcal{L}$ sheaf of vector fields leaving Ω invariant

$$\mathcal{L} \longrightarrow \underline{T} \xrightarrow{j_\infty} \underline{J_\infty(T)}$$

and now I can look at the sub R-module L of $\underline{J_\infty(T)}$ generated by the [illegible] image of this.

Thus say $L \subset J_\infty(T)$ is a subbundle

J_∞

$\boxed{\mathcal{O}_X \otimes_{R} \mathcal{L}}$

(i) sheaf of $\mathcal{O}_X$ modules

(ii) define the bracket as ~~[illegible]~~ operators on fns.

May 67 Illusie

Lecture tomorrow

~~[illegible]~~

1. Correction — no distinguished generators for $\Omega(pt)$

2. Affine categories + groupoids

rings ~~[illegible]~~

Suppose given a functor $(\text{rings}) \xrightarrow{\mathcal{C}} \text{Cat}$ which is representable

$$\text{Ob } \mathcal{C}(R) = \text{Hom}_{(\text{rings})}(A, R)$$

$$\text{Fl } \mathcal{C}(R) = \text{Hom}_{\text{"}}(P, R)$$

$$A \underset{t}{\overset{s}{\rightrightarrows}} P \xrightarrow{\varepsilon} A \qquad \Delta : \cancel{R} P \to P {}_t\!\otimes_{A\,s} P \quad \text{map of } A, A \text{ alg.}$$

Call $(A, P, s, t, \varepsilon, \Delta)$ an affine category. An affine groupoid = ~ inversion $i : P \to P$

$\text{Com}(A, P)$ = cat of comodules M

$$M \xrightarrow{\Delta} P \otimes_A M$$

tensor category $\begin{cases} \text{unitary} \\ \text{assoc} \\ \text{comm.} \end{cases}$ $\quad 1 = A \quad \Delta = 1 \otimes (?)$

3. $\mathfrak{a}$: tensor cat $\begin{cases} \text{unitary} & 1 \\ \text{assoc.} \\ \text{comm.} \end{cases}$ $\quad 1 \otimes X \simeq X \otimes 1 \simeq X$, $\quad X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$, $\quad X \otimes Y \simeq Y \otimes X$

$h : \mathfrak{a} \to \text{Mod } A$ tensor functor $\begin{cases} h1 \simeq A \\ hX \otimes_A hY \to h(X \otimes Y) \end{cases}$

forgetful functor. $\text{Com}(A, P) \to \text{Mod } A$.

Define $\mathcal{C} = \underline{\text{End}}^{\otimes} h : (\text{Rings}) \longrightarrow \text{Cat}$

$$\begin{cases} \text{Ob } \mathcal{C}(R) = \text{Hom}(A, R) \\ \text{Hom}_{\mathcal{C}(R)}(u, v) = \underset{\text{Hom}^{\otimes}(\mathfrak{a}, \text{Mod } R)}{\text{Hom}^{\otimes}}(R_u \otimes_A h, R_v \otimes_A h) \end{cases}$$

$\mathcal{C}' = \underline{\mathrm{Aut}}^{\otimes} h$

$$\mathrm{Hom}_{\mathcal{C}'(R)}(u,v) = \mathrm{Isom}_{\underline{\mathrm{Hom}}^{\otimes}(\mathcal{a},\, \mathrm{Mod}\, R)}\left(R_u \otimes_A h,\ R_v \otimes_A h\right)$$

Suppose $\underline{\mathrm{End}}^{\otimes} h$ representable i.e. $\exists\, A \xrightarrow{s} P \xleftarrow{t} A$

$\ni \quad \forall \quad R \underset{v}{\overset{u}{\leftleftarrows}} A$

$$\cancel{\mathrm{Hom}^{\otimes}(h, h) \simeq \mathrm{Hom}_{A\otimes A\ \mathrm{alg}}(P, R)}$$

~~this means $\exists$ canonical~~

$$\mathrm{Hom}_{A,A\ \mathrm{algs}}(P, R) \cong \mathrm{Hom}^{\otimes}_{/R}\left(R_u \otimes_A h,\ R_v \otimes_A h\right)$$

$$\| $$

$$\mathrm{Hom}^{\otimes}_{/A}\left(h,\ {}_uR_v \otimes_A h\right)$$

i.e. $\exists$ canonical ~~functor~~ morphism in $\underline{\mathrm{Hom}}^{\otimes}(\mathcal{a}, \mathrm{Mod}\, A)$.

$$\gamma : h \longrightarrow {}_sP_t \otimes_A h$$

: not quite ~~correct~~ general enough. One wants (A,P) to act on $h : \mathcal{a} \to \mathrm{Mod}\, A'$ when given $A \to A'$ and $\forall X \in \mathcal{a}$

$$hX \longrightarrow P \otimes_A hX$$

In particular have

$$\begin{tikzcd} & \mathrm{Com}(A,P) \arrow[d] \\ \mathcal{a} \arrow[ur, dashed] \arrow[r] & \mathrm{Mod}\, A \end{tikzcd}$$

__Defn__: Say that (A,P) __acts__ on $h : \mathcal{a} \to \mathrm{Mod}\, A$ if there is given dotted arrow. Of course one gets a morphism ~~(functor)~~

$$(A,P) \longleftrightarrow \underline{\mathrm{End}}^{\otimes} h$$

---

4. $f : (\underline{A}, P) \longrightarrow (\underline{A}', P')$ morphism of affine cats.

fully faithful $=$ $A' \otimes_A P \otimes_A A' \xrightarrow{\sim} P'$

equivalence: $\mathrm{ff} + \exists\ g : A' \longrightarrow A$ and $\theta : P \to A \ni$ $\theta : id_A \cong gf$

Suppose $f : (A, P) \longrightarrow (A', P')$ is a morphism and (A, P) acts on h.

$$\begin{tikzcd} & \mathcal{C}om(A,P) \arrow[r] \arrow[d] & \mathcal{C}om(A',P') \arrow[d] \\ \mathcal{A} \arrow[ur] \arrow[r] & \mathrm{Mod}\,A \arrow[r] & \mathrm{Mod}\,A' \end{tikzcd}$$

so A', P' acts on $h' = A' \otimes_A h$. If $(A, P) = \underline{\mathrm{End}}^{\otimes} h$ (resp. $\underline{\mathrm{Aut}}^{\otimes} h$) then $(A', P') = \mathrm{End}^{\otimes} h'$ (resp. $\underline{\mathrm{Aut}}^{\otimes} h'$) $\Longleftrightarrow$ f is fully faithful.

<u>Suppose f equivalence.</u> Then choosing g, θ

$$A \otimes_{g\,A'} h' = A \otimes_{g\,A'} (A' \otimes_{f\,A} h) = A_{gf} \otimes_A h \overset{\theta}{\simeq} h$$

5. Ω. $\underline{\mathrm{Aut}}^{\otimes} \Omega$

I want to apply the preceding to

$$\Omega^{cv} = \Omega : (\mathrm{Man})^{\circ} \longrightarrow \mathrm{Mod}\, A \qquad A = \Omega(pt).$$

If R is a (graded anti-commutative) ring ~~that~~ a morphism $v : \Omega(pt) \longrightarrow R$ same as $F(X,Y) = \sum' a_{k\ell} X^k Y^\ell$, $a_{k\ell} \in R^{2-2k-2\ell} = R_{-2+2k+2\ell}$. Let

$$G(R) = \left\{ \varphi(X) = \sum_{n\geq 0} r_n X^{n+1} \;\middle|\; r_n \in R_{2n},\ r_0 \in R^* \right\}.$$

group under composition.

