Theorem: (Harish-Chandra). If $\mathcal{V}$ is an irreducible $\mathfrak{g}_{\mathbb{k}}$ module then for some irreducible $M_{\mathbb{k}}$ module $\mathcal{I}$, $\mathcal{V}$ is a composition quotient of $\mathcal{I}(\mathcal{I})$.

Proof: Let $\Lambda$ be an irreducible finite $\mathfrak{k}$ module such that $\mathfrak{g} = \text{Hom}_{\mathfrak{k}}(\Lambda, \mathcal{V}) \neq 0$. Then $\mathfrak{g}^0$ is an irreducible $\Omega^0_{\Lambda}$ module. We shall make the following assumption which will be verified later:

Hypothesis: There is an irreducible $M_{\mathbb{k}}$ module $\mathcal{I}$ such that $\mathfrak{g}^0$ is the restriction of $\text{Hom}_{M_{\mathbb{k}}}(\Lambda, \mathcal{I})$ under the natural homomorphism $\square: \Omega^0_{\Lambda} \to U(\mathfrak{g}) \otimes \text{Hom}_{M_{\mathbb{k}}}(\Lambda, \Lambda)$.

Consider diagram

\[
\begin{array}{ccc}
(\text{U(g)}_{\mathbb{k}} \otimes \Lambda)_{\Omega^0_{\Lambda}} & \xrightarrow{\alpha} & I(\mathcal{I}) \\
\downarrow \beta & & \\
\mathcal{V} & \xrightarrow{\psi} & \\
\end{array}
\]

Definition of $\beta$: Recall $\mathfrak{g} = \text{Hom}_{\mathfrak{k}}(\Lambda, \mathcal{V})$ so

\[
\beta(\psi \otimes \lambda \otimes \omega) = \psi \cdot \omega(\Lambda).
\]

Definition of $\alpha$: By universal property of LHS $\alpha$ given by a map

\[
\xi \rightarrow \text{Hom}_{\mathfrak{k}}(\Lambda, I(\mathcal{I}))
\]

of $\Omega^0_{\Lambda}$ modules.
But $\text{Hom}_K(\Lambda, I(I)) \rightarrow \text{Hom}_M(\Lambda, I)$ is a diformorphism.

The map

$$\Omega^i \rightarrow U(\alpha) \otimes \text{Hom}_M(\Lambda, I),$$

and by hypothesis we can find such a diform, non-zero:

$$\xi \rightarrow \text{Hom}_M(\Lambda, I).$$

So we get $\alpha$ which is non-zero.

Problem is to construct a map

$$\text{Hom}_K(\Lambda, V) \rightarrow \text{Hom}_M(\Lambda, I),$$

which is a diformism for the map

$$\Omega^i \rightarrow U(\alpha) \otimes \text{Hom}_M(\Lambda, I).$$

But you are now reduced to the following lemma:

**Lemma:** Suppose $R \rightarrow S$ map of rings, $R
\Lambda_1$ irreducible $R$ module. Then there is a non-zero diform $\Lambda_1 \rightarrow \Lambda_2$ where $\Lambda_2$ is an irreducible $S$
module.

**Proof:** Choose an irreducible quotient of $S \otimes_R \Lambda_1$, which is possible since $\Lambda_1 \sim R/I \Rightarrow S \otimes_R \Lambda_1 \sim S/I \infty$. Conclude $S \otimes_R \Lambda_1$ might be zero.
\[ F : \Omega^\Lambda \rightarrow U(\omega) \otimes \text{Hom}_m(\Lambda, \Lambda). \]

\[ \text{ring hom.} \]

\[ \text{End}_{\text{g}}(\lambda \otimes \text{End}_{\text{g}}) \rightarrow \text{End}_{\text{g}}(\lambda \otimes 1 \otimes \text{g}) \]

Now an irreducible module over \( U(\omega) \otimes \text{Hom}_m(\Lambda, \Lambda) \) is of the form \( \lambda \otimes \text{Hom}_m(\Lambda, \Lambda) = \text{Hom}_m(\Lambda, \lambda) \). In the case where \( I(\lambda) \) is irreducible, then I know in principle how to define an isomorphism

\[ I(\lambda \otimes \sigma) \rightarrow I(\lambda \otimes \sigma), \quad \sigma = \lambda \otimes \tau. \]

and of course the three diagrams

\[ \lambda (\sigma) = \lambda - \frac{1}{2} \]

\[ \lambda - 1 = \lambda' \cdot \lambda \]

which of course means that

\[ \text{Hom}_m(\Lambda, \lambda) \otimes \text{g} \rightarrow \text{Hom}_m(\Lambda, \lambda) \otimes \text{g} \]

as \( \Omega^\Lambda \) modules.

In other words

\[ \lambda(\sigma) = \lambda - \frac{1}{2} \]

Recall \[ \lambda(\sigma) = \lambda' \cdot \lambda \]

\[ \boxed{1} \]

\[ \boxed{1} \]
Claim $F$ an $\mathcal{O}_\Lambda$ isom.

\[ \text{Hom}_M(\Lambda, \Lambda) \otimes \lambda \otimes g \rightarrow \text{Hom}_M(\Lambda, \Lambda) \otimes \lambda \otimes g \]

Let a group $W$ act on a ring $R$.

Let $V$ be an $R$ module.

Let $s \in W$. Then $s$ defines

\[ U(\alpha) \otimes \text{Hom}_M(\Lambda, \Lambda) \rightarrow U(\alpha) \otimes \text{Hom}_M(\Lambda, \Lambda) \]

\[ p \otimes \varphi \rightarrow p^s \otimes \varphi^s \]

where $\varphi^s(\lambda) = \varphi(\alpha_5^{-1} \lambda) \varphi(\alpha_5 \lambda)$

all $\lambda m \mu m^{-1} = \mu$ this is independent of the choice of $\alpha_5$

and

$\varphi^s(\lambda) = ?$
Conjecture: $F: \Omega_\Lambda \to [\mathcal{U}(\mathfrak{o}) \otimes \text{Hom}_\mathbb{M}(\Lambda, \Lambda)]^W$ is an isomorphism.

Definition of $F$:

$$\Omega_\Lambda \xrightarrow{\text{End}_g(\mathcal{U}(\mathfrak{o}) \otimes \Lambda)} \text{End}_{\mathbb{M}_{\mathcal{O}}} \mathcal{U}(\alpha) \otimes \Lambda \xrightarrow{\rho_{\mathcal{O}}} \mathcal{U}(\alpha) \otimes \text{Hom}_\mathbb{M}(\Lambda, \Lambda)$$

$$F \mathcal{U}(\alpha) \otimes \text{Hom}_\mathbb{M}(\Lambda, \Lambda) \xrightarrow{\psi} \mathcal{U}(\alpha) \otimes \text{Hom}_\mathbb{M}(\Lambda, \Lambda)$$

Evidence:

We know that generically $I(s \circ g) \sim I(s \circ g)$, hence $\lambda \circ \log \otimes \text{Hom}_\mathbb{M}(\Lambda, \nu) \sim \lambda \circ \log \otimes \text{Hom}_\mathbb{M}(\Lambda, \nu_q)$. 

$G(\Omega_\Lambda) \subset \mathcal{U}(\alpha) \otimes \text{Hom}_\mathbb{M}(\Lambda, \Lambda)$ right modules.

$$\lambda \otimes \text{Hom}_\mathbb{M}(\Lambda, \nu) = \lambda \circ \log \otimes \text{Hom}_\mathbb{M}(\Lambda, \nu_q)$$

as $F(\Omega_\Lambda)(\subset \mathcal{U}(\alpha) \otimes \text{Hom}_\mathbb{M}(\Lambda, \Lambda)$) right modules.

$P(A+g) = \beta(A)$

$\mathcal{N}(A) \circ \rho(A) = (\beta A)(\lambda)$

$A^2 - A \mapsto (A + \frac{1}{2})^2 - (A + \frac{1}{2}) = A^2 - \frac{1}{4}$

$\beta A = A + g(A)$
Be Cautious.

We know that \( F(\mathfrak{A}) \subset \mathcal{U}(\mathfrak{g}) \otimes \text{Hom}_M(M, M) \) is such that there is an isom

\[ \lambda \otimes \text{Hom}_M(M, M) \cong \lambda^g \otimes \text{Hom}_M(M, M^g) \]

right module.

We know that \( \mathfrak{g} \) acts on \( \mathcal{U}(\mathfrak{g}) \otimes \text{Hom}_M(M, M) \) by

\[ s(P \otimes y) = (sP) \otimes sy \]

where \( (sP)(\lambda) = P(s^{-1}\lambda) \)

\[ (sy)^s = \alpha_s y \alpha_i \]

\[ (\lambda \otimes v)^a_s (t) = (\lambda \otimes v)(\alpha_s^{-1} t \alpha_s) \quad t \in MA. \]

\[ = (\lambda \otimes v)(\alpha_s^{-1} m \alpha_s \cdot e^{\alpha_s^{-1} \alpha_s}) \quad t = me^a \]

\[ = \mathcal{U}(\alpha_s^{-1} m \alpha_s) \cdot e^{\lambda(\alpha_s^{-1} \alpha_s)} \]

Problem: Calculate the module structure of the right \( \mathcal{U}(\mathfrak{g}) \otimes \text{Hom}_M(M, M) \) module and we define

\[ V^s \]

by

\[ (P \otimes y)(v)^s = (P \otimes y)(v) \]
Then \( V^s = \lambda^s \otimes \text{Hom}_M(\Lambda, \nu^{\otimes s}) \).

Applying \( \psi : V^s \to W \) same as a map \( \overline{\psi} : V \to W \)

\[ \overline{\psi}(z \varphi) = \psi(z \varphi^s) = z^s \varphi(z). \]

so I want a map

\[ \chi : \lambda \otimes \text{Hom}_M(\Lambda, \nu) \to \lambda^s \otimes \text{Hom}_M(\Lambda, \nu^{\otimes s}) \]

such that \( \chi(z \varphi) = \chi(z) \varphi^s \).
Definition: of $F : \Omega_\Lambda \rightarrow U(\omega) \otimes \text{Hom}_M(\Lambda, \Lambda)$

$\Omega_\Lambda = \text{End}_G( U(\omega)(g, \Lambda) ) \circ \text{End}_M( U(\omega)(\Lambda)) = U(\omega) \otimes \text{Hom}_M(\Lambda, \Lambda)$

$\beta(A) = A + g(A)$.

$g = \frac{1}{2} \sum_{\alpha \in \Sigma} \alpha$.

Conjecture: $F$ induces an isomorphism

$\Omega_\Lambda \rightarrow [U(\omega) \otimes \text{Hom}_M(\Lambda, \Lambda)]^W$.

This should not be that difficult.

Alternative definition of $F$. Know that

$U(\omega)(g)^k \rightarrow \Omega_\Lambda$ onto

$a \mapsto (u \circ \lambda \mapsto u \circ a \circ \lambda)$

$\psi_a(\omega \circ \lambda) = u \circ a \circ \lambda$

$\psi_a(\omega \circ \lambda) = (u \circ a \circ \lambda) = \psi_b(\psi_a(u \circ \Lambda))$.

Take $U(\omega)(g)^k$?

$U(\omega(g)) \rightarrow U(\omega(g)) \otimes U(\omega(g)) \circ (\Lambda) \sim U(\omega) \otimes \text{Hom}(\Lambda, \Lambda)$. 
Given $u \in U(g)^k$, write

\[ u = u_+ + u_0 + u_- \]

where

\[ u_+ \in U(g) \]
\[ u_0 \in U(\mathfrak{g}) \otimes \text{Hom}(\Lambda, \Lambda) \]
\[ u_- \in U(g) \otimes \text{Hom}(\Lambda, \Lambda) \]

\[ U(g) \otimes \text{Hom}(\Lambda, \Lambda) \]

\[ 0 \to J(\Lambda) \to U(\mathfrak{g}) \to \text{Hom}(\Lambda, \Lambda) \to 0 \]

\[ V \xrightarrow{id} V \]

\[ U(\mathfrak{g}) \otimes \text{Hom}(\Lambda, \Lambda) \]

\[ U(\mathfrak{g}) U(\mathfrak{g}) \otimes U(\mathfrak{g}) \]

\[ U(\mathfrak{g}) \otimes \text{Hom}(\Lambda, \Lambda) \]

\[ \text{Given } \pi \]

\[ \text{Let } \pi : U(g) \rightarrow U(\mathfrak{g}) \otimes \text{Hom}(\Lambda, \Lambda) \]

\[ U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g}) \]

\[ \varepsilon \otimes \text{id} \otimes \pi \]

Then claim that if $u \in U(g)^k$, $\pi$ is a homomorphism.

\[ \pi(u \cdot V) \]
\[ U(y) \stackrel{\pi_i}{\longrightarrow} U(\omega)U(\bar{z}). \]

\[ U \equiv u_0 \mod \, \nu U(y) + U(y) \delta(\lambda) \]

where \( u_0 \in U(\omega) \)

\[ U(\omega) = T + \delta(\lambda) \]

\[ u - u_0 \in \nu U(y) + U(y) \delta(\lambda) \]

\[ v - v_0 \in \]

\[ U v - u_0 v_0 = (u - u_0) v + u_0 (v - v_0) \]

want work...

\[ U(y)^k \rightarrow U(y)^k U(y) \]

\[ U(y)^k \]
Let $v \in U(g)^k$.

$$\psi_v : \Lambda \to \Lambda \otimes U(g) \otimes \Lambda \to U(\Lambda) \otimes \Lambda$$

$$\psi_v(\lambda) = v \otimes \lambda.$$

$$U \otimes U(\Lambda) \otimes U(\Lambda) \otimes U(\mathbb{k})$$

$$\Lambda \to (\varepsilon \otimes \text{id} \otimes \text{id}) \circ : \Lambda$$

So take $v$ and apply $\varepsilon \otimes \text{id}$ to get in $U(\Lambda) U(\mathbb{k})$.

