January 12, 1967

Summary: If $Q_n = P(1+O(1)/P^n)$, then we have for $f: P(1+L) \rightarrow X$, that

$$f_\ast 1 = \sum_{n=0}^{\infty} a_n c_1(L)^n$$

where

$$[Q_n] = \sum_{k=0}^{n} a_k [P^{n-k}]$$

This determines the $a_k$ in terms of the $[P^n]$ and the $[Q^n]$; but it would be nice to have a formula in terms of the $[P^n]$ alone.

$$\Phi [Q_n] = \sum_{k=0}^{n} \phi(a_k)$$

however $\Phi$ have $Q \rightarrow P^n \rightarrow P^n$ is an iterated projective bundle, hence $\Phi [Q_n] = 1$.

$$\phi(a_k) = \begin{cases} 1 & k = 0 \\ 0 & k > 0 \end{cases}$$

$$[Q_0] = [P^1]$$

$$[Q_1] = [P^1]^2 + a_1$$
Calculations of Chern numbers for \( Q_n = P(1 + 0(1)/P^n) \).

Let \( f: P(1+L/P^n) \to P^n \) where \( L = 0(1) \). Working in ordinary homology, we have \( c_i(L) = H \) where \( H^*(P^n) = \mathbb{Z}[H]/(H^{n+1}) \).

\[
T_{P^n} = \frac{(n+1)L}{1}
\]

\[
c(T_{P^n}) = (1 + H)^{n+1}
\]

\[
T_f = f^*(1+L) \otimes 0(1)/0
\]

\[
c(T_f) = c(f^*(1+L) \otimes 0(1))
\]

\[
= c(0(1)) \cdot c(f^*L \otimes 0(1))
\]

\[
= (1 + \bar{y})(1 + f^*H + \bar{y})
\]

\[
= 1 + (2\bar{y} + f^*H)
\]

From now on we drop \( f^* \).

\[
c(T_{Q_n}) = c(T_{P^n}) c(T_f) = (1 + H)^{n+1}(1 + 2\bar{y} + H)
\]

\[
c_1(T_{Q_n}) = (n+2)H + 2\bar{y}
\]

\[
\sum_{i=0}^{n+1} c_i(T_{Q_n}) = (n+2)H + 2\bar{y}
\]

\[
\sum_{i=0}^{n+1} c_i(T_{Q_n}) = a^{n+1} + \frac{b^{n+1} - a^{n+1}}{b - a} (b - a)
\]

\[
= a^{n+1} + \frac{b^{n+1} - a^{n+1}}{b - a} (b - a)
\]

Now \( b - a = 2\bar{y} \) and \( \bar{y} = -H \mod \bar{y} \)

\[
b \equiv nH \mod \bar{y}
\]
\[ c_1(T_{\alpha n})^{n+1} = \frac{(nH)^{n+1} - ((n+2)H)^{n+1}}{-2H} \cdot 2^g \]

\[ = \{ (n+2)^{n+1} - n^{n+1} \} H^n \cdot 2^g \]

\[ \int_{Q_n} c_1(T_{\alpha n})^{n+1} = \{ (n+2)^{n+1} - n^{n+1} \} \int_{H^n} f^g \]

\[ = \{ (n+2)^{n+1} - n^{n+1} \} \]

\( (= 56, n=2) \)

On the other hand

\[ c_1(T_{p^1 \times p^n}) = \mathbb{P}^n \cdot c_1(T_{p^1}) + \mathbb{F}^n \cdot c_1(T_{p^n}) \]

\[ = 2^g + (n+1)H \]

\[ c_1(T_{p^1 \times p^n})^{n+1} = 2^g \cdot ((n+1)H)^n \cdot (n+1) \]

\[ = 2^g \cdot ((n+1)H)^n \cdot (n+1) \]

\[ = 2^g \cdot (n+1)^{n+1} H^n \cdot 2 \]

\[ \int_{P^1 \times P^n} c_1(T_{p^1 \times p^n}) = 2(n+1)^{n+1} \]

\( (= 54, n=2) \)

Thus \[ [Q_2] \neq [P_1][P_2] \].

However for \( n = 1 \)

\[ \int_{Q_1} c_1(T_{\alpha 1})^2 = 9 - 1 = 8 \]

\[ \int_{Q_1} c_2(T_{\alpha 1}) = \chi(Q_1) = 4 \]

\[ \int_{P_1} c_1(T_{p_1}) = 2 \cdot 2 = 8 \]

\[ \int_{P_2} c_2(T_{p_2}) = \chi(P_2)^2 = 4 \]

and therefore by Milnor one knows that \[ [Q_2] = [P_1]^2 \].
X manifold, L line bundle on X.

\[ \begin{array}{c}
X \xrightarrow{i_1} P(1+L) \xleftarrow{i_2} X \\
\text{normal bundle } L^{-1} \quad \text{normal bundle } L
\end{array} \]

\( i_2(x) \) = the line \( C(1,0) \) in the fibre over \( x \)
\( i_1(x) \) = the line \( 0 \oplus L(x) \).

\( \mathcal{O}(-1) = \{ (u,v) \mid u \text{ line in } 1+L, u \cdot v \neq 0 \} \).

\( i_2^* \mathcal{O}(-1) = \{ u \mid u \in C \oplus 0 \text{ over } X \} = 1 \) trivial
\( i_1^* \mathcal{O}(-1) = \{ u \mid u \in C \oplus L \} = L \).

\[ \begin{aligned}
& \therefore \begin{cases} 
L_1^* \mathcal{O}(1) = L^{-1} \\
L_2^* \mathcal{O}(1) = 1
\end{cases} \\
& (l_1)_* \cdot 1 = c_1(\mathcal{O}(1)) \quad \text{defn} \quad \mathbb{Z}
\end{aligned} \]

\( i_1(x) \) is the place where \( \mathcal{O}(-1) = \mathcal{O} + f^*L \) or equivalently where \( \mathcal{O}(-1) \rightarrow f^*(1+L) \rightarrow f^*1 \) vanishes, or equivalently where the canonical section of \( \mathcal{O}(1) \) vanishes. Therefore

\[ \begin{array}{c}
(l_1)_* \cdot 1 = c_1(\mathcal{O}(1)) \\
\text{defn} \quad \mathbb{Z}
\end{array} \]

Similarly \( i_2(x) \) is the place where \( \mathcal{O}(-1) = f^*1 + \mathcal{O} \) or equivalently where \( \mathcal{O}(-1) \rightarrow f^*(1+L) \rightarrow f^*L \) is zero, or where the canonical section of \( \mathcal{O}(1) \otimes f^*L \) is zero. Thus

\[ (l_2)_* \cdot 1 = c_1(\mathcal{O}(1) \otimes f^*L) . \]
hence \( f_*(\tilde{\mathfrak{a}}) = 1 \)
\[ f_* \circ (\mathfrak{c}(L) \circ f^* L) = 1 \]

Now we know from exact sequence

\[ \Omega^*_X(L) \xrightarrow{(f_*)} \Omega^*(P(1+L)) \xrightarrow{\iota_*^*} \Omega^*(X) \]

that \( \Omega^*(P(1+L)) \) is free as an \( \Omega^*(X) \) module with basis 1 and \((L)_x 1\). Similarly it has basis 1, \((L)_x 1\).

\[ \Omega^*(P(1+L)) = \Omega^*(X).1 + \Omega^*(X).\tilde{\mathfrak{a}} \]

Write \((L)_x 1 = c_1(\mathfrak{c}(L) \circ f^* L) = A + B \cdot \mathfrak{a} \) \( B \in \Omega^X \) \( A \in \Omega^X \). 

so now apply \( \iota_*^* \)

\[ \iota_*: \circ = (\mathfrak{a})^* (L)_x 1 = A + B \cdot \mathfrak{a} \]

\[ \iota_*: c_1(L) = (L)_x 1 = A + B \cdot \mathfrak{a} \]

Therefore

\[
\begin{align*}
A &= c_1(L) \\
\mathfrak{c}(L) B &= -c_1(L)
\end{align*}
\]

Apply \( f_* \)

\[ 1 = f_*(A + B \mathfrak{a}) = A \cdot f_* 1 + B \cdot f_*(\mathfrak{a}) = A \cdot f_* 1 + B \]
Now multiply by \( c_i(L^{-1}) \) and find

\[
c_i(L^{-1}) = c_i(L^{-1}) c_i(L) f_*(1) \mathcal{O} - c_i(L).
\]

However,

\[
f_*(1) = \sum_{k=0}^{\infty} a_k c_i(L)^k, \quad a_k \in \mathbb{Q}^{2k-2}(L^k)
\]

so

\[
c_i(L) + c_i(L^{-1}) - c_i(L^{-1}) \sum_{k=0}^{\infty} a_k c_i(L)^{k+1} = 0
\]

\[
\therefore \quad c_i(L^{-1}) = \frac{-c_i(L)}{1 - \sum_{k=0}^{\infty} a_k c_i(L)^{k+1}}
\]

\[\times\text{ manifold, } L_1, L_2 \text{ line bundles over } X\]

\[
\begin{align*}
\xrightarrow{\ell_1} & \quad \mathbb{P}(L_1 + L_2) \quad \xleftarrow{\ell_2} \quad X \\
\uparrow \mathbb{P}(L_1) & \quad \text{normal bundle } L_1^* \otimes L_2 \quad \text{normal bundle } L_2^* \otimes L_1
\end{align*}
\]

\[
\mathcal{O}(L^{-1}) = \{ (\mathcal{O}_L, x) \mid L \subset L_1 + L_2, \mathcal{O} \in L \}
\]

\[
\ell_1^* \mathcal{O}(L^{-1}) = L_1 \quad \ell_1^* \mathcal{O}(1) = L_1^*
\]

\[
\ell_2^* \mathcal{O}(L^{-1}) = L_2 \quad \ell_2^* \mathcal{O}(1) = L_2^*
\]
$L_1(x)$ is where $O(-1) \subset L_1$, or where $O(-1) \rightarrow f^*(L_1 + L_2) \rightarrow f^* L_2$
vanishes, or where the canonical section of $O(1) \otimes f^* L_2$ is zero

\[
\begin{align*}
(L_1)_* 1 &= c_1(\mathcal{O}(1) \otimes f^* L_2) \\
(L_2)_* 1 &= c_1(\mathcal{O}(1) \otimes f^* L_1)
\end{align*}
\]

\[
\begin{align*}
\tilde{z} &= A + B (L_1)_* 1 \\
\bar{z} &= \overline{A + B (L_2)_* 1}
\end{align*}
\]

\[
\begin{align*}
L_1^* &= c_1(L_1^{-1}) = A + B c_1(L_1^{-1} \otimes L_2) \\
L_2^* &= c_1(L_2^{-1}) = A + B c_1(L_2^{-1} \otimes L_1)
\end{align*}
\]

\[
\begin{align*}
f_* : f_*(\tilde{z}) &= A f_*(1) + B \\
f_* : f_*(\bar{z}) &= \overline{A f_*(1) + B}
\end{align*}
\]

According to our preceding calculation:

\[
c_1(L_2^{-1} \otimes L_1) \left\{ 1 - c_1(L_1^{-1} \otimes L_2) f_*(1) \right\} = -c_1(L_1^{-1} \otimes L_2)
\]

where $g : \mathbb{P}(1 + L_1^{-1} \otimes L_2) \rightarrow X$

but $g$ is isomorphic to $f$.

\[
c_1(L_2^{-1} \otimes L_1) = \frac{-c_1(L_1^{-1} \otimes L_2)}{1 - c_1(L_1^{-1} \otimes L_2) f_*(1)}
\]

\[
f_*(1) = \sum_{k=0}^{\infty} a_k c_1(L_1^{-1} \otimes L_2)^k.
\]
Now set
\[ z = c_1 (L_1^{-1} \otimes L_2) \]
\[ z' = c_1 (L_2^{-1} \otimes L_1) \]
\[ x = c_1 (L_1^{-1}) \]
\[ y = c_1 (L_2^{-1}) \]

\[ z + z' = z z' f_k (1) \]
\[ f_k (1) = \sum_{k=0}^{\infty} a_k z^k \]

\[
\begin{cases}
  x = A + B z \\
  y = A \\
  f_k (z) = A f_k (1) + B \\
\end{cases}
\]

\[ x = \overline{A} \]
\[ y = \overline{A + B z'} \]
\[ f_k (z) = \overline{A f_k (1) + B} \]

\[ z z' y f_k (1) + z' (x - y) = z z' x f_k (1) + z (y - x) \]

\[ (y - x) [z z' f_k (1)] = (y - x)(z + z') \]

Therefore we obtain nothing new.
Where did the calculation go wrong?

$X$ manifold, $L$ a complex line bundle over $X$.

$s$ section of $L$, transversal to $y_{2}$, $y = s^{-1} 0$.