Given φ and v recall $\exists!$ mult. char class

$$\tilde{\varphi} : K \longrightarrow R_v \otimes_A \Omega$$

$$\tilde{\varphi}(L) = \sum_{n\geq 0} r_n \otimes c_1^{\Omega}(L)^n$$

and $\exists$ ~~unique additive natural transformation~~

$$\hat{\varphi} : \Omega \longrightarrow R_v \otimes_A \Omega$$

$$\hat{\varphi}(f_* x) = f_* \left(\hat{\varphi} x \cdot \tilde{\varphi}(\nu_f) \right)$$

By R-R $\hat{\varphi}$ unique ring hom $\ni$ $\hat{\varphi}(c_1 L) = \sum r_n \otimes c_1^{\Omega}(L)^{n+1}$.

Proposition: Given $u, v : R \longrightarrow \Omega$ with group law F_u, F_v

$$\mathrm{Isom}^{\otimes}(R_u \otimes_A \Omega,\ R_v \otimes_A \Omega) \cong \{ \varphi \in G(R) \mid \varphi * F_v = F_u \}.$$

Proof: Given φ consider

$$\hat{\varphi} : \Omega \longrightarrow R_v \otimes_A \Omega$$

$$\hat{\varphi}(c_1 L) = \sum r_n \otimes c_1(L)^{n+1}$$

$\hat{\varphi}$ extends ! to a R-hom.

$$\hat{\hat{\varphi}} : R \otimes_{u\,A} \Omega \longrightarrow R_v \otimes_A \Omega$$

where u is composite

$$\begin{tikzcd} A \arrow[r, "F^{\Omega}", "\hat{\varphi}"'] & R_v \otimes_A \Omega(pt) & R \arrow[l, "in_1", "\cong"'] \end{tikzcd}$$

$F_u \otimes 1$ $\qquad$ F_u

To calculate F_u. $\qquad \varphi'(X) = \varphi(X) \otimes 1$

$$c_1'(L) = 1 \otimes c_1 L = c_1^{\Omega_v}(L)$$

$$\hat{\varphi}(c_1 L) = \varphi'(c_1' L)$$

$$\Rightarrow (\hat{\varphi} F^{\Omega})(\hat{\varphi} c_1' L, \hat{\varphi} c_1' L_0) = \hat{\varphi}(c_1(L \otimes L_0))$$

$$= \varphi'(F^{\Omega_v}(c_1' L, c_1' L_0))$$

$$\Rightarrow (\hat{\varphi} F^{\Omega})(\varphi' X, \varphi' Y) = \varphi'(F^{\Omega_v}(X, Y)).$$

$$\hat{\varphi} F^{\Omega} = \varphi' * F^{\Omega_v} \qquad F^{\Omega_v} = in_2 F^{\Omega}$$

$$F^{\Omega_v} = 1 \otimes F^{\Omega}$$

$$= F_v \otimes 1$$

$$F_u \otimes 1 = (\varphi \otimes 1) * (F_v \otimes 1)$$

$$\boxed{F_u = \varphi * F_v}$$

$$\underline{F_u = \varphi * F_v = \varphi * \psi * F_w = (\varphi \circ \psi) * F_w}$$

$$\begin{tikzcd} \Omega_u \arrow[r, "\hat{\hat{\varphi}}"] \arrow[rr, bend right, "\widehat{\widehat{\varphi\psi}}"'] & \Omega_v \arrow[r, "\hat{\hat{\psi}}"] & \Omega_w \end{tikzcd}$$

$$\hat{\hat{\psi}}\left(\hat{\hat{\varphi}}(c_1 L)\right) = \hat{\hat{\psi}}(\varphi\, c_1 L) = \varphi\, \hat{\hat{\psi}}\, c_1 L$$
$$= (\varphi \circ \psi)^{\hat{\hat{}}}\, c_1 L$$

$\therefore$ $\hat{\hat{\varphi}}$ is an isomorphism inverse $(\varphi^{-1})^{\hat{\hat{}}}$.

$$\{\varphi \in G(R) \mid \varphi * F_v = F_u\} \longrightarrow \mathrm{Isom}^{\otimes}(\Omega_u, \Omega_v)$$
$$\hookrightarrow \qquad \ni \Theta$$

$$\begin{tikzcd} R_u \otimes_A \Omega \arrow[r, "\simeq"] & R_v \otimes_A \Omega \\ \Omega \arrow[u, "1\otimes"] \arrow[ur, "\Theta_0"'] & \end{tikzcd}$$

with $\Theta : R_u \otimes_A \Omega \xrightarrow{\simeq} R_v \otimes_A \Omega$.

$$\Theta_0\, c_1(L) = \sum_{n\geq 0} r_n \otimes c_1 L^{n+1} \qquad r_n \in R_{2-}$$

To show $r_0 \in R^*$ let $\iota : pt \to \mathbb{P}^1$

$$\begin{tikzcd} 0 \arrow[r] & \Omega_u(pt) \arrow[r, "\iota_*"] \arrow[d, "r_0"] & \Omega_u(\mathbb{P}^1) \arrow[r, "\iota^*"] \arrow[d, "\Theta\,\cong"] & \Omega_u(pt) \arrow[r] \arrow[d, "\cong\,\Theta"] & 0 \\ 0 \arrow[r] & \Omega_v(pt) \arrow[r, "\iota_*"] & \Omega_v(\mathbb{P}^1) \arrow[r, "\iota^*"] & \Omega_v(pt) \arrow[r] & 0 \end{tikzcd}$$

$$\Theta\, \iota_* 1 = \Theta c_1 \mathcal{O}(1) = r_0\, c_1 \mathcal{O}(1) + 0 = r_0\, \iota_* 1 \qquad \therefore r_0 \in R^*$$

$$\left\{ \begin{array}{l} (\underset{\sim}{\mathrm{Aut}}^{\otimes}\Omega)(R) \quad = \quad \text{category} \quad \text{objects: formal gp. law } F \\ \qquad\qquad\qquad\qquad\qquad \mathrm{Hom}(F, F') = \{\varphi \in G(R) \mid \varphi^* F' = F\} \end{array} \right.$$

~~$\mathrm{Spec}\,\Omega(pt) \times G \longrightarrow \mathrm{Spec}\,\Omega(pt)$~~

~~Now suppose that we do the above for $\Omega_{(p)}$~~

~~$\Omega_{(p)}$ $\Omega_{(p)}$~~ __Example 1:__ __Over $\mathbb{Q}$.__

Work with $\Omega_{\mathbb{Q}}$ $\quad \underline{(A_{\mathbb{Q}}, P_{\mathbb{Q}}) \sim (\mathbb{Q}, \mathbb{G}_m)}$

$$\Omega_{\mathbb{Q}} \simeq (A_{\mathbb{Q}}) \otimes_{\mathbb{Q}} (\mathbb{Q} \otimes_{A_{\mathbb{Q}}} \Omega_{\mathbb{Q}})$$

$$A_{\mathbb{Q}} \longrightarrow \underline{\mathbb{Q}} \longrightarrow A_{\mathbb{Q}}$$

