January 12, 1969

Summary: If $Q_n = \mathbb{P}(1+O(1)/\mathbb{P}^n)$, then we have for $f: \mathbb{P}(1+L) \longrightarrow X$, that

$$f_* 1 = \sum_{n=0}^{\infty} a_k \, c_1(L)^k \qquad a_k \in \Omega^{-2k-2}(pt)$$

where

$$[Q_n] = \sum_{k=0}^{n} a_k \, [\mathbb{P}^{n-k}].$$

This determines the $a_k$ in terms of the $[\mathbb{P}^n]$ and the $[Q^n]$; ~~also~~ but it would be nice to have a formula in terms of the $[\mathbb{P}^n]$ alone.

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$$\Phi[Q_n] = \sum_{k=0}^{n} \Phi(a_k).$$

however have $Q \longrightarrow \mathbb{P}^n \longrightarrow pt.$ is an iterated projective bundle, hence $\Phi[Q_n] = 1$.

$$\therefore \ \Phi(a_k) = \begin{cases} 1 & k=0 \\ 0 & k>0. \end{cases}$$

$$[Q_0] = [\mathbb{P}^1]$$

$$[Q_1] = [\mathbb{P}^1]^2 + a_1$$

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## Calculations of Chern numbers for $Q_n = \mathbb{P}(1+\mathcal{O}(1)/\mathbb{P}^n)$.

Let $f : \mathbb{P}(1+L/\mathbb{P}^n) \longrightarrow \mathbb{P}^n$ where $L = \mathcal{O}(1)$. Working in ordinary homology, we have $c_1(L) = H$ where $H^*(\mathbb{P}^n) = \mathbb{Z}[H]/(H^{n+1})$.

$$T_{\mathbb{P}^n} = (n+1)L/1$$

$$c(T_{\mathbb{P}^n}) = (1+H)^{n+1}$$

$$T_f = f^*(1+L)\otimes\mathcal{O}(1)/\mathcal{O}$$

$$\begin{aligned}
c(T_f) &= c(f^*(1+L)\otimes\mathcal{O}(1)) \\
&= c(\mathcal{O}(1))\cdot c(f^*L\otimes\mathcal{O}(1)) \\
&= (1+\xi)(1+f^*H+\xi) \\
&= 1+(2\xi+f^*H)
\end{aligned}
\qquad \boxed{\xi^2+(f^*H)\xi = 0}$$

From now on we drop $f^*$.

$$c(T_{Q_n}) = c(T_{\mathbb{P}^n})\, c(T_f) = (1+H)^{n+1}(1+2\xi+H)$$

$$c_1(T_{Q_n}) = (n+2)H + 2\xi$$

$$\begin{aligned}
c_1(T_{Q_n})^{n+1} &= ((n+2)H+2\xi)^{n+1} \\
&= a^{n+1} + \frac{b^{n+1}-a^{n+1}}{b-a}(b-a) \\
&= \cancel{a^{n+1}}_{0} + (b^n + b^{n-1}a + \dots + ba^{n-1} + a^n)(b-a)
\end{aligned}
\qquad
\begin{aligned}
a &= (n+2)H \\
b &= (n+2)H + 2\xi
\end{aligned}$$

Now $b-a = 2\xi$ and $\xi = -H \mod \xi$

$b \equiv nH \mod \xi$

$$c_1(T_{Q_n})^{n+1} = \frac{(nH)^{n+1} - ((n+2)H)^{n+1}}{-2H} \cdot 2\xi$$

$$= \left\{(n+2)^{n+1} - n^{n+1}\right\} H^n \xi.$$

$$\therefore \int_{Q_n} c_1(T_{Q_n})^{n+1} = \left[(n+2)^{n+1} - n^{n+1}\right] \int_{\mathbb{P}^n} H^n \, \underline{f_* \xi}$$

$$= \left[(n+2)^{n+1} - n^{n+1}\right] \qquad (= 56, \; n=2)$$

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On the other hand

$$c_1(T_{\mathbb{P}^1 \times \mathbb{P}^n}) = pr_1^* c_1(T_{\mathbb{P}^1}) + f^* c_1(T_{\mathbb{P}^n})$$

$$= 2\xi + (n+1) H \qquad \begin{cases} \xi^2 = 0 \\ H^{n+1} = 0 \end{cases}$$

$$c_1(T_{\mathbb{P}^1 \times \mathbb{P}^n})^{n+1} = 2\xi \cdot ((n+1)H)^n \cdot (n+1)$$

$$= 2(n+1)^{n+1} H^n \xi$$

$$\int_{\mathbb{P}^1 \times \mathbb{P}^n} c_1(T_{\mathbb{P}^1 \times \mathbb{P}^n}) = 2(n+1)^{n+1}. \qquad (= 54, \; n=2)$$

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Thus $\boxed{[Q_2] \neq [\mathbb{P}_1][\mathbb{P}_2].}$

However for $n = 1$

$$\int_{Q_1} c_1(T_{Q_1})^2 = 9 - 1 = 8 \qquad \int_{Q_1} c_2(T_{Q_1}) = \chi(Q_1) = 4$$

$$\int_{\mathbb{P}_1^2} c_1(T_{\mathbb{P}_1^2}) = 2 \cdot 2^2 = 8 \qquad \int_{\mathbb{P}_1^2} c_2(T_{\mathbb{P}_1^2}) = \chi(\mathbb{P}_1)^2 = 4$$

and therefore by Milnor one knows that $\boxed{[Q_1] = [\mathbb{P}_1]^2.}$

$X$ manifold, $L$ line bundle on $X$.

$$\begin{array}{ccccc} X & \xrightarrow{i_1} & \mathbb{P}(1+L) & \xleftarrow{i_2} & X \\ & \uparrow & & \uparrow & \\ & \text{normal bundle } L^{-1} & & \text{normal bundle } L & \end{array}$$

$$i_2(x) = \text{the line } \mathbb{C}(1,0) \text{ in the fibre over } x$$

$$i_1(x) = \text{the line } 0 \oplus L(x).$$

$$\mathcal{O}(-1) = \{(v,\ell) \mid \ell \text{ line in } 1+L,\ v \in \ell\}.$$

$$i_2^* \mathcal{O}(-1) = \{ v \mid v \in \mathbb{C} \oplus 0 \text{ over } x\} = 1 \quad \text{trivial}$$

$$i_1^* \mathcal{O}(-1) = \{ v \mid v \in 0 \oplus L \} = L.$$

$$\therefore \begin{cases} i_1^* \mathcal{O}(1) = L^{-1} \\ i_2^* \mathcal{O}(1) = 1 \end{cases}$$

$i_1(X)$ is the place where $\mathcal{O}(-1) = 0 + f^*L$ or equivalently where $\mathcal{O}(-1) \hookrightarrow f^*(1 \oplus L) \longrightarrow f^*1$ vanishes, or equivalently where the canonical section of $\mathcal{O}(1)$ vanishes. Therefore

$$(i_1)_* 1 = c_1(\mathcal{O}(1)) \overset{\text{defn}}{=\!=} \xi$$

Similarly $i_2(X)$ is the place where $\mathcal{O}(-1) = f^*1 \oplus 0$, or equivalently where $\mathcal{O}(-1) \longrightarrow f^*(1 \oplus L) \longrightarrow f^*L$ is zero, or where the canonical section of $\mathcal{O}(1) \otimes f^*L$ is zero. Thus

$$(i_2)_* 1 = c_1(\mathcal{O}(1) \otimes f^*L).$$

hence
$$f_*(\xi) = 1$$
$$f_*\, c_1(\mathcal{O}(1)\otimes f^*L) = 1$$

Now we know from split exact sequence

$$\begin{tikzcd}
\Omega^*_{pr/X}(L) \arrow[r, "(j_1)_*"] & \Omega^*(\mathbb{P}(1+L)) \arrow[r, "\iota_1^*"] & \Omega^*(X) \\
\Omega^{*-2}(X) \arrow[u, "\wr"', "0_*"] \arrow[ur, "(\iota_2)_*"] & &
\end{tikzcd}$$

that $\Omega^*(\mathbb{P}(1+L))$ is free as an $\Omega^*(X)$ module with basis $1$ and $(\iota_2)_*1$. Similarly it has basis $1, (\iota_1)_*1$.

$$\Omega^*(\mathbb{P}(1+L)) = \Omega^*(X)\cdot 1 + \Omega^*(X)\cdot\xi$$

Write

$B \in \Omega^0(X)$

~~$A, B \in \Omega^*(X)$~~

$A \in \Omega^2(X)$

$$(\iota_2)_*1 = c_1(\mathcal{O}(1)\otimes f^*L) = A + B\cdot\xi$$

so now apply

$\iota_1^*$: $\quad 0 = (\iota_1)^*(\iota_2)_*1 = A + B\cdot\iota_1^*\xi = A + B\cdot c_1(L^{-1})$

$\iota_2^*$: $\quad c_1(L) = \iota_2^*(\iota_2)_*1 = A + B\,\iota_2^*\xi = A$

Therefore
$$\boxed{\begin{aligned} A &= c_1(L) \\ c_1(L^{-1})\,B &= -c_1(L) \end{aligned}}$$

apply $f_*$

$$1 = f_*(A + B\xi) = A\cdot f_*1 + B\cdot f_*(\xi) = A\cdot f_*1 + B$$

Now multiply by $c_1(L^{-1})$ and find

$$c_1(L^{-1}) = c_1(L^{-1})c_1(L)f_*(1) - c_1(L).$$

$$\boxed{c_1(L) + c_1(L^{-1}) - c_1(L^{-1})c_1(L)f_*(1) = 0}$$

However

$$f_*(1) = \sum_{k=0}^{\infty} a_k\, c_1(L)^k \qquad a_k \in \Omega^{-2k-2}(pt)$$

so

$$c_1(L) + c_1(L^{-1}) - c_1(L^{-1})\sum_{k=0}^{\infty} a_k\, c_1(L)^{k+1} = 0$$

$$\therefore \quad \boxed{c_1(L^{-1}) = \frac{-c_1(L)}{1 - \sum_{k=0}^{\infty} a_k\, c_1(L)^{k+1}}}$$

---

$X$ manifold, $L_1, L_2$ line bundles over $X$

$$\underset{\mathbb{P}(L_1)}{X} \xrightarrow{i_1} \mathbb{P}(L_1 + L_2) \xleftarrow{i_2} X$$

normal bundle $L_1^{-1} \otimes L_2$ normal bundle $L_2^{-1} \otimes L_1$

$$\mathcal{O}(-1) = \{(v, \ell) \mid \ell \subset L_1 + L_2,\ v \in \ell\}.$$

$$i_1^* \mathcal{O}(-1) = L_1 \qquad i_1^* \mathcal{O}(1) = L_1^{-1}$$

$$i_2^* \mathcal{O}(-1) = L_2 \qquad i_2^* \mathcal{O}(1) = L_2^{-1}$$

$l_1(X)$ is where $\mathcal{O}(-1) \subset L_1$, or where $\mathcal{O}(-1) \longrightarrow f^*(L_1+L_2) \longrightarrow f^*L_2$ vanishes, or where the canonical section of $\mathcal{O}(1) \otimes f^*L_2$ is zero

$$(l_1)_* 1 = c_1(\mathcal{O}(1) \otimes f^*L_2)$$
$$(l_2)_* 1 = c_1(\mathcal{O}(1) \otimes f^*L_1)$$

$$\xi = A + B(l_1)_* 1 \qquad\qquad \xi = \bar{A} + \bar{B}(l_2)_* 1$$

$$l_1^* : \quad c_1(L_1^{-1}) = A + B\, c_1(L_1^{-1} \otimes L_2) \qquad\qquad c_1(L_1^{-1}) = \bar{A}$$

$$l_2^* : \quad c_1(L_2^{-1}) = A \qquad\qquad c_1(L_2^{-1}) = \bar{A} + \bar{B}\, c_1(L_2^{-1} \otimes L_1)$$

$$f_* : \quad f_*(\xi) = A \cdot f_*(1) + B \qquad\qquad f_*(\xi) = \bar{A} f_*(1) + \bar{B}$$

According to our preceding calculation

$$c_1(L_2^{-1} \otimes L_1)\left\{1 - c_1(L_1^{-1} \otimes L_2)\, g_*(1)\right\} = -c_1(L_1^{-1} \otimes L_2)$$

where $\quad g : \mathbb{P}(1 + L_1^{-1} \otimes L_2) \longrightarrow X$

but $g$ is isomorphic to $f$.

$$c_1(L_2^{-1} \otimes L_1) = \frac{-c_1(L_1^{-1} \otimes L_2)}{1 - c_1(L_1^{-1} \otimes L_2)\, f_*(1)}$$

$$f_*(1) = \sum_{k=0}^{\infty} a_k\, c_1(L_1^{-1} \otimes L_2)^k.$$

Now set
$$z = c_1(L_1^{-1} \otimes L_2)$$
$$z' = c_1(L_2^{-1} \otimes L_1)$$
$$x = c_1(L_1^{-1})$$
$$y = c_1(L_2^{-1}).$$

$$z + z' = zz' f_*(1)$$
$$f_*(1) = \sum_{k=0}^{\infty} a_k z^k$$

$$\begin{cases} x = A + Bz \\ y = A \\ f_*(\xi) = A f_*(1) + B \end{cases}$$

$$x = \bar{A}$$
$$y = \bar{A} + \bar{B} z'$$
$$f_*(\xi) = \bar{A} f_*(1) + \bar{B}$$

$$zz' y f_*(1) + z'(x-y) = zz' x f_*(1) + z(y-x)$$

$$\cancel{(x-y)(z+z')}$$

$$(y-x)[zz' f_*(1)] = (y-x)(z+z')$$

Therefore we obtain nothing new

# Where an old calculation went wrong.

$X$ manifold, $L$ a complex line bundle over $X$

$s$ section of $L$ transversal to $0$, $Y = s^{-1}0$.

Then there is a map

$$\theta : X \times \mathbb{C} \longrightarrow L$$

$$\theta(x, \lambda) = \lambda s(x)$$

which is an isomorphism over $X - Y$. Consequently there is an isomorphism

$$\theta : (X-Y) \times \mathbb{P}^1 \longrightarrow \mathbb{P}(1+L) \,|\, X-Y$$

$$(x, \lambda_0, \lambda_1) \longmapsto \lambda_0 + \lambda_1 s(x)$$

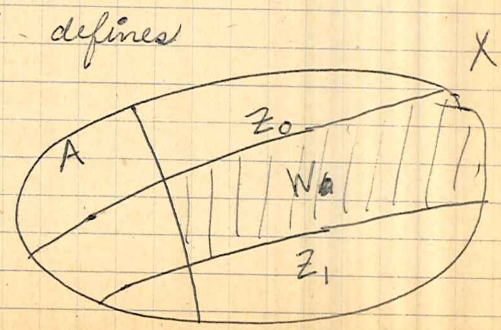
Therefore if $f : \mathbb{P}(1+L) \longrightarrow X$ and $g : X \times \mathbb{P}^1 \longrightarrow X$ are the canonical maps, we should have that

$$f_* 1 - g_* 1 = i_* \alpha$$

where $i : Y \hookrightarrow X$ and $\alpha \in \Omega^{-4}(Y)$. We propose to determine $\alpha$.

