

Date: On $K_2(F)$ where F is a field April 22, 1969

p prime

Theorem: $\left. \begin{array}{l} p \neq \text{char } F \quad \text{cd}_p F \leq 1 \\ \text{or } p = \text{char } F, [F:F^p] \leq p \end{array} \right\} \Rightarrow K_2(F) = pK_2(F).$

Key lemmas: Suppose E is a finite F -algebra $a = x^m$ $b = N_{E/F} y$
 $x, y \in E$, then $\{a, b\} \in mK_2(F)$. $(\{, \} : F^* \times F^* \rightarrow K_2(F) \text{ universal symbol})$

Proof: $\{a, b\} = \{a, N y\} = N\{a, y\}$ (proj. formula of Bass)
 $= N\{x^m, y\} = (N\{x, y\})^m$.

Proof of thm. $p \neq \text{char } F$, ~~more generally suppose m is prime to p~~
~~prime to $\text{char } F$. ~~Claim we may assume that $\mu_m \subseteq F$.~~~~
~~Claim we may assume $\mu_p \subseteq F$.~~ In effect if $F' = F[\mu_p]$
then $[F', F] = m$ is prime to p , F' has $\text{cd}_p \leq 1$ and by proj formula

$$K_2(E)/pK_2(F) \xrightarrow{\sim} K_2(F')/pK_2(F).$$

If $\mu_p \subseteq F$, then $E = F(a^{1/p})$ is cyclic over F ^{(of order p (if $a^{1/p} \in F$ trivial))} so by periodicity

$$F^*/N_{E/F}(E^*) \simeq H^2(E/F, E^*) = \text{Ker } \text{Br}(F) \rightarrow \text{Br}(E)$$

Thus $F^*/N_{E/F} E^* \hookrightarrow {}_p\text{Br}(F)$. But by Hilbert 90

$$H^2(F, \mu_p) = {}_p\text{Br}(F)$$

is zero as $\text{cd}_p F \leq 1$. Thus b is a norm from $F(a^{1/p})$ and so can use key lemma.

If $p = \text{char } F$ + $[F:F^p] \leq p$, then either $a \in F^p$ (trivial)

or $F^{1/p} = F[a^{1/p}]$ so that $b = y^p = Ny$ so done again.

The theorem admits an almost converse.

Examples of symbols:

$$\textcircled{1} \quad K_2(F) \longrightarrow \Omega_{F/F_0}^2 \quad F_0 \text{ prime field}$$

$$\{a, b\} \longmapsto \frac{da}{a} \wedge \frac{db}{b} = (a, b)_{\text{diff}}$$

In characteristic p , $(a, b)_{\text{diff}} = 0 \implies da, db$ dependent
 $\implies b^{1/p} \in F(a^{1/p}) \implies \{a, b\} \in pK_2(F)$. This shows that $[F: F^p] \leq 1$
 necessary for $K_2(F) = pK_2(F)$.

$\textcircled{2}$ Suppose m prime to char F . Then

$$F^*/(F^*)^m \xrightarrow[\cong]{\delta} H^1(F, \mu_m)$$

so get $(a, b)_m = \delta b \circ \delta a \in H^2(F, \mu_m^{\otimes 2})$. Tate shows
 this is a symbol as follows. First he shows that if E is
 a finite F -algebra, then

$$\begin{array}{ccc} K_2(E) & \xrightarrow{\lambda_m^E} & H^2(E, \mu_m^2) \\ \downarrow N & & \downarrow \\ K_2(F) & \xrightarrow{\lambda_m^F} & H^2(F, \mu_m^2) \end{array}$$

commutes. Then one takes $E = F[x]/(x^m - a)$ and notes that

$$1 - a = N_{E/F}(1 - x)$$

Thus $(a, 1-a)_m$ is divisible by m so is zero.

The basic conjecture

$$\lambda_m: K_2(F)/mK_2(F) \xrightarrow{?} H^2(F, \mu_m^2) \quad \text{is injective}$$

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If $\mu_m \subset F$, then

$$H^2(F, \mu_m^2) = H^2(F, \mu_m) \otimes \mu_m = {}_m\text{Br}(F) \otimes \mu_m$$

and

(Carps
Loceux) $(a, b)_m = (a, b)_z \otimes z$ z gen of μ_m .

where $(a, b)_z$ is the "cyclic" algebra $x^m = a, y^m = b, yxy^{-1} = zx$.

One knows well that

$$(a, b)_m \neq 0 \implies b \in N F(a^{1/m})^* \\ \xrightarrow[\text{lemma}]{\text{key}} \{a, b\} \in {}_m K_2(F).$$

Thus

$$\lambda_m : K_2(F)/{}_m K_2(F) \longrightarrow H^2(F, \mu_m^2)$$

is injective on decomposable elements, hence injective if $\text{Ker } \lambda_m$ generated by decomposable

Lemma (linear alg): X, Y v.s. over a field k , then any bilinear map $X \otimes Y \rightarrow k$ has this property.

Cor: λ_m injective if m is a prime $\neq \text{char } F$ and if $\dim_{\mathbb{F}_p} H^2(F, \mu_p^2) \leq 1$, more generally if $\dim_p \text{Br}(F[\zeta_p]) \leq 1$

Using this Tate proves

Thm: F local field (\mathbb{R}, \mathbb{C} , p-adic) then for all m

$$K_2(F)/{}_m K_2(F) \xrightarrow{\cong} \mu_F/{}_m \mu_F$$

~~... ..~~

Thm: $\lambda: K_2(F) \longrightarrow \hat{\mu}_F$ Hilbert symbol (surjective)

- $\text{Ker } \lambda$ divisible
- $\text{Ker } \lambda$ generated by $\{[u, u]\}$ for arb. small open U in F
- Any continuous symbol $F^* F^* \rightarrow A$, A loc. comp factors through μ_m (C. Moore).

Later remarks:

$$K_2(F) = F^* \otimes F^* / \text{subgp gen. by } [a] \otimes [1-a]$$

$$[a] = a \text{ if } a \in F^*$$

$$[a] = 0 \text{ if } a \in F - F^*$$

In effect the antisymmetry follows from

$$f \neq 1 \Rightarrow 1 = \{f^{-1}, 1-f^{-1}\} = \{f^{-1}, (f-1)f^{-1}\} = \{f, 1-f\}^{-1} \{f^{-1}, -f^{-1}\} = \{f, -f\}$$

(true also for $f=1$ by bilinearity).

whence

$$1 = \{fg, -fg\} = \text{~~... \{f, g\} \{g, f\} \{g, -g\}~~}$$

$$= \{f, -fg\} \{g, -fg\} = \{f, -f\} \{f, g\} \{g, f\} \{g, -g\}$$

$$= \{f, g\} \{g, f\}.$$

Lemma: If $T^m = f$ splits completely in $F[T]$, all f then $K_2(F)$ is uniquely divisible by m .