Then there is a map

$\Theta: X \times C \rightarrow L$

$\Theta(x, \lambda) = \lambda s(x)$

which is an isomorphism over $X - Y$. Consequently there is an isomorphism

$\Theta: \left( X \right) \times \mathbb{P}^1 \rightarrow \mathbb{P}((1 + L)|_{X - Y})$

$(x, \lambda, \lambda_1) \mapsto \lambda_0 + \lambda_1 s(x)$

Therefore if $f: \mathbb{P}(1 + L) \rightarrow X$ and $g: X \times \mathbb{P}^1 \rightarrow X$

are the canonical maps, we should have that

$f \circ i - g \circ i = i \circ x$

where $i: Y \hookrightarrow X$ and $x \in \Omega^d(Y)$. We propose to determine $x$.

General situation: Given a manifold $X$ and two proper oriented manifolds $Z_0$ and $Z_1$ over $X$ and an isomorphism $\Theta: Z_0 \times X - A \rightarrow Z_1 \times X - A$ over $X - A$ respecting orientation, one forms $W = Z_1 \times I \rightarrow X - A$ and defines

$\varphi: (-Z_0) \cup Z_1 \rightarrow \partial W$

to be identity on $Z_1$ and $\Theta$ on $Z_0$.

This defines an element of $\Omega_A(X)$. 
By the usual excision process we can move everything inside of a neighborhood of \( A \). If \( A \) is an submanifold, then one excises into a tubular neighborhood \( N \) and throws away \( W \) taking \( Z \rightarrow N \rightarrow A \), and this gives the isomorphism \( \Omega(A) \cong \Omega(A) \).

The thing to note is that as \( \Theta \) is an isomorphism we take a tubular nbhd \( N \) of \( A \) and glue \( -Z_0 \vert N \) and \( Z_1 \vert N \) together over \( \partial N \) by means of the isomorphism \( \Theta: Z_0 \vert \partial N \cong Z_1 \vert \partial N \).

**Original situation.** Here \( Z_0 = X \times \mathbb{P}^2 \), \( Z_1 = \mathbb{P}(-1+L) \) and \( \Theta \) is as described above. As the situation is local we assume that \( X \cong L \mid Y \) and that \( L = X \times Y(L \mid Y) \). Let \( E = L \mid Y \). Have to glue

\[-(X \times \mathbb{P}^2)\mid \partial E \quad \text{and} \quad \mathbb{P}(-1+L)\mid \partial E \cong \partial E \times \mathbb{P}(1+E)\]

by means of the isomorphism

\[SE \times \mathbb{P}^2 \longrightarrow SE \times \mathbb{P}(1+E) \]

\[(z, (0, l)) \quad \longrightarrow \quad (z, (0, l))\]
Problem: What is a complex manifold?

Not the same as conjugation

If \( X \) is an almost complex manifold so that \( T_X \) is endowed with a complex structure, let \( X \) denote the complex manifold given by \( X = X \) as manifolds and the cpx. structure on \( T_X \) is conjugate to that of \( X \), i.e.

If \( \overline{v} \in T_X \) corresponds to \( v \in T_X \), then

\[ \lambda \overline{v} = \overline{\lambda v} \]

If \( E \) is a complex vector bundle, then \( E \cong E' \) as far as homotopy goes, since if \( \langle , \rangle \) is a hermitian metric get \( \varphi : E \to E' \) \( \varphi(v)(w) = \langle v, w \rangle \). It's clear that \( E' \) is very seldom stably equivalent to \( -E \).

This is why you get wrong answer before. You glued \( DE \) and \( DE \) together to form \( P(I+\mathbb{E}) \) where you should have glued \( -DE \) and \( DE \) to get the fiber suspension of \( SE \), the boundary of the disk in \( 1_\mathbb{R} \oplus \mathbb{E} \).

Get formula for \( q_1 \): Take \( O(1) \) over \( S^2 \) get a projective bundle \( P(1+O(1)/S^2) \), however we now put in the weakly complex structure on \( S^2 \) so that it's the boundary of \( O(3) \). Thus get an weakly cpx. structure on \( P(1+O(1)/S^2) \) different from usual.
Given \( L/X \) to glue \( P^2 + P(1+L) \) by \( L-X \).

\[
\begin{array}{c}
\text{DL} \\ \text{SL} \end{array} \quad \beta' \\
\text{DL} \quad \beta \\
\text{SL} \quad \gamma
\end{array}
\]

\[
\alpha (u, \lambda) = (u, \lambda u).
\]

\[
\beta(u) = C(1, u).
\]

\[
\gamma(u) = C(1, u^*)
\]

where \( u^* \) unique \( \langle u^*, u \rangle = 1 \)

\[
\beta'(u, \lambda)(x) = (\text{the line } C(1, u) \text{ at } \lambda x) \quad \text{the linear function on this line given by } \mu((1, x)) = x
\]

\[
\gamma'(u, v) = (\text{the line } C(1, v^*) \text{ at } u)
\]

Locally suppose \( \sigma = x \sigma \) \( u^* = \sigma x \) \( \lambda^{-1} x^* \leftarrow \langle \sigma x, x^* \rangle (1, x) \leftarrow \langle x^*, x \rangle \)

\[
\mu(1, \sigma x^*) = \langle u, x^* \rangle \sigma^{-1} \quad \text{or} \quad \mu(\sigma, x^*) = \langle u, x^* \rangle
\]

showing that \( \mu \) depends nicely as \( x \to 0 \).
January 13, 1969: Riemann-Roch thm. for $\Omega$:

A universal formula

$$ c_1(L_1 \otimes L_2) = \sum_{k, \ell \geq 0} b_{k\ell} c_1(L_1)^k c_1(L_2)^\ell $$

where $b_{k\ell} \in \Omega^*(pt)$. This follows by determination of $\Omega^*(\mathbb{P}^n \times \mathbb{P}^m)$. As multiplication of line bundles is associative, etc., it follows that

$$ F(X, Y) = \sum_{k, \ell \geq 0} b_{k\ell} X^k Y^\ell $$

is a formal group law over $\Omega^*(pt)$. Hence over $\Omega^*(pt) \otimes \mathbb{Q}$ there is a power series $\psi(X)$ such that

$$ \psi\left( \sum b_{k\ell} X^k Y^\ell \right) = \psi(X) + \psi(Y) $$

$$ \psi(0) = 0, \quad \psi'(0) = 1 $$

To find $\psi$ differentiate with respect to $Y$ and set $Y = 0$.

$$ \psi'(F(X, 0)) \cdot \frac{\partial F(X, 0)}{\partial Y} = \psi'(0) = 1 $$

$$ \psi'(X) = \frac{F'(X, 0)^{-1}}{F'(X, 0)} = \left[ \sum b_{k\ell} X^k \right]^{-1} $$

$$ \psi(X) = \int_0^X \left[ \sum b_{k\ell} X^k \right]^{-1} dX $$

Claim that $\psi(F(X, Y)) = \psi(X) + \psi(Y)$.

This is true mod $Y^2$. Observe that as functions of $Y$, they coincide for $Y = 0$, hence (char 0) - enough to show their derivatives coincide for $Y = 0$.  

...
are equal, i.e.

\[ \psi'(F(X,Y)) F_2(X,Y) = \psi'(Y). \]

By assumption, true for \( Y = 0 \), i.e.

\[ \psi'(X) F_2(X,0) = 1 \]

Also

\[ F(X, F(Y,Z)) = F(F(X,Y), Z) \]

so applying \( \frac{\partial}{\partial Z} \) and setting \( Z = 0 \), yields

\[ F_2(X,Y) F_2(Y,0) = F_2(F(X,Y), 0). \]

\[ \psi''(Y) = F_2(Y,0)^{-1} = F_2(F(X,Y), 0)^{-1} F_2(X,Y) = \psi''(F(X,Y)) F_2(X,Y), \]

which proves the claim.

Now set

\[ \psi(x) = e^{\psi(x)} \]

so that

\[ \begin{cases} 
\psi(F(X,Y)) = \psi(x) \psi(y) \\
\psi(0) = 1, \quad \psi'(0) = 1 
\end{cases} \]

and define

\[ \text{ch}(L) = \psi(c_1(L)) \in \Omega^*(X) \quad \text{(if dim } X < \infty) \]

\[ \text{ch}(L_1 \otimes L_2) = \text{ch}(L_1) \text{ch}(L_2) \]

\[ \text{ch}(L_0) = 1 + c_1(L) + \ldots \]
Let \( \Phi: \Omega^* \to K \) be the canonical map. Then

\[
\Phi(c_1(L_1 \otimes L_2)) = 1 - L_1^{-1}L_2^{-1} = (1-L_1^{-1}) + (1-L_2^{-1}) - (1-L_1^{-1})(1-L_2^{-1})
\]

\[
= \Phi(c_1(L_1)) + \Phi(c_1(L_2)) - \Phi(c_1(L_1)) \cdot \Phi(c_1(L_2))
\]

\[
\therefore \quad F(x, y) = x + y - xy \quad \text{in this case so}
\]

\[
\Phi(x) = (1-x)^{-1}, \quad \Psi(x) = -\log(1-x)
\]

for this formal group. By uniqueness of these functions we find that

\[
\Phi(c_1(L)) = L = \Psi(\Phi(c_1(L)))
\]

Therefore if

\[
\Phi(x) = 1 + x + a_2x^2 + \ldots \quad a_i \in \Omega^*(pt)
\]

\[
\Phi(a_i) = 1
\]

Similarly if \( \varepsilon: \Omega^* \to H^* \) is the canonical map

\[
\varepsilon(ch_L) = ch^H_L.
\]

Therefore

\[
\varepsilon(\Phi(x)) = e^x
\]

i.e.

\[
\varepsilon(a_i) = \frac{1}{i!}
\]

Now by means of the splitting principle we can extend \( ch \) to a ring homomorphism

\[
ch: K(X) \to \Omega(X) \otimes \mathbb{R}
\]
which is compatible with $f^*$ and a section of $\xi$. Compatibility with $f_*$ doesn't hold, and leads to a Riemann-Roch thm:

$$\text{ch}(f_!(x)) \cdot \text{Todd}(\Theta_x) = f_* (\text{ch}_x \cdot \text{Todd}(\Theta_x)),$$

where $x \in K_{pr/y}(Y)$ and $f: X \to Y$, and where Todd is a multiplicative extension of a characteristic class given on line bundles by a power series with leading term $1/n!$.

Claim it is enough (granted a workable formalism of $K$-theory with support) to prove this formula, where $f_*$ is the inclusion of the zero section of a line bundle. Thus can rewrite the formula

$$\text{ch}(f_!(x)) = f_* (\text{ch}_x \cdot \text{Todd}(\Theta_x))$$

where it suffices for compositions. Factoring $f$ into an inclusion $X \hookrightarrow Y \times V \xrightarrow{pr_2} Y$ one reduces to the case where $f$ is either $i: X \to V$ or $\pi: V \to X$ and $V$ is a vector bundle, over $X$. Here however we have that

$$L_*: \Omega(X) \xrightarrow{\sim} \Omega_{pr_2/x}(V) \quad L^*: \Omega(V) \xrightarrow{\sim} \Omega(X)$$

$$L_!: K(X) \xrightarrow{\sim} K_{pr_2/x}(V) \quad L^!: K(V) \xrightarrow{\sim} K(X)$$

are isomorphisms. Thus given $x \in K(X)$ have $x = L^! \sigma$ (where $\sigma = \pi^! x$), so

$$\text{ch}(L^! x) = \text{ch}(L_!(\sigma \epsilon V)) = \text{ch}_V \cdot (\text{ch} L_! x)$$

$$L_*(\text{ch}_x \cdot \text{Todd}(V)^{-1}) = \text{ch}_V \cdot L^*(\text{Todd}(V)^{-1}).$$
reducing to proving that

\[ \text{ch}_x(1) = i_*((\text{Todd } V)^{-1}) \]

which implies

\[ \text{ch}(L \cdot 1) = (\text{ch}_x(1)) (\text{Todd } V)^{-1} \]

Similarly, if \( v \in K(V) \), have \( v = \frac{1}{x} \cdot x \) and

\[ \text{ch}(\pi_1 V) = \text{ch} x \]

\[ \pi_x (\text{ch}(L \cdot x) \pi^* \text{Todd } V) = \pi_x (\text{ch}(L \cdot 1) \pi^* \text{Todd } V) \]

reducing to

\[ \pi_x (\text{ch}(L \cdot 1)) \cdot \text{Todd } V = 1 \]

which follows from (1) by applying \( \pi_x \). By the splitting principle and fact that \( L \) and \( L \cdot x \) commute with smooth base change, we may assume \( V \) splits and reduce to cohomology 1, i.e. when \( V \) is a line bundle \( L \). Formula (2) shows us that

\[ \text{Todd } (L) = \frac{c_1(L)}{\text{ch}(1-L^{-1})} \]

and conversely this implies (2). Here's a lousy argument that

\( (2) \Rightarrow (1) \): Write left and right sides of (1) as \( i_* a, i_* b \), getting then \( (c_1 x)_* a = c_1 x_* a = c^* i_* b = (i_* a)_* b \) using (2), so
\[ c_1(L) (a-b) = 0. \] But \( a-b \) is a power series in \( c_1(L) \) and \( c_1(O(1)) \) is a non-zero divisor in \( \Omega^*(P^n) = \Omega^*[c_1(O(1))] \). \( \therefore a = b \).

This proves Riemann-Roch.