General situation: Given a manifold $X$ and two proper oriented manifolds $Z_0$ and $Z_1$ over $X$ and an isom. $\theta : Z_0 | X - A \xrightarrow{\sim} Z_1 | X - A$ over $X - A$ respecting orientation, one forms $W = Z_1 \times I \xrightarrow{f_1 pr_1} X - A$ and defines

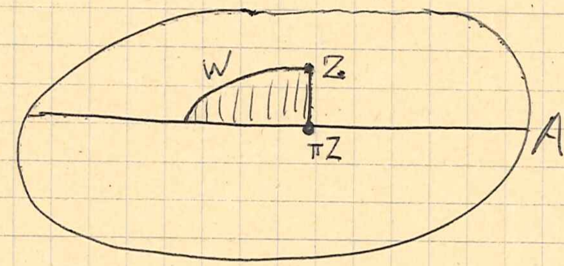
$$\varphi : (-Z_0) \amalg Z_1 \longrightarrow \partial W$$




to be identity on $Z_1$ and $\theta$ on $Z_0$.

This defines an element of $\Omega_A(X)$.

By the usual excision process we can move everything inside of a neighborhood of $A$. If $A$ is an oriented submanifold, then one excises into a tubular neighborhood $N$ and throws away $W$ taking $Z \longrightarrow N \xrightarrow{\pi} A$, and this gives the isomorphism $\Omega(A) \simeq \Omega_A(X)$.




~~We return to original situation~~ The thing to note is that as $\theta$ is an isomorphism we take a tubular nbd $N$ of $A$ and glue $-Z_0|N$ and $Z_1|N$ together over ~~[illegible]~~ $\partial N$ by means of the isomorphism $\theta : Z_0|\partial N \simeq Z_1|\partial N$.

<u>Original situation</u>: Here $Z_0 = X \times \mathbb{P}^2$, $Z_1 = \mathbb{P}(1+L)$ and $\theta$ is as described above. ~~[illegible]~~ As the situation is local we assume that ~~[illegible] $\mathbb{D}(L|Y)$~~ $X \simeq L|Y$ and that $L \simeq X \times_Y (L|Y)$. Let $E = L|Y$. Have to glue

$$-(X \times \mathbb{P}^2)|\mathbb{D}E \quad \text{and} \quad \mathbb{P}(1+L)|\mathbb{D}E \simeq \mathbb{D}E \times_Y \mathbb{P}(1+E)$$

$$\begin{array}{c} -(X \times \mathbb{P}^2)|\mathbb{D}E \\ \wr\wr \\ -\mathbb{D}E \times_Y (Y \times \mathbb{P}^2) \end{array}$$

by means of the isomorphism

$$\mathbb{S}E \times \mathbb{P}^2 \longrightarrow \mathbb{S}E \times_Y \mathbb{P}(1+E)$$

$$(z, (\lambda_0, \lambda_1)) \longmapsto (z, (\lambda_0, \lambda_1 z))$$

~~[struck out]~~ Problem: What is — ~~[struck out]~~ a complex manifold?

Not the same as conjugation

If $X$ is an almost complex manifold so that $\tau_X$ is endowed with a complex structure, let $\bar{X}$ denote the almost complex manifold ~~[struck out]~~ given by $\bar{X} = X$ as manifolds and the cx. structure on $\tau_{\bar{X}}$ is conjugate to that of $X$, i.e.

If $\bar{v} \in \tau_{\bar{X}}$ corresp. to $v \in \tau_X$, then

$$\lambda\bar{v} = \overline{\bar{\lambda} v}.$$

~~If dim X = 1, then~~

If $E$ is a complex vector bundle, then $\bar{E} \simeq E'$ as far as homotopy goes since if $\langle\,,\rangle$ is a hermitian metric get $\varphi : \bar{E} \longrightarrow E'$ $\quad \varphi(\bar{e})(u) = \langle u, e\rangle$. It's clear that $E'$ is very seldom stably equivalent to $-E$.

This is why you got wrong answer before. You glued $\overline{\mathbb{D}E}$ and $\mathbb{D}E$ together to form $\mathbb{P}(1+E)$ where you should have glued $-\mathbb{D}E$ and $\mathbb{D}E$ to get the fiber suspension of $\mathbb{S}E$, the boundary of the disk in $1_{\mathbb{R}} \oplus E$.

Get formula for $a_1$: Take $O(1)$ over $S^2$ get a projective bundle $\mathbb{P}(1+O(1)/S^2)$; however now put on the weakly complex structure on $S^2$ so that its the boundary of $D^3$. Thus get an weakly cx. structure on $\mathbb{P}(1+O(1)/S^2)$ different from usual.

Given $L/X$. to glue $\mathbb{P}^2$ + $\underline{\mathbb{P}(1 \oplus L)}$ by can isom. on $\underline{L - X}$.

$$\begin{tikzcd}
 & \mathbb{D}L \times \mathbb{C} \arrow[rr, "\beta'"] & & \mathcal{O}(+1) \\
 & \mathbb{D}L \arrow[rr, "\beta"] & & \mathbb{P}(1+L) \\
\mathbb{S}L \times \mathbb{C} \arrow[uur, hook] \arrow[rr, "\alpha"] & & \mathbb{D}L \times_X L \arrow[uur, "\gamma'"] \arrow[d] & \\
\mathbb{S}L \arrow[uur] \arrow[rr, hook] & & \mathbb{D}L \arrow[uur, "\gamma"] &
\end{tikzcd}$$

$$\alpha(v, \lambda) = (v, \lambda v).$$

$$\beta(v) = \mathbb{C}(1, v).$$

$$\gamma(v) = \mathbb{C}(1, v^*)$$

where $v^*$ unique $\ni$ $\langle v, v^* \rangle = 1$

~~where~~

$$\beta'(v, \lambda) = (\text{the line } \mathbb{C}(1,v), \text{ the linear function on this line given by } \mu((1,v)) = \lambda)$$

$$\gamma'(v, w) = (\text{the line } \mathbb{C}(1,v^*), \text{ the linear function given by } \mu((1,v^*)) = \langle w, v^* \rangle).$$

locally suppose $v = as$ $\quad s \neq 0 \quad$ $v^* = (\bar{a})^{-1} s^*$ $\quad$ ~~$\langle w, (\bar{a})^{-1}s^* \rangle\, (1, (\bar{a})^{-1}s^*)$~~

$$\mu(1, \bar{a}^{-1}s^*) = \langle w, s^* \rangle\, \bar{a}^{-1} \quad \text{or} \quad \mu(\bar{a}, s^*) = \langle w, s^* \rangle$$

showing that $\mu$ ~~depends~~ varies nicely as $a \to 0$.

January 13, 1969: Riemann-Roch thm. for $\Omega$:

$\exists$ universal formula

$$c_1(L_1 \otimes L_2) = \sum_{k,l\geq 0} b_{kl}\, c_1(L_1)^k c_1(L_2)^l$$

where $b_{kl} \in \Omega^*(pt.)$. This follows by determination of $\Omega^*(\mathbb{P}^n \times \mathbb{P}^m)$. As multiplication of line bundles is associative, etc. it follows that

$$F(X,Y) = \sum_{k,l\geq 0} b_{kl} X^k Y^l$$

is a formal group law over $\Omega^*(pt.)$. Hence over $\Omega^*(pt.)\otimes\mathbb{Q}$ there is a power series $\psi(X)$ such that

$$\boxed{\begin{gathered}\psi\left(\sum b_{kl} X^k Y^l\right) = \psi(X) + \psi(Y) \\ \psi(0) = 0, \quad \psi'(0) = 1\end{gathered}}$$

To find $\psi$ differentiate wrt $Y$ and set $Y=0$.

$$\psi'(F(X,0)) \cdot \frac{\partial F}{\partial Y}(X,0) = \psi'(0) = 1$$

exists since $b_{01} = 1$.

$$\psi'(X) = F_2(X,0)^{-1} = \left[\sum_k b_{k1} X^k\right]^{-1}$$

$$\boxed{\psi(X) = \int_0^X \left\{\sum b_{k1} X^k\right\}^{-1} dX}$$

Claim that

$$\psi(F(X,Y)) = \psi(X) + \psi(Y)$$

This is true mod $Y^2$. Observe that as functions of $Y$, they coincide for $Y=0$, hence (char 0) enough to show their derivatives wrt $Y$

are equal, i.e.

$$\psi'(F(X,Y))\,F_2(X,Y) \overset{?}{=} \psi'(Y).$$

By assumption, true for $Y=0$, i.e.

$$\psi'(X)\,F_2(X,0) = 1$$

Also

$$F(X,F(Y,Z)) = F(F(X,Y),Z)$$

so ~~[illegible]~~ applying $\frac{\partial}{\partial Z}$ and setting $Z=0$, yields

$$F_2(X,Y)\,F_2(Y,0) = F_2(F(X,Y),0).$$

or

$$\psi'(Y) = F_2(Y,0)^{-1} = F_2(F(X,Y),0)^{-1}F_2(X,Y) = \psi'(F(X,Y))F_2(X,Y),$$

which proves the claim.

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Now set

$$\varphi(X) = e^{\psi(X)}$$

so that

$$\begin{cases} \varphi(F(X,Y)) = \varphi(X)\varphi(Y) \\ \varphi(0) = 1\,, \quad \varphi'(0) = 1 \end{cases}$$

and define

$$ch(L) = \varphi(c_1(L)) \in \Omega^*(X) \qquad (\text{if } \dim X < \infty).$$

$$\boxed{\begin{aligned} ch(L_1\otimes L_2) &= ch(L_1)\,ch(L_2) \\ ch(L) &= 1 + c_1(L) + \cdots\cdots \end{aligned}}$$

Let $\Phi : \Omega^{*U} \longrightarrow K$ be the canonical map. Then

$$\Phi(c_1(L_1 \otimes L_2)) = 1 - L_1^{-1} L_2^{-1} = (1 - L_1^{-1}) + (1 - L_2^{-1}) - (1 - L_1^{-1})(1 - L_2^{-1})$$

$$= \Phi(c_1(L_1)) + \Phi(c_1(L_2)) - \Phi(c_1(L_1)) \cdot \Phi(c_1(L_2))$$

$$\therefore \; F(X, Y) = X + Y - XY \qquad \text{in this case so}$$

$$\bar{\varphi}(X) = (1 - X)^{-1} \qquad \bar{\psi}(X) = -\log(1 - X)$$

for this formal group. By uniqueness of these functions we find that

$$\Phi(\varphi(c_1(L)) = L = \bar{\varphi}(\Phi c_1(L))$$

Therefore if

$$\varphi(X) = 1 + X + a_2 X^2 + \cdots \qquad a_i \in \Omega^*(pt)$$

$$\boxed{\Phi(a_i) = 1}$$

Similarly if $\varepsilon : \Omega^* \longrightarrow H^*$ is the canonical map

$$\varepsilon(ch^{\Omega} L) = ch^{H} L.$$

or

$$\varepsilon(\varphi(X)) = e^X$$

i.e.

$$\boxed{\varepsilon(a_i) = \frac{1}{i!}}$$

Now by means of the splitting principle we can extend ch to a ring homomorphism

$$ch : K(X) \longrightarrow \Omega^U(X) \otimes \mathbb{Q}$$

which is compatible with $f^*$ and is a section of $\Phi$. Compatibility with $f_*$ doesn't hold and leads to a Riemann-Roch thm:

$$ch(f_!(x)) \cdot \mathrm{Todd}(\Theta_Y) = f_*\left(ch\, x \cdot \mathrm{Todd}(\Theta_X)\right),$$

where $x \in K_{pr/Y}(X)$, and $f: X \longrightarrow Y$, and where Todd is a multiplicative extension of a characteristic class given on line bundles by a power series with leading term 1.

Claim it is enough (granted a workable formalism of $K^{+\Omega}$ theory with supports) to prove ~~this~~ formula where $f$ is the inclusion of the zero section of a line bundle. Thus can rewrite the formula

$$ch(f_! x) = f_*\left(ch\, x \cdot \mathrm{Todd}(\Theta_f)\right)$$

whence one sees it behaves for compositions. Factoring $f$ into an inclusion $X \longrightarrow Y \times V \xrightarrow{pr_1} Y$ one reduces to the case where $f$ is either $i: X \longrightarrow V$ or $\pi: V \longrightarrow X$ and $V$ is a vector bundle over $X$. Here however we have that

$$\iota_*: \Omega(X) \xrightarrow{\sim} \Omega_{pr/X}(V) \qquad \iota^*: \Omega(V) \xrightarrow{\sim} \Omega(X)$$

$$\iota_!: K(X) \xrightarrow{\sim} K_{pr/X}(V) \qquad \iota^!: K(V) \xrightarrow{\sim} K(X)$$

are isomorphisms. Thus given $x \in K(X)$ have $x = \iota^! v$ (where $v = \pi^! x$), so

$$ch(\iota_! x) = ch(\iota_!(\iota^! v)) = ch\, v \cdot (ch\, \iota_! 1)$$

$$\iota_*(ch\, x \cdot \mathrm{Todd}(V)^{-1}) = ch\, v \cdot i_*((\mathrm{Todd}(V))^{-1}).$$

reducing to proving that

(1) $$\boxed{ch\, i_!1 = i_*((\text{Todd } V)^{-1})}$$ , which implies

~~so that~~

(2) $$\boxed{ch(\iota^!\iota_!1) = (\iota^*\iota_*1)\,(\text{Todd } V)^{-1}}$$ .

Similarly if $v \in \underset{pr/X}{K}(V)$, have $v = \iota_! x$ and

$$ch(\pi_! v) = ch\, x$$

$$\pi_*\big(ch(\iota_! x)\ \pi^*\text{Todd } V\big) = \pi_*\big(ch(\iota_! 1)\ \pi^* ch\, x \cdot \text{Todd } V\big)$$

$$= \pi_*(ch(\iota_! 1))\, ch\, x \cdot \text{Todd } V$$

reducing to

$$\pi_*(ch(\iota_! 1)) \cdot \text{Todd } V = 1$$

which follows from (1) by ~~thus we have to prove (2)~~ applying $\pi_*$. By the splitting principle and fact that $\iota_!$ and $\iota_*$ commute with smooth base change ~~[illegible]~~ we may assume $V$ splits and reduce to codimension 1 i.e. when $V$ is a line bundle $L$. Formula (2) shows us that

(3) $$\boxed{\text{Todd}(L) = \frac{c_1(L)}{ch(1-L^{-1})}}$$

and conversely ~~[illegible]~~ this implies (2). Here's a lousy argument that (2) $\Rightarrow$ (1): ~~[illegible]~~ Write left and right sides of (1) as $\iota_* a$, $\iota_* b$, ~~[illegible]~~ then ~~[illegible]~~ $(\iota^*\iota_*1)\cdot a = \iota^*\iota_* a = \iota^*\iota_* b = (\iota^*\iota_*1)\cdot b$ using (2), so

$c_1(L)\,(a-b)=0$. But ~~universally~~ $a-b$ is a universal power series in $c_1(L)$ and $c_1(\mathcal{O}(1))$ is a non-zero divisor in $\Omega^*(\mathbb{P}^\infty) = \Omega^*[[c_1(\mathcal{O}(1))]]$. $\therefore a=b$.
This proves Riemann–Roch.