$\mathbb{Q} \otimes_A \Omega$ retract of Ω

$H^*(X, \mathbb{Q})$

$$\boxed{\Omega^*_{\mathbb{Q}}(X) \simeq \Omega(pt)_{\mathbb{Q}} \otimes_{\mathbb{Z}} H^*(X, \mathbb{Z})}$$

Example 2: Unoriented cobordism $\mathfrak{N}^*(X)$

$$\begin{cases} F^{\mathfrak{N}}(X,Y) = \sum_{k,\ell} a_{k\ell} X^k Y^\ell & a_{k\ell} \in \mathfrak{N}^{1-k-\ell}(pt). \\ F^{\mathfrak{N}}(X,X) = 0 & \text{law of height } \infty \end{cases}$$

$\mathrm{Aut}^{\otimes} \mathfrak{N}^*$ to each R of char 2

laws $a_{k\ell} \in R_{k+\ell-1}$ of height ∞.

$\mathrm{Autst}^{\otimes}(\mathfrak{N}^*)$ coordinates changes in $G_1(R)$.

$$\boxed{\mathbf{F}. \quad \exists!\ \ell(X) = X + \sum_{\substack{n \neq 2^i - 1 \\ n \geq 0}} a_n X^{n+1} \qquad a_n \in R_n.}$$

$\mathfrak{N}^*(X)$.

$$\begin{array}{ccc} L & \longrightarrow & \mathfrak{N}^*(pt) \\ \downarrow & & \\ \mathbb{Z}/2 & & \end{array}$$

$$\mathbb{Z}/2 \underset{L}{\otimes} \mathfrak{N}^* = \mathfrak{M}^*$$

$$\mathfrak{M}^* \underset{\mathbb{Z}/2}{\otimes} L \xrightarrow{\sim} \mathfrak{N}^*$$

(a) L poly ring gen. $\neq 2^i - 1$

$\mathfrak{N}^*(pt)$ by Thom.

$$\boxed{\begin{array}{l} \mathfrak{M}^*(pt) = \mathbb{Z}/2 \\ \mathfrak{N}^*(pt) \xleftarrow{\sim} L. \end{array}}$$

$$\mathfrak{M}^*(X) = H^*(X, \mathbb{Z}/2)$$

$$\mathfrak{N}^*(X) \cong \mathfrak{N}^*(pt) \otimes H^*(X, \mathbb{Z}/2)$$

Alg. geometry.

$\mathcal{V}$ schemes smooth quasi-projective / k field.

~~[illegible]~~ 0. Let $Y \longmapsto Q(Y)$, $f \longmapsto f^*$ is a contravariant functor on $\mathcal{V}$ values in (rings)

I. $Y \mapsto Q(Y)$. $f \mapsto f_*$ f proper cov. values in Ab.

II. tr. base change for f_*

III. $Q(Y) \xrightarrow{\sim} Q(\mathbb{A}^1 \times Y)$

IV. $Q(Y_1 \amalg Y_2) \xrightarrow{\simeq} Q(Y_1) \times Q(Y_2)$

V. $(f_1 \times f_2)_*(x \boxtimes y) = f_{1*}x \boxtimes f_{2*}y$

VI. If L line bundle let $c_1(L) = i^* i_* 1$. Then $Q(\mathbb{P}E)$ ~~[illegible]~~ free.

VII. $Y \subset X$ closed ~~[illegible]~~ immersion in $\mathcal{V}$ $j: U \to X$ comp.

$$Q(Y) \xrightarrow{i_*} Q(X) \xrightarrow{j^*} Q(U) \to 0 \quad \text{exact}$$

VIII. $i: Y \to X$ closed imm. $i^* i_* y = (i^* i_* 1) \cdot y$

Let Ω be the initial object of the category of such Q.

Theorem 1: Over $\mathbb{Q}$ $\exists$ universal theory $\Omega_{\mathbb{Q}}$ and infact $\exists$

$$\Omega_{\mathbb{Q}}(X) \xleftarrow{\sim} \mathbb{Q}[P_1, P_2, \ldots] \otimes K(X) \quad : \quad Ch$$

compatible with f^*. #

Proof: By axioms have Chern classes and a formal group law in Q F^Q

C2

Then $\exists$ $\ell(F(X,Y)) = \ell X + \ell Y$ and $\ell X = \sum' p_n \frac{X^{n+1}}{n+1}$

and I can define

$$ch : K(X) \longrightarrow Q(X)$$

unique additive ext. of

$$ch\, L = e^{\ell(c_1^Q L)}$$

ch is a ring homomorphism. Now define

$$\boxed{Ch : \mathbb{Q}[P_1, \ldots] \otimes_{\mathbb{Z}} K(X) \longrightarrow Q(X)}$$

$$P_i \otimes 1 \longmapsto p_i$$

$$1 \otimes x \longmapsto ch\, x$$

$$[Ch, f^*] = 0$$

$$ch(c_1^K(L)) = ch(1 - L^{-1}) = 1 - e^{-\ell(c_1^Q L)}$$

let $\text{Todd}\, X = \frac{X}{1 - e^{-\ell(+X)}} = \frac{X}{1 - e^{-\sum P_n \frac{X^{n+1}}{n+1}}}$

Then define $f_!$ in $\mathbb{Q}[P] \otimes K$ by

$f_!(r \otimes x) = r\, f_!(x)$ $\quad r \in \mathbb{Q}[P],\ x \in K$

$$f_!(x) = f_*(x \cdot \text{Todd}^{-1}(\nu_f))$$

Todd L

$$\mathrm{Ch}\,\{c_1^K(L)\} = \frac{1-e^{-\ell(c_1^Q L)}}{c_1^Q L}\cdot c_1^Q L$$

$$= T(c_1^Q L)\cdot c_1^Q L.$$

Therefore define

$$f_!(x) = f_*(x\cdot \Phi(\nu_f))$$

$$c_1^!(L) = T(L)\cdot c_1^K(L)$$

$$\mathrm{ch}\, c^K L = \mathrm{ch}(1-L^{-1}) = 1-e^{-\ell(c_1^Q L)}$$

Set $\gamma(X)$ = inverse of $1-e^{-\sum_{n\geq 0} p_n \frac{X^{n+1}}{n+1}}$

and define $f_*^\gamma x = f_*(x\cdot\tilde\gamma(\nu_f))$

where $\tilde\gamma(L) = \dfrac{\gamma(c_1^K L)}{c_1^K L} = \sum a_n (c_1^K L)^n$ if $\gamma(X) = \sum a_n X^{n+1}$

Then

$$c_1^\gamma L = i^* L_*^\gamma 1 = i^*(1\cdot\tilde\gamma(c_1^* L))$$

$$= c_1 L\cdot\tilde\gamma c_1 L = \gamma(c_1^K L)$$

and

$$\mathrm{ch}\, c_1^\gamma L = (\mathrm{ch}\,\gamma)(\mathrm{ch}\, c_1^K L) = \left(1-e^{-\sum p_n \frac{X^{n+1}}{n+1}}\right)^{-1}\circ\left(\;\right)_{=c_1^Q L}$$

~~so~~ set $\Omega = Q[P] \otimes_{\mathbb{Z}} K$ with $f_*^{\Omega} = f_*^{\gamma}$.