Proof: F^* divisible by $m \xrightarrow{!} F^* \otimes F^*$ uniquely divisible by m
 (seems to use structure theory of ^{fg}abelian groups) Rests ~~on~~ showing that

$${}_m(A \otimes B) = \text{Im } {}_m A \otimes B + A \otimes {}_m B \quad \text{true by passage to limits}$$

Thus enough to show that R gen. by $[a] \otimes [1-a]$ is divisible by m
 but if $T^m - a = \prod (T - x_i)$ then

$$[a] \otimes [1-a] = \sum [a] \otimes [1-x_i] = \sum m[x_i] \otimes [1-x_i] \in mR.$$

Corollary: F algebraically closed $\implies K_2(F)$ is a \mathbb{Q} vector space.

April 27, 1969.

On category schemes.

§1. A, A cogebras and bigebras.

1.1 Let A be a ring not necessarily commutative. By an A, A cogebras we mean an A -bimodule P endowed with maps of A -bimodules

$$P \xrightarrow{\varepsilon} A$$

$$P \xrightarrow{\Delta} P \otimes_A P$$

satisfying the counit and coassociativity identities. Here the tensor product \otimes is taken with respect to the right A -module structure on the first factor; it is endowed with the left (resp. right) A -module structure coming from the left (resp. right) A -module structure of its ~~second~~ first (resp. second) factor.

1.2. Example. Let $T: \text{Mod } A \rightarrow \text{Mod } A$ be a (co?) triple endowed with structural maps

$$\begin{cases} T \rightarrow \text{id} \\ T \rightarrow TT \end{cases}$$

If T is compatible with inductive limits, then

$$T(M) = P \otimes_A M$$

where $P = T(A)$ is an A -bimodule. Then P is an A, A cogebras. ~~We shall prove in the next section of this example that~~

1.3. By a comodule for an A, A cogebras P we mean an A -module M endowed with a map of A -modules

$$M \xrightarrow{\Delta} P \otimes M$$

such that the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\Delta} & P \otimes M \\ & \searrow \cong & \downarrow \varepsilon \otimes \text{id} \\ & & A \otimes M \end{array}$$

$$\begin{array}{ccc} M & \xrightarrow{\Delta} & P \otimes M \\ \downarrow \Delta & & \downarrow \Delta \otimes \text{id} \\ P \otimes M & \xrightarrow{\text{id} \otimes \Delta} & P \otimes P \otimes M \end{array}$$

commute. The category $\text{Com}(P)$ of P -comodules forms an additive category which is abelian if P is right A -flat.

Proposition 1.3.1: The forgetful functor

$$h: \text{Com}(P) \longrightarrow \text{Mod } A$$

has as right adjoint ~~the~~ $g(M) = P \otimes M$.

Proof: ~~Given~~ Given a P -comodule X and a map $X \xrightarrow{u} M$ of A -modules, the corresponding map of P -comodules is

$$X \xrightarrow{\Delta} P \otimes X \xrightarrow{\text{id} \otimes u} P \otimes M.$$

The triple associated to the pair

$$\begin{array}{ccc} \text{Com}(P) & \xrightleftharpoons[h]{g} & \text{Mod } A \end{array}$$

is

$$T_M = hgM = P \otimes M.$$

~~This gives P as the right adjoint of h .~~

1.4. Suppose that

$$A \xrightleftharpoons[g]{h} \text{Mod } A$$

are additive adjoint functors, \mathcal{A} being an additive category. ~~Then~~
 Suppose that hg commutes with inductive limits so that

$$hg(M) = P \otimes M$$

for an A, A cogebra P . Then ~~there~~ for any $X \in \text{Ob } \mathcal{A}$
 there ~~is an~~ adjunction map

$$hX \longrightarrow hghX = P \otimes hX$$

and hence h induces a functor

$$\mathcal{A} \xrightarrow{\tilde{h}} \text{Com}(P).$$

Using the faithfully flat descent arguments one obtains the following conditions for \tilde{h} to be an equivalence.

Theorem 1.4.1: Let ~~\mathcal{A}~~ \mathcal{A} be an abelian category
 and let

$$\mathcal{A} \begin{array}{c} \xrightarrow{h} \\ \xleftarrow{g} \end{array} \text{Mod } A$$

be adjoint functors. Assume that h is exact and faithful
 and that ~~hg~~ g commutes with inductive
 limits. Then the pair (\mathcal{A}, h) is equivalent to the category
 of P -comodules and the forgetful functor for some A, A cogebra P which
 is right A -flat.

Proof sketch: hg commutes with lim's so

$$hgM = P \otimes M$$

where P is an A, A cogebra, and we have

$$\tilde{h}: \mathcal{A} \longrightarrow \text{Com}(P).$$

Note that P is right flat since h, g are exact. \tilde{h} is fully faithful; this follows from the fact that for $X \in \text{Ob } \mathcal{A}$

$$X \longrightarrow ghX \rightrightarrows ghghX$$

is exact (apply h , it becomes contractible). \tilde{h} is essentially surjective, since if $X = \text{Ker } gM \rightrightarrows ghgM$, then $\tilde{h}X \cong M$.

1.5. Here's another way of recovering P from

$$h: \text{Com}(P) \longrightarrow \text{Mod } A$$

not using g . Let F be an A, A -module and let

$$h_F: \text{Com}(P) \longrightarrow \text{Mod } A$$

$$h_F(M) = F \otimes M$$

Then

Prop. 1.5.1:

$$\text{Hom}(h, h_F) \cong \text{Hom}_{A, A\text{-mod}}(P, F)$$

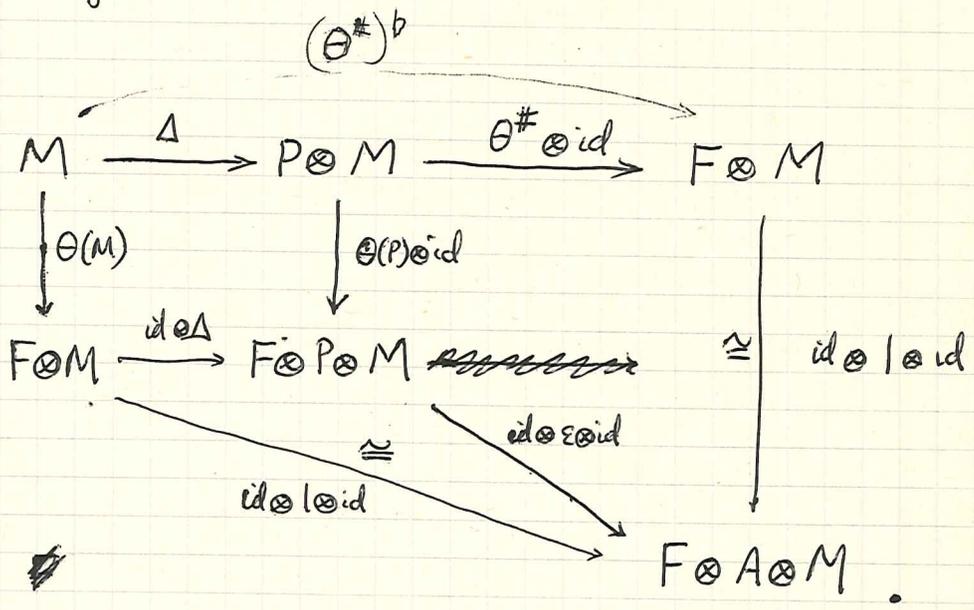
Proof: Given $\theta: h \rightarrow h_F$ define $\theta^\#$ to be the composition

$$P \xrightarrow{\theta(P)} F \otimes P \xrightarrow{\text{id} \otimes \varepsilon} F \otimes A \cong F$$

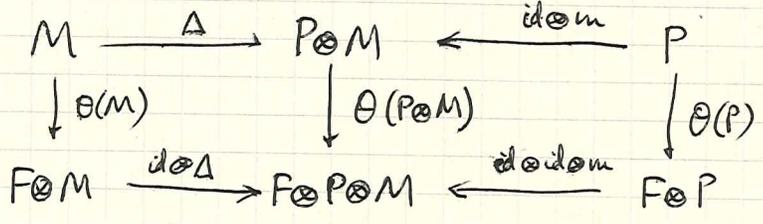
~~Map~~ This is obviously a left A -module homomorphism. It is also a right one since right multiplication by a is an endo. of P in $\text{Com}(P)$. Conversely given $\varphi: P \rightarrow F$ an A, A -hom. define $\phi(M)$ to be the composition

$$M \xrightarrow{\Delta} P \otimes M \xrightarrow{\varphi \otimes \text{id}} F \otimes M$$

Obviously $(\varphi^b)^\# = \varphi$. To show $(\theta^\#)^b = \theta$ we ~~use the diagram~~ use the diagram



Everything is clearly commutative except the upper left square. To see this one commutes note that $\Delta: M \rightarrow P \otimes M$ is a morphism in $\text{Com}(P)$ (the map $M \rightarrow ghM$) so the first square of



is commutative. For any element m of M the map $\text{id} \otimes m: P \rightarrow P \otimes M$ is a map in $\text{Com}(P)$, hence the second square commutes for all $M \in M$. Thus $\theta(P \otimes M) = \theta(P) \otimes \text{id}_M$ and so we are done.

Corresponding to the identity $h \rightarrow h = h_A$ we get from 1.5.1 the map

$$\varepsilon: P \rightarrow A$$

Δ corresponds and to the composition

$$h \xrightarrow{c} P \otimes h \xrightarrow{id \otimes \varepsilon} P \otimes P \otimes h$$

\parallel
 h_P

where $c: h \rightarrow h_P$ is the canonical map corresponding to $id_P: P \rightarrow P$.

~~Remark 1.5.2:~~

Remark 1.5.2: So given a category \mathcal{A} (not nec. additive) and a functor

$$h: \mathcal{A} \rightarrow \text{mod } A$$

such that

$$F \text{ mod } \rightarrow \text{Hom}(h, h_F)$$

is representable, we obtain a A, A coalgebra. Conversely any P is obtained in this way with $\mathcal{A} = \text{Com } P$ and $h = \text{forgetful functor}$.

One can next ask when

$$\tilde{h}: \mathcal{A} \rightarrow \text{Com}(P)$$

is an equivalence. Necessary and sufficient conditions are due to Beck for an arbitrary triples. Sufficient practical conditions are given by theorem 1.4.1.

1.6. Let $P(A)$ be the category of projective A -modules of finite type, let \mathcal{A} be an additive category and let

$$h: \mathcal{A} \rightarrow P(A)$$

be an additive functor. We suppose that the functor $X \mapsto \text{Hom}(hX, A)$ is ind-representable. Then in fact there are adjoint functors

$$\text{Ind } \mathcal{A} \begin{matrix} \xleftarrow{h} \\ \xrightarrow{g} \end{matrix} \text{Ind } P(A) \stackrel{\text{by Lazard}}{=} \text{Flat}(A)$$

In effect the category $\text{Ind } \mathcal{A}$ is closed under direct sums, retracts, and filtered inductive limits, hence ~~the category of M in $\text{Flat}(A)$ for which $X \mapsto \text{Hom}(hX, M)$ is ind-representable is closed under direct sums, retracts, and filtered inductive limits, so ~~is~~ is also of $\text{Flat}(A)$.~~ Moreover

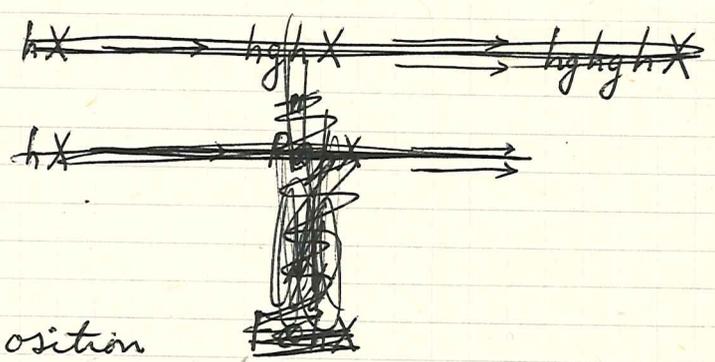
$$(1.6.1) \quad hgM \xrightarrow{\sim} hgA \otimes M$$

~~is~~ for the same reason. (One sees that g is compatible with filtered ~~is~~ inductive limits since for all $X \in \text{ob } \mathcal{A}$

$$\begin{aligned} \left(\varinjlim_i g M_i \right)(X) &= \varinjlim_i \text{Hom}(hX, M_i) = \text{Hom}(hX, \varinjlim_i M_i) \\ &= g(\varinjlim_i M_i)(X). \end{aligned}$$