---

Apply Riemann-Roch to $\mathbb{P}^n$.

$$\Theta_{\mathbb{P}^n} = (n+1)\mathcal{O}(1)/\mathcal{O}$$

$$\mathrm{Todd}(\Theta_{\mathbb{P}^n}) = \left(\mathrm{Todd}\ \mathcal{O}(1)\right)^{n+1}$$

$$f_*\left\{ch\ \mathcal{O}(q)\cdot \mathrm{Todd}(\Theta_{\mathbb{P}^n})\right\} = ch\left\{f_!\ \mathcal{O}(q)\right\}$$

where $f: \mathbb{P}^n \longrightarrow pt.$ $\qquad f_!\,\mathcal{O}(q) = \chi(\mathcal{O}(q)) = \binom{q+n}{n}$.

Let $H = c_1(\mathcal{O}(1))$ so that $\Omega^*(\mathbb{P}^n) = \Omega^*(pt)[H]/H^{n+1}$

$$f_*(H^i) = \begin{cases} [\mathbb{P}^{n-i}] & 0\le i\le n \\ 0 & i>n \end{cases}$$

$$\mathrm{Todd}\ \mathcal{O}(1) = \frac{c_1(\mathcal{O}(1))}{ch(1-\mathcal{O}(1)^{-1})} = \frac{H}{1-\frac{1}{\varphi(H)}} = \frac{H\varphi(H)}{\varphi(H)-1}$$

$$ch\ \mathcal{O}(q) = \varphi(H)^q$$

Thus

$$f_*\left\{\varphi(H)^q\left(\frac{H\varphi(H)}{\varphi(H)-1}\right)^{n+1}\right\} = \binom{q+n}{n}$$

$$\varphi(H) = 1 + a_1 H + a_2 H^2 + \cdots \qquad a_1 = 1$$

$$\frac{\varphi(H)-1}{H} = a_1 + a_2 H + \cdots$$

$$f_* \left\{ \varphi(H)^{g+n+1} \left( \frac{\varphi(H)-1}{H} \right)^{-n-1} \right\} = \binom{g+n}{n}$$

In particular

$$\boxed{f_* \left\{ \left( \frac{H}{\varphi(H)-1} \right)^{n+1} \right\} = \binom{-1}{n} = (-1)^n}$$

From this equation we can recursively compute the coefficients $a_i$ as polynomials in the $[\mathbb{P}^k]$:

__$n=0$:__ $\quad \frac{1}{a_1} = 1 \implies \boxed{a_1 = 1}$

__$n=1$:__ $\quad f_* \left( \frac{1}{1+a_2 H} \right)^2 = f_* (1 - 2a_2 H) = [\mathbb{P}^1] - 2a_2 = -1$

$$\boxed{a_2 = \frac{1 + [\mathbb{P}^1]}{2}.} \qquad \begin{array}{l} \Phi(a_2) = 1 \\ \varepsilon(a_2) = \frac{1}{2} \end{array}$$

__$n=2$:__ $\quad f_* \left( \frac{1}{1 + a_2 H + a_3 H^2} \right)^3 \equiv f_* \left( 1 + (-3)(a_2 H + a_3 H^2) + \frac{(-3)(-4)}{2} (a_2 H)^2 \right)$

$$= f_* \left( 1 - 3a_2 H - 3a_3 H^2 + 6 a_2^2 H^2 \right)$$

$$= \mathbb{P}^2 - 3 a_2 \mathbb{P}^1 - 3a_3 + 6 a_2^2 = 1$$

$$3a_3 = \mathbb{P}^2 - 3\left( \frac{1+\mathbb{P}^1}{2} \right) \mathbb{P}^1 + 6 \left( \frac{1+\mathbb{P}^1}{2} \right)^2 - 1$$

$$6a_3 = 2\mathbb{P}^2 - 3[\mathbb{P}^1 + \cancel{(\mathbb{P}^1)^2}] + 3(1 + 2\mathbb{P}^1 + \cancel{(\mathbb{P}^1)^2}) - 2$$
$$= 2\mathbb{P}^2 + 3\mathbb{P}^1 + 1$$

Therefore

$$\boxed{a_3 = \frac{2[\mathbb{P}^2] + 3[\mathbb{P}^1] + 1}{6}}$$

$\Phi(a_3) = 1$ ✓

$\varepsilon(a_3) = \frac{1}{3!}$ ✓

$n=3$: I calculated that

$$\boxed{a_4 = \frac{1 + 6[\mathbb{P}_1] + 11[\mathbb{P}_2] + 6[\mathbb{P}_3]}{24} + \frac{[\mathbb{P}_1]^2 - [\mathbb{P}_2]}{8}}$$

<u>Conjecture:</u>

$$a_n \equiv \frac{(1+[\mathbb{P}_1])\cdots(1+(n-1)[\mathbb{P}_1])}{n!} \qquad \text{mod } \left([\mathbb{P}_n] - [\mathbb{P}^1]^n\right)_{n \geq 0}$$

True see page 10:

Note that if $f^{(n)} : \mathbb{P}^n \longrightarrow pt$ then

$$f^{(n)}_*(H^i) = \operatorname{res}\Big\{ H^i \cdot H^{-n-1} \sum_{j\geq 0} P_j H^j \cdot dH \Big\}$$

hence

$$f^{(n)}_*\left\{ \left( \frac{H\varphi(H)}{\varphi(H)-1} \right)^{n+1} \right\} = 1 \qquad \text{all } n \geq 0$$

means that

$$\operatorname{res}\left\{ \left( \frac{\varphi(H)}{\varphi(H)-1} \right)^{n+1} \cdot \sum P_j H^j \, dH \right\} = 1 \qquad \text{for all } n\geq 0$$

Now as $\varphi(H) = 1 + H + \cdots$ there is a change of variable

$$\bar{H} = \frac{\varphi(H)-1}{\varphi(H)}$$

$$H = \xi(\bar{H})$$

and so using invariance of residue we have

$$\operatorname{res}\left\{ \bar{H}^{(-n-1)} \sum_{j\geq 0} P_j \xi(\bar{H})^j \, \xi'(\bar{H}) \, d\bar{H} \right\} = 1 \qquad \text{for all } n\geq 0$$

i.e.

$$\sum P_j \xi(\bar{H})^j \xi'(\bar{H}) \, d\bar{H} = 1 + \bar{H} + \bar{H}^2 + \cdots = \frac{1}{1-\bar{H}}.$$

Integrating

$$\sum_{j\geq 0} P_j \frac{\xi(\bar{H})^{j+1}}{j+1} = -\log(1-\bar{H})$$

or

$$\boxed{e^{\sum_{j\geq 0} P_j \frac{H^{j+1}}{j+1}} = \varphi(H)}$$

(Myshenko ?)

<u>Check</u>:
$$\begin{cases} \varepsilon(\varphi(H)) = e^{H} & \checkmark \\ \Phi(\varphi(H)) = \dfrac{1}{1-H} & \checkmark \end{cases}$$

Riemann-Roch for projective space $\mathbb{P}^n$:

$$f_*^{(n)}\left\{\varphi(H)^{g}\left(\frac{H\varphi(H)}{\varphi(H)-1}\right)^{n+1}\right\} = \operatorname{res}\left\{\varphi(H)^{g}\left(\frac{H\varphi(H)}{\varphi(H)-1}\right)^{n+1} H^{-n-1}\sum p_j H^j\, dH\right\}.$$

$$= \operatorname{res}_{H=0}\left\{\varphi(H)^{g}\left(\frac{\varphi(H)}{\varphi(H)-1}\right)^{n+1}\frac{\varphi'(H)\,dH}{\varphi(H)}\right\} \qquad \varphi(H) = 1+X$$

$$= \operatorname{res}_{X=0}\left\{(1+X)^{g+n}\frac{dX}{X^{n+1}}\right\} = \binom{g+n}{n} \quad \checkmark$$

---

$$\boxed{ch(L) = e^{\sum_{j\geq 0} p_j \frac{c_1(L)^{j+1}}{j+1}}}$$

In particular if $p_j = a^j$

$$ch(L) = e^{\sum_{j\geq 0} a^j \frac{c_1(L)^{j+1}}{j+1}} = e^{-\frac{1}{a}\log(1-a c_1(L))}$$

$$= (1-a c_1(L))^{-1/a} = 1 + c_1(L) + \frac{\left(-\frac{1}{a}\right)\left(-\frac{1}{a}-1\right)}{2}(a c_1(L))^2 + \cdots$$

$$= 1 + c_1(L) + \frac{1+a}{2!}c_1(L)^2 + \frac{(1+a)(1+2a)}{3!}c_1(L)^3 + \cdots\cdots$$

January 18-20, 1969:

Let $\mathcal{V}_G$ be the category of $G$-manifolds where $G$ is a compact Lie group.

Problem: If $X$ is a $G$-manifold, then is there is a finite dimensional representation $V$ of $G$ and an equivariant embedding $X \longrightarrow V$?

Counterexample: Let $H_n = \mu_{2^n} \subset S^1 = G$ and let $X = \coprod_{n \geq 0} G/H_n$. If $V$ is a finite dimensional representation of $G$, then $V = V_1 \oplus \cdots \oplus V_n$ where the $V_i$ are 1-dimensional given by **characters** $z v = z^{m_i} v$ if $v \in V_i$, $z \in G$. If $v = (v_1, \ldots, v_n) \in V$, then the stabilizer of $v$ is $\{z \in G \mid z^{m_i} = 1 \text{ if } v_i \neq 0\} \subset \{z \in G \mid z^{m_1 \cdots m_n} = 1\} = \mu_{m_1 \cdots m_n}$, provided $v \neq 0$. Thus if $X \subset V$ we would have $\mu_{2^q} \subset \mu_{m_1 \cdots m_n}$ for all $q$ which is impossible.

---

This counterexample arises because $X$ has infinitely many orbit types.

Definition: An orbit type is an isomorphism class of transitive $G$ manifolds, or what comes to the same thing a **conjugacy** class of closed subgroups in $G$.

Proposition: If $X$ is a compact $G$-manifold, then the set of orbit types is finite.

Proof: By induction on the dimension of $X$: Consider orbit of each point and choose an equivariant tubular neighborhood. As

$X$ is compact there is a finite covering by such tubular nbds. Each tubular nbd has ~~[illegible]~~ for its orbit types the zero section and those of the sphere bundle which has finitely many orbit types, since it is a compact manifold of 1 lower dimension.

---

It is clear that an embeddable $G$-manifold has only finitely many orbit types which leads to:

<u>Conjecture:</u> If $X$ is a $G$-manifold with only finitely many orbit types, then $X$ may be embedded in a finite dimensional representation of $G$.

<u>Case 1.</u> $X$ has only one orbit type given by a normal subgroup $H$. Then $Q = G/H$ acts freely on $X$ and so one is reduced to the case where $G$ acts freely on $X$. Let $Y = X/G$ and let $Y \longrightarrow W$ be an embedding where $W$ is a vector space. Since $Y$ is finite dimensional, the principal $G$-bundle over $Y$ given by $X$ is induced by an equivariant map $X \longrightarrow E$ where $E$ is ~~compact~~ a principal bundle which is compact. ( Embed $G \hookrightarrow U(n)$ form $X \times_G U(n)$, comes from ~~[illegible]~~ vector bundle over $Y$ which is induced from Stieffel manifold $U(n+N)/U(N) = E$ for some $N$). Now equivariant embeddings of compact manifolds are easy; one takes an embedding and by Peter-Weyl approximates the embedding functions by representative functions, then uses that any map sufficiently close to an embedding is an embedding.

Therefore we get an equivariant embedding $E \hookrightarrow V$. It is then clear that $X \hookrightarrow Y \times E \hookrightarrow W \times V$ is an equivariant embedding of $X$.

Case 2: $X$ has only one orbit type. Let $H$ be a closed subgroup of $G$ such that $X = G \cdot X^H$. Then if $NH$ is the normalizer of $H$ we have $NH$ acts freely on $X^H$ and

$$G \times_{NH} X^H \xrightarrow{\sim} X$$

~~By case 1 we get an $NH$-equivariant embedding of $X^H$ into $W$ hence a $G$-equivariant embedding of $X$ into $G \times_{NH} W$, which is a homogeneous vector bundle over $G/NH$.~~ As in case 1 one chooses an $NH$-principal bundle map $X^H \longrightarrow E$, with $E$ compact and an embedding $NH \backslash X^H \hookrightarrow W$. This gives an embedding

$$G \times_{NH} X^H \hookrightarrow G \times_{NH} (W \times E) \cong W \times (G \times_{NH} E)$$

and so choosing a $G$-embedding $G \times_{NH} E \longrightarrow V$ one is done.

Case 3: Suppose $X$ is the interior of a compact $G$-manifold $\bar{X}$ with boundary. Construct a collar around $\partial X$ and a smooth function $\varphi$ on $\bar{X}$ representing distance from $\partial X$ in collar and constant outside collar. Then get $X \xrightarrow{(\varphi, i)} \mathbb{R}_+ \times \mathbb{R}^n$ an embedding which we can make equivalent. Now use a diffeo of $\mathbb{R}^+$ with $\mathbb{R}$ to get an equivariant embedding of $X$ in $\mathbb{R} \times V$, where $G$ acts trivially on $\mathbb{R}$.

Example: An equivariant bundle $G \times_H V$. Here can take $\varphi$ to be (distance from $0$)$^2$.

Related problem: Let $E$ be an equivariant bundle over $X$. If $X$ has finitely many orbit types, then is $E$ a quotient of a equivariant bundle of the form $X \times V$, where $V$ is a representation of $G$?