$\boxed{Ch : \Omega \longrightarrow Q}$

Prop: Suppose Q over Q as above. Then $f_*^{Q} 1 = 1$ for any blowup $\Rightarrow$ $F^{Q}(X,Y) = X + Y + \beta XY$ some $\beta \in Q(pt)$.

May, 67

Write up a (small) version of operations — goal = complete proofs of your assertions about operations in ΩT.

1. Key result is that you can recover all operations from ring operations —

Suppose $h : \mathcal{a}^\circ \longrightarrow \text{Mod } A$ is a tensor functor i.e. (we are) given

$$\begin{cases} A \to h1 \\ hX \otimes_A hY \longrightarrow h(X \otimes Y) \end{cases}$$

If $\exists$ A module P_t $\ni$

$$\text{Hom}_{\mathbb{Z}}(h, F \otimes_A h) \simeq \text{Hom}_A(P_t, F)$$

then P has the ~~structure~~ of an affine category.

2. Criterion that P exist: h ind representable by Künneth objects

$$h(X) = \varinjlim_i \text{Hom}_{\mathcal{a}}(X, E_i)$$

where $h(E_i)$ finitely gen. proj. A-module and

$$\begin{cases} h(E_i) \otimes_A h(X) \xrightarrow{\sim} h(E_i \otimes X) \\ A \xrightarrow{\sim} h(1) \end{cases}$$

$$\text{Hom}_{\mathbb{Z}}(h, M \otimes_A h) = \varprojlim_i M \otimes_A h(E_i)$$

$$= \text{Hom}_{A^\circ}(\varinjlim h(E_i), M)$$

3. Operations in ΩT!

Affine categories and ~~algebra~~ operations in generalized cohomology theories.

$\boxed{\text{1. Cogebras.}}$

Definition: A-dicogebra.

$$\left\{\begin{array}{l} \varepsilon : P \longrightarrow A \\ \Delta : P \longrightarrow P \otimes_A P. \end{array}\right.$$

Com(P) left comodules.

$$M \longrightarrow P \otimes_A M.$$

Example: Two structures coincide

$\boxed{\text{2. Endomorphisms of a functor } h : \mathcal{a} \longrightarrow \text{Mod } A.}$

Assume $\exists$ right A-module $P \ni$

$$\mathrm{Hom}_{A^\circ}(P, M) \xrightarrow{\sim} \mathrm{Hom}_{\underline{\mathrm{Hom}}(\mathcal{a}^\circ, \mathcal{Ab})}(h, M \otimes h)$$

Then P is an A-dicogebra and h induces

$$h' : \mathcal{a} \longrightarrow \mathrm{Com}\, P.$$

Prop: If $f : \mathrm{Com}\, P \longrightarrow \mathrm{Mod}\, A$ is the forgetful functor then $\mathrm{End}(f) = P$.

Proof: ✓

~~The identity morphism of h gives rise to a A-bimodule ~~module~~ map~~

~~$\varepsilon : P \longrightarrow A$~~

~~and the ~~map~~ composition~~

~~$h \longrightarrow P \otimes h$~~

(3.) Sufficient conditions for P to exist

Proposition: Suppose

$$h(X) = \varinjlim \mathrm{Hom}(X, E_i)$$

where $h(E_i) \in \mathcal{P}(A)$. Then $P = \varinjlim h(E_i)^{\vee}$.

(4) ~~[illegible]~~ ~~Bialgebras~~ Tensor products

A commutative

$P * P'$

$\mathrm{Com}(P) \times \mathrm{Com}(P') \longrightarrow \mathrm{Com}(P * P')$

~~[illegible]~~

(5.) Affine categories ~~+ groupoids.~~

define ⊗ algebras over A. | represent functors, antipode or inversion

$\mathrm{Com}(P) =$ a tensor category. ass. comm. unit. ⊗

Prop: $h : \mathcal{A} \longrightarrow \mathrm{Mod}\,A$ ⊗-functor, then $\underline{\mathrm{End}}^{\otimes} h$ is an affine category.

(6) Affine groupoids and existence of an antipode

~~[illegible]~~ Cohomology operations

$\mathcal{A}$ stable finite homot. cat

$h^*: \mathcal{A}^\circ \longrightarrow \text{grMod } h^*(pt)$

h gen. coh. theory with products

$$\boxed{h^*(X) = \varinjlim_i {}^*\{X, E_i\} \quad \Big| \quad \text{where } h^*(E_i) \text{ proj. f.t.}}$$

~~[illegible]~~

$$P = \varinjlim h^*(E_i)^\vee$$

moreover $\exists$ canonical map

$$h^*(X) \longrightarrow P \otimes_A h^*(X)$$

$$h^*(X) = \varinjlim_i \text{Hom}^*(X, E_i)$$

$$\longrightarrow \varinjlim_i \underset{\text{Modgr}(h^*pt)}{\text{Hom}}(h^*E_i, h^*X)$$

$$\cong \varinjlim_i h^*_*(E_i)' \otimes_{h^*(pt)} h^*(X)$$

$$P \otimes_A h^*(X).$$

Conclusion: If F is a gr $h^*(pt)$ module, then

<u>Definition</u>: Given ~~and A, P as above~~ an affine category (A, P) we say that two morphisms $A \xrightarrow{u} R$ and $v: A \longrightarrow S$ are **equivalent** if the extended affine categories $(R, R_u \otimes_A P \otimes_A R_u)$ and $(S, S_v \otimes_A P \otimes_A S_v)$ are equivalent.

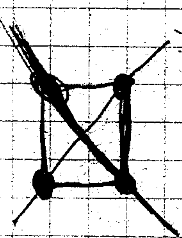
The reason for this definition is as follows. Let $h: \mathcal{Q} \longrightarrow \mathrm{Com}\, P$ be a tensor functor. Then the extended functors h_u and h_v determine each other. In effect we are given

$$f: R \longrightarrow S$$

$$f_1: R \otimes P \otimes R \longrightarrow S \otimes P \otimes S$$

$$\begin{tikzcd} C_2 \arrow[r] \arrow[dr, "f.f."'] & C_3 \arrow[d, "f.f."] \\ & C_1 \end{tikzcd}$$

The basic question is it representa $\boxed{\text{no}}$



__Lemma:__ P has unique ring structure $\ni$

$\gamma : h \longrightarrow P \otimes h$ compatible with products i.e.

$$
\begin{tikzcd}
h(X) \otimes h(Y) \arrow[r, "\gamma \otimes \gamma"] \arrow[d] & (P \otimes hX) \otimes (P \otimes hY) \arrow[d] & \\
& P \otimes P \otimes hX \otimes hY \arrow[d] \arrow[r] & P \otimes hX \otimes hY \arrow[dl, "\#"] \\
h(X \otimes Y) \arrow[r, bend right, "\gamma"] & P \otimes h(X \otimes Y) &
\end{tikzcd}
$$

commutes. Moreover P is commutative + associative with unit. Finally if R is an A,A alg, then a map of A,A modules ${}_1P_2 \longrightarrow {}_1R_2$ is a ring homomorphism iff $h \longrightarrow R \otimes h$ is compatible with tensor product.