Proposition 1.6.1: Let $h: \mathcal{A} \rightarrow P(A)$ be an additive functor such that $X \mapsto \text{Hom}(hX, A)$ is ind-representable,

Let $P = hg(A)$. Then P is an A, A comodule ~~which~~ which is flat as a left A -module. Moreover by 1.6.1 P is an A, A coalgebra in a natural way. Finally I claim that the formula 1.5.1 holds. In effect given $\varphi: P \rightarrow F$ one defines $\varphi^b: h \rightarrow h_F$ ~~as follows. But we note that it suffices to define φ^b on hX for X of the form $P \otimes hX$. In effect there is a natural~~



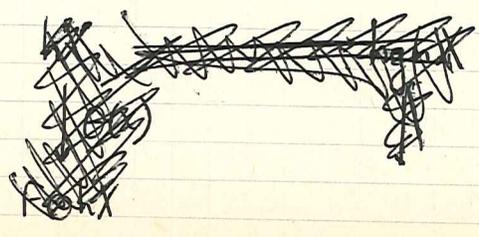
to be the composition

$$hX \longrightarrow hghX \simeq P \otimes hX \xrightarrow{\varphi \otimes \text{id}} F \otimes hX.$$

Given $\theta: h \rightarrow h_F$ ~~it~~ it extends to the ind categories so we can define $\theta^\#$ to be the composite

$$P = hg(A) \xrightarrow{\theta(gA)} F \otimes hg(A) \xrightarrow{\text{id} \otimes \epsilon} F \otimes A \simeq F.$$

Again $(\varphi^b)^\# = \varphi$ is obvious. To show $(\theta^\#)^b = \theta$ we follow the proof of 1.5.1, the key point being to show that



$$\theta(hghX) \text{ and } \theta(gA) \otimes \text{id} : P \otimes hX \longrightarrow F \otimes P \otimes hX \text{ coincide,}$$

or more precisely that

$$\begin{array}{ccc} hgA \otimes hX & \xrightarrow{\cong} & hghX \\ \downarrow \theta(gA) \otimes \text{id} & & \downarrow \theta(g hX) \\ F \otimes hgA \otimes hX & \xrightarrow{\cong} & F \otimes hghX \end{array}$$

commutes. ~~Again the proof~~ We may replace hX by any M in $\text{Flat}(A)$ and again fix an element $m \in M$ and use the same argument as before. Thus we have proved

Proposition 1.62: Let $h: A \rightarrow \mathcal{P}(A)$ be an additive functor such that $X \mapsto \text{Hom}(hX, A)$ is ind-representable, say

$\text{Hom}(hX, A) = \varinjlim \text{Hom}(X, E_i)$
 $P = \varinjlim h(E_i)$ is an A, A cogebra and i
 Then for all A, A modules F we have a canonical isomorphism

$$\text{Hom}(h, h_F) \cong \text{Hom}_{A, A \text{ mod}}(P, F).$$

In particular P is an A, A -cogebra in a natural way.

Of course h induces a functor $\tilde{h}: A \rightarrow \text{Com}(P) \cap \mathcal{P}(A)$. We would like to know when \tilde{h} is an equivalence of categories, ~~Unfortunately~~ but to carry out the descent argument of 1.4 I need ~~the hypotheses of 1.4.1~~ the hypotheses of 1.4.1. This forces me to assume A is a field.

Theorem:

1.7. Let A be a field, ~~and~~ let \mathcal{A} be an abelian category, and let

$$h: \mathcal{A} \longrightarrow \text{Mod}(A)$$

be a faithful exact functor. Then the pair (\mathcal{A}, h) is equivalent to the pair $(\text{Com}(P) \cap \text{Mod}(A), \text{forget})$ where P is the A, A coalgebra given by

$$(1.7.1) \quad \text{Hom}_{A, A \text{ mod}}(P, F) = \text{Hom}(h, h_F).$$

Moreover the category of such functors h is equivalent to the category of A, A coalgebras.

Proof: ~~Every object of \mathcal{A} is of finite length, hence every object of $\text{Ind } \mathcal{A}$ is strictly ind-representable.~~ In fact $\text{Ind } \mathcal{A}$ is the locally noetherian category ~~associated~~ associated to \mathcal{A} by Gabriel. As h is exact and \mathcal{A} is abelian $X \mapsto \text{Hom}(hX, V)$ is left exact, hence ind-representable. Thus we are in the situation of 1.6 and we have adjoint functors

$$\text{Ind } \mathcal{A} \begin{array}{c} \xrightarrow{h} \\ \xleftarrow{g} \end{array} \text{Mod}(A)$$

$$hgV \simeq hgA \otimes V$$

where $hgA = P$ is an A, A coalgebra. ~~Note~~ ^{Claim} (the extended) that h is exact and faithful; as h is a left adjoint it suffices to show h transforms a non-zero injection into a non-zero injection. This follows from the fact that every ind-representable functor is strictly ind-representable.

By 1.4.1

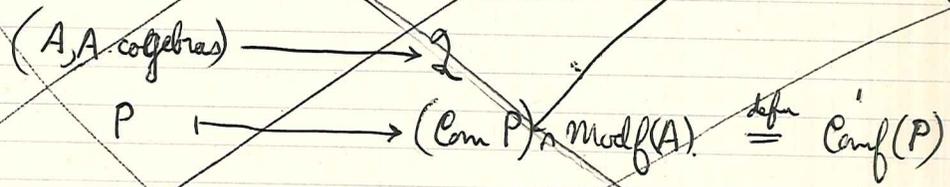
$$h: \text{Ind } A \longrightarrow \text{Com}(P)$$

is an equivalence of categories. Thus

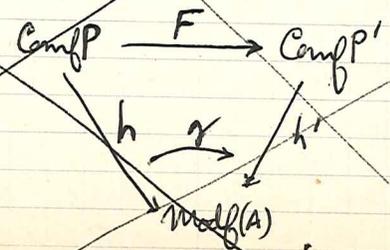
$$\tilde{h}: \mathcal{A} \longrightarrow \text{Com}(P) \cap \text{Modf}(A)$$

is fully faithful. The small \tilde{h} is essentially surjective since a finite dimensional P -module is ~~obtained~~ as a noetherian object of $\text{Com}(P)$ hence ~~exists~~ corresponds to a noetherian object of $\text{Ind } A$, which by Gabriel is isomorphic to an object of \mathcal{A} . Thus the small \tilde{h} is an equivalence of categories. By 1.6.2 P is the A, A -cogebra satisfying 1.7.1, so the first assertion of the theorem is proved.