---

Case 2': If $X$ has a single orbit type associated to a (closed) ~~space~~ subgroup $H$ of $G$, so that

$$X \simeq G \times_{NH} X^H$$

then a $G$-bundle over $X$ is the same as a $NH$-bundle on $X^H$ and since $NH/H$ acts freely on $X^H$ the same as a $H$-bundle $E$ over $NH \backslash X^H$ (uses lemma below). But $H$ acts trivially on $Y = NH \backslash X^H$, so an $H$ bundle is just a representation of $H$ in a vector bundle $E$ on $Y$. May assume $E$ complex whence

$$E \simeq \bigoplus_i \mathrm{Hom}_H(W_i, E) \otimes W_i,$$

where $W_i$ runs over the irreducible repns. of $H$. Now write the bundles $\mathrm{Hom}_H(W_i, E)$ as quotients of trivial bundles whence $E$ is a quotient of $Y \times V$, $V$ a representation of $H$, and so

Example to show ~~[illegible]~~ that X have finitely many orbit types is necessary: $G = S^1$, $H_n = \mu_{2^n}$, $W_n = \mathbb{C}$ with standard $\mu_{2^n}$ action. $E = \coprod_{n\geq 1} G \times_{H_n} W_n \longrightarrow \coprod_{n \geq 1} G/H_n$

If E has a finite dimensional generating $\overbrace{\text{subspace}}^{\text{invariant}}$ $\overbrace{\text{we}}^{\text{of sections}}$ can decompose it into 1-dimensional representations. Let $f \in \Gamma(E) = \prod_n \Gamma(G/H_n, G\times_{H_n} W_n)$ $f = (f_n)$, where $f_n \in \mathrm{Hom}_{H_n}(G, W_n)$, and suppose $z * f = z^m f$, i.e.

$$f_n(zx) = z^m f_n(x) \qquad \text{all } z, x \in S^1, \text{ int } n$$

But

$$f_n(zx) = z f_n(x) \qquad \text{if } z \in \mu_{2^n}$$

Therefore $f_n \neq 0 \Rightarrow z^m = z$ all $z \in \mu_{2^n} \Longrightarrow 2^n | m-1$.
Hence only finitely many $f_n$ are $\neq 0$ so we can't have a finite dimensional generating invariant subspace of sections.

---

Case 4: G finite. Here both conjectures are true.
In effect if E is a G-bundle over X with a generating ~~[illegible]~~ finite-dimensional subspace $V \subset \Gamma(E)$, then $G \cdot V$ is also finite dimensional. Similarly for an embedding.

---

~~[illegible]~~ It is clear that we must also assume that the equivariant bundle E itself has only finitely many orbit type since if $E \hookrightarrow \underline{X} \times V$ ~~[illegible]~~ is a subbundle ~~[illegible]~~ and if $X \hookrightarrow W$ is an embedding, then $E \hookrightarrow X \times V \hookrightarrow W \times V$ is an embedding.

Lemma: Let $(X, \partial X)$ be a $G$-manifold with boundary (and with finitely many orbit types). Let $\alpha : \partial X \longrightarrow V$ be an equiv. embedding with $V$ a representation of $G$. Then $\exists$ an equivariant embedding $\beta : X \longrightarrow \mathbb{R} \times V \times V'$ ~~such that~~ $\beta = (\beta_1, \beta_2, \beta_3)$ such that $\beta_2, \beta_3$ are constant near $\partial X$ and $\beta_2 | \partial X = \alpha$, $\beta_3 | \partial X = 0$.

Sketch of Proof: Induction on no. of orbit types. If there is a single orbit type we know how to proceed. Then

$$X = G \times_{NH} X^H$$

where $NH/H$ acts freely on $X^H$. Embed $NH \backslash X^H \hookrightarrow \mathbb{R}_+ \times \mathbb{R}^n$ and also find a map $X^H \longrightarrow E$ where $E$ is a compact manifold on which $NH/H$ acts freely. Then

$$X = G \times_{NH} X^H \longrightarrow (NH \backslash X^H) \times (G \times_{NH} E) \hookrightarrow \mathbb{R}_+ \times \mathbb{R}^n \times V$$

where $G \times_{NH} E \hookrightarrow V$ is an equivariant embedding.

If there is more than one orbit type, let $G/H$ be minimal ($H$ maximal) among the orbit types that occur and let $Y = G \cdot X^H$ be the union of the orbits of $X$ of this type. Then $Y$ is a closed invariant submanifold of $X$; let $N$ be an invariant tubular nbd. By an argument to be given below we can embed the normal bundle $\nu$ of $Y$ in $X$ ~~into~~ as a subbundle of a bundle $Y \times V$ where $V$ is a representation of $G$. This gives us an equivariant embedding $N \hookrightarrow Y \times DV \hookrightarrow W \times DV$

$\hookrightarrow \mathbb{R}_+ \times W \times V$, if $Y$ closed. Now extend embedding on $\partial N$ to $X - \mathrm{Int}(N)$ by induction hypothesis and piece together, UGH. Must also contend with corners.

Lemma: If $E$ is an equivariant bundle over a (with finitely many orbit types) $G$-manifold $X$ ~~with finitely many orbit types~~, then $E$ is a subbundle of $X \times V$ for some representation of $G$.

Proof: Induction on number of orbit types of $X$. Enough to consider case of 1-orbit type since we have to find a finite diml. inv. generating subspace of sections, hence if $X = X_1 \cup_A X_2$ where $A = \partial X_1 = \partial X_2$ and if $V_i$ generates $E/X_i$, then using a collar (around $A$) can assume $V_i \subset \Gamma(X, E)$ generates over $X_i$ whence $V_1 + V_2$ generates over $X$.

~~Now suppose $X$ has one orbit type of type $G/H$ so~~

~~$$X = G \times_{NH} X^H$$~~

~~where $NH/H$ acts freely on $X^H$. Let~~

~~$$E = G \times_{NH} E'$$~~

~~where $E'$ is a $NH$-bundle on $X^H$. If $E' \subset X^H \times V'$ with $V'$ a rep. of $NH$, then~~

~~$$E \subset G \times_{NH} (X^H \times V')$$~~

Now assume $X$ has one orbit type belonging to $H$, so

$$X \simeq G \times_{NH} X^H$$

By hypothesis $E$ has finitely many orbit types, hence $\exists$ a representation $W$ of $G$ such that for each $x \in X^H$ there is a surjection $W \twoheadrightarrow E_x$ of $H$-modules (one must show that finitely many orbit types $\Longrightarrow$ finitely many different isotropy representations (see ~~[illegible]~~ below)). Hence $\exists$ surjection

$$G/H \times W \longrightarrow G \times_H E_x$$

of $G$-bundles over the orbit $Gx$ and hence if we choose a finite set of generating sections for the bundle on $G\backslash X = NH\backslash X^H$ whose sections are

$$\mathrm{Hom}_G(W, \Gamma(X, E))$$

then we have written $E$ as a quotient of $\cancel{X} \times W^n$. QED.

---

The correct hypothesis appears to be that the conjugacy classes of stabilizers + isotropy representations form a finite set. Perhaps if $V_i$ is a family (of bounded dimension) of representations of $G$ and if the (set of) orbit types of $\coprod V_i$ ~~are~~ is finite then only finite many ~~[illegible]~~ inequivalent representations occur. Yes.

<u>Proposition</u>: Let $G$ be a compact Lie group and let $V_i$ $i \in I$ be a family of representations of $G$ in finite dimensional real vector spaces. Assume that

(i) $\dim V_i \leq N$ for all $i$

(ii) $\bigcup_i$ (orbit types of $V_i$) finite

Then the set of isomorphism classes of the family $\{V_i\}$ is finite.

<u>Proof</u>: Let $R_{\mathbb{R}}(G)$ be the representation ring of (real) representations of $G$. We have to show that $\{[V_i] \mid i \in I\}$ in $R_{\mathbb{R}}(G)$ is finite. If $G^0$ is the connected component of $G$, then the restriction map $R_{\mathbb{R}}(G) \longrightarrow R_{\mathbb{R}}(G^0)$ is finite to one†; hence may assume $G$ connected. ~~If $T$ is a maximal torus of $G$, then $R_{\mathbb{R}}(G) \to R_{\mathbb{R}}(T)$~~ Also ~~complexification~~ extension and restriction of scalars from $\mathbb{R}$ to $\mathbb{C}$ define maps

$$R_{\mathbb{R}}(G) \underset{\varphi}{\overset{\psi}{\rightleftarrows}} R(G)$$

with $\psi\varphi = 2$. As $R_{\mathbb{R}}(G)$ is without torsion, $\varphi$ is injective. But (the set of) orbit types of $\mathbb{C} \otimes_{\mathbb{R}} V \cong V + V$ is finite, since the orbits of $V+V$ under $G \times G$ are of form $G/H_1 \times G/H_2$ which is compact and hence its orbit types under $G$ are finite in numbers. Thus we may assume the representations are complex.

Let $T$ be a (maximal) torus. Then $R(G) \longrightarrow R(T)$ is injective so may assume that $G = T$. (Again use lemma: If $G$ acts on $X$

with finitely many orbit types + if $\varphi: H \to G$ is a homomorphism then $X$ has finitely many $H$ orbit types.)

We may also assume the representations $V_i$ are irreducible: In effect ~~[illegible]~~

$R(G) \simeq \bigoplus_{\chi} \mathbb{Z}\chi$ where $\chi$ runs over irreducibles so each $[V_i] = \sum a^i_\chi \chi$ with $a^i_\chi \geq 0$ and $\sum_i a^i_\chi \leq N$. For the set $\{[V_i]\}$ to be finite is therefore the same as the set $\{\chi \mid a^i_\chi \neq 0 \text{ for some } i\}$ to be finite. On the other hand the orbit types of a subrepresentation are contained in those of the representation. (Here use (†), otherwise false because we could take $V_i = \mathbb{C}^i$ with trivial action).

Thus we have a torus $T$ and a set of characters $\chi_i$ of $T$. The orbit type of a character $\chi$ is the subgroup $\operatorname{Ker}\chi$ plus $T$. Thus $\bigcup \operatorname{Ker}\chi_i$ is finite. But $\operatorname{Ker}\chi_i$ determines $\chi_i$ up to signs since $\operatorname{Aut} S^1 = \mathbb{Z}/2\mathbb{Z}$. $\therefore$ $\{\chi_i\}$ is finite. QED.

---

† This assertion is false (take $G$ finite). ~~[illegible]~~ However ~~[illegible]~~

if the $V_i$ restricted to $G^0$ form finitely many isomorphism classes then as $V_i \subset \operatorname{Map}_{G^0}(G, V_i)$ the $V_i$ over $G$ are all subreps of finitely many representations. But the set of isomorphism classes of subrepresentations of a given representation is finite.

Jan 22, 69

Let $\tilde{c}(E) = (\mathrm{rg}\, E, c_1 E, c_2 E, \dots)$ and let $P_i$ be the power series with coefficients in $\Omega^*(pt.)$ such that

$$c_i(E \otimes F) = P_i(\tilde{c}(E), \tilde{c}(F)) \qquad i \geq 0.$$

Then

<u>Conjecture</u>: Let $G$ be a compact Lie group. Then $\tilde{c} : R(G) \longrightarrow \Omega^*_G(pt.)$ is universal with respect to maps of $R(G)$ into an $\Omega^*(pt.)$-algebra such that

$$\begin{cases} c_i(E+F) = \sum_{j+k} c_j(E) c_k(F) \\ \mathrm{rg}(E+F) = \mathrm{rg}\, E + \mathrm{rg}\, F \\ c_i(E \otimes F) = P_i(\tilde{c}(E), \tilde{c}(F)) \qquad i > 0 \\ \mathrm{rg}(E \otimes F) = \mathrm{rg}\, E \cdot \mathrm{rg}\, F \end{cases}$$

<u>Conjecture'</u>: $\tilde{c} : K^*(X) \longrightarrow \Omega^*(X)$ is also universal when $H_*(X)$ has no torsion.

(It might be necessary to add in the formulas for $\lambda_i$, e.g.

$$\tilde{c}_j(\lambda_i E) = Q_{ij}(\tilde{c}(E)) \dots)$$

The conjecture over $\mathbb{Q}$. In this case ~~we~~ want a universal character such that

$$ch(E+F) = ch\,E + ch\,F$$
$$ch(E\cdot F) = ch\,E \cdot ch\,F$$

Thus we want an $\Omega^*_{\mathbb{Q}}(pt)$-algebra $A$ with a ring homomorphism

$$ch : K(X) \longrightarrow A$$

which is universal. It follows that $A \simeq \Omega^*_{\mathbb{Q}}(pt) \underset{K^*(pt)}{\otimes} K^*(X)$ which is true.

---

Recall

$$ch\,L = e^{\sum_{j\geq 1} P_{j-1} \frac{c_1(L)^j}{j}}$$

Recall also the Bergman formulas:

$$\left(1 + \sum_{n\geq 1} a_n t^n\right) = e^{\sum_{j\geq 1} \omega_j \frac{t^j}{j}} = \prod_{n=1}^{\infty} (1 - x_n t^n)^{-1}$$

where the $\omega_j$ are the phantom coordinates and the $x_n$ are the Bergman-Witt coordinates related by

$$\omega_j = \sum_{d|j} d\, x_d^{j/d}$$

<u>Conjecture</u>: Let $Q_j \in \Omega^*(pt) \otimes \mathbb{Q}$ given by

$$P_{j-1} = \sum_{d|j} d\, Q_d^{j/d}$$

Then the $Q_j$ $j\geq 2$ $\in \Omega^*(pt)$ and form a polynomial system of generators.

Problem with this conjecture is that $\dim P_{j-1} = j-1$ hence $Q_j$ not homogeneous.

---

A consequence of the conjecture is that $\Omega^*(pt)$ is isomorphic to the coordinate ring of the universal Witt scheme hence has two natural maps

$$\psi^a, \psi^m : \Omega^*(pt) \longrightarrow \Omega^*(pt) \otimes \Omega^*(pt)$$

given by

$$\begin{cases} \psi^a(P_i) = P_i \otimes 1 + 1 \otimes P_i & i > 0 \\ \psi^m(P_i) = P_i \otimes P_i & i \geq 0. \end{cases}$$

---

Conjecture false because for $j = 2$ it says that

$$P_1 = 2Q_2 + 1$$

however ~~[illegible]~~. $P_1$ generates $\Omega^{-2}(pt)$.

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# Steenrod operations in cobordism

Let $\alpha \in \Omega^q(X)$ be represented by $f : Z \longrightarrow X$ proper + oriented of codimension $q$. Let $G \longrightarrow \Sigma(k)$, symmetric group on $k$ letters be a homomorphism (of finite gps) and let

$\beta \in \Omega^G_r(pt, \sigma^q)$ = bordism classes of equivariant maps $W \longrightarrow pt$ where $W$ is compact oriented of dim $r$ on which $G$ acts freely so as to change orientation by $\sigma^q$, where $\sigma : G \longrightarrow \Sigma(k) \xrightarrow{\text{sign}} \pm 1$.