Apply Riemann-Roch to \( P^n \):

\[ \Theta_{\mathbb{P}^n} = (n+1)O(1)/O \]

\[ \text{Todd}(\Theta_{\mathbb{P}^n}) = (\text{Todd } O(1))^{n+1} \]

\[ f_* \{ \text{ch } O(b) \cdot \text{Todd}(\Theta_{\mathbb{P}^n}) \} = \text{ch} \{ f_! O(b) \} \]

where \( f : \mathbb{P}^n \to \) pt. \( f_! O(b) = \chi(O(b)) = \binom{q+n}{n} \).

Let \( H = c_1(O(1)) \) so that \( \Omega^* (P^n) = \Omega^* (pt)[H]/H^{n+1} \)

\[ f_* (H^i) = \begin{cases} [P^{n-i}] & \text{for } i \leq n \\ 0 & \text{for } i > n \end{cases} \]

\[ \text{Todd } O(1) = \frac{\prod c_1(O(1))}{\text{ch}(1-O(1)^{-1})} = \frac{H}{1 - \frac{1}{\varphi(H)}} = \frac{\varphi(H)}{\varphi(H) - 1} \]

\[ \text{ch } O(1) = \varphi(H)^{\frac{q}{2}} \]

Thus

\[ f_* \left\{ \varphi(H)^{\frac{q}{2}} \left( \frac{H \varphi(H)}{\varphi(H) - 1} \right)^{n+1} \right\} = \binom{q+n}{n} \]
\[
\psi(H) = 1 + a_1 H + a_2 H^2 + \ldots \quad \text{if } a_i = 1
\]

\[
\frac{\psi(H) - 1}{H} = a_1 + a_2 H + \ldots
\]

\[
f_x \left\{ \psi(H)^{n+1} \left( \frac{\psi(H) - 1}{H} \right)^{-n-1} \right\} = \left( \frac{8 + n}{n} \right)
\]

In particular

\[
f_x \left\{ \left( \frac{H}{\psi(H) - 1} \right)^{n+1} \right\} = \binom{-1}{n} = (-1)^n
\]

From this equation, we can recursively compute the coefficients \( a_i \) as polynomials in \( H \):

\[
n = 0: \quad \frac{1}{a_1} = 1 \quad \Rightarrow \quad a_1 = 1
\]

\[
n = 1: \quad f_x \left( \frac{1}{1 + a_2 H} \right)^2 = f_x \left( 1 - 2a_2 H \right) = [P^2] - 2a_2 = -1
\]

\[
a_2 = \frac{1 + [P^1]}{2}
\]

\[
\Phi(a_2) = \frac{1}{C(a_2)} = \frac{1}{2}
\]

\[
n = 2: \quad f_x \left( \frac{1}{1 + a_2 H + a_3 H^2} \right)^3 \equiv f_x \left( 1 - (3a_2 H + a_3 H^2) + \frac{(-3)(-1)}{2} (a_2 H)^2 \right)
\]

\[
= f_x \left( 1 - 3a_2 H - 3a_3 H^2 + C a_2^2 H^2 \right)
\]

\[
= P^2 - 3a_2 P^1 - 3a_3 + 6a_2^2 = 1
\]

\[
3a_3 = P^2 - 3 \left( \frac{1 + P^1}{2} \right) + 6 \left( \frac{1 + P^1}{2} \right)^2 - 1
\]

\[
C a_3 = 2P^2 - 3 \left[ P^1 \left( \frac{1 + P^1}{2} \right) \right] + 3 \left( 1 + 2P^1 + (P^1)^2 \right) - 1
\]

\[
= 2P^2 + 3P^2 + 1
\]
Therefore

\[
a_3 = \frac{2[p^2] + 3[p]^3 + 1}{6}
\]

\[\bar{c}(a_3) = 1 \checkmark\]
\[c(a_3) = \frac{1}{3!} \checkmark\]

\[n=3; I\ calculated\ that\]

\[
a_4 = \frac{1 + 6[p_1] + 11[p_2] + 6[p_3]}{24} + \frac{[p_1]^2 - [p_2]}{8}
\]

**Conjecture:**

\[
a_n \equiv \frac{(1 + [p_1]) \cdots (1 + (n-1)[p_1])}{n!} \mod ([p_n]^n - [p_1]^n)_{n \geq 0}
\]

True see page 10:
Note that if \( f^{(n)}: \mathbb{P}^n \rightarrow \mathbb{P}^1 \) then

\[
f^{(n)}_*(H^i) = \text{res} \left\{ H^i \cdot H^{-n-1} \sum_{j=0}^{n-1} P_j H^j \cdot dH \right\}
\]

hence

\[
f^{(n)}_* \left\{ \left( H \varphi(H) \right)^{n+1} \right\} = 1 \quad \text{all } n \geq 0
\]

means that

\[
\text{res} \left\{ \left[ \frac{\varphi(H)}{\varphi(H)-1} \right]^{n+1} \cdot \sum_{j=0}^{n-1} P_j H^j \cdot dH \right\} = 1 \quad \text{forall } n \geq 0
\]

Now as \( \varphi(H) = 1 + H + \cdots \) there is a change of variable

\[
\overline{H} = \frac{\varphi(H) - 1}{\varphi(H)}
\]

\[
H = \varphi(\overline{H})
\]

and so using invariance of residue we have

\[
\text{res} \left\{ \overline{H}^{-(n+1)} \sum_{j=0}^{n-1} P_j \varphi(\overline{H})^j \overline{\varphi(H)} d\overline{H} \right\} = 1 \quad \text{for all } n \geq 0
\]

i.e.

\[
\sum_{j=0}^{n-1} P_j \varphi(\overline{H})^j \overline{\varphi(H)} d\overline{H} = 1 + \overline{H} + \overline{H}^2 + \cdots = \frac{1}{1-H}
\]

Integrating

\[
\sum_{j=0}^{\infty} P_j \varphi(\overline{H})^{j+1} = -\log(1-H)
\]

or

\[
\sum_{j=0}^{\infty} P_j \frac{H^{j+1}}{j+1} = \varphi(H)
\]
Check:
\[
\begin{aligned}
\epsilon(\varphi(H)) &= e^H \\
\delta(\varphi(H)) &= \frac{1}{1-H}
\end{aligned}
\]

Hilbert–Roch for projective space \( \mathbb{P}^n \):

\[
\frac{1}{n!} \int \left\{ \varphi(H) \frac{e^{H \varphi(H)}}{\varphi(H) - 1} \right\}^{n+1} = \lim_{R \to \infty} \left\{ \varphi(H) \frac{e^{R \varphi(H)}}{\varphi(H) - 1} \right\}^{n+1} R^{-n-1} \int \frac{\varphi(H)^n}{\varphi(H) - 1} dH
\]

= \lim_{R \to \infty} \left\{ \varphi(H) \frac{e^{R \varphi(H)}}{\varphi(H) - 1} \right\}^{n+1} \frac{\varphi'(H) dH}{\varphi(H)}

= \lim_{x \to 0} \left\{ (1+x)^{b+2^n} \right\}^{n+1} \frac{d(x^n)}{x^{n+1}} = \binom{b+n}{n}

\[
\text{ch}(L) = e^{\sum_{j=0}^{n} \frac{p_j}{j+1}}
\]

In particular if \( p_j = a^j \):

\[
\text{ch}(L) = e^{\sum_{j=0}^{n} a^j \frac{c_j(L)^{j+1}}{j+1}} = e^{-\frac{1}{a} \log(1 + ac_j(L))}
\]

\[
= (1 + ac_j(L))^{-\frac{1}{a}} = 1 + c_j(L) + \frac{(\frac{-1}{a})(\frac{-1}{a})(-1)}{2}(ac_j(L))^2 + ...
\]

\[
= 1 + c_j(L) + \frac{1+a}{2} c_j(L)^2 + \frac{(1+a)(1+2a)}{3} c_j(L)^3 + .......
\]
Let $V_G$ be the category of $G$-manifolds where $G$ is a compact Lie group.

**Problem:** If $X$ is a $G$-manifold, then there is a finite dimensional representation $V$ of $G$ and an equivariant embedding $X \hookrightarrow V$?

**Counterexample:** Let $H_n = \mathbb{Z}/2\mathbb{Z} \subset S^4 = G$ and let $X = \bigsqcup_{n \geq 0} G/H_n$. If $V$ is a finite dimensional representation of $G$, then $V = V_1 \oplus \cdots \oplus V_n$ where the $V_i$ are 1-dimensional given by characters $\chi = \chi^m$ if $\chi V_i$, $\chi \in G$. If $v = (v_1, \ldots, v_n) \in V$, then the stabilizer of $v$ is $\{ g \in G | \chi^m = 1 \}$ provided $v \neq 0$.

Thus if $X \subset V$ we would have $\chi^{2m} \neq \chi^{m_1 \cdots m_n}$ for all $g$, which is impossible.

This counterexample arises because $X$ has infinitely many orbit types.

**Definition:** An orbit type is an isomorphism class of transitive $G$-manifolds, or what comes to the same thing, a conjugacy class of closed subgroups in $G$.

**Proposition:** If $X$ is a compact $G$-manifold, then the set of orbit types is finite.

**Proof:** By induction on the dimension of $X$. Consider orbit of each point and choose an equivariant tubular neighborhood. As
X is compact there is a finite covering by such tubular nbs. Each tubular nbhd has for its orbit types the zero section and those of the sphere bundle which has finitely many orbit types, since it is a manifold of 1 lower dimension.

It is clear that an embeddable G-manifold has only finitely many orbit types which leads to:

Conjecture: If X is a G-manifold with only finitely many orbit types, then X may be embedded in a finite dimensional representation of G.

Case 1. X has only one orbit type given by a normal subgroup H. Then Q = G/H acts freely on X and so one is reduced to the case where G acts freely on X. Let Y = X/G and let Y → W be an embedding where W is a vector space. Since Y is finite dimensional the principal G-bundle over Y given by X is induced by an equivariant map X → E where E is a principal bundle which is compact. (Embed G → U(n) form X × U(n), comes from vector bundle over Y which is induced from Stiefel manifold U(n+N)/U(n) = E for some N). Now equivariant embeddings of compact manifolds are easy; one takes an embedding and by Peter-Weyl approximates the embedding functions by representative functions, then uses that any map sufficiently close to an embedding is an embedding.
Therefore we get an equivariant embedding $E \hookrightarrow V$. It is then clear that $X \leftarrow Y \times E \rightarrow W \times V$ is an equivariant embedding of $X$.

**Case 2:** $X$ has only one orbit type. Let $H$ be a closed subgroup of $G$ such that $X = G \times H$. Then if $NH$ is the normalizer of $H$ in $G$ we have $NH$ acts freely on $X^H$ and

$$G \times_{NH} X^H \rightarrow X$$

By case 1 we get an equivariant embedding of $X^H$ into $V$. As in case 1 one chooses an $NH$-principal bundle map $X^H \rightarrow E$, with $E$ compact and an embedding $\varphi : NH \backslash X^H \rightarrow W$. This gives an embedding

$$G \times_{NH} X^H \rightarrow G \times_{NH} (W \times E) \cong W \times (G \times_{NH} E)$$

and so choosing a $G$-embedding $G \times_{NH} E \rightarrow V$ one is done.

**Case 3:** Suppose $X$ is the interior of a compact $G$-manifold $\bar{X}$ with boundary. Construct a smooth collar around $\partial X$ and a function on $X$ representing distance from $\partial X$ in collar and constant outside collar. Then get $X \xrightarrow{\psi(i)} \mathbb{R}^+ \times \mathbb{E}^n$ an embedding which we can make equivariant. Now use a diffeo of $\mathbb{R}^+$ with $\mathbb{R}$ to get an equivariant embedding of $X$ in $\mathbb{R} \times V$, where $G$ acts trivially on $\mathbb{R}$. 
Example. An equivariant bundle $G \times H V$. Here we can take $\mu$ to be $(\text{distance from } 0)^2$.

Related problem: Let $E$ be an equivariant bundle over $X$. If $X$ has finitely many orbit types, then is $E$ a quotient of an equivariant bundle of the form $X \times V$, where $V$ is a representation of $G$?