Then apply map $U(\mathbb{k}) \to \text{Hom}(\Lambda, \Lambda)$.

Thus seems to be correct.

$$v \mapsto v \lambda$$

$$U \otimes U \to \text{Hom}(\Lambda, \Lambda) \times \to (\lambda \mapsto x \lambda)$$

$$\pi \nu \nu = \pi(\pi \nu).$$

$$\pi \nu = (\varepsilon \otimes \text{id}) u.$$
problem is to make $W$ act.

Take an element $\alpha \in W$. Then make $\alpha$ act. 

**First problem** — how to show image of $F$ lies where it should. H-C's method is to define a

$$D \psi = \langle \gamma(D), e^\lambda \rangle \psi$$

and to show that $\psi \lambda = \psi$. Then

$$\langle \gamma(D), e^{s\lambda} \rangle = \langle \gamma(D), e^\lambda \rangle$$

$$\langle \gamma(D)^s, e^\lambda \rangle$$

... $\gamma(D)^s \gamma(D)$ have same values and so are equal.

next he defines considers the filtered map

$$\Omega^L \xrightarrow{F} [U(\omega) \otimes \text{Hom}_M(\lambda, \lambda)]^W$$

$$\text{gr} \Omega^L = \text{Hom}_{f,k} (S(\phi) \otimes \lambda, S(\phi) \otimes \lambda) = \text{Hom}_k (\Lambda, S(\phi) \otimes \lambda)$$

$$= J \otimes \text{Hom}_k (\Lambda, \mathcal{H} \otimes \lambda)$$
\[ \Omega_{\Lambda} \rightarrow [S(\phi) \otimes \text{Hom}_\mathbb{M}(\Lambda, \Lambda)]^W \]

\[ [S(\phi) \otimes \text{Hom}(\Lambda, \Lambda)]^k \cong [S(\phi) \otimes \text{Hom}_\mathbb{M}(\Lambda, \Lambda)]^N \]

Suppose \( \text{Hom}_\mathbb{M}(U(\phi) \otimes \Lambda_1, U(\phi) \otimes \Lambda_2) \)

then tensoring with \( U(\phi)^W \rightarrow \mathcal{O} \) get.

\[ \left[ U(\phi)^W \otimes \text{Hom}_\mathbb{M}(\Lambda_1, \Lambda_2) \right] \]

try!! \quad \text{Hom}

for \( \mathfrak{sl}(2, \mathbb{R}) \) how does \( W \) act on

\[ \text{Hom}_\mathbb{M}(\Lambda_1, \Lambda_2) \]

\[ \text{ie} \quad \text{given } \alpha_s \quad \phi \in \text{Hom}_\mathbb{M}(\Lambda_1, \Lambda_2) \]

\[ \text{ie. } \alpha_s \in \mathcal{K} \text{ so } \]

\[ \alpha_s \phi \alpha_s^{-1} = \pm \phi. \]

\( \alpha_s \) acts trivially if \( \sigma_1 \equiv \sigma_2 \) \( (2) \)

by \( -1 \) if \( \sigma_1 \equiv \sigma_2 + 1 \) \( (2) \)
Therefore in even differences we would get
\[ U(\alpha)^W \otimes \text{Hom}(\Lambda_1, \Lambda_2) \]
so if compatible with composition
\[ [H \otimes \text{Hom}(\Lambda, \Lambda)]^k \rightarrow [H \otimes \text{Hom}_M(\Lambda, \Lambda)]^W \]

In Helgason is given a proof that
\[ S(\phi)^k \rightarrow S(\alpha)^W \]
using a mixture of techniques. Techniques
in injectivity s-s els. all conjugate to things in or
\[ S(\alpha) \text{ integral over } J \text{ direct calc.} \]
\[ S(\alpha)^W \rightarrow J \]
But have same q.f. by S theory.

\[ [S(\phi) \otimes \text{Hom}(\Lambda, \Lambda)]^k \rightarrow [S(\alpha) \otimes \text{Hom}(\Lambda, \Lambda)]^N \]

In injectivity. Given \[ \sum_i P_i \otimes q_i \]
assume that
\[ \sum_i P_i(\alpha) \cdot q_i(\lambda) = 0 \quad \text{all } \lambda \in \Lambda \quad a \in \alpha' \]
\[ \Rightarrow P_i(\alpha) = 0 \quad \text{all } a \in \alpha \]
But \[\sum_i p_i(p^*) \circ p_i(1) \neq 0\]

Then move by \(K\)

maybe this proves injectivity in general!! It still

have a map. Ought to work.

write this up.

why onto? The method of Chevalley is to show that the map is onto integral \& birational and hence an isomorphism as \(S(p)^k\) is normal.

Proof: That \(S(p)^k\) is normal. Let \(z \in S(p)^k\).

be integral over \(S(p)^k \Rightarrow z \in S(p)^k\).

\[p^k z = p^k z^k = q^k = q \Rightarrow z^k = 1 \Rightarrow z \in S(p)^k\.

Therefore consider the map

\[S(p)^k \circ \text{Hom}(A,L) \rightarrow [S(a) \circ \text{Hom}(A,L)]^N\]
which is gotten by restriction.

\[ S(\phi) \otimes \text{Hom}(\Lambda, \Lambda) \cong \Gamma(\phi', \Lambda \otimes \text{Hom}(\sigma) \Lambda) \]

**General fact:** for any finite \( k \) module \( \Lambda \)

\[ \left[ S(\phi) \otimes \Lambda \right]^k \xrightarrow{\sim} \left[ S(\alpha) \otimes \Lambda \right]^N \]

\[ \Gamma(\phi', \Lambda \otimes \Lambda)^k \xrightarrow{\sim} \Gamma(\alpha', \Lambda \otimes \Lambda)^N \]

why? 

The general problem:

\[ \Gamma(\phi') \not\xrightarrow{\sim} \Gamma(\phi)^k \]

not an isom.

but this is Rallis maybe:

\[ \left[ S(\phi) \otimes \Lambda \right]^k \xrightarrow{\sim} \left[ S(\alpha) \otimes \Lambda \right]^N \答疑 \]

\[ \mathcal{J} \otimes \left[ \Lambda \otimes \Lambda \right]^k \]
Have to prove that

\[
\text{Res} : \quad \left[ S(\mathfrak{p}) \otimes \Lambda \right]^k \xrightarrow{\cong} \left[ S(\mathfrak{o}) \otimes \Lambda \right]^N
\]

\[
\text{Hom}_k(\Lambda, S(\mathfrak{p})) \xrightarrow{\cong} \text{Hom}_N(\Lambda, S(\mathfrak{o})).
\]

\[
\text{method: \quad} 1 \otimes_J S(\mathfrak{p})
\]

\[
\Gamma(\mathfrak{p}', \Theta \otimes \Lambda)
\]

over the good set the fibers

Let \( U \) be the set of regular elts of \( \mathfrak{p}' \):

\[
\Gamma(U, \Theta \otimes \Lambda)^K
\]

idea is that

\[
\mathfrak{p}'_{\text{reg}} \quad \xrightarrow{\mathfrak{o}'_{\text{reg}}} \quad \mathfrak{o}'_{\text{reg}}/W
\]

so that clearly

\[
\Gamma(\mathfrak{p}'_{\text{reg}}, \Theta \otimes \Lambda)^K \xrightarrow{\cong} \Gamma(\mathfrak{o}'_{\text{reg}}, \Theta \otimes \Lambda)^N
\]

Thus get an isomorphism at generic point.
We have to redo certain ideas of H-C & Kostant.

The adjoint representation: Want to show that

\[(S(g) \otimes \Lambda)^N \sim (S(h) \otimes \Lambda)^N\]

Thus the 0 weight must occur in \( \Lambda \).

\[S(g) \otimes N \rightarrow S(h) \]

Note that \( S(g) \otimes N \rightarrow S(h) \) proves that

by means of the H-C method.

The Chevalley method: injectivity

\[S(g) \otimes N \text{ integrally closed} \checkmark\]

\[S(h) \text{ integral over } S(g) \otimes N\]

Quotient fields are the same.

Extend Scalars to modules. Thus I have a finite group \( W \) acting on \( X \) and two

\[\text{there is a module } M \text{ over } A^N \text{ such that } A \otimes_A M \rightarrow M\]

\[X \xrightarrow{f} X/W \]

descends since flat.
\[ S(h) \quad S(h) \otimes \Lambda^h \]
\[ S(h)^W \quad (S(h) \otimes \Lambda^h)^W \]

is a faithfully flat descent

however, look at the image of \( S(g) \otimes \Lambda^y \)

+ can you show that

\[ (S(g) \otimes \Lambda^y)^g \otimes S(h)^g \quad \sim \quad S(h) \otimes \Lambda^h \]

auto.?

\[ (S(g) \otimes \Lambda^y)^g \rightarrow S(h) \otimes \Lambda^h \]

Show that image contains \( \Lambda^h \).

i.e. that \( \not= 0 \) weight space

This gives simple proof that \( S(g)^g \rightarrow S(h)^W \) is faithful.

We know

but

\[ S(g)^g \otimes S(h)^W \]
Why is \((S(g) \circ \Lambda)^g \circ S(h) \rightarrow S(h) \circ \Lambda^h)\) onto?

\((S(g) \circ \Lambda)^K \circ S(\alpha) \rightarrow S(\alpha) \circ \Lambda^M)\) onto?

Take an \(M\) invariant \(\Lambda\). Somehow want to induce?

Look at \(0\) weight space!!! of \(\Lambda\) construct enough to show

\[\Lambda^h = (S(g) \circ \Lambda)^g + hS(h) \Lambda^h\]

Take a \(0\) weight element + use Nakayama

\[\Lambda\]

\[S(g)^g \uparrow \text{ff.} \]

\[S(g)^g\]
Conjecture: $(S(f) \otimes N)^k \sim (S(\alpha) \otimes N^m)^W$

evidence for conjecture - true if $N=1$

In fact both sides are free $S(f)^k = S(\alpha)^W$ modules of the same rank: left $(H \otimes N)^k = N^m$ (Halls)

right: by Chevalley $S(\alpha)$ free of rank $|W|$ over $S(\alpha)^W$ and $S(\alpha)/F \simeq$ group ring of $W$. So $(S(\alpha) \otimes N^m)^W \sim N^m$.

How to prove the conjecture.

Given that $g$ is a semi-simple Lie algebra with $g = k+p$. Let $\alpha$ be a maximal abelian semi-simple subspace.

Choose $z \in \alpha$ so that $T_z = (\text{ad } z)^2$ is as regular as possible.

$$S = J \otimes H$$ follows from Chevalley thm.

$$S(g) \sim J$$

To show that $S$ free over $J$.

To show that

$S(g)$ free over $S(g)^g$

using that

$S(h)$ free over $S(h)^W$
Why is $S(\gamma) \xrightarrow{\text{free over}} S(\gamma)^W$ ?

Injectivity by conjugacy theorem, which are free over $k$.

Thus, if $\tilde{R}$ is the orthogonal of $\alpha$ in $f$, we can filter $R$ by powers of $\tilde{R}$ and because things are graded, the filtration is finite in each degree, and hence $R$ is free over $k$.

Then must know that $S(\gamma)$ integral over $S(\gamma)$.
Prove that $S(g)^G \rightarrow S(h)^W$

(a) injectivity.

(b) $S(h)$ integral over $S(g)^G$. The point is to examine the char. poly of $ad x$

Set $P(x, T) = \det (T-\text{ad} x)$

$$= T^n + p_1(x)T^{n-1} + \ldots + p_n(x)$$

where $p_i(x) \in S(g)^G$. One then see that the eigenvalues of $ad h$

Thus $\lambda(h)^n + p_1(h)\lambda(h)^{n-1} + \ldots + p_n(h) = 0$

all $x$ and $\lambda$.

By Cayley Hamilton, $P(h, \text{ad} h) = 0$

$P(H, \lambda(H)) = 0$ for all $H$

$\Rightarrow \lambda$ integral over $S(g)^G$

$\Rightarrow S(g)^G \rightarrow S(h)^W$ integral okay.

$\Rightarrow S(g)^G \rightarrow S(h)$ integral.
Next have to know that

\[ \mathbf{S}(\mathfrak{o})^g \hookrightarrow \mathbf{S}(\mathfrak{h})^W \]

is an iso for quotient fields. This is easy because \( g \)

by the conjugacy theorem every regular orbit intersects \( \mathfrak{h} \) transversally at \( W \) orbit. Thus a function on \( \mathfrak{h} \) defines a \( g \) function on \( \mathfrak{o} \).

\vspace{1cm}

Lemma: let \( D \) be the regulator of \( g \) and let \( f \in \mathbf{S}(\mathfrak{o})[\frac{1}{p}] \)

be \( g \) invariant. Then if \( f/\mathfrak{h} \) is regular, \( f \) is regular.

\vspace{1cm}

Proof: \( \mathbf{S}(\mathfrak{o})^g \) is a unique factor domain, so write \( f = \frac{u}{D^g} \)
in lowest terms, then if \( f \neq 0 \), \( f \) is not regular on \( \mathfrak{h} \) unless \( u \) vanishes to order \( j \) on \( \mathfrak{h} \).