Then representing $\beta$ by $W \longrightarrow pt$, ~~clear~~ move and form fibre product

$$\begin{tikzcd}
Q \arrow[r] \arrow[d, "\text{codim } kq"'] & Z^k \arrow[d, "f^k"] \\
W \times X \arrow[r, "\text{move } \Delta pr_2"', "\text{transversal to } f^k"'] \arrow[d, "\text{dim } r"', "pr_2"] & X^k \\
X &
\end{tikzcd}$$

Then $G \backslash Q \longrightarrow X$ is proper oriented of codimension $kq - r$ ~~[illegible]~~ and will be denoted $\beta \backslash \alpha^k$. Thus have a cohomology operation

$$\Omega^q(X) \longrightarrow \Omega^{kq-r}(X)$$
$$\alpha \longmapsto \beta \backslash \alpha^k$$

defined for each element

$$\beta \in \Omega^G_r(pt, \sigma^q)$$

~~Cobordism theory in algebraic geometry~~

~~Properties of cohomology~~

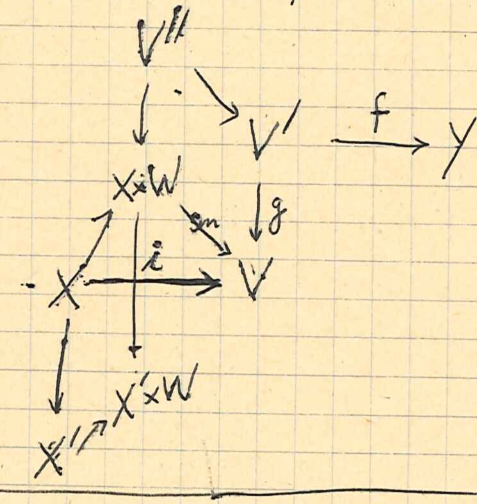
Lesson from equivariant cobordism theory is that any element $\alpha \in Hom_{m}(X,Y)$ is representable

$$\begin{tikzcd} & V' \arrow[r, "f"] \arrow[d, "g"] & Y \\ X \arrow[r, "i"] & V & \end{tikzcd}$$

$g$ proper-oriented

$$\alpha = i^* g_* f^*$$

Proof: Closed under composition



Feel certain that

$$Hom_{m_G}(X,Y) = \varinjlim_{X \to V} \Omega(V \times Y)_{pr/V}$$

where $\Omega$ is the bordism group with not necessarily free action, and where $V$ runs over cat with objects $X \to V$ and maps $V \to V'$ smooth under $X$.

## Character as a natural transformation of cohomology theories:

We have defined

$$\mathrm{ch}: K(X) \longrightarrow \Omega_{\mathbb{Q}}^{ev}(X)$$

and now extend it to $K^{-g}(X)$ $g \geq 0$ by making

$$\begin{tikzcd}
\tilde{K}^{-g}(X) \arrow[r, "\mathrm{ch}"] \arrow[d, "\simeq\ (\text{defn. of } \tilde{K}^{-g})"] & \Omega^{\mathrm{parity}(g)}(X) \arrow[dd, "\simeq\ (\text{canonical isom given by } u\cup \text{ where } u \in \Omega^{g}(S^g, pt.) \text{ is the generator})"] \\
\tilde{K}^{0}(S^g \wedge X) \arrow[d, equal] & \\
K(S^g \times X, * \times X) \arrow[r, "\mathrm{ch}"] & \Omega_{\mathbb{Q}}^{ev}(S^g \times X, * \times X)
\end{tikzcd}$$

(forget ~ notation and instead keep track of which spaces have basepoints)

commutatives. Claim that ch is compatible with Bott periodicity, e.g.

$$\begin{tikzcd}
K(X) \arrow[r, "\mathrm{ch}"] \arrow[d, "\beta\cup"] & \Omega_{\mathbb{Q}}^{ev}(X) \arrow[d, "\text{canonical isom given by cupping with generator } u \in \Omega^2(S^2, pt.)"] \\
K(S^2 \times X, * \times X) \arrow[r, "\mathrm{ch}"] & \Omega_{\mathbb{Q}}^{ev}(S^2 \times X, * \times X)
\end{tikzcd}$$

where $\beta \in K(S^2, pt.)$ is the Bott class $\mathcal{O}(1) - 1$

However $\mathrm{ch}\,\beta = 1 + c_1(\mathcal{O}(1)) - 1 = c_1(\mathcal{O}(1)) = u$

(at least up to sign conventions which we shall not check here)

because $\Omega^2(S^2, pt.) \cong \mathbb{Z}$. Therefore

$$\boxed{\mathrm{ch}: K^{\pm}(X) \longrightarrow \Omega_{\mathbb{Q}}^{\pm}(X) \text{ is a morphism of } \mathbb{Z}/2\mathbb{Z} \text{ graded coh. theories}}$$

Therefore we get the folk theorem that the extension of the character to an $\Omega^*_{\mathbb{Q}}$ linear map

$$\boxed{\Omega'_{\mathbb{Q}}(pt) \otimes_{\mathbb{Z}} K^{\pm}(X) \xrightarrow{\sim} \Omega^{\pm}_{\mathbb{Q}}(X)}$$

is an isomorphism of cohomology theories.

## Steenrod operations on complex projective spaces:

$$H^*(CP^\infty, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}\,[x] \qquad \deg x = 2$$

$$P_t(x) = x + t x^p$$

in general $\quad P^i : H^q \longrightarrow H^{q+2i(p-1)} \quad$ is determined by stability and

$$P^i x = x^p \qquad \text{if degree } x = 2i.$$

Thus

$$P_t(x^k) = (P_t x)^k = (x + t x^p)^k$$

$$= x^k\left(1 + \binom{k}{1} t x^{p-1} + \binom{k}{2} t^2 x^{2p-2} + \cdots\right)$$

$$\therefore \quad \boxed{P^i x^k = \binom{k}{i} x^{k+2i(p-1)}} \qquad P^1(x^k) = k x^{k+2(p-1)}$$

$\neq 0$ if $k \not\equiv 0$ $(p)$.

<u>Lemma</u>: If

$$k = a_0 + a_1 p + \cdots \qquad 0 \le a_i < p$$
$$i = b_0 + b_1 p + \cdots \qquad 0 \le b_i < p$$

are the expressions for $k$ and $i$ with basis $p$, then

$$\binom{k}{i} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1} \cdots \qquad \text{mod } p$$

<u>Proof</u>:

$$(1+X)^k = (1+X)^{a_0 + a_1 p + \cdots}$$

$$\equiv (1+X)^{a_0}(1+X^p)^{a_1} \cdots \qquad (\text{mod } p$$

$$= \left(\sum_{b_0 \le a_0} \binom{a_0}{b_0} X^{b_0}\right)\left(\sum_{b_1 \le a_1} \binom{a_1}{b_1} X^{b_1}\right) \cdots\cdots$$

QED.

Jan 25, 69

<u>Witt ring reviewed:</u>

$A$ ring, $W(A)$ is the ring (in fact $\lambda$-ring) functor of $A$ defined as follows:

$$W(A) \simeq 1 + A[[t]]^+ \qquad \text{as abelian groups}$$

with multiplication and $\lambda$-operations given by "universal polynomials" determined by the rules

$$(1+at)\circ(1+bt) = (1+abt)$$

$$\lambda^i(1+at) = \begin{cases} 1 & i=0 \\ 1+at & i=1 \\ 0 & i\geq 2 \end{cases}$$

Alternatively over the rationals we can use the formulas

$$1+\sum a_n t^n = e^{-\sum_{n\geq 1} w_n \frac{(-t)^n}{n}} = e^{\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} w_n t^n}$$

and say that addition ~~in W(A) is~~ (resp. mult.) in $W(A)$ is given by addition (resp. mult.) componentwise of $\underline{w} = (w_1, w_2, \ldots\ldots)$. To describe the $\lambda$'s on the $w$'s is harder instead (as we are already over $\mathbb{Q}$) we may describe the Adams operations, given by formulas

$$\lambda_t(E) = e^{\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \psi^n(E) t^n} \qquad \psi^0(E) = \text{rang } E$$

Therefore

$$\psi^n(w) = (i \longmapsto w_{ni}) \qquad \text{since } \psi^n(L) = L^n$$

$$\Rightarrow \psi^n(1+at) = 1+a^n t$$

Summary:

$$\begin{cases} \underline{w}' + \underline{w}'' = (i \longmapsto w_i' + w_i'') \\ \underline{w}' \circ \underline{w}'' = (i \longmapsto w_i' w_i'') \\ \psi^n(\underline{w}) = (i \longmapsto w_{ni}) \end{cases}$$

(denoted $F_n$ by Cartier-Mumford).

Another description of $\psi^n$ or $F_n$ is to consider

$$\begin{tikzcd} 1 + A[[t]]^+ & 1 + A[[\bar{t}]]^+ \arrow[l, hook'] \\ t^n & \bar{t} \arrow[l, mapsto] \\ W(A) \arrow[uu, "\simeq"] & W(A) \arrow[l, "\psi^n"'] \arrow[uu, Rightarrow] \end{tikzcd}$$

This enables one to define $V_n$ by

$$\begin{tikzcd} 1 + A[[t]]^+ \arrow[r, "\text{norm}"] & 1 + A[[\bar{t}]]^+ \\ W(A) \arrow[u, equal] \arrow[r, "V_n"] & W(A) \arrow[u, equal] \end{tikzcd}$$

$$V_n(1+at) = 1 + a^n t^n$$

$$\left\{ V_n(\underline{w}) = \left(i \longmapsto \begin{cases} 0 & n \nmid i \\ w_{n/i} \cdot n & \text{otherwise} \end{cases}\right). \right.$$

(of course $V_n$ doesn't define an operation in K-theory, but it might be the transpose of $F_n = \psi^n$ for some kind of Poincaré duality)

$$\boxed{F_n V_n(\underline{w}) = n\underline{w}}$$

Let $W_p(A) \subset W(A)$ be "given by" ?

$$w_i = 0 \quad \text{if} \quad i \text{ is not a power of } p.$$

It is necessary to assume that $A$ is a $\mathbb{Z}_{(p)}$ ($\mathbb{Z}$ localized at the ideal $(p)$) - algebra, in order that $W_p(A)$ be defined. If $A$ is torsion-free we have

$$W_p(A) = W_p(A \otimes \mathbb{Q}) \cap W(A) \subset W(A \otimes \mathbb{Q}).$$

Claim $W_p(A)$ is a sub $\lambda$-ring of $W(A)$.

Proof:

First assume $A$ over $\mathbb{Q}$.

Let $\underline{v} = (v_0, v_1, \ldots) \in W_p(A)$ with inclusion $I: W_p(A) \longrightarrow W(A)$ given by

$$I\underline{v} = \left(i \longmapsto \begin{cases} v_a & \text{if } i = p^a \\ 0 & \text{otherwise} \end{cases}\right)$$

Then

$$\psi^n(\underline{I\underline{v}}) = \begin{cases} \left(i \longmapsto \begin{cases} v_a & \text{if } ni = p^a \\ 0 & \text{otherwise} \end{cases}\right) \\ 0 \quad \text{if } n \neq p^a \text{ for some } a. \end{cases}$$

Set $\psi^n \underline{v} = 0$ if $n \neq p^a$ some $a$, $\psi^{p^a}\underline{v} = (i \longmapsto v_{a+i})$.

$$I(\psi^{p^a}\underline{v}) = \left(i \longmapsto \begin{cases} (\psi^{p^a}\underline{v})_b & \text{if } i = p^b \\ \parallel & \\ \underline{v}_{b+a} & \text{if } i = p^b \ (\Leftrightarrow ni = p^{a+b}) \end{cases}\right)$$

so one sees these are equal. When $A$ torsion-free, it follows that $W_p(A)$ is stable under all $\lambda$-operations, since both $W_p(A \otimes \mathbb{Q})$ and $W(A)$ inside of $W(A \otimes \mathbb{Q})$ are. But any $A$ is a quotient of a torsion free one.

If $k$ is a perfect ring of characteristic $p$, $W_p(k)$ is its Witt ring. It is characterized as a ring $R$ complete for the $p$-adic topology such that $p$ is a non-zero divisor, and such that $R/pR \xrightarrow{\sim} k$. The isomorphism between such a ring $R$ and $W_p(k)$ is constructed as follows: Let $s: k \longrightarrow R$ be the Teichmüller section. Then get isomorphism

$$\underline{x} = (x_0, x_1, \ldots) \longmapsto s(x_0) + s(x_1^{p^{-1}})p + s(x_2^{p^{-2}})p^2 + \cdots$$

where $\underline{x} = (x_0, x_1, \ldots)$ is the description of $W_p(k)$ by Witt coordinates related to phantom coordinates $w_i$ ($= {}^{\text{th}}w_{p^i}$) by

$$w_0 = x_0$$
$$w_1 = x_0^p + p x_1$$
$$w_2 = x_0^{p^2} + p x_1^p + p^2 x_2$$
$$\cdots$$

<u>Problem</u>: We know that $W_p(k) \xrightarrow{\sim} R$ is a $\lambda$-ring. What are the $\lambda$-operations?

It suffices to determine $\psi$ operations. Let $\underline{x} = (x_0, \ldots$ be a Witt vector and let $\underline{x}'$ be $\psi^p \underline{x}$. Claim that

$$\underline{x}' = \psi^p \underline{x} = (x_0^p, x_1^p, \ldots\ldots)$$

In effect in terms of the phantom coordinates $\underline{w}' = (w_1, w_2, \dots)$ if $\underline{w} = (w_0, w_1, \dots)$. Thus ~~in some~~ choosing a torsion free ring over $k$ we have formulas

$$x_0' = x_0^p + p x_1 \quad \Longrightarrow \quad x_0' \equiv x_0^p \quad (p)$$

$$x_0'^p + p x_1' = x_0^{p^2} + p x_1^p + p^2 x_2 \quad \Longrightarrow \quad x_1' \equiv x_1^p \quad (p)$$

etc.

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~~Thus~~ Conclude ~~$\phi$~~

$$\boxed{\begin{array}{ll} \text{On} \quad W_p(k) & \text{char } k = p \\ \psi^n = 0 & (n,p) = 1 \\ \psi^p(x_0, x_1, \dots) = (x_0^p, x_1^p, \dots) & \end{array}}$$

Thus $\psi^p$ on $A$ is just the ~~[illegible]~~ unique lifting of Frobenius on the residue field $k$. The $\lambda$-operations are given by

$$\frac{1}{\lambda_t(a)} = e^{\sum_{n\geq 1} (\psi^p)^n(a) \cdot \frac{t^{p^n}}{p^n}}$$

In the case $k = \mathbb{F}_p$, then $\psi^p = id$ so we have

$$\frac{1}{\lambda_t(a)} = e^{a \sum_{n\geq 1} \frac{t^{p^n}}{p^n}} = \prod_{(p,n)=1} (1-t^n)^{-a\mu(n)/n}$$

which checks since $\mathbb{Z}_p$ is a binomial ring so that $(1+z)^a$ is defined and since the power series with $a = 1$ has coefficients in $\mathbb{Z}_{(p)}$.

January 27-28, 1969:

## $\Omega(\mathbb{P}E)$:

Theorem If $E$ is a complex vector bundle of dimension $n$ over $X$, then $\Omega(\mathbb{P}E) = \Omega(X)[\xi]/(\xi^n - f^*c_1(E')\xi^{n-1} + \cdots + (-1)^n f^* c_n(E'))$, where $\xi = c_1(\mathcal{O}(1))$.