Case 2': If $X$ has a single orbit type associated to a closed subgroup $H$ of $G$, so that

$$X = G \times \frac{X}{NH} H$$

then a $G$-bundle over $X$ is the same as a $NH$-bundle on $X^H$ and since $NH/H$ acts freely on $X^H$ the same as a $H$-bundle over $NH \times H$ (see lemma below). But $H$ acts trivially on $Y = NH \times H$, so an $H$-bundle is just a representation of $H$ in a vector bundle on $Y$. May assume $E$ complex whence

$$E \cong \bigoplus_{i} \text{Hom}(W_i, E) \otimes W_i$$

where $W_i$ runs over the irreducible reps of $H$. Now write the bundles $\text{Hom}_H(W_i, E)$ as quotients of trivial bundles whence $E$ is a quotient of $Y \times V$, $V$ a representation of $H$, and so
Example to show that $X$ have finitely many orbit types is necessary: $G = S^1$, $H_n = \mu_{2^n}$, $W_n = \mathbb{C}$ with standard $\mu_{2^n}$ action.

$$E = \bigsqcup_{n=1}^{\infty} G \times_{H_n} W_n \to \bigsqcup_{n=1}^{\infty} G / H_n$$

If $E$ has a finite dimensional generating subspace, we can decompose it into 1-dimensional representations. Let $f \in \Gamma(E) = \bigsqcup_{n=1}^{\infty} \Gamma(G / H_n, G \times_{H_n} W_n)$, $f = (f_n)$, where $f_n \in \text{Hom}_{H_n} (G, W_n)$, and suppose $z \cdot f = z^m f$, i.e.,

$$f_n (z \cdot \alpha) = z^m f_n (\alpha) \quad \text{all } z \in \mu_{2^n} \text{ int } n$$

But

$$f_n (z \cdot \alpha) = z f_n (\alpha) \quad \text{if } z \in \mu_{2^n}$$

Therefore $f_n \neq 0 \implies z^m = z \quad \text{all } z \in \mu_{2^n} \implies 2^n / m - 1$. Hence only finitely many $f_n$ are $\neq 0$ so we can't have a finite dimensional generating invariant subspace of sections.

Case 4: $G$ finite. Here both conjectures are true. In effect if $E$ is a $G$-bundle over $X$ with generating finite-dimensional subspace $V \subset \Gamma(E)$, then $G \cdot V$ is also finite dimensional. Similarly for an embedding.

It is clear that we must assume that the equivariant bundle $E$ itself has only finitely many orbit types since if $E \hookrightarrow X \times V$ is a subbundle, and if $X \hookrightarrow W$ is an embedding, then $E \hookrightarrow X \times V \hookrightarrow W \times V$ is also embedding.
Lemma: Let $(X, \partial X)$ be a $G$-manifold with boundary. Let $\varphi : \partial X \to V$ be an embedding with $V$ a representation of $G$. Then there is an equivariant embedding $\beta : X \to R_+ \times R^n \times V$ such that $\beta_2|\partial X = \varphi$, $\beta_3|\partial X = 0$.

Sketch of Proof: Induction on no. of orbit types. If there is a single orbit type, we know how to proceed. Then

$$X = G \times_{NH} X^H$$

where $NH/H$ acts freely on $X^H$. Embed $NH \times X^H \hookrightarrow R_+ \times R^n \times V$ and also find a map $X^H \to E$ where $E$ is a compact manifold on which $NH/H$ acts freely. Then

$$X = G \times_{NH} X^H \to (NH \backslash X^H) \times (G \times_{NH} E) \to R_+ \times R^n \times V$$

where $G \times_{NH} E \hookrightarrow V$ is an equivariant embedding.

If there is more than one orbit type, let $G/H$ be minimal (H maximal) among the orbit types that occur, and let $Y = G \cdot X^H$ be the union of the orbits of $X$ of this type. Then $Y$ is a closed invariant submanifold of $X$; let $N$ be an invariant tubular nbhd. By an argument to be given below, we can embed the normal bundle $\nu$ of $Y$ in $X$ as a subbundle of a bundle $Y \times V$ where $V$ is a representation of $G$. This gives us an equivariant embedding $N \hookrightarrow Y \times VD \hookrightarrow W \times DV$. 

This gives us an equivariant embedding $N \hookrightarrow Y \times VD \hookrightarrow W \times DV$.
\[ \rightarrow \mathbb{R}^+ \times \mathcal{W} \times \mathcal{V} \text{ if } Y \text{ closed. Now extend embedding on } \mathcal{E}N \text{ to } X - \text{Int}(N) \text{ by induction hypothesis and piece together, } \mathcal{U}_H. \]

Must also contend with corners. (With finitely many orbit types)

Lemma: If \( E \) is an equivariant bundle over a \( G \)-manifold \( X \) with finitely many orbit types, then \( E \) is a subbundle of \( X \times V \) for some representation of \( G \).

**Proof:** Induction on number of orbit types of \( X \). Enough to consider case of \( 1 \)-orbit type since we have to find a finite diml. inv. generating subspace of sections, hence if \( X = X_1 \cup X_2 \) where \( A = \partial X_1 = \partial X_2 \) and if \( V_i \) generates \( E|_{X_i} \), then using a collar around \( A \), can assume \( V_i \subseteq P(X,E) \) generates over \( X_i \) where \( V_1 + V_2 \) generates over \( X \).

Now suppose \( X \) has one orbit type of type \( G/H \) so

\[ X = \frac{G \times_{NH} X^H}{\text{where } NH/H \text{ acts freely on } X^H} \]

Let

\[ E = G \times_{NH} E' \]

where \( E' \) is a \( NH \)-bundle on \( X^H \). If \( E' \subseteq X^H \times V' \) with \( V' \) a rep. of \( NH \), then

\[ E \subseteq G \times_{NH} (X^H \times V') \]
Now assume \( X \) has one orbit type belonging to \( H \), so

\[
X \cong G \times_{NH} X^H
\]

By hypothesis \( E \) has finitely many orbit types, hence \( \exists \) a representation \( W \) of \( G \) such that for each \( x \in X^H \) there is a surjection \( W \twoheadrightarrow E_x \) of \( H \)-modules (one must show that finitely many orbit types \( \Rightarrow \) finitely many different isotropy representations \( \Rightarrow \)). Hence \( \exists \) surjection

\[
G/H \times W \twoheadrightarrow G_H \times E_x
\]

of \( G \)-bundles over the orbit \( Gx \) and hence \( \exists \) if we choose a finite set of generating sections for the bundle on \( Gx = NH \backslash X^H \) whose sections are

\[
\text{Hom}_G(W, \Gamma(X, E))
\]

then we have written \( E \) as a quotient of \( X \times W \).

QED.

The correct hypothesis appears to be that the conjugacy classes of stabilizers + isotropy representations form a finite set. Perhaps if \( V \) is a family of representations of \( G \) and if the orbit types of \( \Pi \times V \) is finite then only finitely many inequivalent representations occur. Yes.
Proposition: Let $G$ be a compact Lie group and let $V_i \in \mathcal{I}$ be a family of representations of $G$ in finite dimensional real vector spaces. Assume that

(i) $\dim V_i \leq N$ for all $i$

(ii) $\bigcup_i$ (orbit types of $V_i$) finite

Then the set of isomorphism classes of the family $\{V_i\}$ is finite.

Proof: Let $R_R(G)$ be the representation ring of $G$. We have to show that $\{[V_i] \mid i \in \mathcal{I}\}$ in $R_R(G)$ is finite. If $G^0$ is the connected component of $G$, then the restriction map $R_R(G) \to R_R(G^0)$ is finite to one; hence we may assume $G$ connected.

Also extension and restriction of scalars from $R$ to $C$ define maps

$$R_R(G) \xrightarrow{\Phi} R(C) \xleftarrow{\Psi} R(G)$$

with $\Psi \Phi = 2$. As $R_R(G)$ is without torsion, $\Phi$ is injective. The set of orbit types of $C \otimes_R V \cong V + V$ is finite, since the orbits of $V + V$ under $G \times G$ are of form $G/H_1 \times G/H_2$ which is compact and hence its orbit types under $G$ are finite in numbers. Thus we may assume the representations are complex.

Let $T$ be a torus. Then $R(G) \to R(T)$ is injective so we may assume that $G = T$. (Again use lemma: If $G$ acts on $X$...
with finitely many orbit types. If \( \varphi: H \to G \) is a homomorphism then \( X \) has finitely many \( H \) orbit types.

We may also assume the representations \( V_i \) are irreducible. In effect \( R(G) = \bigoplus X \) where \( X \) runs over irreducibles so each \( [V_i] = \sum a_i^j X \) with \( e_a^j \geq 0 \) and \( \sum a_i^j \leq N \). For the set \( \{[V_i]\} \) to be finite is therefore the same as the set \( \{X \mid a_i^j \neq 0 \text{ for some } i\} \) to be finite. On the other hand the orbit types of a subrepresentation are contained in those of the representation. (Here use (i), otherwise false because we could take \( V_i = C^i \) with trivial action).

Thus we have a torus \( T \) and a set of characters \( \chi_i \) of \( T \). The orbit type of a character \( \chi \) is the subgroup \( \ker \chi \) plus \( T \). There \( U \ker \chi_i \) is finite. But \( \ker \chi_i \) determines \( \chi_i \) up to signs since \( \text{Aut} S^1 = \mathbb{Z}/2\mathbb{Z} \). \( \{\chi_i\} \) is finite. QED.

This assertion is false (take \( G \) finite). However

if the \( V_i \) restricted to \( G^0 \) form finitely many isomorphism classes then as \( V_i \in \text{Map}_{G^0}(G, V) \) the \( V_i \) over \( G \) are all subrep. of finitely many representations. But the isomorphism classes of subrep. of a given representation is finite.
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Let $\bar{c}(E) = (r_0 E, c_1 E, c_2 E, \ldots)$ and let $P_i$ be the power series with coefficients in $\Omega^*(pt.)$ such that

$$c_i(E \otimes F) = P_i(\bar{c}(E), \bar{c}(F)) \quad i \geq 0.$$  

Then

Conjecture: Let $G$ be a compact Lie group. Then $\bar{c}: R(G) \rightarrow \Omega^*_G(pt.)$ is universal with respect to maps of $R(G)$ into an $\Omega^*(pt.)$-algebra such that

$$c_i(E + F) = \sum_{j+k=i} c_j(E) c_k(F)$$

$$rg(E + F) = rg E + rg F$$

$$c_i(E \otimes F) = P_i(\bar{c}(E), \bar{c}(F)) \quad i \geq 0$$

$$rg(E \otimes F) = rg E \cdot rg F.$$

Conjecture: $\bar{c}: K^*(X) \rightarrow \Omega^*(X)$ is also universal

when $H_*(X)$ has no torsion.

(It might be necessary to add in the formulas

for $\lambda_2$, e.g.

$$c_i(\lambda_2 E) = Q_{ij}(\bar{c}(E)).$$)
The conjecture over $\mathbb{Q}$. In this case we want a universal character such that
\[ \text{ch}(E+F) = \text{ch}E + \text{ch}F \]
\[ \text{ch}(E \cdot F) = \text{ch}E \cdot \text{ch}F \]
Thus we want an $\Omega_{\mathbb{Q}}^*(pt)$-algebra with a homomorphism
\[ \text{ch} : K(X) \rightarrow \mathbb{Q}A \]
which is universal. It follows that $A \cong \Omega_{\mathbb{Q}}^*(pt) \otimes K(X)$\[ \text{whi}c\text{h is true.} \]

Recall
\[ \text{ch}L = e^{\sum_{g \geq 1} \frac{c_g(L)}{g}} \]
Recall also the Bergman formulas:
\[ (1 + \sum a_n t^n) = e^{\sum_{n \geq 1} \frac{a_n t^n}{n}} = \prod_{n=1}^{\infty} (1 - x_n t^n)^{-1} \]
where the $w_i$ are the phantom coordinates and the $x_n$ are the Bergman-Witt coordinates related by
\[ w_i = \sum d x^i \frac{d}{d x^i} \]

Conjecture: Let $Q_i \in \Omega_{\mathbb{Q}}^*(pt) \otimes \mathbb{Q}$ given by
\[ P_{i-1} = \sum d \frac{d Q_{i-1}}{d x^i} \]
Then $Q_i \in \Omega_{\mathbb{Q}}^*(pt)$ and form a polynomial system of generators.
Problem with this conjecture is that $\dim P_{j-1} = j - 1$
hence $Q_j$ not homogeneous.

A consequence of the conjecture is that $\Omega^*(pt)$
is isomorphic to the coordinate ring of the universal
Witt scheme hence has two natural maps

$$\varphi^a, \varphi^m : \Omega^*(pt) \rightarrow \Omega^*(pt) \otimes \Omega^*(pt)$$
given by

$$\begin{align*}
\varphi^a(p_i) &= p_i \otimes 1 + 1 \otimes p_i & i > 0 \\
\varphi^m(p_i) &= p_i \otimes p_i & i \geq 0.
\end{align*}$$

Conjecture false because for $j = 2$ it says that

$$P_1 = 2Q_2 + 1$$

however $P_1$ generates $\Omega^2(pt)$. 
Steenrod operations in cobordism

Let $\alpha \in \Omega^g(X)$ be represented by $f: Z \to X$ proper and oriented of codimension $g$. Let $G \to \Sigma(k)$, symmetric group on $k$ letters be a homomorphism and let

$$\beta \in \Omega^g_k(pt, \sigma^k) = \text{bordism classes of maps } \quad \begin{array}{c} W \\ \to \quad \text{pt} \end{array} \quad \text{where } W \quad \text{compact oriented of dim } n \quad \text{on which } G \text{ acts freely so as to change orientation by } \sigma^k: G \to \Sigma(k) \sym \pm 1.$$

Then representing $\beta$ by $W \to \text{pt}$, class move and form fibre product

$$\begin{array}{ccc}
Q & \xrightarrow{\text{mod } kG} & Z^k \\
\downarrow \dim kG & \downarrow \leftarrow \downarrow f^k & \downarrow \\
W \times X & \to & X^k \\
\dim k & \xrightarrow{p_2} & \text{transferral to } f^k \\
X & \xrightarrow{\text{move } 4p_2} & X^k
\end{array}$$

Then $G \setminus Q \to X$ is proper and will be denoted $\beta | x^k$. Thus have a cohomology operation

$$\Omega^g(X) \xrightarrow{\alpha} \Omega^k_{g-n}(X) \xrightarrow{\beta} \Omega^k_{g-n}(X)$$

defined for each element $\beta \in \Omega^g_k(pt, \sigma^k)$. 
Lesson from equivariant cobordism theory is that any element \( \alpha \in \text{Hom}_m(X,Y) \) is representable.

\[
\begin{array}{c}
\forall' f \rightarrow Y \\
\downarrow g \\
X \\
\downarrow i \\
i \rightarrow V
\end{array}
\]

\( g \) proper-oriented

\[
\alpha = i^*g^*f^*
\]

Proof: Closed under composition

\[
\begin{array}{c}
\forall' f \rightarrow Y \\
\downarrow g \\
V \\
\downarrow f \\
f \rightarrow Y
\end{array}
\]

Feel certain that

\[
\text{Hom}_m(X,Y) = \lim_{x \rightarrow V} I(V \times Y)
\]

where \( I \) is the bordism group with \( \mathbb{Z} \) not necessarily free action, and where \( V \) runs over set with objects \( X \rightarrow V \) and maps \( V \rightarrow V' \) smoothly under \( X \).
Character as a natural transformation of cohomology theories.