Next I have to try to prove

\[ (\mathbf{S}(\mathfrak{o}) \otimes \Lambda)^g \rightarrow (\mathbf{S}(\mathfrak{h}) \otimes \Lambda)^N \]

Again one can see that this should be an isomorphism when tensored with \( \mathbf{C} \left[ \frac{1}{D} \right] \) so one writes so take \( x \in (\mathbf{S}(\mathfrak{h}) \otimes \Lambda)^N \) and write \( D^g x = \beta \) where \( \beta \in \mathbf{S}(\mathfrak{o}) \otimes \Lambda \)

with \( j \) least. Thus

\[ x = \frac{\beta}{D^g} |_{\mathfrak{h}}. \]
\[
\frac{\beta}{D^4} \text{ is regular on } h \quad \text{by assumption}
\]

and thus \[ \beta \equiv \varphi D \mod m \]

\[
\text{Hm. vanishing}
\]

Go back to functions.

Given \( f \) invariant on \( h \) and we can put

\[ D^4 f = \text{res } u \quad \text{where } u \in S(\varphi) \]

assume \( \gamma \) least. Then

\[ \frac{u}{D} \quad \text{restricted to a reg. fn. on } h \]

ie

\[ u = D^2 v + \sum_{N} \theta \quad \theta \in I(h) \]

first note that if \( \nu \) invariant \( \mod m \)

then can modify to \( \theta \) \( N \) invariant.

So can also assume \( \theta \in N \text{ invariants in } I(h) \)

By \( D + I(h) \) transversely, so
\[(S(\mathfrak{g}) \otimes \Lambda)^N \rightarrow (S(\mathfrak{g}) \otimes \Lambda)^N\]

This is onto:

(a) show \( \circ C[\frac{1}{d}] \) it is onto; here \( D = \prod_{\alpha \in \Delta} \alpha \)

(b) suppose given \( s \in (S(\mathfrak{g}) \otimes \Lambda)^N \). Think of \( s \) as an \( N \)-invariant polynomial function on \( \mathfrak{g} \) with values in \( \Lambda \). By (a)

\[ T = D^s \]

where \( t \) is a \( \mathfrak{g} \)-equivariant function on \( \mathfrak{g} \) with values in \( \Lambda \).

We may assume that \( g \) is least, i.e. that \( \frac{t}{D} \) is not regular on \( \mathfrak{g} \). This means that at some and hence most points of \( \mathfrak{g} \) where \( D = 0 \) we have \( t \neq 0 \). I want to conclude that there is a point \( x \in \mathfrak{g} \) with \( D(x) = 0 \) and \( t(x) \neq 0 \). However, the generic points of \( \{ x \in \mathfrak{g} \mid D(x) = 0 \} \) is not \( \mathbb{K} \)-conjugate to an element of \( \mathfrak{g} \).

**The Hypothesis \( \mathfrak{g} \)-invariant**

Why true for functions? \( t \) is a \( \mathfrak{g} \)-invariant function on \( \mathfrak{g} \); \( t \) restricted to \( \mathfrak{g} \)-Sing is non-zero.

Why is it true that \( t \) restricted to \( \mathfrak{g} \)-Sing is non-zero?

**Idea:** Choose an orbit \( \mathcal{O} \) such that \( t \) restricted to this orbit is \( \neq 0 \); \( \mathcal{O} \subset \mathfrak{g}_c \).

**The Hypothesis**

Idea is that \( \mathcal{O} \) is a complete intersection and its singular locus is codim 2.
so $\frac{1}{t}$ which is regular on generic orbit must extend to a reg. fcn. on $\overline{O}$; then $t \neq 0$ on $\overline{O}$.

Be more precise and work in H-C proof:

\[
\int_{\text{int closed}} S(p) \rightarrow S(0)^N
\]

regular on $\infty$

Sheaf of functions on $p$ values in $\Lambda$

which are invariant + have poles singularities along $D$

\[ t = Ds + g \quad g \in I(\infty) \]

Given $t : \overline{O} \rightarrow \Lambda$ invariant.

Question can $t$ be $\neq 0$ generically 0 on sing. set.

This won't work: Take a function $f$ on $\overline{O}$ vanishing on sing set let $V$ be its space of translates; then get

\[ t : \overline{O} \rightarrow V' \]

\[ x \quad (f \rightarrow g(x)) \]

equivariant + $t = 0$ on sing. set, but is not generally 0.
Can you make same technique work globally? Take the affine variety $\text{Psing}$ and a function $f$ on $\text{Psing}$ which is non-zero yet vanishes on the semi-simple elements of $\text{Psing}$ ($\exists$ a non-semi-simple quasi-regular elt. of $\text{Psing}$, call it $z$; then nearby any element is quasi-reg. + non-semi-simple; so can choose $f \ni f(z) \neq 0$ yet $f(z.s.) = 0$). The space of $K$ translates of $f$ is finite dimensional — call it $V$ and consider the map $t: \text{Psing} \rightarrow V'$

$$v \mapsto (f \mapsto f(x)) \quad f \in V$$

Then $t$ is a reg. function on $\text{Psing}$ with values in $V'$ which is non-zero yet whose restriction to $\text{Ps.s.}$ $\text{Psing}$ (in particular at $z$) is 0.

Can $t$ be lifted to a fn. on $\mathfrak{p}$.

$$\text{Hom}(V, S(\mathfrak{p})) \rightarrow \text{Hom}(V, S(\mathfrak{p})/I(0)) \rightarrow 0$$

Yes!!! Thus there exists a polynomial function $t: \mathfrak{p} \rightarrow V'$

$k$ invariant, such that it vanishes on the irregular semi-simple elements, and yet is non zero on regular non-semi-simple regular when restricted to $\mathfrak{c}$, etc.
Summary: The conjecture that

\[ (S(\mathcal{P}')) \otimes N \rightarrow (S(\mathcal{Q}')) \otimes N \]

is an isomorphism is false in general.

Proof: Consider in \( S(\mathcal{P}') \) the set of elements which

let \( \mathcal{P}' \subseteq \mathcal{P} \) be the subvariety of singular elements

and let \( Z \in \mathcal{P}' \) be a quasi-regular element. The set of

such \( Z \) is open in \( \mathcal{P}' \) so there is a non-zero function

\( f \) on \( \mathcal{P}' \) which vanishes on all non quasi-

regular elements and in particular the semi-simple elts of \( \mathcal{P}' \).

Lift \( f \) to an element \( f' \) of \( S(\mathcal{P}') \) and let \( V \) be the

\( K \) subspace of \( S(\mathcal{P}') \) generated by \( f' \) and let \( \Lambda = V' \) so that

we get an invariant element \( V \subseteq (S(\mathcal{P}') \otimes N)^k \)

corresponding to the inclusion \( V \rightarrow S(\mathcal{P}') \).

Let \( D \) be the "discriminant function" on \( \mathcal{P} \) so that \( D \) is the

irreducible element of \( S(\mathcal{P}) \) with \( D \) zero for zeroes

and \( D \) is invariant. Then by constructing \( X \) is a

polynomial function on \( \mathcal{P} \) which is \( k \) invariant, which

vanishes for sing semi-simple elements, and which is non-zero

for some singular elements. Hence \( X = D' \) vanishes on \( \mathcal{P}' \) so

\( X/D \) is a regular func. on \( \mathcal{O} \) with

values in \( \Lambda \), necessarily \( N \) invariant. But as \( D \) is

injective if \( X/D \in \text{Im} \) \( \Rightarrow X/D \in (S(\mathcal{P}') \otimes N)^k \)

which is

false since \( X = 0 \) where \( D = 0 \).
More explicitly for $sl(2, \mathbb{R})$. Here $\rho = (X, Y)$

$k = (H)$ where $H \cdot X = X$
$H \cdot Y = -Y.$

so $e^{tH} \cdot X = e^{tX}$
$e^{tH} \cdot Y = e^{-tY}$

and $\alpha = (X + Y)$. Consider

$\frac{1}{12} A$

$N = e^{\pi i n H}$

$(S(\rho) \otimes X)^k \quad (S(\alpha) \otimes X)^N$

$C[X_Y] \otimes X$

$C[A^2] \cdot A \otimes X$

$(S(\rho) \otimes (X^2))^k 
\quad (S(\alpha) \otimes (X^2))^N$

$C[X_Y] \cdot Y^2 \otimes X^2$

$C[A^2] \cdot X^2$

map takes

\[
\begin{align*}
X & \mapsto A \\
Y & \mapsto A
\end{align*}
\]

so image is $C[A^2] A^2 \otimes X^2$ not equal !!!!
Conjecture: There is an analytic function $D$ such that if we stay away from the zeros of $D$ we get an isomorphism of categories.

We know that

$$\text{Hom}_g \left( U(g)_1 \otimes V_1, U(g)_2 \otimes V_2 \right) \rightarrow \left[ U(\mathfrak{g}) \otimes \text{Hom}_m (V_1, V_2) \right]$$

is injective and hopefully we can get it to be Weyl invariant?

try sl(2, \mathbb{R}) for Weyl invariance!

Image of $F$ in $U(\mathfrak{g}) \otimes \text{Hom}(N\sigma N, N\sigma + 1)$ is

$$\begin{cases}
(A - \sigma - \frac{1}{2}) \otimes \phi_{\sigma + 1} \\ (A + \sigma) \otimes \phi_{\sigma + 1}
\end{cases} \rightarrow \begin{cases}
(A - \sigma - \frac{1}{2}) \otimes \phi_{\sigma + 1} \\ (A + \sigma + \frac{1}{2}) \otimes \phi_{\sigma + 1}
\end{cases}$$

Action of $W$?

$W$ acts by $-1$ on $\text{Hom}(N\sigma N, N\sigma + 1)$

$$A \rightarrow -A$$

$$(A - \sigma - \frac{1}{2}) \otimes \phi_{\sigma + 1} \rightarrow (A - \sigma - \frac{1}{2}) \otimes \phi_{\sigma + 1}$$

$$(A + \sigma + \frac{1}{2}) \otimes \phi_{\sigma + 1} \rightarrow (A + \sigma + \frac{1}{2}) \otimes \phi_{\sigma + 1}$$
(A^2+\frac{1}{4})(A-\sigma-\frac{1}{2}) \psi_{\sigma-\frac{1}{2}} \psi^\sigma \psi^\sigma \psi_{\sigma+\frac{1}{2}} = A^2 - (\sigma+\frac{1}{2})^2 - (\sigma-\frac{1}{2})^2.

Thus \psi^{\sigma+\frac{1}{2}} \psi^{\sigma-\frac{1}{2}} \psi_{\sigma+\frac{1}{2}} \psi_{\sigma-\frac{1}{2}} = A^2 - (\sigma-\frac{1}{2})^2.

\Delta = H^2 + \frac{1}{4} + 2YX

\Lambda - \Lambda^2 = \sigma^2 + \Lambda^2 - \Lambda^2 \Lambda^2 - \Lambda

\Lambda_1 \oplus \Lambda_2

\text{End}_q (U(\alpha) \otimes \Lambda_1 \otimes \Lambda_2) \rightarrow U(\alpha) \otimes \text{Hom}_q (\Lambda_1 \otimes \Lambda_2, \Lambda_1 \otimes \Lambda_2)

\text{Weyl acts on each piece}

U(\alpha) \otimes (\Lambda_1, \Lambda_2)_m

U(\alpha) \otimes (\Lambda_1, \Lambda_2)_m

\text{under W?}
Euclidean case: \( \mathfrak{g} = \mathfrak{sl}_2 \), or, \( \mathfrak{g} = \text{null of } \mathfrak{g} \text{ in } \mathcal{E} \).

End of \( \{U(g) \otimes \Lambda\} \) \( \rightarrow \) \( (S(\alpha) \otimes \text{Hom}_M(\Lambda, \Lambda))^W \)

\[ \text{Hom}_k (\Lambda, S(\mathfrak{g}) \otimes \Lambda) \]

\[ \downarrow \]

\[ \left[ S(\mathfrak{g}) \otimes \text{Hom}(\Lambda, \Lambda) \right]^k \rightarrow \left[ S(\alpha) \otimes \text{Hom}(\Lambda, \Lambda) \right]^N \]

An isomorphism off \( D \). \( \otimes \mathbb{C}[D^{-1}] \). Clean

Corollary: Nice description of irreducible \( \mathfrak{g} \) modules whose support doesn't intersect \( D = 0 \).

Return to \( \text{sl}_2 (2, \mathbb{R}) \).

Category consists of objects \( \Sigma \) \( \in \mathcal{C} \)

\( \text{Hom}(\sigma, \tau) = 0 \) unless \( \sigma - \tau \in \mathbb{Z} \)

\( \text{Hom}(\sigma, \sigma + n) \) free module over \( \mathbb{C}[A^2] \) with a generator \( (X_+)^n \) \( n > 0 \)

\( (X_-)^n \) \( n < 0 \)
such that
\[ X_+ X_- = \left[ A^2 - \left( \sigma + \frac{1}{2} \right)^2 \right] \text{id}_\sigma \quad \text{in} \ \text{Ham}(\sigma, \sigma). \]
\[ X_- X_+ = \left[ A^2 - \left( \sigma - \frac{1}{2} \right)^2 \right] \text{id}_\sigma \]

check:
\[ X_+ X_- X_+ = -2\sigma \text{id}_\sigma \quad \text{OKAY, because} \]
\[ X_+ = \sqrt{2} Y \]
\[ X_- = \sqrt{2} X. \]

The fundamental idea: Choose a morphism
\[ C[A^2] \to \mathbb{R} \]

extend the base and calculate the resulting category

It's clear that we want \( z^2 \neq (\sigma + \frac{1}{2})^2 \).
\[ z^2 \neq (\sigma_0 + n + \frac{1}{2})^2 \]
or \[ z^2 \neq (\sigma_0 + \frac{1}{2} + n)^2. \]

Take a function such as
\[ z \]

\( R \) is an algebra of entire \( \mathbb{C} \) functions flat over \( C[A^2] \).
\[
\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)
\]

we want
\[
A^2 \neq (\sigma_0 + \frac{1}{2} + n)^2
\]

what is the analytic fn with zeroes
\[
\sigma_1 = \sigma_0 + \frac{1}{2}
\]

\[
(\pm \sigma_1) + n \quad n \in \mathbb{Z}
\]

Fix some point as the origin \(0\).