Proof: One shows by a standard induction (see below) on $n$ that

$$\Omega(\mathbb{P}^n) = \Omega[H]/(H^{n+1}) \qquad H = c_1(\mathcal{O}(1))$$

It follows by a Mayer-Vietoris argument that $\Omega(\mathbb{P}E)$ is a free module over $\Omega(X)$ with basis $1, \xi, \ldots, \xi^{n-1}$. Define $c_i(E')$ by the relation

(1) $$\xi^n - f^*c_1(E')\xi^{n-1} + \cdots + (-1)^n f^*c_n(E') = 0.$$

Then $E \mapsto c_i(E')$ is functorial and since $f^*$ is injective for $f: \mathbb{P}E \longrightarrow X$ and any $E$, to prove formulas

$$c(E'+F') = c(E')c(F')$$

may assume $E', F'$ split.

Thus assume $E$ split, $E = L_1 + \cdots + L_n$. Then let $H_j = \mathbb{P}(L_1 + \cdots + \hat{L}_j + \cdots + L_n) \subset \mathbb{P}E$ and $i_j$ the inclusion. Then $H_j$ is the zero set of the section

$$\mathcal{O} \longrightarrow f^*E \otimes \mathcal{O}(1) \longrightarrow f^*(L_j) \otimes \mathcal{O}(1)$$

so

$$i_{j*}1 = c_1(f^*L_j \otimes \mathcal{O}(1))$$

As $\bigcap H_j = \phi$, we have

$$\prod_{j=1}^{n} c_1(f^*L_j \otimes \mathcal{O}(1)) = 0.$$

Recall $\quad c_1(M\otimes N) = F(c_1(M), c_1(N)) = c_1(M) + c_1(N)\left(1 + \overset{c_1(M)}{G}(c_1(M), c_1(N))\right)$

for two line bundles $M, N$. Thus

$$c_1(\mathcal{O}(1)) - c_1(f^*L_j^{-1}) = c_1(\mathcal{O}(1)\otimes f^*L_j)\left[1 + c_1(f^*L_j^{-1})\, G(c_1(f^*L_j^{-1}), c_1(\overbrace{\mathcal{O}(1)\otimes f^*L_j}))\right]$$

so

$$(2) \qquad \prod_{j=1}^{n} \left(\xi - f^* c_1(L_j^{-1})\right) = 0.$$

Comparing coefficients of the relation we have

$$(3) \qquad c_i(E') = \sum_{j_1<\cdots<j_i} c_1(L'_{j_1}) \cdots\cdots c_1(L'_{j_i})$$

proving Whitney sum formula.

For purposes of equivariant cobordism theory we cannot use Mayer-Vietoris so following seems useful. Assume $E = L_1 + \cdots + L_n$. Then as before we have the relation (2). To show that $1, \xi, \ldots, \xi^{n-1}$ form a basis for $\Omega(\mathbb{P}E)$ as a $\Omega(X)$ module we use induction. Let $F = L_1 + \cdots + L_{n-1}$, $L = L_n$.

$$\mathbb{P}F \xrightarrow[f]{i} \mathbb{P}E \xleftarrow{j} \mathbb{P}L = X$$

normal bundle $\mathcal{O}(1)\otimes L$ $\qquad$ normal bundle $\underline{Hom}(L, F) = \sum_{i<n} L^{-1}\otimes L_i$.

<u>We will assume that</u>

$$\Omega(X) \xrightarrow{j_*} \Omega(\mathbb{P}E) \xrightarrow{i^*} \Omega(\mathbb{P}F)$$

(It can probably be proved in $\Omega_G$ theory)

<u>is exact.</u> It then follows that it is split exact since $i^*$ is onto by induction + $f_* j_* = \mathrm{id}$. Now $j(X)$ is where the section

$$\mathcal{O} \longrightarrow f^*E\otimes\mathcal{O}(1) \longrightarrow L_i \otimes \mathcal{O}(1)$$

~~[illegible]~~ vanish for $i<n$ and as these are transversal

$$j_* 1 = \prod_{i<n} c_1(L_i \otimes \mathcal{O}(1))$$

Thus $\Omega(\mathbb{P}E)$ has a $\Omega(X)$- basis consisting of

$$\begin{cases} \xi^i & 0 \le i < n-1 \\ \prod_{i<n} c_1(L_i \otimes \mathcal{O}(1)) & \end{cases}$$

But

$$\prod_{i<n}\left(\xi - c_1(f^*L_i')\right) = \prod_{i<n}\left(c_1(\mathcal{O}(1)\otimes f^*L_i)\left[1 + c_1(f^*L_i')\, G(f^*L_i', \mathcal{O}(1)\otimes f^*L_i)\right]\right)$$

nilpotent, hence [ ] is a unit in $\Omega(\mathbb{P}E)$.

Thus $\Omega(\mathbb{P}E)$ has basis

$$\xi^i \qquad 0 \le i < n-1$$

$$\prod_{i<n}\left(\xi - c_1(f^*L_i')\right)$$

hence also the basis $\xi^i$ $0 \le i < n$, which was to be proved.

Problem: Calculate $f_* : \Omega(\mathbb{P}E) \longrightarrow \Omega(X)$.

As $\Omega(\mathbb{P}E) = \Omega(X)[\xi] \big/ (\xi^n - f^*c_1(E')\xi^{n-1} + \cdots)$
it is enough to know $f_*\xi^i$ $\quad 0 \le i < n$. One knows that there are universal formulas

$$f_*\xi^i = P_{in}(c_1E, \dots, c_nE)$$

since $\Omega(BU(n)) = \Omega[[c_1, \dots, c_n]]$. To determine these formulas, we may assume $E = L_1 + \cdots + L_n$. Set $F = L_1 + \cdots + L_{n-1}$, $L = L_n$.

$$\mathbb{P}F \xrightarrow{\ i\ } \mathbb{P}E \xleftarrow{\ j\ } \mathbb{P}L = X$$

normal bundle $\mathcal{O}_F(1) \otimes L$      normal bundle $L^{-1} \otimes F$

$$i_*1 = c_1(\mathcal{O}(1) \otimes L)$$

$$j_*1 = c_{n-1}(\mathcal{O}(1) \otimes F) = \prod_{i<n} c_1(\mathcal{O}(1) \otimes L_i)$$

where we leave off $f^*$ for convenience.

(inefficient see page 7)

$$\xi - c_1(L') = c_1(\mathcal{O}(1) \otimes L)\underbrace{\left[1 + c_1(L')\,G(c_1(L'), c_1(\mathcal{O}(1) \otimes L))\right]}_{\alpha}$$

$$= i_*1 \cdot \alpha = i_*(i^*\alpha)$$

Therefore

$$\xi - c_1(L') = \iota_* \left\{ 1 + c_1(L')\, G(c_1 L', c_1(\mathcal{O}_F(1) \otimes L)) \right\}$$

and so we "know" $f_*(\xi^{i+1} - c_1(L')\xi^i)$ as $\overset{\text{by induction}}{}$ "know" $g_*$ where $g : \mathbb{P}F \longrightarrow X$. Similarly

$$\prod_{i<n} (\xi - c_1(L_i')) = j_* \left\{ \prod_{i<n} (1 + c_1(L_i')\, G(c_1 L_i', c_1(L \otimes L_i^{?}))) \right\}.$$

So we "know" $f_*$ of

$$\prod_{i<n} \{c_1(L') - c_1(L_i')\} = \prod_{i<n} (\xi - c_1(L_i')) + (\xi - c_1(L'))(\text{something})$$

However $\underbrace{\text{knowing}}$ $\prod_{i<n} (c_1(L') - c_1(L_i')) \cdot f_* 1$ implies we know $f_* 1$ since universally the coefficient is a non-zero divisor. Therefore in principle we can recursively determine $f_*$.

<u>Example</u>: $n=2$, $E = L_1 + L_2$. Let $Q(X,Y) = X - Y + \cdots$ such that $c_1(L \otimes M^{-1}) = Q(c_1(L), c_1(M)) = F(c_1(L), I(c_1(M)))$ where $c_1(M^{-1}) = I(c_1(M))$. Then

$$c_1(\mathcal{O}(1) \otimes f^* L_i) = F(\xi, I(c_1(L_i)))$$

Now want to write this as a $\Omega(X)$ linear combination of $1, \xi$ using the relation

$$(\xi - c_1(L_1'))(\xi - c_1(L_2')) = 0$$

But

$$f(X) \equiv \frac{a f(b) - b f(a)}{a-b} + \frac{f(a) - f(b)}{a-b} X \qquad \text{mod } (X-a)(X-b).$$

$\therefore$

$$c_1\left(\mathcal{O}(1) \otimes f^* L_1\right) = \frac{X F(Y, I(X)) - Y F(\cancel{X, I(X)})}{X-Y}$$

$$+ \frac{F(\cancel{X, I(X)}) - F(Y, I(X))}{X-Y} \xi$$

where $X = c_1(L_1')$, $Y = c_1(L_2')$. Similarly

$$c_1\left(\mathcal{O}(1) \otimes f^* L_2\right) = \frac{X F(\cancel{Y, I(Y)}) - Y F(X, I(Y))}{X-Y}$$

$$+ \frac{F(X, I(Y)) - F(\cancel{Y, I(Y)})}{X-Y} \xi$$

Use that $f_*$ of both are 1, $F(X, I(X)) = 0$ & get

$$1 = \frac{X F(Y, IX)}{X-Y} f_* 1 + \frac{-F(Y, IX)}{X-Y} f_* \xi$$

$$1 = \frac{-Y F(X, IY)}{X-Y} f_* 1 + \frac{F(X, IY)}{X-Y} f_* \xi$$

Solving

$\therefore$

$$\boxed{\begin{aligned} f_* 1 &= \frac{1}{F(X, IY)} + \frac{1}{F(Y, IX)} \\ f_* \xi &= \frac{X}{F(X, IY)} + \frac{Y}{F(Y, IX)} \end{aligned}}$$

and in general

$$\boxed{f_* \xi^q = \frac{X^q}{F(X, IY)} + \frac{Y^q}{F(Y, IX)} \qquad q \geq 0}$$

The general case: $E = L_1 + \cdots + L_n$.

$$\prod_{j \neq i} c_1(\mathcal{O}(1) \otimes f^* L_j) = (\mathbb{P}L_i \longrightarrow \mathbb{P}E)_* 1.$$

$$= \prod_{j \neq i} F(\xi, IX_j) = \prod_{j \neq i} \frac{\xi - X_j}{X_i - X_j} F(X_i, IX_j) \qquad \boxed{X_i = c_1(L_i^\vee)}$$

where the last follows since $\prod_{i=1}^{n}(\xi - X_i) = 0$. ~~and the division~~ (In effect by division algorithm ~~if~~ $G(\xi) = Q(\xi)\prod(\xi - X_i) + R(\xi)$ uniquely with degree $R(\xi) < n$ and $R(\xi)$ is determined by the values $R(X_i)$). Thus we get the equations

$$1 = \left(\prod_{j \neq i} \frac{F(X_i, IX_j)}{X_i - X_j}\right)\left[f_* \xi^{n-1} - c_1(F_i') f_* \xi^{n-2} + \cdots\cdots \right]$$

where $F_i' = \sum_{j \neq i} L_j$, which can be used to ~~successively~~ solve for the $f_* \xi^i$ $0 \leq i < n$. Solution given by

$$\boxed{f_* \xi^q = \sum_{i=1}^{n} \frac{X_i^q}{\prod_{j \neq i} F(X_i, IX_j)}} \qquad X_i = c_1(L_i')$$

In effect

$$f_* \xi^{q+n-1} - c_1(F_k') f_* \xi^{q+n-2} + \cdots + (-1)^{n-1} c_n(F_k') f_* \xi^q$$

$$= \sum_i \frac{X_i^q \left( X_i^{n-1} - c_1 F_k' X^{n-2} + \cdots \right)}{\prod_{j \neq i} F(X_i, IX_j)} = \sum_i \frac{X_i^q \prod_{j \neq k} (X_i - X_j)}{\prod_{j \neq i} F(X_i, IX_j)}$$

$$= \frac{\prod_{j \neq k} (X_k - X_j)}{\prod_{j \neq k} F(X_i, IX_j)}$$

QED.

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Remark: Notice that the right hand side of this formula must be a power series in the $X_i$. ~~[illegible]~~

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Observe that

$$f_* \xi^q = \sum_{i=1}^{n} \frac{X_i^q}{\prod_{j \neq i} F(X_i, IX_j)}$$

$$= \text{total res} \left\{ \frac{Z^q \, dZ}{\prod_{j=1}^{n} F(Z, IX_j)} \right\}$$

Example: $\quad F(X,Y) = X+Y-aXY$

$$IY = \frac{-Y}{1-aY}$$

$$F(X,IY) = \frac{X-Y}{1-aY}$$

$$f_*(\xi^q) = \sum_{i=1}^{n} \prod_{j\neq i} \frac{\left(\frac{1}{a} - X_j\right)}{X_i - X_j} \quad a^{n-1} X_i^q$$

But recall Lagrange interpolation formula

$$P(Z) = \sum_{i=1}^{n} \cdot \prod_{j\neq i} \left(\frac{Z - X_j}{X_i - X_j}\right) \cdot P(X_i) \qquad \text{if } P \text{ is a poly of degree} < n.$$

Therefore taking $\quad P(Z) = a^{n-1} Z^q \qquad q < n \quad$ we have

$$\boxed{f_*(\xi^q) = a^{n-1-q} \qquad 0 \leq q \leq n-1}$$

In particular if $a = 0$ (cohomology) or $a = 1$ (K-theory) we get old formulas. (recall $\xi = 1 - \mathcal{O}(-1)$ in K-theory).

# $\boxed{\text{Formulas for } K_G(\mathbb{P}V)}$

$V$ G-vector bundle over a G-space $X$

$$\boxed{K_G(\mathbb{P}V) = K_G(X)[T]/(\lambda_{-T}(V))}$$

where $T =$ the class of $\mathcal{O}(1)$.

The relation comes from $\underbrace{\text{the}}_{\text{dualizing}}$ sequence

$$\cdots \longrightarrow \Lambda^2 V' \otimes \mathcal{O}(-2) \longrightarrow V' \otimes \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow 0$$

to obtain

$$0 \to \mathcal{O} \longrightarrow V \otimes \mathcal{O}(1) \longrightarrow \Lambda^2 V \otimes \mathcal{O}(2) \longrightarrow \cdots$$

or

$$1 - T\lambda^1(V) + T^2\lambda^2(V) - \cdots\cdots = 0$$

i.e.

$$\boxed{\lambda_{-T}(V) = 0}$$

Thus $K_G(\mathbb{P}V)$ is free as a $K_G(X)$ module with basis $1, T, \ldots, T^{n-1}$ where $n = \dim V$. Perhaps it is better to write out the relation in the form

$$T^n - \lambda^1(V')T^{n-1} + \lambda^2(V')T^{n-2} - \cdots = 0.$$

---

$$f_* : K_G(\mathbb{P}V) \longrightarrow K_G(X) \quad \text{given by}$$

$$f_* T^i = \begin{cases} [S_i V'] & \text{for } i \geq 0 \\ 0 & \text{for } -n < i < 0 \\ (-1)^{n-1} S_{-n-i} V \otimes \Lambda^n V & \text{for } i \leq -n \end{cases}$$

$\Gamma_{-n-i}$ in char $\neq 0$.