We have defined
\[ \text{ch}: K(X) \to \Omega^\omega(X) \]
and now extend it to \( K^g(X) \) \( g \geq 0 \) by making
\[ K^g(X) \to \Omega^\text{parity}(g)(X) \]

(\( \text{ch} \) is given)

\[ K^0(S^3 \times X) \]
\[ K^1(S^3 \times X) \]
\[ K^2(S^3 \times X) \]

commutative. Claim that \( \text{ch} \) is compatible with Bott periodicity, e.g.

\[ K(X) \to \Omega^\omega(X) \]
\[ K(S^3 \times X) \to \Omega^\omega(S^3 \times X) \]
\[ \beta \in K(S^3 \times X) \]

where \( \beta \) is the Bott class. However \( \text{ch} \beta = \frac{1}{2} \pi_c(O(1)) -1 = c_1(O(1)) = u \)

(at least up to sign conventions which we shall not check here)

because \( \Omega^\omega(S^3 \times X) \cong \mathbb{Z} \). Therefore

\[ \text{ch}: K^g(X) \to \Omega^\omega(X) \]

is a morphism of \( \mathbb{Z}/2\mathbb{Z} \) cohomology.
Therefore we get the folk theorem that the extension of the character to an $\mathbb{Q}^*$ linear map

\[
\begin{array}{ccc}
\Omega^+(pt) \otimes \mathbb{Z} K^+(X) & \xrightarrow{\sim} & \Omega^+(X) \\
\end{array}
\]

is an isomorphism of cohomology theories.
Steenrod operations on complex projective spaces:

\[ H^*(\mathbb{C}P^\infty, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z} [x] \quad \text{deg } x = 2 \]

\[ P^t_*(x) = x + tx^p \]

in general \( p^i: H^8 \rightarrow H^{8+2i(p-1)} \) is determined by stability and

\[ p^i x = x^p \quad \text{if degree } x = 2i. \]

Thus

\[ P^t_*(x^k) = (P^t_*(x))^k = (x + tx^p)^k \]

\[ = x^k \left( 1 + \binom{k}{1} tx^{p-1} + \frac{1}{2} t^2 x^{2p-2} + \cdots \right) \]

\[ p^i x^k = \binom{k}{i} x^{k-2i(p-1)} \]

\[ p^i(x^k) = k x^{k+2i(p-1)} \]

\[ \neq 0 \quad \text{if } k \neq 0 \quad (p). \]

Lemma. If \( k = a_0 + a_1 p + \cdots \quad 0 \leq a_i < p \)

\( i = b_0 + b_1 p + \cdots \quad 0 \leq b_i < p \)

are the expressions for \( k \) and \( i \) with basis \( p \), then

\[ \binom{k}{i} \equiv (a_0)(a_1) \cdots \mod p \]

Proof:

\[ (1 + x)^k = (1 + x)^{a_0 + a_1 p + \cdots} \]

\[ = (1 + x)^{a_0}(1 + x^p)^{a_1} \cdots \quad \pmod{p} \]

\[ = \left( \sum_{b_0 \leq a_0} (a_0) X^{b_0} \right) \left( \sum_{b_1 \leq a_1} (a_1) X^{b_1} \right) \cdots \]

\[ \text{QED.} \]
Witt ring reviewed.

A ring, \( W(A) \) is the ring (in fact \( A \)-ring) functor of \( A \) defined as follows:

\[
W(A) = 1 + A[[t]]^+ \quad \text{as abelian groups with multiplication and} \ A \text{-operations given by "universal polynomials" determined by the rules}
\]

\[
(1 + at)(1 + bt) = (1 + abt)
\]

\[
\lambda^n(1 + at) = \begin{cases} 
1 & n = 0 \\
1 + at & n = 1 \\
0 & n \geq 2
\end{cases}
\]

Alternatively over the rationals we can use the formulas

\[
1 + \sum a_n t^n = e^{\sum \frac{a_n}{n} \psi^n(t^n)}
\]

and say that addition (resp. mult.) in \( W(A) \) is given by addition (resp. mult.) componentwise of \( \nu^\sigma = (\nu_1^\sigma, \nu_2^\sigma, \ldots) \). To describe the \( \lambda \)'s on the \( \nu \)'s is harder instead (as we are already over \( \mathbb{Q} \)) we may describe the Adams operators, \( \psi^n \) given by formulas

\[
\psi^n(E) = e^{\sum \frac{(-1)^{n-1}}{n} \psi^n(E)t^n}
\]

\[
\psi^0(E) = \text{rank } E
\]

Therefore

\[
\psi^n(\nu^n) = (i \mapsto \nu^n(i)) \quad \text{since } \psi^n(L) = L^n \\
\Rightarrow \psi^n(1 + at) = 1 + a^n t
\]
Summary:
\[
\begin{align*}
\omega'_i + \omega''_i &= (i \mapsto \omega'_i + \omega''_i) \\
\omega'_i \cdot \omega''_i &= (i \mapsto \omega'_i \cdot \omega''_i) \\
\psi^n(\omega) &= (i \mapsto \omega^n_{-i})
\end{align*}
\]

Another description of \( \psi^n = F_n \) is to consider
\[
\begin{array}{ccc}
1 + A[[t]]^+ & \xrightarrow{\text{norm}} & 1 + A[[t]]^+ \\
\uparrow & & \uparrow \\
W(A) & \xrightarrow{\psi^n} & W(A)
\end{array}
\]

This enables one to define \( V_n \) by
\[
1 + A[[t]]^+ \xrightarrow{\text{norm}} 1 + A[[t]]^+ \xrightarrow{V_n} W(A)
\]
\[
V_n(1 + at) = 1 + at^n
\]
\[
\left\{ \begin{array}{c}
V_n(\omega) = (i \mapsto \lambda^{n+i}_{W_i} \omega \text{ otherwise})
\end{array} \right.
\]

(of course \( V_n \) doesn't define an operation in K-theory, but it might be the transpose of \( F_n = \psi^n \) for some kind of Poincaré duality)
Let $W_p(A) \subset W(A)$ be "given by"
\[ u_i = 0 \quad \text{if } i \text{ is not a power of } p. \]

It is necessary to assume that $A$ is a $\mathbb{Z}_p$ (all localizations at the ideal $(p)$) algebra, in order that $W_p(A)$ be defined. If $A$ is torsion-free we have
\[ W_p(A) = W_p(A \otimes \mathbb{Q}) \cap W(A) \subset W(A \otimes \mathbb{Q}). \]

**Claim:** $W_p(A)$ is a sub $\lambda$-ring of $W(A)$.

**Proof:**

First assume $A$ over $\mathbb{Q}$.

Let $U = (u_0, u_1, \ldots) \in W_p(A)$ with inclusion $I: W_p(A) \rightarrow W(A)$ given by
\[ I \sigma = (i \mapsto \begin{cases} u_i & \text{if } i = p^a \\ 0 & \text{otherwise} \end{cases}) \]

Then $\psi^n(I \sigma) = (i \mapsto \begin{cases} u_i & \text{if } ni = p^a \\ 0 & \text{otherwise} \end{cases})$.

Let $\psi^n \sigma = 0$ if $n \neq p^a$ for some $a$, $\psi^n \sigma = (i \mapsto \sigma_{p^a i})$.

Then $I(\psi^n \sigma) = (i \mapsto \begin{cases} \sigma_{p^a i} & \text{if } i = p^a \\ (\psi^n \sigma)_b & \text{if } i = p^b \\ u_{b+a} & \text{if } i = p^b (\Rightarrow ni = p^{a+b}) \end{cases})$. 
so one sees these are equal. When A is torsion-free, it follows that \( W(A) \) is stable under all \( \lambda \)-operations, since both \( W_p(A \otimes \mathbb{Q}) \) and \( W(A) \) inside of \( W(A \otimes \mathbb{Q}) \) are. But any \( A \) is a quotient of a torsion-free one.

If \( k \) is a perfect ring of characteristic \( p \), \( W_p(k) \) is its Witt ring. It is characterized as complete for the p-adic topology such that \( p \) is a non-zero divisor, and such that \( R/pR \to k \). The isomorphism between such a ring \( R \) and \( W_p(k) \) is constructed as follows: let \( \sigma_k(k) \to R \) be the Teichmüller section. Then get isomorphism

\[
\psi = (x_0, x_1, \ldots) \mapsto s(x_0) + s(x_1^p) + s(x_2^p^2) + \ldots
\]

where \( \psi = (x_0, x_1, \ldots) \) is the description of \( W_p(K) \) by Witt coordinates related to phantom coordinates \( \psi_c \) (\( = W_p(k) \)) by

\[
\begin{align*}
\omega_0 &= x_0 \\
\omega_1 &= x_0^p + px_0 \\
\omega_2 &= x_0^{p^2} + px_1^p + p^2x_2
\end{align*}
\]

Problem: We know that \( W_p(k) \to R \) is a \( \lambda \)-ring. What are the \( \lambda \)-operations?

It suffices to determine \( \psi \) operations. Let \( x = (x_0, \ldots) \) be a Witt vector and let \( x' = \psi^p(x) \). Claim that

\[
x' = \psi^p(x) = (x_0^p, x_1^p, \ldots)
\]
In effect in terms of the phantom coordinates $\omega' = (\omega_1', \omega_2', \ldots)$ if $\omega = (\omega_0, \omega_1, \ldots)$. Thus choosing a torsion free ring over $k$ we have formulas

$$x_0' = x_0^p + px_1, \quad \Rightarrow \quad x_0' = x_0^p \quad (p)$$

$$x_0' + px_1' = x_0^{p^2} + px_1^p + p^2x_2, \quad \Rightarrow \quad x_0' \equiv x_1^p \quad (p)$$

etc.

Thus concludes $\psi$

\[ \begin{array}{c|c|c}
\text{On} & W_p(k) & \text{char } k = p \\
\psi^n & \equiv 0 & (a, p) = 1 \\
\psi^p (x_0, x_1, \ldots) & = (x_0^p, x_1^p, \ldots) \\
\end{array} \]

Thus $\psi$ on $A$ is just the unique lifting of Frobenius on the residue field $k$. The $\lambda$-operations are given by

$$\frac{1}{\lambda_t(a)} = e^{\sum \frac{(\psi)^n}{n!} \cdot \frac{tp^n}{p^n}}$$

In this case $k = \mathbb{F}_p$, then $\psi^p = id$ so we have

$$\frac{1}{\lambda_t(a)} = e^{\sum \frac{tp^n}{p^n}} = \prod_{(p,n) = 1} \left(1 - t^n\right)^{-a \mu(n)/n}$$

which checks since $\mathbb{Z}_p$ is a binomial ring so that $(1 + t)^a$ is defined and since the power series with $a = 1$ has coefficients in $\mathbb{Z}_p$. 
Theorem. If $E$ is a complex vector bundle of dimension $n$ over $X$, then
$$\Omega(\mathcal{P}E) = \Omega(X)^{[3]} / \langle r^n - f^*c_1(E)^n + \ldots + (-1)^n f^*c_n(E) \rangle,$$
where $r = c_1(\mathcal{O}(1))$.

Proofs. One shows by a standard induction on $n$ that
$$\Omega(E^n) = \Omega[H]/(H^{n+1}),$$
where $H = c_1(\mathcal{O}(1))$.

It follows by a Mayer-Vietoris argument that $\Omega(\mathcal{P}E)$ is a
free module over $\Omega(X)$ with basis $\mathcal{Z}^{0,0} \ldots \mathcal{Z}^{0,n}$. Define
$c_i(E)$ by the relation
$$\mathcal{Z}^{0,n} - f^*c_1(E) \mathcal{Z}^{0,n-1} + \ldots + (-1)^n f^*c_n(E) = 0.$$
Then $E \to c_i(E)$ is functorial and since $f^*$ is injective
for $f: \mathcal{P}E \to X$ and any $E$, to prove formulas
$$c(E') = c(E') c(F')$$
may assume $E', F'$ split.