~~For the second assertion we must make precise the category \mathcal{Q} whose objects are the faithful exact h with target $\text{Modf}(A)$. Thus we define a morphism from $h: \mathcal{A} \rightarrow \text{Modf}(A)$ to $h': \mathcal{A}' \rightarrow \text{Modf}(A')$ to be a pair consisting of an additive functor $F: \mathcal{A} \rightarrow \mathcal{A}'$ together with an isomorphism $h' \circ F \cong h$ modulo the isomorphisms of F with other functors. Then we have a functor~~



Suppose given



Modulo ~~the isomorphisms~~ an isomorphism of

For the second assertion we must know that for any A, A cogeбра P there are enough objects in $\text{Com}(P) \cap \text{mod}(A) \stackrel{\text{def}}{=} \text{Comf}(P)$. Now in fact $\text{Comf}(P)$ is the ^(full subcategory of) Noetherian objects of $\text{Com}(P)$ and $\text{Com}(P) \cong \text{Ind Comf}(P)$ in virtue of the following

Lemma ^(1.7.2): Any P -comodule M is the union of its subcomodules which are of finite type as A -modules.

Proof: Suppose that e_i is a basis for P as a right A -module and let ~~$\varphi_i: M \rightarrow M$~~ $\varphi_i: M \rightarrow M$ be defined by

$$\Delta m = \sum e_i \otimes \varphi_i(m) \in A \otimes M$$

~~Then~~ if $a \in A$ and $s(a)e_i = \sum_j e_j t(a_{ij})$

where s (resp. t) refers to the left (resp. right) A -module structure of P , then

$$\Delta(am) = \sum_{i,j} e_j \otimes a_{ij} \varphi_i(m).$$

This shows that $\varphi_i(am)$ is an A -linear combination of the $\varphi_i(m)$. Thus if V is a finite dimensional subspace of M and $V = \sum A v_k \quad 1 \leq k \leq m$, then

$$\bar{V} = \sum_i A \varphi_i(V) = \sum_{k,i} A \varphi_i(v_k)$$

is also finite dimensional. But \bar{V} is a subcomodule of M , indeed writing

$$\Delta e_i = \sum_j e_j \otimes e_k t(a_{jk}^i)$$

we have that

$$\sum_i e_i \otimes \Delta \varphi_i(\sigma_k) = (id \otimes \Delta) \Delta \sigma_k = (\Delta \otimes id) \Delta \sigma_k = \sum_j e_j \otimes e_k \otimes \sum_{i,j,k} \varphi_i(\sigma_k)$$

showing that $\Delta \varphi_i(\sigma_k) \in P \otimes V$. QED for the lemma.

We now make precise the category of faithful exact functors $h: \mathcal{A} \rightarrow \text{Mod}(A)$. ^{used in thm 1.7} ~~Start with~~ Start with the 2-category whose objects are such pairs (\mathcal{A}, h) , whose 1-morphisms ~~from~~ ^{from} (\mathcal{A}, h) to (\mathcal{A}', h') are pairs (F, u) where $F: \mathcal{A} \rightarrow \mathcal{A}'$ is an additive functor and u is an isomorphism $h \rightarrow h' \circ F$, and whose 2-morphisms from (F, u) to (F_1, u_1) are isomorphisms of functors $\Theta: F \rightarrow F_1$ such that

$$\begin{array}{ccc} h'F(x) & \xrightarrow{h(\Theta(x))} & h'F_1(x) \\ \downarrow u & & \downarrow u_1 \\ & & h(x) \end{array}$$

commutes for all $x \in \text{Ob } \mathcal{A}$. The category with objects (\mathcal{A}, h) and isomorphism classes of 1-maps (F, u) is the category \mathcal{C} we have in mind. Now we have a functor

$$\begin{array}{ccc} (\mathbb{A}\mathbb{A} \text{ algebras}) & \longrightarrow & \mathcal{C} \\ P & \longmapsto & \text{Comf}(P) \end{array}$$

It's pretty clear that what we have done above shows this functor is an equivalence of categories. QED theorem.

1.8. ^(from now on) Suppose that A is a commutative ring. If P and Q are A, A cogbras, we let $P * Q$ be the A, A -cogbra

$$P * Q = P \otimes_{A \otimes A} Q$$

$$P \otimes_{A \otimes A} Q \xrightarrow{\Delta * \Delta} (P \otimes_A P) \otimes_{A \otimes A} (Q \otimes_A Q)$$

$$P \otimes_{A \otimes A} Q \xrightarrow{\epsilon * \epsilon} A \otimes_{A \otimes A} A = A$$

$$\downarrow$$

$$(P \otimes_{A \otimes A} Q) \otimes_A (P \otimes_{A \otimes A} Q)$$

(The upper is ~~not~~ universal for multilinear maps $\Phi: P \times P \times Q \times Q \rightarrow A$ such that

~~$$\Phi(a, b, c, d) = \Phi(a, c, b, d)$$

$$\Phi(a, b, d, c) = \Phi(a, b, c, d)$$

$$\Phi(b, a, c, d) = \Phi(a, b, c, d)$$~~

$$\Phi(xa, y, z, w) = \Phi(x, ay, z, w)$$

$$\Phi(x, y, za, w) = \Phi(x, y, z, aw)$$

$$\Phi(ax, y, z, w) = \Phi(x, y, az, w)$$

$$\Phi(x, ya, z, w) = \Phi(x, y, z, wa)$$

The lower satisfies the additional conditions

$$\Phi(xa, y, z, w) = \Phi(x, y, za, w)$$

$$\Phi(x, ay, z, w) = \Phi(x, y, z, aw)$$

which are equivalent granted the others. This accounts for the fact that there are 4 tensors over A in the upper and 5 in the lower)

Suppose that

$$h: A \longrightarrow \text{Mod } A$$

$$h_1: A_1 \longrightarrow \text{Mod } A$$

(not nec. additive)

are functors whose endos. are represented by P and Q .