The last formula results from Serre duality

$$H^i(E)' = H^{n-1-i}(E'\otimes\omega).$$

tangent bundle $\Theta = \mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}\otimes V/\mathcal{O}(-1))$

$$= \mathcal{O}(1)\otimes V/\mathcal{O}$$

$$0 \to \mathcal{O} \to \mathcal{O}(1)\otimes V \to \Theta \to 0$$

$$\therefore \quad 0 \to \Omega \to \mathcal{O}(-1)\otimes V' \to \mathcal{O} \to 0$$

$$\therefore \omega = \Lambda^{n-1}\Omega = \Lambda^n[\mathcal{O}(-1)\otimes V'] = \mathcal{O}(-n)\otimes\Lambda^n V'.$$

$$H^q(\mathcal{O}(i))' = H^{n-1-q}(\mathcal{O}(-i)\otimes\mathcal{O}(-n)\otimes\Lambda^n V')$$

$\therefore$ if $i \le -n$

$$f_*\mathcal{O}(i) = (-1)^{n-1}H^{n-1}(\mathcal{O}(i))$$

$$= (-1)^{n-1}\{H^0(\mathcal{O}(-i-n))\otimes\Lambda^n V'\}'$$

$$= (-1)^{n-1}(S_{-n-i}V'\otimes\Lambda^n V')'$$

---

As $f_*(\mathcal{O}(i))$ satisfies the **difference equation**

$$f_*(T^i) - \lambda^1(V')f_*(T^{i-1}) + \cdots\cdots + (-1)^n\overbrace{f_*}^{\lambda^n(V')}(T^{i-n}) = 0$$

it admits an expansion in terms of exponentials

$$f_*(T^i) = \sum_{j=1}^{n} c_j(\alpha_j)^i$$ ?

where the $\alpha_j$ are the roots of the equation $\sum_{i=0}^{n}(-1)^i\lambda^i(V')X^i$

$f_*(T^i)$ is not a polynomial function of $i$ unless $V'$ is stably trivial.

---

$$H^*_G(\mathbb{P}V) = H^*_G(X)[\xi]\big/\left(\xi^n + f^*c_1(V)\,\xi^{n-1} + \cdots + f^*c_n(V)\right) \qquad \xi = c_1(\mathcal{O}(1))$$

Check if $V = E_1 \oplus \cdots \oplus E_n$ sum of line bundles

For each $i$ we get a divisor $H_i$ where $\mathcal{O} \otimes E_i' \longrightarrow \mathcal{O} \otimes V' \longrightarrow \mathcal{O}(1)$ fails to be surjective, or equivalently where the section

$$\mathcal{O} \longrightarrow \mathcal{O}(1) \otimes V \longrightarrow \mathcal{O}(1) \otimes E_i$$

$$\boxed{\begin{array}{c} H_i = \mathbb{P}(E_1 + \cdots + \hat{E}_i + \cdots E_n) \\ \subset \mathbb{P}V \end{array}}$$

vanishes. The cohomology class of this divisor is

$$c_1(\mathcal{O}(1) \otimes E_i) = \xi + c_1(E_i).$$

$\boxed{\text{have left out } f^*c_1(E_i)}$

As intersection of these hyperplanes is zero, we get

$$\prod_{i=1}^{n} \left(\xi + c_1(E_i)\right) = \xi^n + c_1(V)\,\xi^{n-1} + \cdots + c_n(V) = 0.$$

Relation might also be written

$$\xi^n - f^*c_1(V')\,\xi^{n-1} + \cdots\cdots + (-1)^n f^*c_n(V') = 0.$$

---

$f_* : H^*_G(\mathbb{P}V) \longrightarrow H^*_G(X)$ is given by

$$f_*(\xi^i) = \begin{cases} 0 & i < n-1 \\ 1 & i = n-1 \\ \sigma_{i-n+1}(V') & i \geq n-1 \end{cases}$$

where $\sigma_i$ are the symmetric functions given by

$$\sum_{i=0}^{\infty} \sigma_i(x_1, \ldots, x_n)\, T^i = \prod_{j=1}^{n} (1 - x_j T)^{-1}$$

e.g. $\sigma_i(x_1, \ldots, x_n) = \sum_{|\alpha|=i} x^\alpha$.

---

<u>Case of cobordism (complex):</u>

We wish to calculate the (complex) cobordism ring of a projective bundle $\mathbb{P}V$. Useful tool is the transformation

$$\Phi : \Omega^*(X) \longrightarrow K(X)$$

defined by $\Phi(f_* 1) = f_! 1$ for $f : Z \longrightarrow X$ oriented + proper. $\Phi$ is compatible with $f_*$, $f^*$ and products and therefore in particular is a ring homomorphism.

If $L$ is a line bundle on $X$ we have $i : X \longrightarrow L$ inducing $i_* : \Omega^*(X) \longrightarrow \Omega^{*+2}_{pr/X}(L)$ and we set

$$i^* i_* 1 = c_1(L) \in \Omega^2(X),$$

the first Conner-Floyd Chern class of $L$. Important:

$$c_1 : \mathrm{Pic}(X) \longrightarrow \Omega^2(X) \quad \text{is } \underline{\text{not}}$$

a homomorphism, since

$$\Phi(c_1(L)) = i^! i_! L = \lambda_{-1}(L') = 1 - L^{-1}$$

is not an additive function of $L$.

The conjecture $\Omega(pt) = \mathrm{Laz}$ + proof $/\mathbb{Q}$
Newton's fmla
good proof that $\Omega(pt)_{\mathbb{Q}} = \mathrm{Laz}_{\mathbb{Q}}$ using
$\Omega_{\mathbb{Q}}(X) =$ univ. Chern ring of $K(X)$.

0

January 29, 1969:

If $A$ is a ring, let $\mathcal{F}(A) = \{F \in A[[X,Y]] \ni$ $F(X,0) = F(0,X) = X, \quad F(X, F(Y,Z)) = F(F(X,Y), Z)$. Then $A \longmapsto \mathcal{F}(A)$ is clearly a representable functor. Given $F$ there is a unique power series $\psi(X) = X +$ higher terms $\in A_{\mathbb{Q}}[[X]] \ni$

$$\psi(F(X,Y)) = \psi(X) + \psi(Y)$$

In fact

$$\psi'(X) = \frac{1}{F_Y(X,0)} \in A[[X]]$$

Writing

$$\psi'(X) = \sum_{i=0}^{\infty} a_i X^i \qquad a_i \in A \qquad a_0 = 1$$

we have

$$\psi(X) = \sum_{i=0}^{\infty} a_i \frac{X^{i+1}}{i+1} \quad \in A_{\mathbb{Q}}[[X]].$$

Conclude: If $(V, F_0)$ represents $\mathcal{F}$ and if

$$\frac{1}{F_{0Y}(X,0)} = \sum_{i=0}^{\infty} a_i X^i$$

then

$$\mathbb{Q}[a_1, a_2, \cdots] \xrightarrow{\sim} V_{\mathbb{Q}}.$$

Consequently if $F(X,Y) \in \Omega^*(pt)[[X,Y]]$ is the series $\ni$

$$c_1(L_1 \otimes L_2) = F(c_1 L_1, c_1 L_2)$$

then there is a unique homomorphism

$$(1) \qquad \varphi : V \longrightarrow \Omega^*(pt)$$

such that $\varphi F_0 = F$. We know that $\varphi a_i = P_i$, hence by Thom

Conjecture: (1) is $V_{\mathbb{Q}} \xrightarrow{\sim} \Omega^*_{\mathbb{Q}}(pt.)$ an isomorphism.

<u>Newton's formulas</u>

Let
$$\psi^q = \sum x_i^q \qquad \text{power sum}$$

$$\lambda^q = \sum_{i_1 < \cdots < i_q} x_{i_1} \cdots x_{i_q} \qquad \text{elementary symmetric fns.}$$

$$-\sum_i \log(1 - tX) = \sum_i \sum_{n \geq 1} \frac{t^n}{n} x_i^n$$

$$-\log \lambda_{-t} = \sum \frac{t^n}{n} \psi^n$$

$$-\log \lambda_t = \sum (-1)^n \frac{t^n}{n} \psi^n$$

$$-\lambda'_t = \lambda_t \sum_{n \geq 1} (-1)^n t^{n-1} \psi^n$$

$$\boxed{\psi^k - \psi^{k-1}\lambda^1 + \psi^{k-2}\lambda^2 - \cdots + (-1)^{k-1}\psi^1\lambda^{k-1} + (-1)^k k \lambda^k = 0}$$

$$\psi^1 = \lambda^1$$
$$\psi^2 = (\lambda^1)^2 - 2\lambda^2$$
$$\psi^3 = (\lambda^1)^3 - 3\lambda^1\lambda^2 + 3\lambda^3$$

January 29, 1969

## Determination of $\Omega_{\mathbb{Q}}(pt.)$

Let $V$ be an algebra over $\mathbb{Q}$ endowed with a formal group law $F$ or equivalently a power series

$$\psi(X) = \sum_{i=0}^{\infty} a_i \frac{X^{i+1}}{i+1} \qquad \text{with} \quad a_i \in V \quad a_0 = 1$$

$$\ni \quad \psi(F(X,Y)) = \psi(X) + \psi(Y)$$

For each manifold $X$ set

(1) $$V(X) = V \otimes_{\mathbb{Z}} K(X);$$

this is ~~[illegible]~~ a contravariant functor from manifolds to $V$-algebras. Let

$$\mathrm{ch}: K(X) \longrightarrow V(X)$$

be the ring homomorphism defined by $\mathrm{ch}\, x = 1 \otimes x$.

We now define Chern classes

$$c_i : K(X) \longrightarrow V(X) \qquad i \geq 0.$$

$$c_0 = 1.$$

If $L$ is a line bundle define $\underbrace{c_1(L) \in V(X)}_{\text{a nilpotent element}}$ by the formula

$$\mathrm{ch}\, L = e^{\psi(c_1(L))}$$

(OKAY because $\mathrm{ch}(1-L)$ is nilpotent in $V(X)$)

$$c_q(L) = 0 \qquad q \geq 2$$

Now if $E$ is any vector bundle on $X$, $V(PE)$ is a free $V(X)$ module with basis $[\mathrm{ch}\, \mathcal{O}(1)]^i$, $0 \leq i < n = \text{rank } E$.

Therefore it also has basis $[ch(\mathcal{O}(1)-1)]^i$ $\quad 0 \le i < n$
and since

$$ch\,(L-1) = \varphi(c_1(L)) - 1 = c_1(L)\underbrace{\{1 + \alpha c_1(L) + \cdots\}}_{\text{unit since } c_1(L) \text{ nilpotent}}$$

$H(PE')$ has basis $c_1(\mathcal{O}(1))^i$ $\quad 0 \le i < n$ over $H(X)$. Hence we may define $c_i(E)$ by the relation

$$\xi^n - f^*c_1 E \cdot \xi^{n-1} + \cdots = 0,$$

where $\xi = c_1(\mathcal{O}(1))$. If $E = L_1 + \cdots + L_n$, then we have the relation

$$\prod_{i=1}^{n} \left(\mathcal{O}(1) - f^* L_i\right) = 0 \qquad \text{in } K(PE')$$

hence applying ch the relation

$$\prod_{i=1}^{n} \left(\varphi(\xi) - f^*\varphi(c_1 L_i)\right) = 0$$

But

$$\varphi(X) - \varphi(Y) = (X - Y)(1 + \underbrace{\cdots\cdots}_{\text{higher order terms which will be nilpotent}})$$

hence

$$\prod_{i=1}^{n} \left[\xi - f^* c_1(L_i)\right] = 0$$

or

$$c_q(E) = \sum_{i_1 < \cdots < i_q}{}' c_1(L_{i_1}) \cdots c_1(L_{i_q}).$$

<u>Digression:</u> We have defined Chern classes using the ~~[illegible]~~ fact that ~~we have~~ K is defined on a category for which we have for each $x \in K(X)$ a representation $x = E - n$ and a splitting map $f : PE \longrightarrow X$, etc. Future work requires us to function with a single $\lambda$-ring K. Thus given $ch : K(X) \longrightarrow V(X)$ we can define ~~[illegible]~~ additive maps $ch_q : K(X) \longrightarrow V(X)$ by the formula

$$ch(\psi^k x) = \sum_{q=0}^{\infty} (k)^q ch_q x \qquad \text{all } k.$$

(This can also be done ~~[illegible]~~ under some nilpotence hypothesis, e.g. in K itself we can write $\psi^k x = \sum_{q=0}^{N} (k)^q x_q \mod F_{N+1}$. Hence $ch_q x = ch(x_q)$, assuming that $ch\, F_{N+1} = 0$.)

$$ch_0 x = rg\, x$$

I claim that there are universal formulas

$$ch_q = P_q(c_1, c_2, \cdots) \qquad q > 0$$

$$c_q = Q_q(ch_1, ch_2, \cdots) \qquad q > 0.$$

~~[illegible]~~ as power series with coefficients in V. To determine these universal formulas one uses the algebraic splitting principle and sets

$$c_t(E) = \prod (1 - tX_i) \qquad E = L_1 + \cdots + L_n$$

$$X_i = c_1(L_i)$$

$$ch(E) = \sum e^{\psi(X_i)}$$

$$ch(\psi^k E) = \sum e^{\psi(c_1(L_i^k))} = \sum e^{k\psi(X_i)}$$

$$= \sum_q \frac{1}{q!} \sum_i \psi(X_i)^q$$

$$\therefore \quad \boxed{ch_q = \frac{1}{q!} \sum_i \psi(X_i)^q}$$

and as the RHS is a symmetric fn. in $X_i$ without constant term $(\psi(X) = X + \cdots)$ (for $q>0$) it can be expressed in terms of the $c_i$, $i>0$. By Newton one knows that

$$\mathbb{Q}[[ch_1, ch_2, \cdots]] = \mathbb{Q}[[\psi X_1, \psi X_2, \cdots]]^{\Sigma} = \mathbb{Q}[[X_1, X_2, \cdots]]^{\Sigma}$$

showing that the $c_i$ are power series in the $ch_i$.

It is not true that $ch_q$ depends only on $c_1, \cdots, c_q$, e.g. for $F(X,Y) = X+Y-aXY$ we have $\psi(X) = -\frac{1}{a}\log(1-aX)$ and

$$ch_2(X) = \tfrac{1}{2}(c_1^2 - 2c_2) + \tfrac{a}{2}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots$$

<u>Remark</u>: Not possible to define

$$ch: K_G(X) \longrightarrow \Omega_G(X) \otimes \mathbb{Q}$$

In effect if $G = S^1$ and $\Omega_G$ is replaced by $H_G$, we have

$$\begin{array}{ccc} K_{S^1}(pt) & & H_G(pt, \mathbb{Q}) \\ \| & & \| \\ \mathbb{Z}[T, T^{-1}] & & H^*(\mathbb{C}P^\infty, \mathbb{Q}) = \mathbb{Q}[X]. \end{array}$$

and $ch\, T = e^X \notin \mathbb{Q}[X]$.