Thus assume $E$ split, $E = L_1 + \ldots + L_n$. Then
let $H_j = P(L_1 + L_2 + \ldots + L_n) \subset \mathcal{P}E$ and $i_j$ the inclusion. Then
$H_j$ is the zero set of the section
$$0 \to f^*\mathcal{O}(1) \to f^*(L_j) \otimes \mathcal{O}(1),$$
so
$$i_j^* 1 = c_1(f^*L_j \otimes \mathcal{O}(1)).$$
As $\cap H_j = \emptyset$, we have
$$\prod_{j=1}^n c_1(f^*L_j \otimes \mathcal{O}(1)) = 0.$$
Recall \[ c_1(M \otimes N) = F(c_1(M), c_1(N)) = c_1(M) + c_1(N)(1 + G(c_1(M), c_1(N))) \]
for two line bundles \( M, N \). Thus

\[
c_1(O(1)) - c_1(F^* L_f^{-1}) = c_1(O(1) \otimes F^* L_f) \left[ 1 + c_1(H^* L_f^{-1}) G(c_1(F^* L_f^{-1}), c_1(O(1))) \right]
\]

so

\[
\prod_{j=1}^n (1 - t c_1(L_{f_j}^{-1})) = 0.
\]

Comparing coefficients of the relation we have

\[
e_x(E) = \sum_{i \in C} c_{x_i}(L_{f_{i_j}}^{1}) \cdots c_1(L_{f_{i_j}})
\]

proving Whitney sum formula.

For purposes of equivariant cobordism theory we cannot use Mayer-Vietoris so following seems useful. Assume \( E = L_1 + \cdots + L_n \). Then as before we have the relation (2). To show that \( 1, f_1, \ldots, f_n \) form a basis for \( \Omega(PE) \) as a \( \Omega(x) \) module we use induction. Let \( F = L_1 + \cdots + L_{n-1}, L = L_n \).

\[
PF \xrightarrow{f} PE \xleftarrow{i} RL = X
\]

We will assume that

\[
\Omega(X) \xrightarrow{f_*} \Omega(PE) \xrightarrow{i^*} \Omega(PF)
\]

is exact. It then follows that it is split exact since \( i^* \) is onto by induction. \( f_* f^* = id \). Now \( f(X) \) is where the section

\[
\mathcal{O} \rightarrow f^* E \otimes \mathcal{O}(1) \rightarrow L_i \otimes \mathcal{O}(1)
\]
vanish for $i < n$ and as these are transversal

$$j \times 1 = \prod_{i=n} c_i (L_i @ O(1))$$

Thus $\Omega(PE)$ has a $\Omega(\mathfrak{g})$-basis consisting of

$$\left\{ \begin{array}{c}
\xi^i \\
\prod_{i=n} c_i (L_i @ O(1))
\end{array} \right\}_{0 \leq i < n-1}$$

But

$$\prod_{i=n} (\xi - c_i (f^* L_i)) = \prod_{i=n} (c_i (O(1) @ f^* L_i) \left[ 1 + c_i (f^* L_i) c_i (L_i @ O(1) @ f^* L_i) \right]$$

nilpotent, hence $I$ is a unit in $\Omega(PE)$.

Thus $\Omega(PE)$ has basis

$$\xi^i \quad 0 \leq i < n-1$$

$$\prod_{i=n} (\xi - c_i (f^* L_i))$$

hence also the basis $\xi^i \quad 0 \leq i < n$, which was to be proved.
Problem: Calculate $f_*: \Omega(PE) \to \Omega(X)$.

As

\[ \Omega(PE) = \Omega(X)[i] / \left( \frac{\partial_r}{\partial_t} f^* (E) \right) \]

it is enough to know $f_*^{\text{ind}} \circ i \circ n$. One knows that there are universal formulas

\[ f_*^{\text{ind}} \circ i = P_{\text{ind}} (q_1 E, \ldots, q_r E) \]

since $\Omega(BV(n)) = \Omega \bigoplus_{c_0, \ldots, c_n} \Omega$. To determine these formulas, we may assume $E = L_1 + \cdots + L_n$. Let $F = L_1 + \cdots + L_{n-1}$, $L = L_n$.

\[ \begin{array}{ccc}
PF & \xrightarrow{i} & PE \\
\text{normal bundle} & \uparrow & \text{normal bundle} \\
\mathcal{O}(1) \otimes L & \xleftarrow{j} & \mathcal{O} \otimes F \\
\end{array} \]

\[ l_* 1 = c_1(\mathcal{O}(1) \otimes L) \]

\[ j_* 1 = c_{n-1}(\mathcal{O}(1) \otimes F) = \prod_{c_n} c_i(\mathcal{O}(1) \otimes L_i) \]

where we leave off $F^*$ for convenience.

\[ \mathfrak{g} - c_1(L') = c_1(\mathcal{O}(1) \otimes L) \left[ 1 + c_1(L') G(c_1(L'), c_1(\mathcal{O}(1) \otimes L)) \right] \]

\[ = l_* 1 \cdot x = l_* (i^* x) \]
Therefore
\[ \mathfrak{q} - c_1(L') = c_\ast \left\{ 1 + c_1(L') G(c_L', c_1(O, L), L) \right\} \]
and so we "know" \( f_\ast (\mathfrak{q} - c_1(L')) \) as \( \ast \) "know" \( g_\ast \) where \( g : PF \to X \). Similarly

\[
\text{TT} (\mathfrak{q} - c_1(L')) = j_\ast \left\{ \text{TT} \left( 1 + c_1(L') G(c_L', q(L') \oplus L) \right) \right\}.
\]

so we "know" \( f_\ast \)

\[
\text{TT} \left\{ c_1(L') - c_1(L') \right\} \xrightarrow{\text{unwinding}} \text{TT} (\mathfrak{q} - c_1(L')) + (\mathfrak{q} - c_1(L')) \text{(something)}
\]

However \( \text{TT} (c_1(L') - c_1(L')) \cdot f_\ast \cdot 1 \) implies we know \( f_\ast \) since

universally the coefficient is a non-zero divisor. Therefore in principle we can recursively determine \( f_\ast \).

**Example.** \( n = 2 \), \( E = L_1 + L_2 \). Let \( Q(X, Y) = X - Y + \ldots \)

such that \( c_1(L \otimes M^{-1}) = Q(c_1(L), c_1(M)) = F(c_1(L), I(c_1(M))) \)

where \( c_1(M^{-1}) = I(c_1(M)) \). Then

\[
c_1 (O(1) \otimes f^* L) = F (\vartheta, I(c_1(L')))
\]

Now want to write this as a \( \Omega(X) \) linear combination of \( 1, 3 \) using the relation

\[
(\mathfrak{q} - c_1(L'))(\mathfrak{q} - c_1(L'))' = 0
\]
\[ f(x) \equiv \frac{ax + b}{a - b} + \frac{f(a) - f(b)}{a - b} x \mod (x-a)(x-b). \]

\[ c_1(\mathcal{O}(1) \otimes f^*_L_1) = \frac{\chi \cdot \mathcal{I}(x) - \chi \cdot \mathcal{I}(y)}{x - y} \]
\[ + \frac{\mathcal{I}(x) - \chi \cdot \mathcal{I}(x)}{x - y} \]

where \( x = c_1(L_1) \), \( y = c_1(L_2) \). Similarly

\[ c_1(\mathcal{O}(1) \otimes f^*_L_2) = \frac{\chi \cdot \mathcal{I}(y) - \chi \cdot \mathcal{I}(y)}{x - y} \]
\[ + \frac{\mathcal{I}(y) - \chi \cdot \mathcal{I}(x)}{x - y} \]

Use that \( f_* \) of both are 1, \( \mathcal{I}(x) = 0 \) + get

\[ 1 = \frac{\chi \cdot \mathcal{I}(y)}{x - y} f_* 1 + \frac{\chi \cdot \mathcal{I}(x)}{x - y} \]
\[ 1 = \frac{\chi \cdot \mathcal{I}(y)}{x - y} f_* 1 + \frac{\chi \cdot \mathcal{I}(x)}{x - y} \]

solving

\[ f_* 1 = \frac{1}{\mathcal{I}(y) - \mathcal{I}(x)} + \frac{1}{\mathcal{I}(x) - \mathcal{I}(y)} \]
\[ f_* \xi = \frac{x}{\mathcal{I}(y) - \mathcal{I}(x)} + \frac{y}{\mathcal{I}(x) - \mathcal{I}(y)} \]
and in general

\[
f_* \xi^\mathcal{B} = \frac{x^\mathcal{B}}{F(x_1, I)} + \frac{y^\mathcal{B}}{F(y_1, I)} \quad \theta \geq 0
\]

\[
\prod_{j \neq i} c_i(\theta(1) \phi^* L_2) = (PL_i \rightarrow PE)_* 1
\]

\[
= \prod_{j \neq i} F(x_j, IX_j) = \prod_{j \neq i} \frac{x_j - x_i}{x_i - x_j} F(x_i, IX_j)
\]

where the last follows since \( \prod_{i=1}^{n} (x_i - x_i) = 0 \).

\[
1 = \left( \prod_{j \neq i} \frac{F(x_j, IX_j)}{X_i - X_j} \right) \left[ f_* \xi^{n-1} c_i(F_i) f_\theta \xi^{n-2} + \ldots \right]
\]

where \( F_i = \sum L_j \), which can be used to solve for the \( f_* \xi^i \), \( 0 \leq i < n \). Solution given by

\[
f_* \xi^\mathcal{B} = \sum_{i=1}^{n} \frac{X_i^\mathcal{B}}{\prod_{j \neq i} F(x_j, IX_j)} \quad X_i = c_i(L_i)
\]
In effect

\[
\sum_i \frac{X_i^2 (X_i - X_j \frac{\prod}{j \neq i} (X_i - X_j))}{\prod_{j \neq k} F(\infty, x_j)} = \sum_i \frac{X_i^2 \prod_{j \neq i} (X_i - X_j)}{\prod_{j \neq i} F(\infty, X_j)}
\]

Q.E.D.

Remark: Notice that the right hand side of this formula must be a power series in the \(X_i\).
Example: \[ F(x, y) = x + y - axy \]
\[ I_y = \frac{-y}{1-ay} \]
\[ F(x, I_y) = \frac{x-y}{1-ay} \]

\[ f_*(\xi^g) = \sum_{i=1}^{n} \prod_{j \neq i} \left( \frac{x_j - x_i}{x_i - x_j} \right) \left( a^{n-1} x_i^g \right) \]

But recall Lagrange interpolation formula:
\[ P(z) = \sum_{i=1}^{n} \frac{z - x_i}{x_i - x_j} \cdot P(x_i) \quad \text{if } P \text{ is a poly of degree } < n. \]

Therefore taking \[ P(z) = a^{n-1} z^g \quad g < n \]
we have

\[ f_*(\xi^g) = a^{n-1-g} \quad 0 \leq g \leq n-1 \]

In particular, if \( a = 0 \) (cohomology) or \( a = 1 \) (K-theory), we get old formulas, (recall \( \xi = 1 - a(-1) \) in K-theory).
Formulas for $K_G(PV)$

$K_G(PV) = K_G(X)[T]/(\lambda_{-T}(V))$

where $T =$ the class of $O(1)$.