Define a new operator \(Y_+\) such that \(Y_+ Y_- = Y_- Y_+ = \text{id.}\).

| Set \(\sigma = 0\) | Change \(X_+\) as an operator \(X_+\) as an operator. |
Proposed Construction: Fix $k$ and consider the category of $k$-modules associated to $\mathcal{V}$ and $\mathcal{W}$, then there is a function of $\mathcal{V}$ and $\mathcal{W}$.

You must define a map

$$\text{Hom}_g (U(\mathfrak{g}) \otimes_k \Lambda_1, U(\mathfrak{g}) \otimes_k \Lambda_2)$$

$$\text{(R \otimes \text{Hom}_M (\Lambda_1, \Lambda_2))^W}$$

where $R$ is a suitable constructed quotient ring of $U(\mathfrak{g})$. 
Conjecture: Let \( Q \) be the quotient field of \( \mathbb{Z} \) and let \( \tilde{Q} \) be the quotient field of \( U(\mathfrak{o})^w \). Then, the following categories are equivalent:

(i) objects: semi-simple finite \( k \) reps \( \Lambda \).
   morphisms \( Q \otimes \text{Hom}_H(U(\mathfrak{o})^w, U(\mathfrak{o})^w) \).

(ii) objects: semi-simple finite \( k \) reps \( \Lambda \).
   morphisms \( \tilde{Q} \otimes \text{Hom}_H(U(\mathfrak{o})^w, U(\mathfrak{o})^w)^w \). 

Either true or false: Try to define a functor \( F \):

\[
F(\sigma) = \sigma \quad \text{clear.}
\]

In (i), have \( X_+X_- \) with relations

\[
X_+X_- = [A^2 - (\sigma + \frac{1}{2})^2] \text{id}_\sigma \quad \text{in } \text{Hom}(\sigma, \sigma)
\]

\[
X_-X_+ = [A^2 - (\sigma - \frac{1}{2})^2] \text{id}_\sigma
\]

In (ii), we have \( Y_+Y_- \) with relations

\[
Y_+Y_- = A^2 \text{id}_\sigma \quad \text{in } \text{Hom}(\sigma, \sigma)
\]

\[
Y_-Y_+ = A^2 \text{id}_\sigma \quad \text{in } \text{Hom}(\sigma, \sigma^+).
\]
Also have to define
\[
F(X_+ (\sigma)) = f(A, \sigma) Y_+ (\sigma) \quad F(p(A^2) \varphi) = P(A^2) F(\varphi)
\]
\[
F(X_- (\sigma)) = g(A, \sigma) Y_- (\sigma)
\]
So that
\[
F(X_+ (\sigma+1), X_+ (\sigma)) = F(X_+ (\sigma+1)) F(X_+ (\sigma))
\]
\[
F \left( \left[ A^2 - \left( \sigma \frac{1}{2} \right)^2 \right] \text{id}_\sigma \right) = g(A, \sigma+1) Y_+ (\sigma+1) \cdot f(A, \sigma) Y_+ (\sigma)
\]
\[
\left[ A^2 - \left( \sigma \frac{1}{2} \right)^2 \right] \text{id}_\sigma = g(A, \sigma+1) f(A, \sigma) A^2 \text{id}_\sigma.
\]
First condition
\[
g(A, \sigma+1) f(A, \sigma) = \frac{A^2 - \left( \sigma \frac{1}{2} \right)^2}{A^2}
\]
\[
F(X_+ (\sigma-1), X_- (\sigma)) = F(X_+ (\sigma-1)) \cdot F(X_- (\sigma))
\]
\[
F \left( \left[ A^2 - \left( \sigma \frac{1}{2} \right)^2 \right] \text{id}_\sigma \right) = f(A, \sigma-1) Y_+ (\sigma-1) \cdot g(A, \sigma) Y_- (\sigma)
\]
\[
\left[ A^2 - \left( \sigma \frac{1}{2} \right)^2 \right] \text{id}_\sigma = f(A, \sigma-1) g(A, \sigma) A^2 \text{id}_\sigma.
2nd condition

\[ f(A, \sigma - 1) \cdot g(A, \sigma) = \frac{A^2 - (\sigma + \frac{1}{2})^2}{A^2} \]

now \( f \) and \( g \) are to be polya in \( A^2 \).

Questions: 0
Conclude: Consider the following categories:

A:  
objects: finite semi-simple k modules
morphism: \[ A(\Lambda_1, \Lambda_2) = \text{Hom}_G (U(g) \otimes_k \Lambda_1, U(g) \otimes_k \Lambda_2) \]

B:  
objects: finite semi-simple k modules
morphism: \[ B(\Lambda_1, \Lambda_2) = k \otimes \left[ U(g) \otimes \text{Hom}_M (\Lambda_1, \Lambda_2) \right]^W \]

where K is the quotient field of \( W(a) \).

Then there is no functor \( F: A \to B \) such that \( F(\Lambda) = \Lambda \).

I am going to calculate this out very carefully for \( sl(2, \mathbb{R}) \).

Let’s work with B. Then can restrict to \( \Lambda \) irreducible \( \Lambda_1 = \sigma \), \( \Lambda_2 = \tau \) where

\[ H |_{\sigma} = \sigma |_{\sigma} \]

Now \( M = \{ e^{2\pi i n H} \mid n \in \mathbb{Z} \} \).
\[ \hat{M} = \{ e^{\pi i n H} \mid n \in \mathbb{Z} \} \]

So \( \text{Hom}_M (\sigma, \tau) \cong \tau - \sigma \)

so \( \text{Hom}_M (\sigma, \tau) \neq 0 \iff \tau - \sigma \in \mathbb{Z} \).

Then \( \hat{M} \) acts by sign whether \( \tau - \sigma \) is odd or even.
On $U(\omega) \cong \mathbb{C}[\mathcal{A}]$ W acts by $-1$. hence

$$\left[U(\omega) \otimes \text{Hom}_M(\mathcal{A}^{\text{op}}, \mathcal{A}^{\text{op}})\right]^w = \begin{cases} \mathbb{C}[\mathcal{A}^2] \otimes \varphi^>_w \quad & \text{even} \\ \mathbb{C}[\mathcal{A}^2] \otimes \varphi^<_w \quad & \text{odd} \end{cases}$$

In particular setting $Y_+^{(\omega)} = A \otimes \varphi^>_w$

$Y_-^{(\omega)} = A \otimes \varphi^<_w$

we have that at least when rational functions $K(\omega)^w_w$ are allowed that

$$B(\omega, 0) = \begin{cases} K(Y_+)^w \quad & n > 0 \\ K(Y_-)^w \quad & n \leq 0 \end{cases}$$

and that

$$Y_-^{(\omega+1)} Y_+^{(\omega)} = A \otimes \varphi^{(\omega+1)}_w = A \otimes \varphi^{(\omega)}_w$$

$$= A^2 \ id_\omega$$

$$Y_-^{(\omega)} Y_+^{(\omega)} = A^2 \ id_\sigma$$

$$Y_+^{(\omega-1)} Y_-^{(\omega)} = A^2 \ id_\sigma$$
Now for $A$. Must calculate

$$\text{Hom}_k(\Lambda_i, U(\mathfrak{g}) \otimes \Lambda_j)$$

Thus

$$\frac{\sum x^\sigma \chi_{\sigma} \Lambda_j}{1}$$

weight $\sigma$

$$\sigma = \lambda - j + \tau$$

so again $\sigma - \tau \in \mathbb{Z}$

observe also that

$$\text{Hom}_k(\sigma, U(\mathfrak{g}) \otimes (\sigma + n))$$ free module over $\mathbb{Z}$ with generator

$$1_\sigma \mapsto y^n 1_{\sigma+n} \quad n > 0$$

$$1_\sigma \mapsto x^n 1_{\sigma+n} \quad n < 0$$

Hence if we let $X_+(\sigma) : 1_\sigma \mapsto y^{1_{\sigma+1}}$

we have

$$X(\sigma) = \begin{cases} 
Z \cdot X_+^n & n > 0 \\
Z \cdot X_-^n & n < 0 
\end{cases}$$

Center gen by $H^2 - H \cdot xy$
and the commutation relation

\[ X_+ (\sigma-1) X_- (\sigma) : \frac{\partial}{\partial \sigma} \rightarrow X_{1_{\sigma-1}} \rightarrow X_{1_{\sigma-1}} \rightarrow X_+ (\sigma-1) X_- (\sigma) \]

\[ XY \rightarrow 1_{\sigma} = \left\{ \frac{\partial}{\partial \sigma} + \frac{1}{2} (H^2 + H) - \frac{1}{2} (\sigma^2 + \sigma) \right\} 1_{\sigma} \]

\[ X_+ (\sigma-1) X_- (\sigma) = \frac{1}{2} \left\{ C - \sigma^2 + \sigma \right\} 1_{\sigma} \]

\[ C = 2 YX + H^2 + H = H^2 + H + 2XY \]

\[ X_- (\sigma+1) X_+ (\sigma) : \frac{\partial}{\partial \sigma} \rightarrow Y_{1_{\sigma+1}} \rightarrow Y_{1_{\sigma+1}} \rightarrow YX 1_{\sigma} \]

\[ YX \rightarrow 1_{\sigma} = \left\{ \frac{\partial}{\partial \sigma} + \frac{1}{2} (H^2 + H) - \frac{1}{2} (\sigma^2 + \sigma) \right\} 1_{\sigma} \]

\[ X_- (\sigma+1) X_+ (\sigma) = \frac{1}{2} \left\{ C - \sigma^2 - \sigma \right\} 1_{\sigma} \]

\[ X_+ (\sigma-1) X_- (\sigma) = \frac{1}{2} \left[ C - \sigma^2 + \sigma \right] 1_{\sigma} \]

\[ X_- (\sigma+1) X_+ (\sigma) = \frac{1}{2} \left[ C - \sigma^2 - \sigma \right] 1_{\sigma} \]
Now define
\[ F(X_+(\sigma)) = f(A^2_3 \sigma) Y_+(\sigma) \]
\[ F(X_-(\sigma)) = g(A^2_2 \sigma) \cdot Y_-(\sigma) \]

Then for \( F \) to be a functor
\[ F(X_+(\sigma-1) \cdot X_-(\sigma)) = F(X_+(\sigma-1)) \cdot F(X_-(\sigma)) \]
\[ \frac{1}{2} [C - \sigma^2 + \sigma id_\sigma] \cdot \frac{f(A^2_3 \sigma-1) \cdot Y_+(\sigma-1) \cdot g(A^2_2 \sigma) \cdot Y_-(\sigma)}{1} \]
\[ \frac{1}{2} \cdot \varphi_\sigma(C - \sigma^2 + \sigma) \cdot id_\sigma = \frac{f(A^2_3 \sigma-1) \cdot g(A^2_2 \sigma) \cdot A^2 \cdot id_\sigma}{1} \]

Define \( \varphi : Z = \mathcal{C}(\mathbb{C}) \rightarrow \mathcal{C}(A^2) \)

\[ F(X_-(\sigma+1) \cdot X_+(\sigma)) = F(X_-(\sigma+1)) \cdot F(X_+(\sigma)) \]
\[ \frac{1}{2} \cdot \varphi_\sigma(C^2 - \sigma^2 + \sigma) \cdot id_\sigma = \frac{g(A^2_3 \sigma+1) \cdot f(A^2_2 \sigma) \cdot id_\sigma}{1} \]
\[ \frac{1}{2} \varphi(C - \sigma^2 + \sigma) = f(A_j^2 \sigma - 1) \ g(A_j^2 \sigma) A^2 \]
\[ \frac{1}{2} \varphi(C - \sigma^2 - \sigma) = g(A_j^2 \sigma + 1) \ f(A_j^2 \sigma) A^2 \]
\[ \frac{1}{2} \varphi(C - \sigma^2 - 2\sigma - 1 + \sigma) = g(A_j^2 \sigma + 1) \ f(A_j^2 \sigma) A^2 \]

Here's your contradiction:

\[ f(A_j^2 \sigma) \ g(A_j^2 \sigma + 1) = F\left(\frac{1}{2} [C - \sigma^2 + (\sigma + 1)] \right) \]

<table>
<thead>
<tr>
<th></th>
<th>( C - \sigma^2 - 2\sigma - 1 + \sigma )</th>
<th>( C - \sigma^2 - \sigma )</th>
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<tbody>
<tr>
<td>( f\left(\frac{1}{2} (C - \sigma^2 - \sigma)\right) )</td>
<td>( C - \sigma^2 - 2\sigma - 1 + \sigma )</td>
<td>( C - \sigma^2 - \sigma )</td>
</tr>
</tbody>
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NO CONTRADITION

\[ f(A_j^2 \sigma) \ g(A_j^2 \sigma + 1) \neq f(A_j^2 \sigma - (\sigma + 1)^2) \ g(A_j^2 \sigma - (\sigma + 1)) \]

See page 13
Feb 10

Situation: You have no theorems of your own yet.