__Proof:__ Uniqueness of ring structure. Suppose given $\mu : P * P \longrightarrow P \ni$ diag. commutes. Consider subcat gen. by E_i; then $hX \otimes hY \xrightarrow{\cong} h(X \otimes Y)$ isom so $\#$. isom so have ~~Boo~~ a transf $\underline{hX} \otimes \underline{hY} \longrightarrow \underline{P \otimes hX \otimes hY}$ which by above corresponds to the map

$$\mu_P : P * P \longrightarrow P$$

shows μ_P unique + how to define it. To show when thus defined, diagram above commutes for all X, Y. But we know it commutes if $X = E_i$ (resp $Y = E_j$) and any element of $\xi \in h(X)$ (resp $\eta \in h(Y)$) comes from $h(E_i)$ by maps $X \longrightarrow E_i$ (resp. ———)

Thus μ_P exists. To show P commutative we compare the maps

$$hX \otimes hY \longrightarrow h(X \otimes Y) \simeq h(Y \otimes X) \longrightarrow P \otimes h(Y \otimes X) \simeq P \otimes hY \otimes hX \simeq P \otimes hX \otimes hY$$

etc.

So the present attempt at proof runs as follows:

Given $\quad {}_1hX \otimes {}_1h'X' \longrightarrow {}_1F_2 \otimes {}_2hX \otimes {}_2h'X'$,

fix X whence by the universal property of γ' it can be expressed uniquely as a composition

$$ {}_1hX \otimes {}_1h'X' \xrightarrow{id \otimes \gamma'} {}_1hX \otimes {}_1P'_2 \otimes {}_2h'X' \xrightarrow{\delta^{(X)}\otimes id} {}_1F_2 \otimes {}_2hX \otimes {}_2h'X' $$

where

$$ \delta^{(X)} : {}_1hX \otimes {}_1P'_2 \longrightarrow {}_1F_2 \otimes {}_2hX $$

is a A-bimodule map, which is natural in X by uniqueness. Thus by the universal property of γ^3, $\delta(X)$ ~~has~~ can be uniquely expressed as a composition

$$ {}_1hX \otimes {}_1P'_2 \xrightarrow{\gamma \otimes id} {}_1P_2 \otimes {}_2hX \otimes {}_1P'_2 \xrightarrow[\sim]{id \otimes T} {}_1P_2 \otimes {}_1P'_2 \otimes {}_2hX \xrightarrow{\mu \otimes id} {}_1F_2 \otimes {}_2hX $$

where $\mu : {}_1P_2 \otimes {}_1P'_2 \longrightarrow {}_1F_2$ is an A-bimodule map. Putting these together we find that θ may be uniquely expressed as the composition

$$ {}_1hX \otimes {}_1h'X' \xrightarrow{\gamma''} ({}_1P_2 \otimes {}_1P'_2) \otimes {}_2hX \otimes {}_2h'X' \xrightarrow{\mu \otimes id \otimes id} {}_1F_2 \otimes {}_2hX \otimes {}_2h'X' $$

where ~~$\mu : {}_1P_2 \otimes {}_1P'_2 \to {}_1F_2$ is~~ γ'' is the composition

$$ {}_1hX \otimes {}_1h'X' \xrightarrow{\gamma \otimes \gamma'} ({}_1P_2 \otimes {}_2hX) \otimes ({}_1P'_2 \otimes {}_2h'X') \xrightarrow{id \otimes T \otimes id} ({}_1P_2 \otimes {}_1P'_2) \otimes ({}_2hX \otimes {}_2h'X') $$

Therefore $End(h \otimes h')$ is representable by ${}_1P_2 \otimes {}_1P'_2$ as claimed ~~qed~~ q.e.

According ~~to~~ prop. it has a dicogebra structure which we ~~left~~ leave

Notes on G's notes of my stuff.

1. Formal cats. | the definition,
2. ~~Gen~~ Complexes of DR in general. | def.
3. Lie algebra assoc. to DR cx. |

~~Also attention to the first integrals questions of generality ither~~

clean up

Operations: $h: \mathcal{A}^{\circ} \longrightarrow \mathrm{Mod}\, A$

assume that $\exists$ ind object $\{E_i \quad i \in I\}$ of $\mathcal{A}$ such that

$$h(X) = \varinjlim_i \mathrm{Hom}_{\mathcal{A}}(X, E_i)$$

and that $h(E_i) \in P(\mathcal{A})$.

work over a base X and establish

$$\varprojlim_{E} \Omega(P(E)) =$$

E

Lemma 1: For every A°-module F

$$\underset{\mathrm{Hom}(\mathcal{A}^{\circ}, ab)}{\mathrm{Hom}}(h, F \otimes h) = \mathrm{Hom}_{A^{\circ}}(P, F)$$

where $P = \varinjlim_i h(E_i)^{\vee}$

$h(E_i)^{\vee} = \mathrm{Hom}_A(h(E_i), A)$

Proof:

$$\underset{\mathrm{Hom}(\mathcal{A}^{\circ}, ab)}{\mathrm{Hom}}(h, F \otimes h) = \varprojlim_i \mathrm{Hom}(\mathrm{Hom}(?, E_i), F \otimes h)$$

$$= \varprojlim_i F \otimes h(E_i) = \varprojlim_i \mathrm{Hom}_{A^{\circ}}(h(E_i)^{\vee}, F)$$

$$= \mathrm{Hom}_{A^{\circ}}(P, F).$$

Question: Does $\exists$ a direct proof.

$$h : \mathcal{Q}^{\circ} \longrightarrow \mathrm{Modgr}(A)$$

$$h(X) \otimes_{A} h(Y) \longrightarrow h(X \wedge Y)$$

suppose that

$$h^*(X) = \varinjlim_{i} \mathrm{Hom}^*(X, E_i)$$

for any A-module F

$$\mathrm{Hom}^*_{\mathbb{Z}}(h^*, F \otimes h^*) = \varprojlim_{i} F \otimes h^*(E_i) \qquad \text{Yoneda}$$

$$= \varprojlim_{i} \mathrm{Hom}_A(h^*(E_i)^\vee, F)$$

$$= \mathrm{Hom}_A(P, F)$$

where $P = \varinjlim_{i} \mathrm{Hom}_A(h^*(E_i), A)$

Structure of P: Call above the right A-module structure

(i) left st.

$$\begin{tikzcd} h \arrow[r, "\gamma"] \arrow[d, "a"] & P \otimes h \arrow[d, "\lambda_a \otimes id"] \\ h \arrow[r, "\gamma"] & P \otimes h \end{tikzcd}$$

Thus P is a bimodule. Check

$$\mathrm{Hom}_A(h, B \otimes h) = \mathrm{Hom}_{A,A}(P, B)$$

(ii) $h \xrightarrow[1\otimes]{} A \otimes h \Rightarrow \varepsilon : P \longrightarrow A$

(iii) $h \xrightarrow{\gamma} P \otimes h \xrightarrow{id \otimes \gamma} P \otimes P \otimes h \Longrightarrow \Delta : P \longrightarrow P \otimes_A P$