The relation comes from the sequence

$$N^2V \otimes O(-2) \rightarrow V' \otimes O(-1) \rightarrow 0 \rightarrow 0$$

to obtain

$$0 \rightarrow 0 \rightarrow V \otimes O(1) \rightarrow N^2V \otimes O(2) \rightarrow \cdots$$

or

$$1 - T \lambda'(V) + T^2 \lambda(2) \rightarrow \cdots$$

i.e.

$$\lambda_{-T}(V) = 0$$

Thus $K_G(PV)$ is free as a $K_G(X)$ module with basis $1, T, \ldots, T^{n-1}$ where $n = \dim V$. Perhaps it is better to write out the relation in the form

$$T^n - \lambda'(V) T^{n-1} + \lambda(2) T^{n-2} \rightarrow \cdots = 0.$$
The last formula results from some duality
\[ H^i(E'^\mathbb{C}) = H^{n-i}(E'^\mathbb{C} \otimes \omega). \]

Tangent bundle \( \Theta = \text{Hom}(O(-1), O \otimes V/O(-1)) \)
\[ = \mathcal{O}(1) \otimes V/O \mathcal{O} \]
\[ \mathcal{O} \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1) \otimes V \rightarrow \Theta \rightarrow \mathcal{O} \]
\[ \Rightarrow \mathcal{O} \rightarrow \Theta(-1) \otimes V \rightarrow \Theta \rightarrow \mathcal{O} \]
\[ \Rightarrow \omega = \Lambda^{n-i} \mathcal{O} = \Lambda^n[O(-1) \otimes V] = O(-n) \otimes \Lambda^n V \]

\[ H^0(O(i))' = H^{n-i-\delta}(O(-i) \otimes O(-n) \otimes \Lambda^n V') \]

\[ f^\ast O(i) = (-1)^{n-i} H^{n-i}(O(a)) \]
\[ = (-1)^{n-i} \left\{ H^0(O(-i-n)) \otimes \Lambda^n V' \right\}' \]
\[ = (-1)^{n-i} \left( \sum_{n-i} V' \otimes \Lambda^n V' \right)' \]

As \( f^\ast (O(i)) \) satisfies the difference equation
\[ f^\ast (T^i) - \sum_{i} \chi^i (V') f^\ast (T^{i-1}) + \cdots + (-1)^n f^\ast (T^{i-n}) = 0 \]

it admits an expansion in terms of exponentials
\[ f^\ast (T^i) = \sum_{j=1}^n c_j (\chi^j)^{i} \]

where the \( \chi_j \) are the roots of the equation \( \sum_{i \geq 0} (-1)^i \chi^i V' \chi^j \)
f_\mathbf{x} (T^i) is not a polynomial function of i unless V' is stably trivial.

\[ H^*_G (\mathbb{P}V) \cong H^*_G (X) \left[ \frac{\mathbb{G}}{(\mathbb{G}^n + \mathbb{G}^n)^n} + \cdots + c_n (V) \right] \]

\[ \mathbb{G} = \mathbb{G}(O(1)) \]

Check if \( V = E_1 \oplus \cdots \oplus E_n \) sum of line bundles
For each i we get a divisor \( H_i \) where \( O \otimes E_i' \to O \otimes V \to O(1) \) fails to be surjective, or equivalently where the section \( O \to O(1) \otimes V \to O(1) \otimes E_i \) vanishes. The cohomology class of this divisor is \( c_i (O(1) \otimes E_i) = \mathbb{G} + c_i (E_i) \).

As intersection of these hyperplanes is zero, we get
\[ \prod_{i=1}^{n} (\mathbb{G} + c_i (E_i)) = \mathbb{G}^n + c_1 (V) \mathbb{G}^n + \cdots + c_n (V) = 0. \]

Relation might also be written
\[ \mathbb{G}^n - c_1 (V) \mathbb{G}^n + \cdots + (-1)^n c_n (V) = 0. \]

\[ \text{f}_*: H^*_G (\mathbb{P}V) \to H^*_G (X), \text{ is given by} \]

\[ \text{f}_* (\mathbb{G}^n) = \begin{cases} 
0 & i < n-1 \\
1 & i = n-1 \\
\sigma_i (V) & i \geq n-1
\end{cases} \]

where \( \sigma_i \) are the symmetric functions given by
\[ \sum_i \sigma_i (x_1, x_2, \ldots, x_n) T^i = \prod_{i=1}^{n} (1-x_i T)^{-1} \]
e.g. \( \sum_{1 \leq i \leq n} x^i \).

Case of cobordism (complex):

We wish to calculate the (complex) cobordism ring of a projective bundle \( PV \). Useful tool is the transformation \( \overline{f} : L(X) \to K(X) \) defined by \( \overline{f}(f^*1) = f_! 1 \) for \( f: Z \to X \) oriented and proper. \( \overline{f} \) is compatible with \( f_* \) \( f^* \) and products and therefore in particular is a ring homomorphism.

If \( L \) is a line bundle on \( X \) we have \( i: X \to L \) inducing \( i_*: \Omega^*(X) \to \Omega^{*+2}_{\text{pr}/X}(L) \) and we set

\[ i^* i_* 1 = c_1(L) \in \Omega^2(X), \]

the first Chern class of \( L \). Important:

\( C_1: \text{Pic}(X) \to \Omega^2(X) \) is not a homomorphism, since

\[ \overline{f}(c_1(L)) = (L^*)^1 = \lambda_1(L) = 1 - L^{-1} \]

is not an additive function of \( L \).

The conjecture $\Sigma^2(\xi) = \lambda z + \text{proof} \alpha$

Newtons finds

good proof that $\Omega(\lambda t) = \lambda z$ using

$\Omega(x) = \text{univ. Chem ring of } K(x)$. If $A$ is a ring, let $F(A) = \{ F \in A[\![X,Y]!] \}$.

$F(x,0) = F(0,x) = X$, $F(x,y,0) = F(x,y)$, $Z$. Then

$A \rightarrow F(A)$ is clearly a representable functor. Given $F$

there is a unique power series $\psi(x) = x^i$ higher terms $\ast$

$\in A_q[[X]]$ \Rightarrow

$$\psi(F(x,y)) = \psi(x) + \psi(y)$$

In fact

$$\psi'(x) = \frac{1}{F_y(X,0)} \in A[[x]]$$

Writing

$$\psi'(x) = \sum_{i=0}^\infty a_i x^i \quad a_i \in A \quad a_0 = 1$$

we have

$$\psi(x) = \sum_{i=0}^\infty \frac{a_i x^{i+1}}{i+1} \in A_q[[X]]$$

Conclude: If $(V,F_0)$ represents $F$ and if

$$\frac{1}{F_0(y)}(X,0) = \sum_{i=0}^\infty a_i X^i$$

then

$$\mathbb{Q}[a_1, a_2, \ldots] \rightarrow V_q$$

Consequently if $F(x,y) \in \Omega^*(\text{pt})[[X,Y]]$ is the series $\ast$

$$c_1(L_1 \otimes L_2) = F(c_1 L_1, c_1 L_2)$$

then there is a unique homomorphism

$$\psi: V \rightarrow \Omega^*(\text{pt})$$

such that $\psi F_0 = F$. We know that $\psi a_i = P_i$, hence by them

Conjecture: $(1)$ is $\psi \sim \Omega^*(\text{pt.})$ an isomorphism.
Newtons formulas

Let \[\psi^q = \sum x_i^q\] power sum
\[\lambda^q = \sum_{i_1 < \ldots < i_q} x_{i_1} \ldots x_{i_q}\] elementary symmetric funs.

\[-\log (1-tx) = \sum_{i=1}^n \sum_{n=1}^\infty \frac{t^n x_i^n}{n} i\]

\[-\log \lambda + = \sum \frac{t^n \psi^n}{n}\]

\[-\log \lambda - = \sum (-1)^n \frac{t^n \psi^n}{n}\]

\[-\lambda' + = \lambda' \sum (-1)^n \frac{t^n \psi^n}{n}\]

\[\psi^k - \psi^{k-1} \lambda' + \psi^{k-2} \lambda'^2 - \ldots + (-1)^{k-1} \psi^1 \lambda^{k-1} + (-1)^k k \lambda^k = 0\]

\[\psi^1 = \lambda'\]
\[\psi^2 = (\lambda')^2 - 2\lambda^2\]
\[\psi^3 = (\lambda')^3 - 3\lambda' \lambda^2 + 3\lambda^3\]
January 29, 1969

Determination of \( \Omega_q(\text{pt}) \)

Let \( V \) be an algebra over \( \mathbb{Q} \) endowed with a formal group law\( \psi \) or equivalently a power series

\[
\psi(x) = \sum_{i=0}^{\infty} a_i \frac{x^{i+1}}{i+1}
\]

with \( a_i \in V \) \( a_0 = 1 \)

\[
\psi(F(x, y)) = \psi(x) + \psi(y)
\]

For each manifold \( X \) set

\[
(1) \quad V(X) = V \otimes_{\mathbb{Z}} K(X); \]

this is a contravariant functor from manifolds to \( V \)-algebras.

Let \( \text{ch} : K(X) \to V(X) \)

be the ring homomorphism defined by \( \text{ch} x = 1 \otimes x \).

We now define Chern classes

\[
c_i : K(X) \to V(X)
\]

\( i \geq 0 \).

\( c_0 = 1 \).

If \( L \) is a line bundle define \( c_i(L) \in V(X) \) by the formula

\[
\text{ch} L = e^\psi(c_i(L))
\]

(OKAY because \( \text{ch} (1-L) \) is nilpotent in \( V(X) \))

\[
c_0(L) = 0 \quad \text{for} \quad i \geq 2
\]

Now if \( E \) is any vector bundle on \( X \), \( V(\text{PE}) \) is a free \( V(X) \) module with basis \( \left[ \text{ch} \mathcal{O}(1) \right]^i \), \( 0 \leq i < n = \text{rank } E \).
Therefore it also has basis \[ \text{ch}(c(1) - 1) \mathcal{E} \text{ is an}\]

and since

\[
\text{ch}(L - 1) = \varphi(c_1(L)) - 1 = c_1(L) \left\{ 1 + a c_1(L) + \ldots \right\}
\]

unit since \( c_1(L) \) is nilpotent.

\( \mathcal{V}(PE) \) has basis \( c_1(0(1))^{i} \) of \( \mathcal{V}(X) \). Hence we may define \( C_\ell(E) \) by the relation

\[
\xi \frac{n}{\ell} - \frac{n}{\ell} c_1(E) \cdot \xi^{n-1} + \ldots = 0,
\]

where \( \xi = c_1(0(1)) \).

If \( E = L_1 + \ldots + L_n \), then we have the relation

\[
\prod_{i=1}^{n} (O(1) - f^* L_i) = 0 \quad \text{in} \ K(PE)
\]

hence applying \( \text{ch} \), the relation

\[
\prod_{i=1}^{n} (\varphi(\xi) - f^* \varphi (c_1 L_i)) = 0
\]

But

\[
\varphi (x) - \varphi (y) = (x-y)(1 + \ldots) \quad \text{higher order terms which will be nilpotent}
\]

hence

\[
\prod_{i=1}^{n} \left( \xi - f^* c_1(L_i) \right) = 0
\]

\[\varphi_\ell (E) = \sum_{i_1 < \ldots < i_\ell} c_1(L_{i_1}) - c_1(L_{i_\ell})\]
Digression: We have defined Chern classes using the fact that \( K \) is defined on a category for which we have for each \( x \in K(X) \) a representative \( x = E - n \) and a splitting map \( f: P(E) \to X \), etc. Further work requires us to function with a single \( 1 \)-ring \( K \). Thus, given \( ch: K(X) \to V(X) \) we can define additive maps \( ch_0: K(X) \to V(X) \) by the formula
\[
ch_0(x) = \sum_{k=0}^{\infty} \frac{\partial^k}{\partial t^k} \text{ch}_k x \quad \text{all } k.
\]
(This can be done \( \square \) under some nilpotent hypothesis, e.g., \( \square \) in \( K \) itself \( \square \) we can write \( \psi^k x = \sum_{b=0}^{N} \frac{\partial^k}{\partial t^k} \text{ch}_b x \mod F_{N+1} \).)

Hence \( ch_0 x = ch(x_0) \), assuming that \( ch F_{N+1} = 0 \).

\[
ch_0 x = \text{rg } x
\]

I claim that there are universal formulas
\[
ch_0 = P_0(\partial_1, \partial_2, \ldots) \quad g > 0
\]
\[
C_0 = Q_0(\text{ch}_1, \text{ch}_2, \ldots) \quad g > 0.
\]

as power series with coefficients in \( V \). To determine these universal formulas one uses the algebraic splitting principle and sets
\[
c_{\delta}(E) = \text{TT} (1-tX_i)
\]
\[
ch(E) = \sum e^{\psi(X_i)}
\]
\[
E = L_1 + \cdots + L_n
\]
\[
X_i = c_i(L_i)
\]
\[ ch(E^k) = \sum e^{\psi(c_i, L_i^k)} = \sum e^{k \cdot \psi(x_i)} \]

\[ = \sum \frac{1}{8!} \sum \psi(x_i)^8 \]

\[ ch(T) = \frac{1}{8!} \sum \psi(x_i)^8 \]

and as the RHS is a symmetric fun. in \( x_i \) without constant term \( (\psi(x) = x + \cdots) \) it can be expressed in terms of the \( c_i, i \geq 0 \). By Newton one knows that

\[ Q[(\psi_1, \psi_2, \cdots)] = Q[[x_1, x_2, \cdots]]^2 = Q[[x_1, x_2, \cdots]]^2 \]

showing that the \( c_i \) are power series in the \( c_i \).

It is not true that \( ch(E) \) depends only on \( c_0, \cdots, c_8, \ldots \) e.g. for \( E(x, y) = x + y - axxy \) we have \( \psi(x) = -\frac{1}{a} \log(1-ax) \) and

\[ ch_g(x) = \frac{1}{2} (c_0^2 - 2c_2) + \frac{a}{2} (c_1^3 - 3c_1c_2 + 3c_3) + \cdots \]

**Remark:** Not possible to define

\[ ch: K_G(x) \rightarrow \Omega_G(x) \otimes Q \]

In effect if \( G = S^1 \) and \( \Omega_G \) is replaced by \( H_G \), we have

\[ K_{S^1}(pt) \quad H_{S^1}(pt, Q) \]

\[ \mathbb{Z}[T, T^{-1}] \quad H^*(OP^2, Q) = \mathbb{Q}[X] \]

and

\[ ch(T) = e^x \otimes Q[X] \]
In fact it is not even clear how to make sense out of the formula

\[ c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)) \]

since the \( c_1(L) \) are no longer nilpotent.