In fact ~~it~~ is not even clear how to make sense out of the formula

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2))$$

since the $c_1(L_i)$ are no longer nilpotent.

---

Now ~~[illegible]~~ for any map $f: X \to Y$ ~~oriented~~ define

$$f^*: V(Y) \longrightarrow V(X)$$

by

$$\begin{cases} f^*(v \cdot y) = v f^*(y) & v \in V \quad y \in K(Y) \\ f^* \operatorname{ch} y = \operatorname{ch} f^! y \end{cases}$$

~~and for any proper oriented map $f: X \to Y$ define~~

$$\cancel{f_*: V(X) \longrightarrow V(Y)}$$

~~by the formula~~

$$\begin{cases} \cancel{f_*(v \cdot x) = v \cdot f_*(x)} \\ \cancel{f_*(\operatorname{ch} x) = \operatorname{ch} f_! x} \end{cases}$$ ~~Dott~~

I claim that for a proper oriented map $f: X \to Y$ it is possible to define

$$f_*: V(X) \longrightarrow V(Y)$$

uniquely so that

$$f_*(v \cdot x) = v f_*(x) \qquad x \in V(X), \ v \in V$$

$$\mathrm{ch}\,(f_! x) = f_*(\mathrm{ch}\, x . \text{Todd}\,\theta_f) \qquad x \in K(X).$$

In effect we have an isomorphism $V \otimes K(X) \xrightarrow{\sim} V(X)$ which we shall again denote by character and we can extend $f_!$ on $K(X)$ to ~~[illegible]~~ $V \otimes K(X)$ so as to be $K$-linear. Let $u(E) \in V \otimes K(X)$ be $\ni$ $\mathrm{ch}\, u(E) = \text{Todd}\, E$ and define

$$f_*(\mathrm{ch}\, z) = \mathrm{ch}\; f_!(z \cdot u(\nu_f)). \qquad z \in V \otimes K(X)$$

Then

$$f_*(\mathrm{ch}\, x \cdot \text{Todd}\,\theta_f) = f_*(\mathrm{ch}\, x \cdot \mathrm{ch}\, u(\theta_f))$$

$$= f_*\, \mathrm{ch}(x \cdot u(\theta_f)) = \mathrm{ch}\; f_!(x \cdot u(\theta_f) \cdot u(\nu_f)).$$

$$= \mathrm{ch}\, f_! x, \quad \text{as claimed.}$$

$u(E)$ is a characteristic class $\underbrace{}_{\text{with}}$ values in $V \otimes K(X)$

$$\begin{cases} u(E+F) = u(E)\,u(F) \\ \mathrm{ch}\, u(L) = \text{Todd}\, L \end{cases}$$

~~If $u(L) = \chi(c_1(L))$ where $\chi(X) \in \Omega_Q[[X]]$, then~~

Let $u(L) = \sum_{n \geq 0} b_n (1 - L^{-1})^n = \chi(1 - L^{-1})$ $b_n \in \Omega_Q(pt)$. Then

$$\mathrm{ch}\, \chi(1 - L^{-1}) = \chi(1 - \varphi(X)^{-1}) = \frac{X}{1 - \varphi(X)^{-1}} \qquad X = c_1(L).$$

so

$$\boxed{\begin{array}{l} u(L) = \chi(1-L^{-1}) \quad \text{where} \\ \{1-\varphi(X)^{-1}\}\,\chi(1-\varphi(X)^{-1}) = X. \end{array}}$$

Thus to find $\chi$ involves inverting $\varphi$. <u>Example:</u> $F(X,Y) = X+Y-aXY$. Then if

$$Y = 1-\varphi(X)^{-1} \qquad \varphi(X) = \frac{1}{1-Y}$$

$$\frac{1}{1-Y} = (1-aX)^{-1/a} \qquad X = \frac{1-(1-Y)^a}{a}$$

$$\therefore \quad \chi(Y) = \frac{1-(1-Y)^a}{aY}$$

$$\boxed{u(L) = \frac{1}{a}\left(\frac{1-L^{-a}}{1-L^{-1}}\right) = \frac{1}{a}\rho^a(L).}$$

where we recall that the Wu class for bundle corresponding to the operation $\psi^k$ is

$$\rho^k(L) = \frac{\psi^k(1-L^{-1})}{1-L^{-1}} = \frac{1-L^{-k}}{1-L^{-1}}.$$

$V(X)$ is therefore a twisted version of the cohomology theory $X \longmapsto V \otimes K(X)$, twisted by the characteristic class Todd. Hence ~~[illegible]~~ there is a unique morphism of cohomology theories

$$\Phi : \Omega_{\mathbb{Q}}(X) \longrightarrow V(X)$$

compatible with ~~Gysin~~ $f^*$ & $f_*$ suchthat $\Phi 1_{pt} = 1_{pt}$.
~~It follows that~~
~~the [illegible]~~

~~[illegible]~~ As $V$ satisfies the splitting principle ($V(\mathbb{P}E)$ free over $V(X)$ etc.) $\Phi$ commutes with Chern classes (one must check that with Gysin defined as above, then $c_1(L) = \iota^* \iota_* 1 \quad \iota : X \to L$). Hence $\Phi : \Omega_{\mathbb{Q}} \to V$ carries the law for $c_1(L \otimes M)$ in $\Omega$ into that for $V$. In particular if the $V$ law is given by $\psi(X) = \sum a_i \frac{X^{i+1}}{i+1}$, then

$$\Phi(P_i) = a_i .$$

Now take $(V, F_0)$ to be the universal formal group law in one variable over $\mathbb{Q}$. $V = \mathbb{Q}[a_1, \dots, a_n, \dots]$ and

$$\psi_0(F_0(X,Y)) = \psi_0(X) + \psi_0(Y)$$

where $\quad \psi_0(X) = \sum_{i \geq 0} a_i \frac{X^{i+1}}{i+1} \qquad a_0 = 1.$

Then we have a unique map

$$\Psi : V \longrightarrow \Omega_{\mathbb{Q}}$$

sending $a_i$ to $P_i$, and $\Psi$ extends to a natural transformation

$$\Psi : V \otimes K(X) \longrightarrow \Omega_{\mathbb{Q}}(X)$$

$$v \otimes x \longmapsto \Psi(v) \cdot ch(x).$$

It is necessary to check that $\Psi$ is compatible with gysin which I think is clear. The point now is that $X \mapsto V(X)$ is that universal recipient for the chern classes so that $\Phi\Psi = id$ while $X \mapsto \Omega_{\mathbb{Q}}(X)$ is the universal cohomology theory so that $\Psi\Phi = id$. Therefore we have proved

**Theorem:** $\Omega_{\mathbb{Q}}(pt) \cong \mathbb{Q}[P_1, P_2, \ldots]$

$$ch : \Omega_{\mathbb{Q}}(pt) \otimes K(X) \xrightarrow{\sim} \Omega_{\mathbb{Q}}(X)$$

$$v \otimes x \longmapsto v \cdot ch\, x$$

$$\begin{cases} ch\, f^! x = f^! ch\, x. \\ ch\, f_! x = f_* (ch\, x \cdot Todd\, \theta_f). \end{cases}$$

where $\begin{cases} ch(L) = e^{\sum P_j \frac{c_1(L)^{j+1}}{j+1}} \\ Todd(L) = c_1(L) / 1 - (ch\, L)^{-1} \end{cases}$

January 30, 1969:

Let $V$ be a ring endowed with a formal group law $F$. ~~[illegible]~~ Let $X \longmapsto V(X)$ be a ~~[illegible]~~ cohomology theory on the category of manifolds with values in $V$-algebras endowed with Gysin homomorphism for $U$-oriented proper maps. We assume $V$ satisfies the splitting principle:

(1) $\quad V(PE)$ is a free module over $V(X)$ with basis $1, \xi, \ldots, \xi^{n-1}$ $\quad \xi = c_1(\mathcal{O}(1))$ $\quad n = \dim E$

and that $\quad (c_1 = i^* i_* 1)$

(2) $$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)).$$

(Note this makes sense since $c_1(\mathcal{O}(1))$ is nilpotent in $V(\mathbb{P}^n)$ hence $c_1(L)$ always nilpotent.)

Then one can define chern classes

$$c_i : K(X) \longrightarrow V(X) \qquad i \geq 0, \quad c_0 = 1$$

so that $c_t : K(X) \longrightarrow 1 + V(X)[[t]]^+$ is the unique natural transf. with

(i) $\quad c_t(x+y) = c_t(x)\, c_t(y)$

(ii) $\quad c_t(L) = 1 + t\, c_1(L)$.

I claim that ~~in terms of the power series~~ there exists unique power series $P_i(c_1, c_2, \ldots; c_1', c_2', \ldots) \in V[[c_1, c_2, \ldots c_1', c_2' \ldots]]$, $Q_{in} \in V[[c_1, \ldots]]$

depending only on $F$ such that

$$c_i(xy) = P_i(c_1(x), \ldots ; c_1(y), \ldots)$$

$$c_i(\lambda^n x) = Q_{in}(c_1(x), \ldots)$$

for all $x, y \in \tilde{K}(X)$ and all $X$. To see this we can argue universally since $V(BU) = \varprojlim V(G_{mn})$ is the power series ring $V[[c_1, \ldots]]$. Here is how one can obtain these power series in principle. Given $x \in \tilde{K}(X)$ (= kernel of $\varepsilon : K(X) \longrightarrow H^0(X, \mathbb{Z})$) it can be written $x = E - n$, $n = \dim E$. Then by splitting may assume $E = L_1 + \cdots + L_n$. Thus

$$x = L_1 + \cdots + L_n - n \qquad c_t(x) = \prod (1 + tX_i)$$

$$y = M_1 + \cdots + M_m - m \qquad c_t(y) = \prod (1 + tY_j)$$

$$X_i = c_1(L_i) \qquad Y_j = c_1(L_j)$$

$$c_t(xy) = \prod_{i,j} c_t((L_i - 1)(M_j - 1)) = \prod_{i,j} \frac{1 + tF(X_i, Y_j)}{(1 + tX_i)(1 + tY_j)}$$

By theorem of elementary symmetric functions the right hand side is a power series in the $c_i(x), c_j(y)$ $1 \le i \le n$, $1 \le j \le m$ depending on $n$ and $m$. However one sees that on going from $n$ to $n-1$ the $(n,m)$th power series goes into the $(n-1,m)$th power series by setting $c_n = 0$. Thus one gets a well-defined power series $P_t$ in $c_1(x), \ldots, c_1(y), \ldots$ which works for all $n, m$. The derivation of the $\lambda$-series is similar but messier:

$$x = L_1 + \cdots + L_n - n$$

$$\lambda_u(x) = \prod_i \lambda_u(L_i - 1) = \prod_i \frac{1+uL_i}{1+u}$$

$$= \prod_i \left\{ 1 - \frac{u}{1+u}(1-L_i) \right\}$$

$$= \prod_{i=1}^{m} \left\{ 1 - u(1-L_i) + u^2(1-L_i) - \cdots \right\}$$

$$= \sum_{j=0}^{\infty} (-u)^j \sum_{p=0}^{j} \sum_{\substack{a_1,\dots,a_p>0 \\ \sum a_i = j}} \sum_{\iota_1<\cdots<\iota_p} (1-L_{\iota_1})\cdots(1-L_{\iota_p})$$

$$\therefore\ \lambda^j(x) = \sum_{p=0}^{j} (-1)^{j-p} \binom{j}{p} \sum_{\iota_1<\cdots<\iota_p} (L_{\iota_1}-1)\cdots(L_{\iota_p}-1)$$

(*) $$c_t(\lambda^j(x)) = \prod_{p=0}^{j} \left\{ \prod_{\iota_1<\cdots<\iota_p} c_t\big((L_{\iota_1}-1)\cdots(L_{\iota_p}-1)\big) \right\}^{(-1)^{j-p}\binom{j}{p}}$$

But

$$c_t\big((M_1-1)\cdots(M_p-1)\big) = c_t\Big( \sum_{j_1\cdots j_q} (-1)^{p-q} M_{j_1}\cdots M_{j_q} \Big)$$

(**) $$= \prod_{I\subset\{1,\dots,p\}} \big(1 + t(X_{j_1} * \cdots * X_{j_q})\big)^{(-1)^{p-|I|}}$$

where $I = \{j_1,\dots,j_q\}$ runs over all subsets of $\{1,\dots,p\}$.

~~[illegible]~~ Combining (*) + (**) one obtains a formula for $c_t(\lambda^j(x))$ as a symmetric power series in $X_1,\dots,X_n$ which can be written as a power series in $c_1,\dots,c_n$. UGH.

We hope to show that $\Omega(X)$ is the universal recipient for a completed Chern class from $K(X)$, at least for manifolds whose homology is torsion-free. Hence

Question: Let $Q(X)$ be the sub $V(pt)$-algebra of $V(X)$ generated by $\operatorname{rg} x$, $c_i(x)$ $i \geq 0$ for all $x \in K(X)$. Is $Q(X)$ stable under $f_*$?

$Q(X)$ is clearly stable under $f^*$. If $f: X \to Y$ is a proper map we may factor it $X \xrightarrow{j} Y \times V \xrightarrow{p} Y$ where $j$ is an embedding. $j_*$ is OKAY because

$$V(X) \xrightarrow{\iota_*} V(N, \dot{N}) \xrightarrow[\text{OKAY}]{g^*} V(Y \wedge V^+)$$

$$\iota_*(x) = \pi^*(x) \cdot \iota_* 1$$

and $\iota_* 1 = c_n(\pi^* \nu)$. However $\pi_*: V(Y \wedge V^+) \longrightarrow V(Y)$ doesn't seem to carry $Q(Y \wedge V^+)$ to $Q(Y)$ since all we know is that if $\alpha = \iota_* \beta \in Q(Y \wedge V^+)$, then $\alpha = \pi^* \beta \cdot \iota_* 1$, i.e. $\beta = \iota^* \left\{ \frac{\alpha}{\iota_* 1} \right\}$. This means that to get something stable under $f_*$ one must permit division of some sort. For example if we work over $\mathbb{Q}$ then $(\operatorname{rg}, c_i(x)) = \hat{c}(x)$ is expressible in terms of $ch_g x$ and conversely, hence $Q(X)$ is generated by $V(pt)$ and $ch\{K(X)\}$; one knows that $f_*$ on $Q(X)$ is then determined by $f_!$ on $K(X)$ and the characteristic classes of $f$.

Answer to question is probably false, in fact Adams claims $\exists$ a finite cx $X$ with $\tilde{K}(X) = 0$ but with $\tilde{H}(X) \neq 0$. ~~Suppose~~ Hence $\Omega(X) \neq 0$ which means that $\exists\ f: Z \to X$ proper + oriented with $f_* 1 \neq 0$ yet $Q(X) = 0$.