Proposition: Let A be a commutative ring and let

$$h : \mathcal{a} \longrightarrow \mathrm{Mod}\, A$$

$$h' : \mathcal{a}' \longrightarrow \mathrm{Mod}\, A$$

be functors such that $\underline{\mathrm{End}}\, h$ (resp. $\underline{\mathrm{End}}\, h'$) is represented by the dicogebra P (resp. P'). Let

$$h \otimes h' : \mathcal{a} \times \mathcal{a}' \longrightarrow \mathrm{Mod}\, A$$

be the functor $X, X' \longmapsto hX \otimes hX'$. Then $\underline{\mathrm{End}}\,(h \otimes h')$ is ~~also~~ represented by the dicogebra $P * P'$ defined as follows

$P * P'$ as an A-bimodule is the tensor product

$$({}_1P_2) \otimes_{A \otimes A} ({}_1P_2).$$

Thus $P * P'$ is generated by elts $p \otimes p'$ $\ni$

$$\begin{cases} ap \otimes p' = p \otimes ap' \\ pa \otimes p' = p \otimes p'a \end{cases}$$

and having the universal property associated with these identities

$$\begin{cases} \varepsilon_{P * P'}(p \otimes p') = \varepsilon_{P}(p)\, \varepsilon_{P'}(p') \\ \Delta_{P * P'} \qquad {}_1P_2 \otimes_{{}_1A, A^2} P' \longrightarrow \end{cases}$$

~~Thm~~ Make this affine category act on Ω as follows.

Given a group law F over a ring R, let

$$\Omega_F = R_u\otimes_A \Omega$$

where $u: A \longrightarrow R$ sends F_{univ} into F. Given another law F' and a power series $p(X) \ni p*F = F'$, one knows there exists a unique multiplicative transformation

$$\hat{p} : \Omega \longrightarrow \Omega_{F'}$$

such that
$$\hat{p}\left(c_1^{\Omega}L\right) = (in_1 p)\left(in_2 c_1^{\Omega} L\right).$$

~~Here~~
$$R \xrightarrow{in_1} \Omega_F \xleftarrow{in_2} \Omega$$

are the inclusions. Moreover one notes that

$$\begin{tikzcd}
F_{univ} \arrow[r, mapsto] & F^{\Omega} \arrow[rr, mapsto] & & (in_1 p)*(in_2 F^{\Omega}) \\
A \arrow[r] \arrow[drr, bend right] & \Omega(pt) \arrow[r, "\hat{p}"] & \Omega_{F'}(pt) & \\
& & R \arrow[u, "in_1"] & F' \arrow[uu, mapsto]
\end{tikzcd}$$

commutes. Consequently $\hat{p}$ induces

$$\Omega_{F'} \longrightarrow \Omega$$

Let $\mathcal{A}$ = suspension category of finite complexes and let h^* be a generalized cohomology theory with products on the category of finite complexes. Let $A = h^*(pt)$. Denote by

$$h : \mathcal{A} \longrightarrow \mathrm{Modgr}(A)$$

the functor induced by $X \mapsto h(X)$. There is a natural transformation

$$h(X) \otimes_A h(Y) \longrightarrow h(X \wedge Y)$$

satisfying some (rather obvious) ~~conditions of~~ associativity unit, and commutativity conditions.

Let $\mathcal{A}$ be the suspension category of finite CW complexes and let h be a generalized cohomology theory with products on the category of finite complexes. Let A be the graded (anti-)commutative ring $h^*(pt)$. Then h may ~~be~~ viewed as an additive functor

$$h : \mathcal{A} \longrightarrow \mathrm{Modgr}(A)$$

endowed with a natural transformation

$$(\quad) \qquad h(X) \otimes_A h(Y) \longrightarrow h(X \wedge Y)$$

satisfying some rather obvious associativity, unit, and commutativity conditions.

~~If R is a commutative (anti-)commutative ring, then let $(\mathrm{End}^{\otimes} h)(R)$ be the category whose objects are~~

~~ring homomorphisms $u: A \longrightarrow R$ and where a morphism from u to v is defined to be a natural transformation θ from h_u~~

If R is a (anti-commutative graded) ring we define a category $(\mathrm{End}^{\otimes} h)(R)$ as follows. For objects we take the sets of morphisms $u: A \longrightarrow R$ of (anti-comm. graded) rings. Given such a u let

$$h_u : \mathcal{a} \longrightarrow \mathrm{Modgr}(R)$$

be the functor $X \longmapsto R_u \otimes_A h(X)$, where R_u denotes R endowed with the A-algebra structure coming via u. Observe that h_u is a tensor functor, i.e. provided with a natural transformation

$$(*) \qquad h_u(X) \otimes_R h_u(Y) \longrightarrow h_u(X \wedge Y).$$

We define a morphism in $(\mathrm{End}^{\otimes} h)(R)$ from u to v to be a stable R-linear natural transformation $\theta: h_u \longrightarrow h_v$ which is compatible with the tensor structure, i.e. the natural transformation $(*)$.

We wish to prove that the functor $\mathrm{End}^{\otimes} h$ from rings to categories is represented by a quasi-bigebra P over A. For this we need the following condition signalled by Adams in connection with generalizations of the Adams spectral sequence [ ].

(**) There is a filtered inductive system E_i in $\mathcal{A}$ such that for all X

$$h(X) = \varinjlim_{i} \pi^*(X, E_i)$$

and such that for each i, $h(E_i)$ is a finitely generated projective A-module.

<u>Theorem</u>: Suppose h is a generalized cohomology theory with products satisfying (**). Then the functor $\mathrm{End}^{\otimes} h$ is represented by a quasi-bigebra P over A. Moreover if M is an A-module, then

$$\mathrm{Homset}^*_{\mathbb{Z}}(h, M \otimes_A h) \simeq \mathrm{Hom}^*_A(P, M)$$

~~§4. Endomorphisms of a tensor functor.~~

~~Suppose that A is commutative and that~~

$$\cancel{h : \mathcal{A}^{\circ} \longrightarrow \mathrm{Modgr}(A)}$$

~~is a functor. Suppose that~~

Proof: (In outline)

Yoneda's lemma

$$\mathrm{Homst}^{*}_{\mathbb{Z}}(h, M \otimes h) = \varprojlim_{i} M \otimes_A h(E_i)$$

$$= \varprojlim_{i} \mathrm{Hom}^{*}_{A}(h(E_i)^{\vee}, M)$$

$$= \mathrm{Hom}^{*}_{A}(P, M)$$

where

$$P = \varinjlim_{i} h(E_i)^{\vee}$$

Let this A-module structure of P be called the right one. To define the left one let $a \in A^{0}$. Then $\exists !\ a^{\#} \ni$

$$\begin{array}{ccc} hX & \xrightarrow{\gamma} & P \otimes hX \\ {\scriptstyle a\cdot}\downarrow & & \downarrow{\scriptstyle a^{\#} \otimes \mathrm{id}} \\ hX & \xrightarrow{\gamma} & P \otimes hX \end{array}$$

Thus P becomes an A-bimodule. **Check that if M is an A-bimodule** then

$$\mathrm{Homst}^{*}_{A}(h, M \otimes h) = \mathrm{Hom}_{A,A}(P, M)$$

In effect $\Longleftarrow$ clear and given $\theta : hX \to M \otimes hX$ left A-linear, let $P \xrightarrow{u} M$ be the right linear map $\ni \theta = (u \otimes id)\gamma$. Then if $a \in A^{\delta}$ $(au \otimes id)\gamma x = a\theta x = \theta ax = (u \otimes id)\gamma ax = (ua \otimes id) \circ \gamma$. $\Longrightarrow$ $u(ap) = a.u(p)$.