Now define for any map \( f: X \to Y \) the map

\[ f^*: V(Y) \to V(X) \]

by

\[
\begin{align*}
    f^*(v \cdot y) &= v \cdot f^*(y) \\
    f^*(ch_1 y) &= ch_1 f^*(y)
\end{align*}
\]

And for any proper oriented map \( f: X \to Y \) define

\[ f_*: V(X) \to V(Y) \]

by the formula

\[
\begin{align*}
    f_*(v \cdot x) &= v \cdot f_*(x) \\
    f_*(ch_1 x) &= ch_1 f_* \cdot x
\end{align*}
\]

I claim that for a proper oriented map \( f: X \to Y \) it is possible to define

\[ f_*: V(X) \to V(Y) \]

uniquely so that

\[ f_*(v \cdot x) = v \cdot f_*(x) \quad x \in V(X), \ v \in V \]
\[ \text{ch} \left( f_! x \right) = f_! \left( \text{ch} x \cdot \text{Todd } \Theta_f \right) \quad x \in K(X). \]

In effect we have an isomorphism \( V \otimes K(x) \cong V(x) \) which we shall again denote by character and we can extend \( f_! \) on \( K(x) \) to \( V \otimes K(x) \) so as to be \( K \)-linear. Let \( u(E) \in V \otimes K(x) \) be \( \exists \) \( \text{ch} u(E) = \text{Todd } E \) and define

\[ f_! \left( \text{ch} \ z \right) = \text{ch} f_! \left( z \cdot u(\Theta_f) \right) \quad z \in V \otimes K(x) \]

Then

\[ f_! \left( \text{ch} x \cdot \text{Todd } \Theta_f \right) = f_! \left( \text{ch} x \cdot \text{ch} u(\Theta_f) \right) \]

\[ = f_! \text{ch} \left( x \cdot u(\Theta_f) \right) = \text{ch} f_! \left( x \cdot u(\Theta_f) \cdot u(\Theta_f) \right) \]

\[ = \text{ch} f_! x, \quad \text{as claimed} . \]

\[ u(E) \text{ is a characteristic class with values in } \ V \otimes K(x) \]

\[ \begin{cases} u(E+F) = u(E)u(F) \\ \text{ch} u(L) = \text{Todd } L \end{cases} \]

Let \( u(L) = \sum_{n \geq 0} b_n \left( 1-L^{-1} \right)^n = \chi(L^{-1}) \quad b_n \in \mathbb{Z}(\mathbb{Q}). \) Then

\[ \text{ch} \chi(L^{-1}) = \chi(1-L^{-1}) = \frac{X}{1-L} \quad X = c_1(L). \]
\[ u(L) = X(1-L^{-1}) \quad \text{where} \]
\[
\{1 - \varphi(X)^{-1}\} X (1 - \varphi(X)^{-1}) = X.
\]

Thus to find \( X \) involves inverting \( \varphi \). Example:

\[ F(x, y) = x + y - axy. \] Then if

\[ y = 1 - \varphi(x)^{-1} \quad \varphi(x) = \frac{1}{1-y} \]

\[
\frac{1}{1-y} = (1-ax)^{-\frac{1}{a}} \quad X = \frac{1 - (1-y)^a}{a}
\]

\[ X(y) = \frac{1 - (1-y)^a}{ay} \]

\[ u(L) = \frac{1}{a} \left( \frac{1 - L^{-a}}{1 - L^{-1}} \right) = \frac{1}{a} \rho^a(L). \]

where we recall that the Wu class for bundle corresponding to the operation \( \gamma^k \) is

\[ \rho^k(L) = \frac{\gamma^k(1-L^{-1})}{1-L^{-1}} = \frac{1-L^{-k}}{1-L^{-1}}. \]
$V(X)$ is therefore a twisted version of the cohomology theory $X \to V \otimes K(X)$ twisted by the characteristic class $\langle \alpha \rangle$. Hence there is a unique morphism of cohomology theories

$$ \phi : \Omega^* \to V $$

compatible with $f^* + f^*$ such that $\phi 1_{X^*} = 1_{X^*}$.

As $V$ satisfies the splitting principle ($V(PE)$ free over $V(X)$ etc) $\phi$ commutes with Chern classes (one must check that with $\phi \circ \phi$ as above, then $c_i(L) = L \otimes c_i 1 : X \to L$). Hence $\phi : \Omega \to V$ carries the law for $c_i(L \otimes M)$ in $\Omega$ into that for $V$. In particular if the $V$ law is given by $y(x) = \sum a_i \frac{X^{i+1}}{i+1}$, then

$$ \phi(p_i) = a_i. $$

Now take $(V, F_0)$ to be the universal formal group law in one variable over $\Omega$. $V = \Omega[a_0, a_1, \ldots]$ and

$$ y_0(F_0(X, Y)) = y_0(X) + y_0(Y) $$

where $y_0(x) = \sum a_i \frac{X^{i+1}}{i+1}$ with $a_0 = 1$.

Then we have a unique map

$$ \phi : V \to \Omega \otimes \Omega $$
sending \( a_i \) to \( P_i \), and \( \mathbf{F} \) extends to a natural transformation

\[
\mathbf{F} : V \otimes K(X) \longrightarrow \Omega^*_Q(X)
\]

\[ v \otimes x \longmapsto \mathbf{F}(v) \cdot \text{ch}(x) \]

It is necessary to check that \( \mathbf{F} \) is compatible with \( g \). I think this is clear. The point is that \( \mathbf{F} \) is universal recipient for the Chern classes so that \( \mathbf{F} \mathbf{F} = \text{id} \) while \( X \mapsto \Omega_Q(X) \) is the universal cohomology theory so that \( \mathbf{F} \mathbf{F} = \text{id} \).

Therefore we have proved

**Theorem**: \( \Omega_Q(\text{pt}) \cong Q[P_1, P_2, \ldots] \)

\[
\text{ch} : \Omega_Q(\text{pt}) \otimes K(X) \longrightarrow \Omega^*_Q(X)
\]

\[ v \otimes x \longmapsto v \cdot \text{ch}(x) \]

\[
\begin{aligned}
\text{ch} f^! x &= f^! \text{ch}(x) \\
\text{ch} f_* x &= f_* (\text{ch}(x) \cdot \text{Todd}(\Theta))
\end{aligned}
\]

where

\[
\begin{aligned}
\text{ch}(L) &= e^{-\sum p_j \frac{c_j(L)^{j+1}}{j+1}} \\
\text{Todd}(L) &= c_1(L)/1-(\text{ch}(L))
\end{aligned}
\]
January 30, 1969:

Let $V$ be a ring endowed with a formal group law $F$. Let $X \rightarrow V(X)$ be a cohomology theory on the category of manifolds with values in $V$-algebras endowed with a gysin homomorphism for $U$-oriented proper maps. We assume $V$ satisfies the splitting principle:

1. $V(PE)$ is a free module over $V(X)$ with basis $1, \beta, \ldots, \beta^{n-1}$ where $\beta = c_1(O(1))$, $n = \dim E$ and that $c_1 = c_1(1)$.

2. $c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2))$.

(Note this makes sense since $c_1(O(1))$ is nilpotent in $V(P^n)$ hence $c_1(L)$ always nilpotent.

Then one can define Chern classes

$c_i : K(X) \rightarrow V(X)$

so that $c_i : K(X) \rightarrow 1 + V(X)[[t]]^+$ is the unique natural transf. with

1. $c_i(x+y) = c_i(x)c_i(y)$
2. $c_i(L) = 1 + t c_i(L)$.

I claim that there exists unique power series $P_i(c_1, c_2, \ldots, c_i', c_i'', \ldots) \in V[[c_1, c_2, \ldots, c_i', c_i'', \ldots]]$, $Q_i \in V[[c_i, c_i', \ldots]]$.
depending only on $F$ such that

$$c_z(xy) = \mathcal{P}_z(c_z(x), \ldots, c_z(y))$$

$$c_z(x^n) = \mathcal{Q}_z^n(c_z(x), \ldots)$$

for all $x, y \in K(K)$ and all $X$. To see this we can argue universally since $V(\mathcal{B}U) = \lim V(G_m)$ is the power series ring $\mathbb{V}[x_1, \ldots]$. Here is how one can obtain these power series in principle. Given $x \in K(K)$ (kernel of $\epsilon : K(K) \to H^0(X, \mathcal{O}_X)$) it can be written $x = E - n$, $n = \dim E$. Then by splitting may assume $E = L_1 + \cdots + L_n$. Thus

$$x = L_1 + \cdots + L_n - n$$
$$y = M_1 + \cdots + M_m - m$$

$$X_i = c_z(L_i) \quad Y_j = c_z(L_j)$$

$$c_z(xy) = \prod_{i,j} c_z((L_i - 1)(M_j - 1)) = \prod_{i,j} \frac{1 + tF(x_i, y_j)}{(1 + tx_i)(1 + ty_j)}$$

By theorems of elementary symmetric functions the right hand side is a power series in the $c_z(x), c_z(y)$ depending on $n$ and $m$. However one sees that on going from $n$ to $n+1$ the power series goes into the $(n+1,m)$ power series by setting $c_n = 0$. Thus one gets a well-defined power series in $c_z(x), c_z(y), \ldots$ which works for all $n, m$. The derivation of the $x$-series is similar but messier.
\[ x = L_1 + \ldots + L_n - n \]

\[ \lambda_n(x) = \prod_i \lambda_n(L_i - 1) = \prod_i \frac{1 + u L_i}{1 + u} \]

\[ = \prod_i \left\{ 1 - \frac{u}{1 + u} (1 - L_i) \right\} \]

\[ = \prod_{i=1}^m \left\{ 1 - u(1 - L_i) + u^2 (1 - L_i) \right\} \]

\[ = \sum_{\delta=0}^{\infty} (-u)^\delta \sum_{p=0}^{\delta} \sum_{a_0, \ldots, a_p \geq 0} \sum_{i_1 < \ldots < i_p} (1 - L_{i_1}) \ldots (1 - L_{i_p}) \]

\[ \therefore \lambda^\delta(x) = \sum_{p=0}^{\delta} (-1)^{d-p} (\delta) \sum_{i_1 < \ldots < i_p} (L_{i_1} - 1) \ldots (L_{i_p} - 1) \]

\[ (\ast) \quad C_t(\lambda^\delta(x)) = \prod_{p=0}^{\delta} \frac{\prod_{i_1 < \ldots < i_p} C_t(L_{i_1} - 1) \ldots (L_{i_p} - 1)}{(-1)^{d-p} (\delta)} \]

But

\[ C_t((M_1 - 1) \ldots (M_p - 1)) = C_t(\sum_{(i, j) \in I} M_{i_j} - M_{i_j}) \]

\[ (\ast\ast) \quad = \prod_{I \subseteq \{1, \ldots, p\}} (1 + t(x_{i_1} \ldots x_{i_p})) \]

where \( I = \{i_1, \ldots, i_p\} \) runs over all subsets of \( \{1, \ldots, p\} \).

Combining \((\ast) + (\ast\ast)\) one obtains a formula for \( C_t(\lambda^\delta(x)) \) as a symmetric power series in \( x_1, \ldots, x_n \) which can be written as a power series in \( c_1, \ldots, c_n \).
We hope to show that $\Omega(X)$ is the universal recipient for a completed Chern class from $K(X)$, at least for manifolds whose homology is torsion-free. Hence

**Question:** Let $Q(X)$ be the sub-$V(pt)$-algebra generated by $rg_x, c_i(x)$ $i \geq 0$ for all $x \in K(X)$. Is $Q(X)$ stable under $f^*$?

$Q(X)$ is clearly stable under $f^*$. If $f: X \to Y$ is a proper map we may factor it $X \xrightarrow{\delta} Y \xrightarrow{j} Y$ where $j$ is an embedding. $j^*$ is OKAY because

$$V(X) \xrightarrow{l^*_x} V(N, \bar{N}) \xrightarrow{j^*} V(YAV^+)$$

and $l^*_x 1 = c_n(\pi^*v)$. However $\pi^*_x: V(YAV^+) \to V(Y)$ doesn't seem to carry $Q(YAV^+)$ to $Q(Y)$ since all we know is that if $\alpha = l^*_x \beta \in Q(YAV^+)$, then $\alpha = \pi^* \beta \cdot l^*_x 1$, i.e. $\beta = l^*_x \{ \frac{\alpha}{l^*_x 1} \}$. This means that to get something stable under $f^*$ one must permit division of some sort. For example if we work over $\mathbb{Q}$ then $(rg, c_i(x)) = c(x)$ is expressible in terms of $ch_x$ and conversely, hence $Q(X)$ is generated by $V(pt)$ and $ch(KX)f^*$ one knows that $f^*$ on $Q(X)$ is then determined by $f^*$ on $K(X)$ and characteristic classes of $f$. 
Answer to question is probably false, in fact Adams claims \( \exists a \text{ finite } \omega X \text{ with } K(X) = 0 \) but with \( H(X) \neq 0 \).

Hence \( \omega(X) \neq 0 \) which means that \( \exists f: \mathbb{Z} \rightarrow X \) proper and oriented with \( f_*1 \neq 0 \) yet \( \omega(X) = 0 \).