Previous structure of $\Lambda$:

Previously we defined a mapping

$$\Theta: \text{Hom}_g(U(\mathfrak{g}) \otimes \Lambda_1, U(\mathfrak{g}) \otimes \Lambda_2) \rightarrow U(\mathfrak{g}) \otimes \text{Hom}_m(\Lambda_1, \Lambda_2)$$

using the functor $V \mapsto \text{Ig}_V$. We hoped that after the map $\Theta: U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ given by $A \cdot g$, the image would land in the Weyl group invariants. This is false for $\text{sl}(2, \mathbb{R})$.

The associated graded map of $\Theta$ is the map

$$[S(\mathfrak{g}) \otimes \text{Hom}(\Lambda_1, \Lambda_2)]^k \rightarrow \frac{\mathbb{C}}{\text{Ig}_v} S(\mathfrak{g}) \otimes \text{Hom}_m(\Lambda_1, \Lambda_2)$$

induced by the map $S(\mathfrak{g}) = S(\mathfrak{g}/k) \rightarrow S(\mathfrak{g}/k + x) = S(\mathfrak{g})$ which arises from the orthogonal projection $p \rightarrow \alpha$ at least for $\text{sl}(2, \mathbb{R})$. One can show that we get a map

$$[S(\mathfrak{g}) \otimes \text{Hom}(\Lambda_1, \Lambda_2)]^k \rightarrow [S(\mathfrak{g}) \otimes \text{Hom}_m(\Lambda_1, \Lambda_2)]^w$$

We hoped this was an isomorphism, but it turned out to be false.

It is injective, and surjective if tensored with $C[1/k]$, when $D$ is discriminant. As $S(\mathfrak{g}) \otimes S(\mathfrak{g})$ both modules are the same rank so we derive Baldio's results on the multiplicities of $\Theta$ as a $K$ representation. It always injective.
Problem: Classify irreducible \( \tilde{g}_k \) modules.

Proposition: Let \( \Lambda \) be a finite \( k \)-module. Then

\[
- : (S(\tilde{g}) \otimes \Lambda)^k \otimes \mathbb{C}[D^{-1}] \longrightarrow (S(\omega) \otimes \Lambda)^k \otimes \mathbb{C}[D^{-1}]
\]

is an isomorphism.

Proof: - is induced by the orthogonal projection \( \rho \rightarrow \omega \); if we identify \( \rho \) with \( \rho^* \) and \( \omega \) with \( \omega^* \) via the Killing form, then we can think of \( - \) as the restriction of functions on \( \rho \) to functions on \( \omega \). Recall that \( - : S(\tilde{g})^k \cong S(\omega)^k \) and that \( D \) is the element of \( S(\rho)^k \) such that

\[
D(\alpha) = \prod_{a \in A} \alpha(a) \quad \alpha \in \omega.
\]

Alternatively we have \( (ad \chi)^2 : \rho \rightarrow \rho \) for \( \chi \in \rho \) and consider the polynomial

\[
det (T - (ad \chi)^2) = T^k + p_1(\chi) T^{k-1} + \cdots + p_k(\chi)
\]

where \( p_j \in S(\rho)^k \). Note that if \( \chi \in \omega \), then we can calculate the eigenvalues as follows. We have

\[
\rho = \omega \oplus \mathfrak{h}
\]

where \( \mathfrak{h} \) has the basis \( e_{\alpha} - e_{\beta_\alpha} \) \( \alpha \in \Sigma^k \).

If \( \chi \in \omega \), then \( \chi \in \mathfrak{h} \), so

\[
(ad \chi)^2 (e_{\alpha} - e_{\beta_\alpha}) = \alpha(\chi)^2 e_{\alpha} - (\alpha(\chi) \beta_\alpha)^2 e_{\beta_\alpha}
\]
Best \( \omega(x) = x(\alpha(x)) = -\alpha(x) \)

so
\[
(ad_x)^2 (e_x - e_\alpha x) = \alpha(x)^2 (e_x - e_\alpha x).
\]

Thus
\[
det(T - (ad_x)^2) = \prod_{x \in \mathcal{B}'} (T - \alpha(x)^2). T^{-} \quad \text{where } l = \dim \alpha.
\]

Thus
\[
p_{\alpha - e}(x) = \prod_{x \in \mathcal{B}'} \alpha(x)^2 = \prod D(x) = \prod D(x).
\]

We now proceed to the proof of the proposition.

An element of \( (S(\mathfrak{g}) \otimes \Lambda)^k \otimes \mathcal{C}[D^{-}] = (S(\mathfrak{g}_D) \otimes \Lambda)^k \) is an algebraic function on \( \text{preg} = \{ x \in \mathfrak{g} | D(x) \neq 0 \} \) with values in \( \Lambda \) which is \( k \)-equivariant. Similarly an element of \( (S(\alpha) \otimes \Lambda)^N \) is an algebraic function on \( \text{preg} \) with values in \( \Lambda \) which is \( N \)-equivariant. Suppose we can show that the quotient of \( \text{preg} \) by \( k \) exists and is isomorphic to the quotient of \( \text{preg} \) by \( N \), then and that these are are faithfully flat descents with descent data generated by the respective groups. Then over \( \text{preg}/k \) I obtain a locally free sheaf whose sections are in \( \mathcal{C}[D^{-}] \) and similarly a locally free sheaf over \( \text{preg}/N \) whose sections are \( (S(\alpha) \otimes \Lambda)^N \). These two sheaves will be isomorphic hence so will their sections.
Problem: Classify irreducible $\mathfrak{g}_k$ modules.

\[ (\mathfrak{g}, k) \longrightarrow (M, \mathbb{C}) \]

\[ V \longrightarrow \mathbb{Y} \]

Exactly as before except now I know how to make the Weyl group act! So now I can define an isom. of \[ I(I) \longrightarrow I(I^s)? \]

\[ \text{Hom}_k (\Lambda, I(I^s)) = \text{Hom}_M (\Lambda, \mathcal{F}) \]

\[ I(I) = \text{ker} \text{Hom}_{M, \mathcal{F}} (U(g), \mathcal{F}) \]

Let $\alpha_s \in \overline{N}$, we want to define a map $\{f(\alpha_s, \mathcal{F}) : I(I) \longrightarrow I(I^s)^{\mathcal{F}}\}$ somehow.

\[ \text{Hom}_{M, \mathcal{F}} (U(g), \mathcal{F}) \longrightarrow \text{Hom}_{M, \mathcal{F}} (U(g), \mathcal{F}^{\alpha_s}) \]

There is an obvious way of proceeding, namely to apply $\alpha_s$ all the way through $i.e.$ send

\[ f \text{ into } \text{ } f \]

which is defined since $\alpha_s \in K$. 
Then of course you get a map

\[ \text{Hom}_{M, \alpha, \pi} (U(g), \pi) \] i.e.

\[ x_s f(xg) = f(xg x_s) = \]

Unfortunately, this doesn't help much with the degeneracy. This clearly works + is correct relative to the map:

\[ \text{End}_g(U(g) \otimes_{\mathbb{R}} M) \rightarrow [U(g) \otimes \text{Hom}_M(M)] \]

Still how do you get irreducibility?

\[ \mathcal{I}(s)/\mathfrak{n}\mathcal{I}(s) = ? \]

Go back: You want to decide when a principal series repn. is completely reducible, i.e. so have to examine the operator maps objects

\[ \text{Hom}_k (\Lambda, \mathcal{I}(s)) = \text{Hom}_M (\Lambda, \mathcal{I}(s)) \]

and the maps between them. One sees that if we are away from the singular locus there is no problem (Theorem of Bruhat) because then we map onto \[ U(g) \otimes \text{Hom}_M(M) \]
However if we are on the singular locus things may still be irreducible!!

We know that every module has its support on an orbit closure. Suppose we are in complex case with adjoint action of $k$ on $k$ and that we have a module supported on the variety of nilpotent which is $\text{invariant}$. There are several nilpotent orbits of nilpotents in addition to the orbits of principal nilpotents and we have to make sure that these do not give rise to quotients. Can you formulate geometrically?

$X$ variety, $Y \subset X$ subvariety, $F$ coherent sheaf on $X$. Then have

\[
\begin{array}{ccc}
F & \rightarrow & F_{\mathcal{O}_Y} \rightarrow 0 \\
\end{array}
\]

Now suppose a group acts on $X, Y, F$. 
\[ f = \frac{g}{k} \rightarrow \frac{g}{k+n} \rightarrow \alpha, \quad \text{so what for?} \]

\[ X = \frac{1}{\frac{1}{2} \sqrt{n-N+1} + A} \rightarrow \frac{1}{\frac{1}{2}} A \]
\[ Y = \frac{1}{\frac{1}{2} \sqrt{n-N+1} + A} \rightarrow \frac{1}{\frac{1}{2}} A \]

\[ \frac{X+Y}{\frac{1}{2}} ; \frac{i \cdot X-Y}{\frac{1}{2}} \]

\[ A \]
\[ B \]

\[ \text{for } \mathbf{A} - i \mathbf{B} = \frac{2X}{\frac{1}{2}} \]

\[ \begin{cases} \text{rank } M = \text{rank } g - \text{rank } \alpha = l-1 \\ \text{rank } K \leq \text{rank } g = 0 \end{cases} \]

\[ \text{two cases for rank } 1 \]

\[ \text{either rank } K = \text{rank } g \implies \text{rank } M = \text{rank } G \]
\[ \text{rank } K = \text{rank } g = 1 \implies \text{rank } K = \text{rank } G \quad \text{[hermitian]} \]

Both Rallis + Hermann seem to imply that if \( G/K \) is of rank 1, then any irreducible rep. of \( M \) occurs at most once in \( G \) as an irreducible rep. of \( K \). Equivalently, this is true if \( V \) is 1-diml. Equivalently, that if we take the sections of \( \hat{G} \otimes F \) and decompose over \( K \) we get a ladder.
\[ \text{Hom}_G(G, \mathbb{I}) = \{ f: G \to \mathbb{I} \mid f(bg) = f(b)f(g) \} \]

\[ \text{Hom}_K(K, \text{Hom}_G(G, \mathbb{I})) = \text{Hom}_M(K, \mathbb{I}) \]

Idea is that such a representation \( \alpha \) is a 1

Claiming that the rep. by spherical harmonics is valid:

\[ \text{Hom}_M(K, \mathbb{I}) \cong \text{Hom}_K(K, \text{Hom}_M(M, \mathbb{I})) \]
Problems
1. Irreducibility of principal series for \( \tilde{\mathfrak{g}} \)
2. Structure of \( \Omega \) and the Weyl action
3. Maximal ideals in \( \mathfrak{u}(\mathfrak{g}) \).

Theorem 1: The irreducible \( \tilde{\mathfrak{g}}, k \)-modules \( V \) are as follows.

As an \( S(p) \) module \( V \) has support on a closed orbit of \( K \) in \( p \), necessarily the orbit of a semi-simple element \( \alpha \).

\( V \) is the space of sections of a homogeneous vector bundle over \( G \times K \alpha \) coming from some irreducible representation of the isotropy group \( M_\alpha \) of \( \alpha \).

Proof: Let \( V \) be an irreducible \( \mathfrak{g}, k \)-module so that \( V \) is a quotient of \( \mathfrak{u}(\tilde{\mathfrak{g}}) \otimes \Lambda \). Then \( V \) is finite type over \( S(p) \) so defines a coherent algebra sheaf over \( p \). The support of \( V \) is closed and \( K \)-stable, call it \( Z \). If \( Y \) is a closed \( K \)-stable subscheme of \( Z \) with ideal \( I \), then \( V/IV \neq 0 \) so \( IV = 0 \) so \( Y = Z \).

Thus \( Z \) must be minimal closed \( K \)-stable, so \( Z \) must be the a closed orbit \( K \alpha \).

The sheaf \( F \) defined by \( V \) comes from \( \mathfrak{g} \) a coherent sheaf \( F \) on \( Z = K/M_\alpha \). As \( Z \) is reduced and the rank of \( F \) is constant \( F \) is a homogeneous vector bundle \( K_x \otimes \mathfrak{g} \) on \( Z \).

Clearly \( V \) must be irreducible.
Conversely given a closed orbit \( K \alpha \) and a homogeneous vector bundle \( K \times M_A \nu \) on \( Z \) we get an irreducible \( \mathfrak{g} \)-module.

Why - how is this done? Suppose I give myself a \( \nu \) with centralizer \( M \) in \( K \) and also an irreducible \( \nu \) of \( M \). Then I have to form the homogeneous bundle

\[
K \times M_A \nu
\]

\[
K/M_A \overset{i}{\rightarrow} \mathfrak{g}^\prime
\]

and take \( i_* (K \times M_A \nu) \) and the representation is then sections of this bundle, i.e.

\[
V = \bigoplus \Gamma(i_* (K \times M_A \nu)) = \text{Hom}_{M_A}(K, \nu)
\]

and where \( \mathfrak{g} \) acts by restriction to the orbit, i.e. an element of \( \mathfrak{g} \) is a function on \( \mathfrak{g}^\prime \) hence also on \( K \mathfrak{g} \), so if

\[
f: K \rightarrow \nu \quad M_A
\]

and \( X \in \mathfrak{g} \), then \( X \) defines \( \tilde{X}: K \rightarrow 1 \) by

\[
\tilde{X}(k) = \langle X, k \alpha \rangle
\]

\[
\tilde{X}(mk) = \langle X, k^{-m^{-\alpha}} \rangle = \langle \text{Ad}(k), \lambda \rangle
\]
and

\[(Xf)(k) = \langle k \cdot X, f \rangle(k)\]
\[= \lambda(k \cdot X) f(k).
\]

ie

\[(Xf)(\delta') = \sum_i \lambda(\delta_i' \cdot X) f(\delta_i'')\]

if \(\Delta \delta = \sum_i \delta_i' \otimes \delta_i''\).