Prop. 1.8.1: Let $h \otimes h_1: A \otimes A_1 \longrightarrow \text{Mod } A$ be the functor
 $X, Y \longmapsto hX \otimes h_1Y$

Then

$$\text{Hom}(h \otimes h_1, (h \otimes h_1)_F) \cong \text{Hom}_{A, A \text{ mod}}(P \times Q, F)$$

sketch

Proof: suppose given $\theta: h \otimes h_1 \rightarrow F \otimes (h \otimes h_1)$. Fixing $x \in hX$ we get a transformation θ_x

$$h_1Y \xrightarrow{x \otimes ?} hX \otimes h_1Y \xrightarrow{\theta} F \otimes hX \otimes h_1Y$$

hence a map

$$\theta_x^\# : P_1 \longrightarrow F \otimes hX$$

~~we obtain~~ Fixing an element $z \in P_1$, we obtain $x \longmapsto \theta_x^\#(z)$, which is a natural transf.

$$: hX \longrightarrow F \otimes hX$$

which is A -linear since $\theta_{ax} = a\theta_x$. Hence we get a map

$$\Phi : P \times P_1 \longrightarrow F.$$

One checks that $\Phi(az, w) = a\Phi(z, w) = \Phi(z, aw)$ and similarly on the right, ~~hence~~ hence we get a map

$$\Phi : P \times P_1 \longrightarrow F$$

of A, A -modules. It remains to check that the composition of functors

induces the cogebra structure on $P * P$, describe in 1.8, but this (sauf erreur) offers no difficulties.

1.8.2. Suppose now $h: \mathcal{A} \rightarrow \text{Mod } A$ is a functor and that \mathcal{A} is endowed with an operation \otimes and a compatibility with h . Thus we have

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{A} & \xrightarrow{\otimes} & \mathcal{A} \\ \searrow h \circ h & \xrightarrow{\mu} & \swarrow h \\ & \text{Mod } A & \end{array}$$

$$\mu: h(X \otimes Y) \simeq hX \otimes hY$$

If $\text{End } h \xrightarrow{\text{is}}$ represented by a cogebra P , then by 1.8.1 $\text{End } h \circ h$ is represented by $P * P$, so the pair (\otimes, μ) gives rise to a cogebra map

$$\mu: P * P \longrightarrow P.$$

If \mathcal{A} is provided with an object 1 and ~~isomorphism~~ as unit isom. $\nu: 1 \otimes X \simeq X$ and if h is provided with a compatibility of this ~~isomorphism~~ structure, i.e.

$$h1 \simeq A$$

$$h(1 \otimes X) \simeq h(X)$$

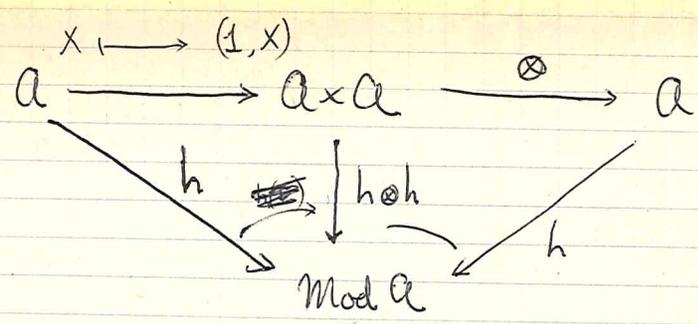
$$\downarrow$$

$$\downarrow$$

$$h1 \otimes hX \simeq A \otimes h(X)$$

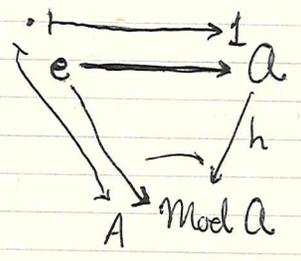
commutes,

then we have ~~isomorphism~~ a composition morphism



(This is $X \mapsto 1 \otimes X$ + isomorphism ~~isomorphism~~)
 $h(1 \otimes X) \simeq h1 \otimes hX \simeq A \otimes hX \simeq hX$)

~~isomorphism~~ which is isomorphic via $\nu: 1 \otimes X \simeq X$ to the identity of (A, h) . The ~~pair~~ pair consisting of 1 and $h1 \simeq A$ can be viewed as a map



hence gives a coalgebra map $A \xrightarrow{\eta} P$. The isomorphism of functors ~~tells us that~~ ^(just described) tells us that

$$P \xrightarrow{\cong} A * P \xrightarrow{\eta \circ \text{id}} P * P \xrightarrow{\mu} P$$

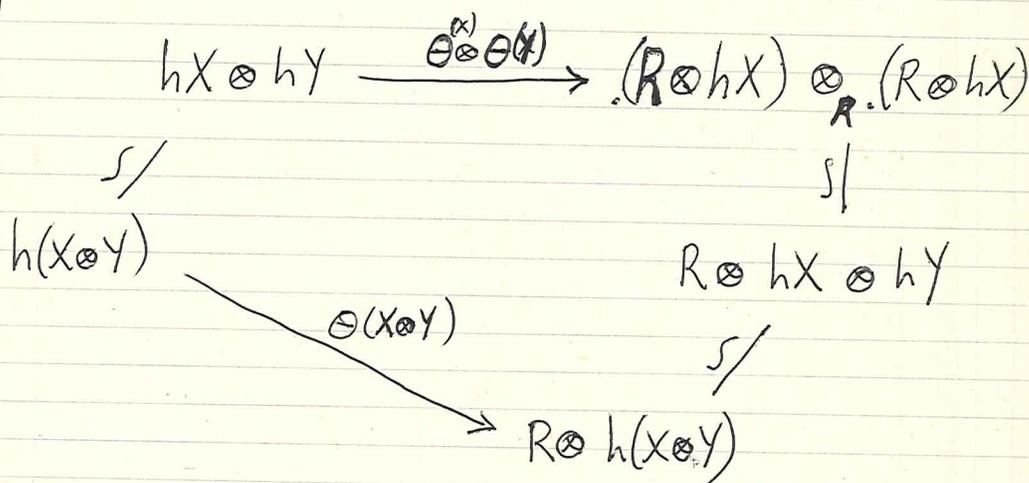
is the identity of P , hence η is a ^{left} unit for P .

Therefore we see that if A is provided with a tensor operation + unit and if h is provided with a compatibility with this structure, then P ~~has a unit.~~ ^{has a unit.} (In a similar manner the existence of (resp. commutativity) ~~isomorphism~~ ^{an associativity isomorphism} for \otimes which is ~~compatible~~ compatible with h will imply that P is associative and commutative as ~~is a~~ a ring.)