Define product structure on P as follows: Start with

$$_1h(X \wedge Y) \xrightarrow{\gamma} {}_1P_2 \otimes_2 h(X \wedge Y).$$

Use $\boxed{A}$ Let $\mathcal{a}' \subset \mathcal{a}$ be the full subcategory containing the $\underbrace{E_i}_{\text{and suspensions}}$. Then

$$\mathrm{Homst}_{\mathbb{Z}}(h|\mathcal{a}', M \otimes h|\mathcal{a}') \cong \mathrm{Hom}_A(P, M)$$

(same argument)

$\boxed{B}$ On this subcategory

$$h(X \wedge Y) \xleftarrow{\sim} hX \otimes_A hY$$

so that we have

$$hX \otimes hY \longrightarrow$$

Claim: $\exists!\, \mu : {}_1P_2 \otimes {}_1P_2 \longrightarrow {}_1P_2 \ni$

$$\begin{tikzcd}
{}_1hX \otimes_1 hY \arrow[r, "\gamma \otimes \gamma"] \arrow[d] & ({}_1P_2 \otimes_2 hX) \otimes ({}_1P_2 \otimes_2 hX) \arrow[r, "\cong", "id \otimes T \otimes id"'] & ({}_1P_2 \otimes {}_1P_2) \otimes ({}_2hX \otimes_2 hY) \arrow[d, "\mu \otimes id"] \\
{}_1h(X \wedge Y) \arrow[r, "\gamma"] & {}_1P_2 \otimes_2 h(X \wedge Y) & {}_1P_2 \otimes ({}_2hX \otimes_2 hY) \arrow[l, "\square"']
\end{tikzcd}$$

To prove the claims we first work on the category $\mathcal{A}'$. Then the map $\square$ is an isom so we have

$$\theta : {}_1hX \otimes {}_1hY \longrightarrow {}_1P_2 \otimes {}_2hX \otimes hY$$

~~Let~~ Fix X; by ~~[illegible]~~ universal property of γ θ comes from

$${}_1hX \xrightarrow{\otimes {}_1P_2} {}_1P_2 \otimes {}_2hX$$

which also comes from

$$\mu : {}_1P_2 \otimes {}_1P_2 \longrightarrow {}_1P_2$$

~~[illegible]~~ It's clear that square // commutes if $X, Y \in \mathcal{A}'$ but holds in general since any element $x \in hX$ is induced by a map $X \longrightarrow E_i$.

So now I've shown μ exists. Next to show μ ring structure. Associativity ~~[illegible]~~: Start with

$$hX \otimes hY \otimes hZ \longrightarrow$$

~~$hX \otimes hY \otimes hZ$~~

~~$h(X \wedge Y \wedge Z)$~~

May 19, 1969

Problem: Given a formal category find an appropriate category of modules.

Example: If A a field + if $\dim_A \Omega < \infty$ then $P = \varprojlim P_n$ is left (also right) pseudo compact and we know by previous work that pseudocompact A modules with P coaction form a good category of modules. More particular if char $A = 0$ + $\Omega = \mathfrak{g}'$ of a Lie algebra then the ~~~~ correct category is pseudocompact $\mathfrak{g}$-modules. This is the dual of the locally noetherian category of $\mathfrak{g}$-modules. The Artinean objects of p.c. $\mathfrak{g}$-mods seem to be the Artin-Rees category I constructed earlier.

Other possibility: Let $\mathcal{C}$ be the topos of covariant functors from $Th(A)$ to $\underline{Sets}$. ~~~~ Then the formal category gives me a category object of $\mathcal{C}$. Observe $\mathcal{C}$ is a ringed topos with $\mathcal{O}(R) = R$; let $\mathcal{E}$ be the fiber category of $\mathcal{O}$-modules over objects of $\mathcal{C}$. Thus if $F \in Ob\, \mathcal{C}$ an $\mathcal{O}_F$-module ~~~~ is a rule associating to $\xi \in F(R)$ an R-module M_ξ such that given $R \to R'$ with $\xi \mapsto \xi' \in F(R')$ then we have a dihomomorphism $M_\xi \to M_{\xi'}$. For example if M is an A-module

$$F(R) = \operatorname{Hom}_{Th}(A, R) \qquad M_\xi = R \underset{\xi\, A}{\otimes} M$$

and these are the "constant $\mathcal{O}$-modules".

Program of research in cobordism theory.

Some basic cleaning up required in

1) Equivariant cobordism | the formal group law picture [projective bundle theorem for a compact non-abelian group

2) Real cobordism [symplectic; self conjugate; SO theory

3) Supports and cobordism theory over a bases.

hypercategories + homotopification. Introducing ~~cobordism~~ cohomology theories with values in triangulated cats.

Critical problem: Find an algebraic model for the category of U motives

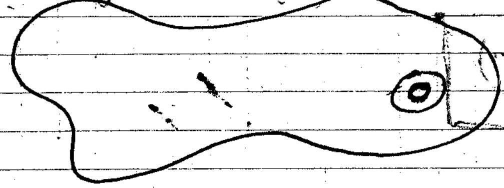
Your key somehow is K theory. You must try to understand everything about cobordism + K theory!

* holds

$P \otimes P$

It might be possible to define multiplication ~~by~~ this way. Still you have to ~~construct a map~~

Relate $\mathrm{Hom}(P, R)$ to a $\mathrm{End}(h_a, h_v)$

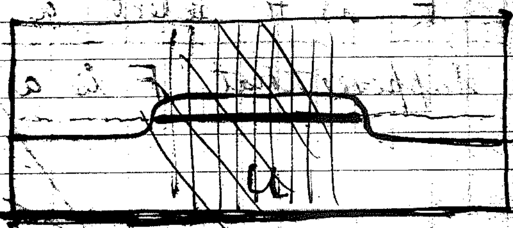


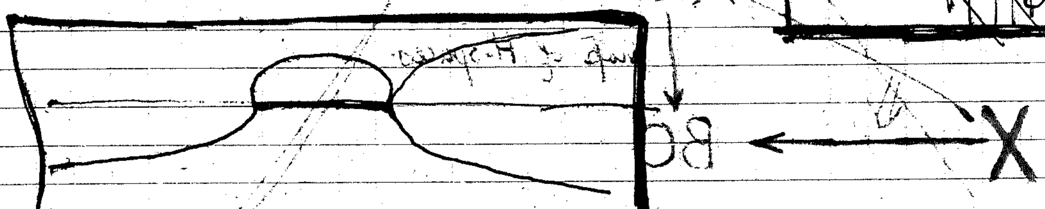
dim $F(P) = (?)$

question whether one can replace Y by something nicer.