This tells me how \(f\) acts and I know how \(k\) acts so I am done!

\[\forall k \in k \quad (Yf)(\delta) = f(\delta Y), \quad \delta \in U(k)\]

\[X \in p \quad (Xf)(\delta) = \lambda(\delta' \cdot X) f(\delta'')\]

\[(Y \cdot X \cdot f)(\delta) = (Xf)(\delta Y) = \lambda(\delta Y' \cdot X) f(\delta Y'')\]

\[(X \cdot Y \cdot f)(\delta) = \lambda(\delta' \cdot X) (Yf)(\delta'') = \lambda(\delta' \cdot X). f(\delta'') Y\]

\[[Y, X] f)(\delta) = \lambda(\delta' \cdot [Y, X]) \cdot f(\delta'')\]
\[ \Delta(\delta Y) = (\delta' \otimes \delta'') (Y \otimes 1 + 1 \otimes Y) \]
\[ = \delta' Y \otimes \delta'' + \delta'' \otimes \delta' Y \]

\[ (Y \ast f)(\delta) = \lambda(\delta' Y \ast X) f(\delta'') + \lambda(\delta' X) f(\delta'') Y \]

\[ (Y \ast f)(\delta) = \lambda(\delta' X) f(\delta'') Y \]

\[ \lambda(\delta' Y \ast X) f(\delta'') = \lambda(\delta' X) f(\delta'') Y \]

\[ \delta' \ast [Y, X] = \delta' \ast (Y \ast X) = \delta' Y \ast X. \]

so it's **OKAY**

now see if you can describe this in the form \( I(\delta) \)?

Mackey says to take the representation of \( MP \) and induce up to the whole group \( \lambda \). \( \lambda \) is a representation of \( P \) + \( \lambda \in \mathfrak{p}' \), idea is that \( \lambda \in \mathfrak{p}' \) but the inclusion \( \mathfrak{p} \subset \mathfrak{p}' \) comes from choosing \( \mathfrak{p} \supset \mathfrak{p} \) done.

Therefore:

\[ \text{Hom} \left( \mathcal{U}(\mathfrak{g}), J \right) \]
\[ \text{M}_{\mathfrak{p}} \]

\[ \text{Hom} \left( \mathcal{U}(\mathfrak{p}) \mathcal{U}(\mathfrak{k}), J \right) = \text{Hom} \left( \mathcal{U}(\mathfrak{k}), J \right) \]
\[ \tilde{f}(x^* \delta) = x^* \cdot \tilde{f}(\delta) = \langle x^*, e^1 \rangle \tilde{f}(\delta). \]

Thus
\[ (x \tilde{f})(x^* \delta) = \tilde{f}(x^* x \delta) = \tilde{f}(x^* x \cdot \delta) + \tilde{f}(x^* (\delta * x)) \]
\[ = \langle x^*, e^1 \rangle \lambda(x) \tilde{f}(\delta) + \lambda(\delta * x) \tilde{f}(1) \]
\[ = \lambda(x) \tilde{f}(x^* \delta) \]

\[ \delta X = \sum_i (\delta^i \ast X) \cdot \delta_i^u. \]
\[ \delta \in U(k) \]
\[ X \in \mathfrak{f}. \]

\[ (x \tilde{f})(x^* \delta) = \tilde{f}(\sum_i \delta^i \ast X \cdot \delta_i^u) \]
\[ = \lambda(\delta \ast X) \tilde{f}(\delta_i^u). \quad \text{OKAY} \]
The problem remains: Suppose I define

\[ I(\lambda) = \ker \text{Hom}_{M_\mathfrak{p}}(U(\mathfrak{g}), \mathfrak{s}) \]

Why is \( I(\lambda) \) irreducible?

**Theorem:** Let \( \lambda \in \mathfrak{p}^\vee \) be such that \( KA \) is closed, and let \( M_\lambda \) be the isotrivial group of \( \lambda \), and let \( U \) be an irreducible rep. f.d. of \( M_\lambda \). Then if \( \mathfrak{s} = \lambda \circ \nu \)

\[ I(\lambda) = \left[ k \text{ finite } \text{Hom}(U(\mathfrak{g}), \mathfrak{s}) \right]^{M_\mathfrak{p}} \]

is an irreducible \( g \cdot k \) module.

**Proof:** At present:

\[ I(\lambda) \cong \left( k \text{ finite } \text{Hom}(U(k), \mathfrak{s}) \right)^{M_\mathfrak{p}} \]

with action given by

\[ (xf)(\delta) = \lambda(\delta^* X) \cdot f(\delta^n) \quad \text{if } X \in \mathfrak{p} \]

\[ (Yf)(\delta) = f(\delta \cdot Y) \quad \text{if } Y \in k. \]

And latter is isomorphic to the sections of the homogeneous vector bundle \( K \times_{M_\mathfrak{p}} \mathfrak{s} \) over \( K/M_\mathfrak{p} \cong K^\lambda \) with its obvious \( K \cdot \mathfrak{p} \) structure.
The latter is clearly irreducible by alg. geometry.

Back to Nakayama:

Basic facts about $K$ action $\alpha$ on $\mathfrak{p}$: TFAE for $x \in \mathfrak{p}$
(i) $Kx$ is isolated in center of $Kx$ closed
(ii) $\text{ad } x$ is semi-simple
(iii) $Kx \cap \mathfrak{a} = \emptyset$

So for adjoint action so that $\mathfrak{a} = \mathfrak{h}$

$$g^\mathfrak{h} = \mathfrak{h} \oplus \sum_{x \in \Delta} g_x^\mathfrak{h}$$

$g^\mathfrak{h}$ semi-simple if $\text{tr } \text{ad } x \text{ ad } y = \langle x, y \rangle$ non-degenerate.

If $\mathfrak{h}$ nilpotent $g^\mathfrak{h}$ get $g^\mathfrak{h} = \bigoplus g_{x}^{\mathfrak{h}}$. Say $\mathfrak{h}$ Cartan in $\mathfrak{h}$ nilpotent $\mathfrak{h}$ is its normalizer.

Then $x$ regular $\iff$ dim $(x)^0$ minimal.

Look at char poly of $\text{ad } x$

$$\det (T - \text{ad } x) = T^n + \ldots + p_n(x)$$

so that for some $\ell$ $p_{\ell-1}(x) \neq 0$ and gives regular elements. It follows that
Examine proofs of conj. thm. for Cartan subalgs.

1. alg.geom one. One consider Zariski open of regular elts. and the space they span their centralizers. One consider the set of Cartan subalgs in Grassmannian - image of reg elts. This set is connected, and the group acts on it. One calculates that the orbits are open by Jacobian criterion, hence must be all.

2. compact one. One takes an element $x$ and minimizes its distance to $\alpha$, i.e.

$$\langle (\text{Ad } x) x, H \rangle = 0 \quad \text{all } H \in \alpha \quad x \in K, \quad \alpha \in \mathfrak{p}.$$ 

$$\Rightarrow \langle x, [x, H] \rangle = 0 \quad \text{all } x \Rightarrow [x, H] = 0 \Rightarrow \alpha \text{ not max abelin}$$

3. topological one. Let $T$ be a max. torus in $K$ so that centralizer of $T = T$ at least for connected part, so one gets weight spaces and so taking a generic member of $T$ one calculates the lefschetz no. by fix pt. formula

$$\frac{K/T}{x \cdot kT} = kT \quad \Rightarrow k \cdot k = T \Rightarrow x \in \mathbb{N}. $$
Killing of $g'$ restricts to a non-deg. form on $g'$. So :

$\text{null}(g') \cap \text{null}(g) = \ker g' \cap \ker g = 0$.

$\ker g \otimes \ker g' = 0$.

So take a Cartan $\mathfrak{g} \otimes \mathfrak{g}$. Then $G = \text{Aut}(\mathfrak{g})$.

And take the weight decamp.

$g = \sum g_i$.

If $\mathfrak{g}$ is its own orth. then $\mathfrak{h}$ is its own reducible.

Since $\mathfrak{g}$ is its own orth. then $\mathfrak{h}$ is its own reducible.

Hence $\mathfrak{g}$ is its own orth.

Consider the weight spaces for $x$ and $

\text{central}(x)$. Have usual $\text{decamp}.$

Then $\mathfrak{h}$ is its own reducible.

Hence $\mathfrak{g}$ is its own orth. and $\mathfrak{h}$ is its own reducible.

Consider the weight spaces for $x$ and $\text{central}(x)$. Have usual $\text{decamp.}$

For each element $x$ of $\mathfrak{g}$, every element has a fixed $\text{elt}$.
So have proved

Then: oq is a (reductive or abelian semi-simple) sub-space of ιq contained in a C.S.

Using only weight spaces and basic properties of reductive algebras. (Not that oq is not a non-deg. inv. abelian form)

\[ oq = k + p \]

\[ \text{or} \]

& max. abelian s.s. in \( p \). Look at K action

\[ x \in p \quad x = s + n \quad m_{qj} \rightarrow s, n \neq p \]

\[ (ad s + n)^2 = (ad s)^2 + 2ad(s ad n + (ad n)^2) \] s.s.

\[ \text{nilp.} \]

\[ 2 ad s ad n + (ad n)^2 = 0 \]
Symmetric space theory.

Of semi-simple $\Theta$ involutions, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, or max abelian subspace of $\mathfrak{p}$ consisting of semi-simple elements of $\mathfrak{g}$. The centralizer of $\mathfrak{a}$ is reductive + $\Theta$ stable + meets $\mathfrak{p}$ in $\mathfrak{a}$ so $= \mathfrak{m} + \mathfrak{a}$, $\mathfrak{m} \subset \Theta$. If $\mathfrak{h}$ is a Cartan of $\mathfrak{g} \cap \Theta$, containing $\mathfrak{a}$, then $\mathfrak{a} + \mathfrak{h} \subset \mathfrak{m} + \mathfrak{a}$ $\Rightarrow$ $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{a}$. Now take roots of $\mathfrak{g}$ with $\mathfrak{a}$ to $\mathfrak{h}$, call them $\Delta$ and $\Theta$ acts on them; $\Delta''$ invariant ones, $\Delta' =$ non-invariant ones. At this stage you have to choose $\Sigma$ carefully so that $\\Delta = \Sigma'' \cup \Sigma' \cup -\Sigma' \cup -\Sigma''$. The set $\Sigma$ be a set of positive roots, so that $\Delta = \Sigma' \cup -\Sigma'$, and let $\Sigma' = \Sigma \cap \Delta'$. Then $\Sigma = \Sigma' \cup \Sigma''$. Claim that if $x \in \Sigma'$, then $\Theta x = -\Sigma'$ not true so have to be careful. So proceed as follows: start with $\mathfrak{m}$, $\Delta''$ and choose $\Sigma''$ a root system for $\mathfrak{m}$ by ordering $\mathfrak{h}_0$, then extend the ordering to $\mathfrak{h}$ order $\mathfrak{a}$ and extend the ordering to $\mathfrak{h}'$ and take the positive roots relative to this ordering (choose a basis $X_1, \ldots, X_n$ for $\mathfrak{a}$ extend to $X_1, \ldots, X_n$ and call $\mu = \Sigma \cdot X_i > 0$ if first non-zero), i.e. $\lambda > 0 \Rightarrow \lambda |\alpha|_0 > 0$.

Any two max abelian subspaces $\mathfrak{a}$ are $K$ conjugate by some differential argument.
Note that $X \in \mathfrak{sl}(2,\mathbb{R}) \implies (\text{ad}X)^2 = 0$.

I understand why any semi-simple elt of $\mathfrak{p}$ is $K$ conjugate to an element of $\mathfrak{a}$. Why does a closed orbit consist of semi-simple elements? Better, given $X \in \mathfrak{p}$ show that $K\alpha \cap \mathfrak{a} \neq \emptyset$.

First try

$$X = \sum_{\alpha \in \Sigma'} c_\alpha (e_\alpha - e_\Theta \alpha) + u$$

$c_\alpha \in \mathbb{C}$, $u \in \mathfrak{a}$.

Thus if we can find an element $H \in \mathfrak{m}$ such that $\alpha(H) > 0$ all $\alpha \in \Sigma'$, we let $t \to \infty$ and we are done. Not often the case e.g. in complex case $\mathfrak{m} = \Delta \mathfrak{h} \subset \mathfrak{h} \times \mathfrak{h}$.

$$\Sigma' = \{ \alpha, 0 \}
\{ 0, -\alpha \}
\{ \alpha \in \Sigma \}.$$}

Thus have $\{ \alpha(H) \}
\{ -\alpha(H) \}$ both appearing over $\mathfrak{m}$.
Fundamental problem: For any \( x \in \mathfrak{p} \) show that \( Kx \cap \mathfrak{a} = \emptyset \).

Try adjoint case - for any \( x \in \mathfrak{g} \) show that \( \overline{Gx} \cap \mathfrak{h} \neq \emptyset \).