Proposition 1.8.3: Let $h: \mathcal{A} \rightarrow \text{Mod } A$ be a functor such that $\text{End } h$ is representable by an A, A -cogebra P . Suppose \mathcal{A} endowed with a unitary associative and commutative tensor product operation ~~with~~ and that h is endowed with a compatibility with this tensor product ^{respect to}; hence P is an A, A -bigebras (commutative). Then for any A, A algebra R

$$\text{Hom}^\otimes(h, h_R) \cong \text{Hom}_{A, A\text{-alg}}(P, R).$$

Proof: Given $\theta: h \rightarrow h_R$, θ is a \otimes functor iff $\forall X, Y$ in \mathcal{A} the diagram



commutes. The upper path from $h \otimes h$ to $R \otimes h \otimes h$ is represented by the composition

$$P * P \xrightarrow{\theta^\# * \theta^\#} R \otimes_{A \otimes A} R \xrightarrow{\mu_R} R.$$

The lower path is represented by

$$P * P \xrightarrow{\mu_P} P \xrightarrow{\theta^\#} R.$$

Thus θ is a tensor functor iff $\theta^\#$ is a ring homomorphism.

Corollary 1.8.4: Let A, h be as in 1.8.3 and let $\text{End}^{\otimes} h$ be the covariant functor from (rings) to Cat given by

$$\text{Ob}(\text{End}^{\otimes} h)(R) = \text{Hom}_{\text{(rings)}}(A, R)$$

$$\text{Hom}_{(\text{End}^{\otimes} h)(R)}(u, v) = \text{Hom}^{\otimes}(h_u, h_v)$$

where if $u: A \rightarrow R$ is a ring homomorphism, then

$$h_u: A \rightarrow \text{Mod } R$$

$$X \mapsto R \otimes_u hX$$

Then $\text{End}^{\otimes} h$ is represented by the co-category objects in rings

$$A \begin{array}{c} \xleftarrow{\varepsilon} \\ \xrightarrow{\text{left}} \\ \xrightarrow{\text{right}} \end{array} P, \quad \Delta: P \rightarrow P \otimes_A P$$

Proof: It suffices to note that a ~~natural~~ natural transformation $h_u \rightarrow h_v$ is the same as one $h \rightarrow R \otimes h$ where R is an A, A -algebra via u, v . Hence by 1.8.3 $\text{Hom}^{\otimes}(h_u, h_v) \cong \text{Hom}_{A, A \text{ alg}}(P, {}_u R_v)$. The rest is checking.

Combining 1.8.4 and ~~1.4.1~~ 1.4.1 we have

Corollary 1.8.5: Let \mathcal{A} be an abelian ^{unitary, associative and comm.} category with tensor product and let $h: \mathcal{A} \rightarrow \text{Mod } A$ be an exact faithful functor ~~compatible~~ compatible with the tensor product. Assume that h has a right adjoint g which commutes with inductive limits.

Then (\mathcal{A}, h) is equivalent to the \otimes -category of P -comodules and the forgetful functor, where P is the A - A bigebra ~~representing~~ representing $\text{End}^{\otimes} h$. Every P which is flat as a right A -module occurs in this way.

Combining 1.8.4 and 1.6.2 we have

Defn: \otimes -category = an additive category endowed with a unitary, assoc., comm. tensor product.

Corollary 1.8.6: If $h: \mathcal{A} \rightarrow P(A)$ is a \otimes -functor where \mathcal{A} is a ~~category~~ \otimes -category and $X \mapsto \text{Hom}_A(hX, A)$ is ind-representable, ~~say~~ say by $\{E_i\}$, then $\text{End}^{\otimes} h$ is represented by an ~~an~~ A, A -bigebra P , given ~~as~~ as a left A -module by

$$P = \varinjlim_i h(E_i).$$

Finally combining 1.8.4 and 1.7 we have

Corollary 1.8.7: Let A be a field, ~~let~~ let \mathcal{A} be an abelian \otimes -category and let $h: \mathcal{A} \rightarrow \text{Mod}(A)$ be a faithful exact tensor functor. Then \mathcal{A} ^(with h) is equivalent to the \otimes -category $\text{Com}(P) \cap \text{Mod} A$, ^(with the forgetful functor) where P is the A, A -bigebra representing $\text{End}^{\otimes} h$. Conversely if P is an A, A -bigebra, then P ~~represents~~ represents $\text{End}^{\otimes} h$ where h is the forgetful functor on $\text{Com}(P) \cap \text{Mod}(A)$.

1.9. In the situation of 1.8.6 we have the following criterion for $\underline{\text{End}}^{\circ}h$ to be a groupoid scheme.

Proposition 1.9.1: Let $h: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$ be as in 1.8.6. Suppose that every object X of \mathcal{A} has a dual DX provided with maps

$$(1.9.2) \quad 1 \xrightarrow{\Phi} DX \otimes X \xrightarrow{\Phi} 1$$

inducing isomorphisms

$$\text{Hom}_{\mathcal{A}}(Z, DX \otimes Y) \cong \text{Hom}_{\mathcal{A}}(X \otimes Z, Y)$$

for all Y, Z in \mathcal{A} . Then $\underline{\text{End}}^{\circ}h$ is an affine groupoid scheme.

Proof: Given $u, v: A \rightrightarrows R$ and $\theta: h_u \rightarrow h_v$, we must show that θ is an isomorphism. By assumption

$$X \cong X \otimes 1 \xrightarrow{id \otimes \Phi} X \otimes DX \otimes X \xrightarrow{\Phi \otimes id} 1 \otimes X \cong X$$

is the identity. Applying h we have that

$$hX \xrightarrow{id \otimes h(\Phi)(1)} hX \otimes hDX \otimes hX \xrightarrow{h\Phi \otimes id} hX$$

is the identity. Thus if

$$h(\Phi)(1) = \sum_{i=1}^n \lambda_i \otimes \sigma_i \quad \begin{matrix} \sigma_i \in hX \\ \lambda_i \in hDX \end{matrix}$$

and if $\langle \sigma, \lambda \rangle = (h\Phi)(\sigma \otimes \lambda) \quad \sigma \in hX, \lambda \in hDX$

then

$$\sigma = \sum_{i=1}^n \langle \sigma, \lambda_i \rangle \sigma_i \quad \text{for all } \sigma \in hX.$$

Similarly

$$DX \cong 1 \otimes DX \xrightarrow{\mathbb{I} \otimes \text{id}} DX \otimes X \otimes DX \xrightarrow{\text{id} \otimes \mathbb{I}} DX \otimes 1 \cong DX$$

is the identity and we find that

$$\lambda = \sum_i \lambda_i \langle v_i, \lambda \rangle \quad \text{for all } \lambda \in hDX.$$

Thus we see that

$$\langle , \rangle : hX \otimes hDX \longrightarrow A$$

is a perfect duality in $\mathcal{P}(A)$ and that $h(\mathbb{I})1$ ^(corresponds to) is the identity ~~transformation~~ transformation under the isomorphism $hX \otimes (hX^\vee) \cong \text{Hom}(hX, hX)$.