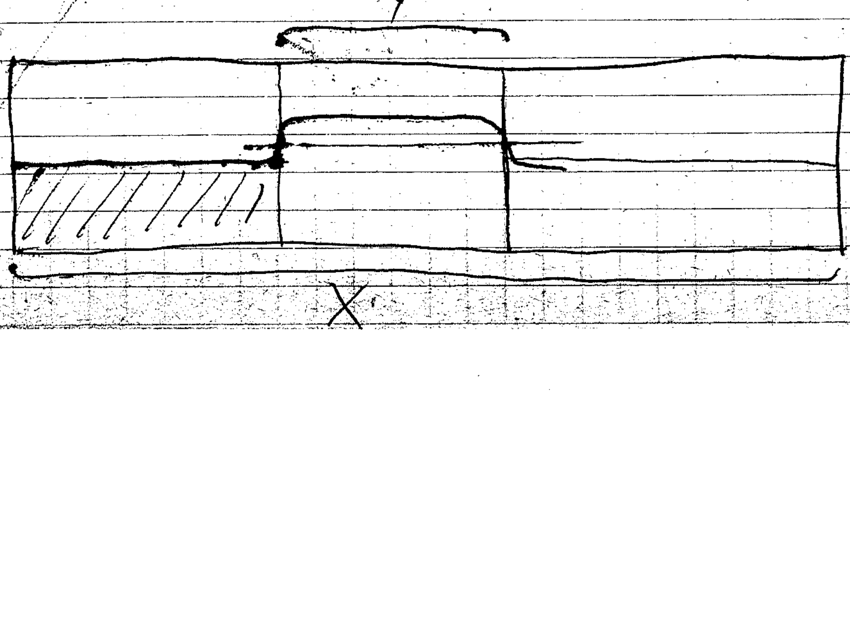
X vector bundle over Y

Thom space





$X \times \mathbb{R}$ $\quad$ $Y \times O$



g acts : $\theta_g : \Omega_G \longrightarrow \Omega_G \otimes_{\mathbb{Z}} R$ hom

also we want

$$(g_1 \cdot g_2) v = g_1 (g_2 v)$$
$$= \chi(g_2) \chi(g_1) v$$

thus $\boxed{\theta_{g_1 g_2} = \theta_{g_1} \circ \theta_{g_2}}$

so need

$$\theta_{g_1} \theta_{g_2} c_1(L) = \theta_{g_1} f_{g_2}(c_1(L))$$
$$= (\theta_{g_1} f_{g_2})(f_{g_1}(c_1(L)))$$

thus I need

$$\boxed{(\theta_{g_1} f_{g_2}) \circ f_{g_1} = f_{g_1 g_2}}$$

~~obvious candidate~~

any ideas: have to produce a ~~unique~~ can element in $F[X] / \prod_{\chi} (X - c_1(\chi))^{n_\chi}$

associated to a homomorphism $\hat{G} \longrightarrow R^*$

need something more

to each χ have $\chi(g) \in R^*$

and χ

need $f(X)$

$\hat{G}$ acts

$=$ a G action on Ω

want a G-action on Ω_G is a $\hat{G}$ grading

$$\Omega_G = \sum V_\chi$$

$$\boxed{V_\chi \cdot V_{\chi'} \subset V_{\chi\chi'}}$$

$$g(v \cdot w) \overset{?}{=} gv \cdot gw$$

$$\| $$

$$\chi(g)\chi'(g) \; v \cdot w$$

$\therefore$ must have

$$\left(\Omega_G \otimes_{\mathbb{Z}} R\right)^* \longleftarrow K_G \qquad ?$$

$$G^\wedge \longrightarrow R^*$$

$$\boxed{Pic_G \longrightarrow \Omega_G \otimes_{\mathbb{Z}} R}$$

need a map (not a hom)

$$Pic_G \longrightarrow \left(\Omega_G \otimes_{\mathbb{Z}} R\right)^*$$

$$Pic_G(X) = \varinjlim_V \,[X, \mathbb{P}\check{V}]$$

hence we need an element of

$$\varprojlim_V \, \Omega_G(\mathbb{P}\check{V}) \otimes_{\mathbb{Z}} R$$

$$= \Big[\underbrace{\left(\Omega_G(pt) \otimes_{\mathbb{Z}} R\right)}_{\Gamma}\{X\}\Big]^*$$

$$\varprojlim_n \left(\Gamma[X] \Big/ \prod_\chi (X - c_1(\chi))^{n_\chi}\right)^*$$

same as something $f(X) \in \Gamma\{X\} \ni$ $f(c_1(\chi)) \in \Gamma^*$ all χ.

grading cobordism by means of rep.

to define an action of G on Ω ie.

to each $g \in G(R)$ want a map

$$u(g) : \Omega_G \longrightarrow \Omega_G \otimes_{\mathbb{Z}} R \qquad \text{auto}$$

$\therefore$ should arise from a ~~char. class~~ ie. ~~an ele~~

a map $K_G \longrightarrow \Omega_G \otimes_{\mathbb{Z}} R$ sends sums to products

$Pic_G(X)$

first example is $E \longmapsto a^{\dim E}$

want something $Pic_G(X) \longrightarrow \Omega_G \otimes_{\mathbb{Z}} R$

G abelian

$$R(G) = \mathbb{Z}[\hat{G}]$$

thus have a hom $\underset{\overset{\|}{K_G(pt)}}{R(G)} \longrightarrow R$

$$\mathbb{Z}[\hat{G}] \longrightarrow R$$
$$\chi \longmapsto \chi(g)$$

$$\Omega_G \otimes_{\mathbb{Z}} R \longleftarrow K_G$$

Chern classes via a sphere bundle argument

$$\begin{array}{c} E \\ \downarrow \\ E-0 \xrightarrow[\pi]{} X \end{array}$$

vector bundle of dim n

~~$\pi^* E = \ldots$~~

$$\pi^* E = 1 + F$$

Given Chern classes ~~of ...~~ of F as known

$$\begin{array}{c} SE \\ \downarrow \pi \\ X \end{array}$$

one has to hope that

$$\pi^* : H(X) \xrightarrow{\sim} H(SE) \text{ in a range}$$

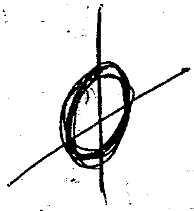
$$SE \xrightarrow{\pi} X \longrightarrow X^E$$

$$\Omega(SE) \longleftarrow \Omega(X) \longleftarrow \Omega(X^E)$$

$$\Omega(X^E) \cong \Omega(X), \quad \Omega(X) \xrightarrow{uc_n(E)} \Omega(X)$$

Conner-Floyd thm. in equivariant cobordism theory

$\Omega_G \longrightarrow \underline{K_G}$ via Atiyah ~~Hir~~ Segal.



$$1 - c_1^{\Omega}(L^\vee) \longleftarrow\!\shortmid L = (1 - c_1^K(L^\vee))$$

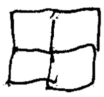
$$\boxed{K_G(pt) \otimes_{L_G} \Omega_G(X)} \longrightarrow K_G(X)$$

has universal property wrt

then L_G should be the universal thing for the equivariant group law ~~X+Y = XY~~

oughta work

What is L_G???????



$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

$g'^* \nu_f \simeq \nu_{f'}$ in strong sense $\therefore$ in weak.