Method I: \( \overline{Gx} \) (alg. geometry) \( \overline{Gx} \) will be a closed subvariety of \( G \times \mathfrak{g} \), hence its ideal \( I \) in \( S(\mathfrak{g}^t) \) will be invariant. Fix \( I \) be a maximal invariant ideal in \( S(\mathfrak{g}^t) \). By my theorem \( I \) will intersect \( S(\mathfrak{g}^t)^0 \) in a maximal ideal. Thus any invariant element of \( I \) will have a zero when restricted to \( \mathfrak{h} \) will have a zero since

\[
\neg: S(\mathfrak{g}^t)^0 \xrightarrow{\sim} S(\mathfrak{h})^W
\]

Now suppose that \( \overline{Gx} \cap \mathfrak{h} = \emptyset \) i.e.

\[
\begin{aligned}
I + J &= S(\mathfrak{g}^t) \\
\text{where } J &= \text{ideal of functions which vanish on } \mathfrak{h}.
\end{aligned}
\]

Kostant's proof: Write \( x = s + n \) and work in the group \( g^s \) which is reductive; this reduces to case where \( s \) is 0. But any nilpotent embeds in a TDS ie \( \exists y \in g^s : [y,n] = n \)

\[
\Rightarrow e^{-ty}n \to 0.
\]

\[
\hat{\lambda} \quad \text{assume } s \in \mathfrak{h}.
\]
Back to Nakayama.

Conjecture: $\forall f \in (g, k)$ module $\Rightarrow \forall x \in V \neq 0$.

Assume $V$ irreducible.

First attempt: Choose $\Lambda$ so that $U(g) \otimes_k \Lambda \neq 0$. Then we know that

$$S(\Phi) \otimes \Lambda \rightarrow V \text{ onto}.$$

The problem is to get hold of the $\alpha$ module structure.

Filter $V$ in the obvious way get a graded module over $S(\Phi)$ whose support at $\infty$ intersect quite nicely.

Let $\mathfrak{g}_0 = \text{center of } \mathfrak{g}$ and consider the normalizers of $\mathfrak{g}_0$ in $\mathfrak{g}$ include $\mathfrak{g}_0$ possibly a lot more.

Try normal form for $\mathfrak{sl}(3, \mathbb{R})$.

$$[e_1^k, e_{kn}] = e_{kn}$$

$$[e_{kn}, e_{nk}] = -e_{kr}$$

Centralizer

Problem is that $R_0 \mid \text{Norm} \subseteq \text{Cent}$
Can you form a nicer filtration than by \( U(g) \)? In other words, we know that \( V \) finitely generated over \( k \) and \( n \), so define a filtration using \( \mathbb{Z}/p^n \) powers of \( n \).

Abelian case again

Another idea: assume \( V/k V = 0 \).

Define an irreducible rep \( \text{Hom}_k(A, V) \) of \( A \).

We have functor \( F: (A, k) \to \) \( k \)-modules.

\[
U(g) \otimes_k \Lambda_2 \to U(g) \otimes_k \Lambda_1 \to U(g) \otimes_k \Lambda_0 \to V \to 0
\]

\[
U(a) \otimes_k \Lambda_2 \to U(a) \otimes_k \Lambda_1 \to U(a) \otimes_k \Lambda_0
\]

\[
\text{Hom} \text{ } \text{Artin-Peas holds}
\]

\[
(U(g) \otimes_k \Lambda_1) \to (U(g) \otimes_k \Lambda_0) \to V \to 0
\]

\[
\text{Then } H_k(A, V)
\]
\[ V = IV \Rightarrow W = IW \text{ all submodules} \]

Suppose \( V \) irreducible \( \neq 0 \), i.e. \( V = \mathbb{C} u/e \).

Then \( IV = \mathbb{C} V \iff I + e = u \iff I \neq 0 \).

If commutative then we can proceed as follows: We obtain an element \( 1 - x \) in the annihilator of \( V \).
Feb 12
maximal ideals in $U(g)$
prime ideals in $U(g)$, $g$ semi-simple

Kostant theory:

$$u: C_\ast \to C_e$$
$$C_\ast \to C_e$$

have trouble where bad values occur.
Using a Cartan subalgebra the bad values occur
when $D \neq 0$.

Basic maps:

$$\psi: S(g) \to S(h)^W$$

$\psi$: $U(g) \to U(h)^W$

recall $\psi$ defined by

$$U(g) \to U(h) \to U(h)^W$$

This enables us to define a map from prime ideals in
$U(g)$ to prime ideals in $U(h)^W$, i.e. orbits of prime ideals
in $h$.

The bad set in the latter case consists of those $\lambda$
such that $2\lambda(h_a) \in \mathbb{Z} - 0$ for some $a \in \Delta$.

Check for $sl(2)$: Let $A$ be Casimir, it has eigenvalue
$|\lambda + g|^2-|g|^2$ in the irreducible rep with dominant wgt $\lambda$, so with
standard base \( H, X, Y \) for \( \mathfrak{sl}(2) \). The root is \( aH \to a \) so \( g = aH \to \frac{1}{2}a \).

What is Killing form. \( \langle H, H \rangle = 2 \)
\( \langle X, Y \rangle = 2 \)

\[
\text{ad}X \text{ad}Y H = [X, Y] = H
\]

\[
\text{ad}X \text{ad}Y X = [X, -H] = [H, X] = X.
\]

\( Y = 0 \).

\[ H \times Y \]

So Casimir is \( \frac{1}{2}(H^2 + XY + YX) \).

\[ \frac{1}{2}(H^2 + H + 2XY) \]

\[ \frac{1}{2}(X^2 + \lambda) \]

In dominant copt. arep of \( \lambda \).

\( H\sigma = 10 \).

Conclude that \( |\mu|^2 = \frac{1}{2} \mu^2 \).

Eigenvalues of \( \Delta \) to avoid are \( \frac{1}{2}\left(\frac{\lambda}{2}\right)^2 + \frac{\lambda}{2} = \frac{1}{8}(\lambda^2 + 2\lambda) \)

\( \ell = 0, 1, 2, \ldots \)

In standard setup

\[ \alpha(H)X = [H, X] = X \]
\[ X_{-\alpha} = \frac{1}{2}Y \]
\[ \alpha(H\alpha) = \frac{1}{2} \]

\[ H\alpha = \frac{1}{2} H \]
Thus must avoid \( \lambda(H) = \frac{\ell}{2} \)

\[ \lambda(H^\ell) = \frac{\ell}{4} \quad \ell = 0, 1, 2, \ldots \]

To make extremely clear...

One define isom. \( V : U(\mathfrak{g}) \otimes V \rightarrow U(\mathfrak{h})^w \)
in such a way that the Weyl char. formula holds. The

\( V \) is the irreducible finite of module with
dominant wgt \( \lambda \) one has a character

\[ \chi_\lambda : U(\mathfrak{g}) \rightarrow \mathbb{C} \]
given by

\[ \chi_\lambda(u) = \frac{1}{\dim V} \cdot \text{tr} \, s(u) \]

Idea is that \( U(\mathfrak{g}) = [U(\mathfrak{g}), U(\mathfrak{g})] + U(\mathfrak{h}) \)
and \( \chi_\lambda \) vanishes on \([U(\mathfrak{g}), U(\mathfrak{g})]\), thus want a formula

\[ \chi_\lambda(h) = \langle h, \prod_{\alpha \in \Phi^+} \frac{e^{\alpha} \cdot \alpha^*}{\det e^{\alpha}} \rangle \quad \text{if } h \in U(\mathfrak{h}) \]

This is the Weyl character formula, as proved in Sophasli.\n
proved first for f.d. reps, then for inf. reps by density as
follows. One defines
\[ V = \bigoplus V_{\lambda} \sigma = V_+ \bigoplus V_{-} \] 

Define \( \beta : \mathfrak{u}(\mathfrak{g}) \to \mathfrak{u}(\mathfrak{h}) \) so that if \( u \in \mathfrak{u}(\mathfrak{g}) \), then

\[ u_{\sigma} \equiv \beta(u)(\lambda) \sigma \pmod{V_+} \]

Then \( \beta : \mathfrak{u}(\mathfrak{g})^g \to \mathfrak{u}(\mathfrak{h}) \) shown to be inj. \( H \to H + g(H) \)

one observes that if \( z \in \mathfrak{u}(\mathfrak{g})^g \), then

\[ \beta(z)(\lambda) = \chi^z_{\lambda}(z) \quad \text{ie} \]

\[ \chi^z_{\lambda}(z) = \langle \beta(z), e^{\lambda} \rangle \]

\[ = \langle \delta(z) e^{\lambda + g} \rangle \]

\[ \delta(z)(\lambda) \]

\[ \delta(z)(\lambda) = \beta(z)(\lambda) \]

\[ P(H + g)(\lambda) = P(H)(\lambda + g). \]

\[ \textbf{NOT CLEAR} \]
1. Want to describe prime ideals in $U(0j)$
Understand situation - orbits in dual - Kolmant's paper. Want the analogous situation.
Features:
(a) A bad set described by vanishing fn.
(b) Parameterization of max ideals by $h^W$.
(c) An analysis of the bad set, showing that there is a unique minimal prime (gen. by max ideal in center) and that there are only finitely many other primes, always of length at most l.
(d) Description of quotient fields of these prime ideals if they exist (Gelfand-Kirillov)

Point is that prime ideals in $U(0j)$ may be "parameterized" by elements of $h^W$ in the same way that orbits in $0j$ can be.
Bad set in the first
Answers for sl(2).

We have a canonical iso-

\[ \gamma : \mathcal{U}(g) \xrightarrow{\otimes} \mathcal{U}(h) \]

\[ \Delta \longrightarrow \frac{1}{2} \left( h^2 - \frac{1}{4} \right) \]

\[ \frac{1}{2} (H^2 + H + \lambda \lambda) \mapsto \frac{1}{2} (H^2 + H) = \frac{1}{2} \left( (H - \frac{1}{2})^2 + H (H - \frac{1}{2}) \right) \]

\[ \frac{1}{2} \left( H^2 - H + \frac{1}{4} + H - \frac{1}{2} \right) \]

\[ \alpha(H) = 1 \quad g = \frac{1}{2} \lambda \]

\[ g(H) = \frac{1}{2} \]

eigenvalue of Casimir in dominant weight cor \( \lambda \) is

\[ \frac{1}{2} \left[ \left( \lambda(H) \right)^2 + \lambda(H) \right] = \frac{1}{2} \lambda + g^2 - |g|^2 \]

where \[ \langle \lambda_1, \lambda_2 \rangle = \frac{1}{2} \lambda_1(H) \lambda_2(H) \]

\[ \text{Claim} \quad \frac{1}{2} \left[ \left( \lambda(H) + \frac{1}{2} \right)^2 - \left( \frac{1}{2} \right)^2 \right] = \frac{1}{2} \left[ \lambda(H)^2 + \lambda(H) \right] \]

This is bad iff \( \lambda(H) = \frac{e}{2} \quad \text{for } e = 0, 1, 2, \ldots \)

\[ H_x = \frac{1}{2} H \]

\[ \therefore \text{iff} \quad \lambda(H) + \frac{1}{2} = \frac{e}{2} \quad \text{for } e = 1, 2, \ldots \]

\[ \text{iff} \quad (\lambda + g)(H_x) = \frac{e}{4} \quad \text{for } e = 1, 2, \ldots \]
\[
\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}
\]

: if
\[
\sin((\lambda + g)/(\lambda + g)) (4\lambda) = 0.
\]

how about
\[
X_\lambda(h) = \left< h, \frac{e^{(\lambda + g)} - e^{-(\lambda + g)}}{e^g - e^{-g}} \frac{g(h)}{\epsilon^{\lambda + g}(h)} \right>
\]

\[
\frac{e^{\lambda + g} - e^{-(\lambda + g)}}{e^g - e^{-g}} \text{ function on } H
\]

\[
\frac{e^{\lambda + g} - e^{-(\lambda + g)}}{e^g - e^{-g}} \text{ a function on the group } \exp h.
\]

\[
\frac{e^{\lambda + g} - e^{-(\lambda + g)}}{e^g - e^{-g}} (\exp H) = \frac{e^{(\lambda + g)H} - e^{-(\lambda + g)H}}{e^{g(H)} - e^{-g(H)}}
\]

Problem: We have a map
\[
\exp: \mathfrak{g} \rightarrow \mathfrak{g} \text{ Aut}(\mathfrak{g})
\]
\[
\exp: h \rightarrow \text{ Aut}(g)
\]

What is the kernel.

\[
e^{\text{Ad}(H)} x_\alpha = e^{\alpha(H)} x_\alpha = \lambda x_\alpha
\]

\[
\iff \alpha(H) \in 2\pi i \mathbb{Z}.
\]

for point for exp:

when \[ H = 4\pi \text{ min } H_x. \]
Want to consider the fn.
\[
\frac{e^{\lambda g} - e^{-(\lambda g)}}{e^g - e^{-g}} \cdot \frac{g(H_0)}{(\lambda^g)(H_0)}
\]
\[
\frac{e^{\lambda g} - e^{-(\lambda g)}}{\lambda^g} \cdot \frac{g}{e^g - e^{-g}}
\]
as a function on \( h \), i.e.
\[
\frac{e^{(\lambda^g)(H)} - e^{-(\lambda^g)(H)}}{(\lambda^g)(H)} \cdot \frac{g(H)}{e^{g(H)} - e^{-g(H)}} = F_\lambda(H)
\]
when is this function singular? Answer: if \( \lambda \) generic, then
this function is singular when
\[
e^{g(H)} - e^{-g(H)} = 0 \quad H \neq 0
\]
\[
e^{2g(H)} = 1
\]
\[
g(H) \in \pi i(\mathbb{Z} - 0).
\]
In my case \( H \) is variable, say
\[
H = t H_\alpha
\]
and \( g(H_\alpha) = \frac{1}{4} \)
so get problems when
\[
H = t H_\alpha \quad t \in 4\pi i(\mathbb{Z} - 0)
\]
bad \( H = 4\pi i n H_\alpha \quad n \neq 0 \)
4. The badness disappears if $\lambda$ such that

\[
\frac{e^{(\lambda + g)(4\pi i n H_\lambda)} - e^{-(\lambda + g)(4\pi i n H_\lambda)}}{(\lambda + g)(4\pi i n H_\lambda)} = 0.
\]

It is important to use $\Gamma$ functions rather than $\sin$.