Now apply θ to the maps \mathbb{I}, \mathbb{I}

$$\begin{array}{ccc} R & \longrightarrow & h_u X \otimes h_u DX & \longrightarrow & R \\ & \searrow & \downarrow \theta(X) \otimes \theta(DX) & \nearrow & \\ & & h_v X \otimes h_v(DX) & & \end{array}$$

observing that the ~~int~~ above paragraph holds with h replaced by h_u and h_v . ~~Thus we have the following situation~~ Thus denoting $h_u X$ by V and $h_v X$ by W , we have a map $\theta(X) = \varphi : V \rightarrow W$ in $\mathcal{P}(R)$ and a map $\psi : V^\vee \rightarrow W^\vee$ such that

$$\begin{array}{ccccc} & & \text{id} & \longrightarrow & V \otimes V^\vee & \xrightarrow{\text{ev}} & R \\ R & & & & \downarrow \varphi \otimes \psi & & \\ & & \text{id} & \longrightarrow & W \otimes W^\vee & \xrightarrow{\text{ev}} & R \end{array}$$

commutes. The second triangle shows that

$$\langle \varphi\sigma, \psi\lambda \rangle = \langle \sigma, \lambda \rangle$$

for all $\sigma \in V, \lambda \in V^\vee$ hence that $\psi^t \varphi = \text{id}_V$. The first triangle shows that

$$\sum_i \varphi e_i \otimes \psi \lambda_i \quad \text{where } \text{id}_V = \sum e_i \otimes \lambda_i$$

is the identity transformation of W , i.e. for all $w \in W, \mu \in W^\vee$

$$\begin{aligned} \langle w, \mu \rangle &= \sum \langle \varphi e_i, \mu \rangle \langle w, \psi \lambda_i \rangle \\ &= \sum \langle e_i, \varphi^t \mu \rangle \langle \psi^t w, \lambda_i \rangle \\ &= \langle \psi^t w, \varphi^t \mu \rangle, \end{aligned}$$

that is, that $\varphi \psi^t = \text{id}_W$. Thus $\varphi = \Theta(X)$ is an isomorphism. QED.

Remarks. 1. The first ^{of the proof} part shows that when ~~every~~ every object of \mathcal{A} has a dual (even though \mathcal{A} isn't additive), then $hX \in P(\mathcal{A})$ and $h(\Theta X) = (hX)^\vee$ for ~~any~~ ^{any} functor $h: \mathcal{A} \rightarrow \text{Mod } A$ compatible with \otimes .

2. If P is ~~an~~ ^{with object ring} an A, A bialgebra with antipode, that is, P is an affine groupoid scheme ~~over~~ A , then define

$$\text{Hom}_A(M, N) \longrightarrow \text{Hom}_A(M, P \otimes N)$$

for two P -comodules M, N ~~by sending f into the composition~~ by sending f into the composition

$$M \xrightarrow{\Delta} P \otimes M \xrightarrow{\text{id} \otimes f} P \otimes N \xrightarrow{\tau \otimes \Delta} (P) \otimes P \otimes N \xrightarrow{\mu \otimes \text{id}} P \otimes N.$$

If $M \in \text{Ob } P(\mathcal{A})$, then this makes $\text{Hom}(M, N)$ into a P -comodule undoubtedly enjoying the properties of an internal Hom. It is

clear (sauf erreur) that ~~then~~ for $M \in \mathcal{P}(A) \cap \text{Com}(P)$ its dual is
 $M^\vee = \text{Hom}_A(M, A)$ with the comodule structure just described.

Comments on rewriting:

Let \mathcal{Q} be the category with objects functors $h: A \rightarrow \text{Mod } A$ and where a morphism $(a, h) \rightarrow (a', h')$ is a pair $F: a \rightarrow a'$ $u: h \cong h'F$ modulo isomorphisms of F with an F' . Then we have

$$\begin{aligned} (A, A \text{ Cog}) &\longrightarrow \mathcal{Q} \\ P &\longmapsto \text{Com } P \end{aligned}$$

Conversely given h suppose $\exists A, A$ module $P \triangleright$

$$\text{Hom}_{A, A \text{ mod}}(P, F) = \text{Hom}(h, h_F)$$

Then show P is an A, A cog and

$$\text{Hom}_2(A, \text{Com}(Q)) = \text{Hom}_{\text{Cog}}(P, Q)$$

|| always

$$\{ \eta \in \text{Hom}(h, h_a) \mid (\text{id} \otimes \eta)\eta = (\Delta \otimes \text{id})\eta \}$$

for tensor products it seems desirable to prove the stronger statement that $\text{Hom}_{A, A \text{ mods.}}(G \otimes P, F) = \text{Hom}(h_G, h_F)$

Now if A commutative and if $\text{End } h \text{ on } a = P$
 $\text{End } h' \text{ on } a' = P'$, then

$\text{End } h \otimes h' \text{ on } a \otimes a'$ is $P * P'$,

the canonical map $h \otimes h' \rightarrow \text{~~h \otimes h'~~} (h \otimes h')_{P * P'}$ being

$$: hX \otimes h'Y \rightarrow (P \otimes hX) \otimes (P' \otimes h'Y) \cong (P \otimes P') \otimes_{A \otimes A} (hX \otimes h'Y)$$

$$\downarrow \\ (P * P') \otimes_A (hX \otimes h'Y)$$

Another method of proving 1.9.1 is as follows:

Given $h: A \rightarrow P(A)$ we can also consider

$$h^\circ: A^\circ \rightarrow P(A), \quad h^\circ(X) = h(X)'. \quad \text{Then if}$$

P represents $\text{End } h$, one shows that P but with left and right A -module structures reversed represents $\text{End } h^\circ$.

Hence if one has a duality functor

$$D: A^\circ \rightarrow A^*$$

with $h(DX) = h(X)'$, then one gets a map $\tau: P \rightarrow P$ reversing left and right A -modules structures.

Concerning 1.7.2 one can prove that P is the union of its lattice of left finite dimensional sub A, A -cogebrae. Indeed we know already that P is the union of its finite dimensional submodules. Pick one V . Then V' is a ^{right} module of over the ring $Q = \text{Hom}_{A^\circ}(P, A)$ with product as in 2.2.6. The annihilator α of V' is an ideal in Q of finite ~~left~~ ^{right} codimension. ~~Then the annihilator in P is a subcogebra of finite right dimension~~ Since V is a submodule of P , V' is a quotient Q° module of Q , hence V' is a quotient of Q/α . So V is a submodule of α^\perp which is a subcogebra of P of finite left dimension.