Loewy gap $\mathfrak{sl}(2, \mathbb{C})$. In this case get pair $(k_0, c)$, $k_0 = \frac{c}{2}$, $c = 0, \ldots, \infty$. Principal series is when $c^2 = (k_0 n)^2$, $n = 1, 2, \ldots$. In case of finite rep with wts $k_0 + \epsilon$, $k_1$, where $c^2 = (k_1 + 1)^2$, relate $c$ to $\lambda, \nu$. $\mathfrak{sl}(2) \times \mathfrak{sl}(2) = \mathfrak{g}$, $k = \text{diag}$. 

$\mathfrak{h} = h_k + \mathfrak{a} = h_1 \times h_2$

If $\mathfrak{h}$ maximal wgt of $\mathfrak{g}$ finite rep module

but $\mathfrak{g} = \bigoplus (\mathfrak{h}, \lambda) = (\mathfrak{h}_1 \times \mathfrak{h}_2)$

as a $\mathfrak{k}$ rep we get weights:

$\lambda = \mathfrak{j}_1 + \mathfrak{j}_2$

$\nu \mathfrak{h} = \mathfrak{j}_1 - \mathfrak{j}_2$

All $\mathfrak{h}$ in $\mathfrak{k}$ as a $\mathfrak{k}$ rep get weights $\nu, \lambda$. 
Here \( \nu + n = \lambda \). \( k_0 = \nu \), \( k_1 = \lambda \). So

\[
C = (\lambda + 1)^n \quad k_0 = \nu
\]

Condition becomes \((\lambda + 1)^2 \neq (\nu + n)^2\). \( n = 1, 2, \ldots \)

\[
\lambda + 1 \neq \nu + n \quad n = 1, 2, \ldots
\]

\[
\lambda + 1 \neq -\nu - n \quad n = 1, 2, \ldots
\]

Use \( \Gamma \) function. \( \frac{1}{\Gamma(\nu)} \) has zeroes \( 0, 1, 2, \ldots \)

\[
\lambda - \nu \neq 0
\]

\[
\nu - \lambda \neq 1 - n \quad n = 0, 1, 2, \ldots
\]

\[
\frac{1}{\Gamma(\nu - \lambda)} \neq 0
\]

and

\[
\lambda + \nu \neq -1 - n
\]

\[
\lambda + \nu + 2 \neq 0
\]

\[
\frac{1}{\Gamma(\nu - \lambda) \Gamma(\lambda + \nu - 2)} \neq 0.
\]
Calculate ideals for $S_2(2)$.

$$\Delta = \frac{1}{2} (H^2 + xy + yx) = \frac{1}{2} (H^2 - H + 2xy)$$

Try to determine when $\Delta - \alpha$ generates a maximal ideal. Let $U(\alpha)/U(\alpha)(\Delta - \alpha) = S(\alpha)/S(\alpha) \Delta = C[x, y, z] / (H^2 + 2xy)$

Let $f(y, H, x)$ be a non-zero polynomial and assume that it is not divisible by $\Delta$. Let $I$ be an ideal containing $\Delta - \alpha$ and let $f \in I$; we wish to show that $f \in U(\alpha) \Delta$. So proceed as follows. If $f$ has degree $n$ consider $\overline{f} \in S_n(\alpha)$

If divisible by $\bar{\Delta} = H^2 + 2xy$, say $\overline{f} = \overline{\Delta} \overline{g}$ where $\overline{g} \in S_{n-2}(\alpha)$

then we have that $\overline{f} - \Delta \overline{g}$ is of degree $n-1$. Then we may assume that $f$ is not divisible by $H^2 + 2xy$. So we take an element $\overline{f}$ of $I$ of least degree not belonging to $U(\alpha) \Delta$. Now we let the group act on $\overline{f}$. All of these elements will belong to $I$. Really we are looking at what happens to $\overline{f} \in S_n(\alpha) = J \otimes H$, where $J = C[\Delta]$ and $H$ = functions on nilp. elements in $g$. Thus $\overline{f} = \sum_{i=0}^{\infty} \overline{\Delta}^i h_i$, degree $h_i = n-2i$

now $h_0 \neq 0$ otherwise $\overline{f} \in S(\alpha) \Delta$.  


Now the thing to observe is that in

$$S_n(g) = \mathbb{H}_n + \sum_{c+f=n; f>0} J_c \otimes H_f$$

$$\text{Hom}(\Lambda, H) = \text{Hom}_M(\Lambda, 1)$$

mult. of O weight space

: each $H_n$ is a different irreducible rep of $g$
so by applying a suitable projection of in $U(g)$ we
can arrange that it contains all of $H_n$ and is therefore

... of finite codimension.

What happens in general? We have $U(g) = \mathbb{Z} \otimes H$

$H = \text{powers of nilpotent elements}$. So

$U(g)/I \leftarrow U(g)/m_2$

$H$

$H$ inherits a peculiar ring structure whose associated
graded ring is an integral domain $\Rightarrow H$ integral

Question: In nilpotent case $H$ is $A_n = \text{diff ops on affine space}$. Here $H$ might be DO on a curved manifold.
Take orbit choose polarization

\[ B/H \rightarrow G/H \rightarrow G/B \]

\[ B/H \text{ manifold homog. bundle} \]

\[ B \text{ acting on } b/H \cong \mathbb{R}^n \text{ affine.} \]

so can you interpret \( \mathbf{U}(g)/M \) as a ring of operators on a line bundle over \( G/B \).

Thus you try to construct an induced module representation over \( G/B \) with correct character.

Wild Conjecture: We know that for any \( \lambda \) of the form

\[ \lambda = \sigma(\lambda_0 + g) - g \quad \sigma \in \Omega \]

the induced rep \( \mathbf{U}(g) \otimes \lambda \) has correct character.

Maybe by taking \( \mathcal{H} \) components of \( \lambda \) and hence all in the orbit and their annihilators we get all the primes this way!!!
Let $\lambda \in h'$ and consider

$$U(g) \otimes \lambda = U(h^-) \otimes \lambda$$

as a $g$-module.

Show that if $\lambda$ is such that

$$\lambda(H_\alpha) \in \text{integral}$$

then for some $\lambda'$ of the form

$$\lambda' + g = \sigma(\lambda + g)$$

what we have?

The weights of $U(g) \otimes \lambda$ are of the form

$$\lambda - \sum n_i \alpha_i, \quad n_i \geq 0$$

Suppose

$$\frac{2}{\alpha(H_\alpha)}$$

is a non-negative integer, $\alpha$ positive root.

Then

$$\sigma_\alpha(\lambda + g) = \lambda - 2 \frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \alpha + g - \alpha$$

$$\sigma_\alpha(\lambda + g) - g = \lambda - 2 \frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \alpha - \alpha$$

$$= \lambda - n \alpha$$

$n$ positive integer.
Thus the obvious necessary condition holds. See if you can show that if
\[
2 \frac{\lambda(H_\alpha)}{\alpha(H_\alpha)} = \ell \quad \text{integer} \geq 0 \quad \text{for some } \alpha \in \Sigma
\]

then \( U(\mathfrak{g}) \otimes \lambda \) is reducible.

Have to find an element in \( U(\mathfrak{g}) \otimes \lambda \)
of weight \( \lambda - (\ell + 1)\alpha \)

There is a combination of \( \lambda \)

Try the obvious, namely
\( X_{-\alpha}^{\ell+1} \otimes \lambda \)

So is \( \overline{x_{\alpha}^\ell x_{-\alpha}^{\ell+1} \otimes \lambda} = 0 \)?

Try \( \ell = 0 \).

\[
X_{\alpha_i}^\ell (X_{-\alpha} \otimes \lambda) = \left[ X_{\alpha_i}^\ell X_{-\alpha} \right] \otimes \lambda \quad \text{not asw}
\]

If \( \alpha_i \) is simple, then
\[
X_{\alpha_i}^\ell (X_{-\alpha} \otimes \lambda) = \left[ X_{\alpha_i}^\ell X_{-\alpha} \right] \otimes \lambda = 0.
\]

If \( j \neq i \)

\[
\{ H_{\alpha_j} \otimes \lambda \}
\]
so if $1 \otimes H_\alpha \lambda = 1 \otimes \lambda(\alpha_c) = 0$. since $\lambda = 0$

```
sl(2) \text{ rank 1} \quad 2 \frac{\lambda(H_\alpha)}{\alpha(H_\alpha)} = 1
```

\[ X_{\phi}(x \otimes \lambda) = \left( H_\alpha^{-1} X_{-\alpha} + X_{-\alpha} H_\alpha \right) \otimes \lambda \]

\[ = \left( -\alpha(H_\alpha) + \frac{2}{\alpha(H_\alpha)} \right) X_{-\alpha} \otimes \lambda \]

Do you need some technique at this point.

we have to consider $S(n^-) \otimes \lambda = S(n^+) \otimes \lambda$

as an $\mathfrak{m}$ module.

\[ U(n^-) \quad \text{using } \mathfrak{g} \text{ operations} \]

so that I take $X \in \mathfrak{m}$ and given

\[ Y^\phi \otimes \lambda \]

Consider

\[ X(Y^\phi \otimes \lambda) = \quad (XY^\phi - Y^\phi X) \otimes \lambda + Y^\phi X \otimes \lambda = 0 \]
\[ \text{ad} X \text{ carries } U(\mathfrak{u}^-) \text{ into } U(\mathfrak{u}^-) \otimes \mathfrak{h}. \]

\[ \text{ad} X \cdot Y \text{ gives } \mathfrak{u}^- \cdot Y \]

derivation either gives a \( \mathfrak{h} \)

or it gives an \( \mathfrak{h} \) which then must moved three to the end.

**Problem:** show that if \[ 2 \frac{\lambda(\mathfrak{h}_x)}{\alpha(\mathfrak{h}_x)} = l \text{ int } \geq 0 \]

that \( U(\mathfrak{g}) \otimes \lambda \) is reducible.

One method is to take

\[
U(\mathfrak{u}^+) \otimes U(\mathfrak{u}^-) \rightarrow U(\mathfrak{g}) \xrightarrow{\beta} U(\mathfrak{h}) \xrightarrow{\lambda} \mathfrak{c}
\]

determine and when non-singular, i.e. we want

\[ U(\mathfrak{u}^-) \otimes \lambda \rightarrow \text{Hom}(U(\mathfrak{u}^+), \lambda) \]

Thus I have to take

\[ X^P Y^Q \]

and write out in the form

\[
\sum \lambda^{PQ} Y^I X^J H^K
\]

and then consider the matrix

\[
P, Q \mapsto \lambda^{PQ} Y^I X^J H^K
\]
and calculate the determinant. If this determinant is non-zero there is one weight vector.

\( \text{sl}(2) \)

\[ x^i y_i = \sum a_{ij} y^j x^i \]

instead try

\[ x^i y_i = \sum a_{ij}^k y^k x^l H^m \]

\[ \sum \frac{\partial^2 x^i}{\partial y^j \partial y^i} = e^{x} e^{y} = e^{x y + H} = e^{(x, y) x} e^{(x, y) y} e^{(x, y) H} \]

\[ X Y = Y X + H \]

\[ X^2 y^2 = X (Y X + H) Y = (Y X + H) (Y X + H) + \text{hell of a mess} \]

\[ \text{hell of a mess} \]
Suppose you know that 
\[ 2 \frac{\lambda(H_\alpha)}{\alpha(H_\alpha)} = \epsilon \] \text{ for } \epsilon \geq 0

then produce a dominant weight vector.

Try geometry. If
\[ 2 \frac{\lambda(H_\alpha)}{\alpha(H_\alpha)} = \epsilon > 0 \quad \text{for all } \alpha \in \Delta, \]

it is known that the irreducible representation is finite dimensional and in fact explicit formula for the maximal ideal is known.

\[ \mathcal{M}_\lambda = \sum_{\alpha \in \Sigma} \mathcal{U}(H_\alpha - \lambda(H_\alpha)) + \sum_{\alpha \in \Sigma} \mathcal{U}X_\alpha + \sum_{\alpha \in \Sigma} \mathcal{U}Y_\alpha^{\lambda(H_\alpha) + 1} \]

where
\[ [H_\alpha, X_\alpha] = 2X_\alpha \]
\[ [H_\alpha, Y_\alpha] = -2Y_\alpha \]
\[ [X_\alpha, Y_\alpha] = H_\alpha \]

and \( \lambda(H_\alpha) = 2 \), so that \( s_\alpha(\lambda) = \lambda - \lambda(H_\alpha)\alpha \)

Also seems to be true that
\[ \mathcal{U}(\pi(-)) \cap \mathcal{M}_\lambda = \sum \mathcal{U}(\pi(-)) Y_\lambda^{\lambda + 1} \]
Proposition: Let $\lambda \in \mathfrak{h}'$ be such that \( 2\frac{\lambda(H_\alpha)}{\alpha(H_\alpha)} = \ell \geq 0 \) for some $\alpha \in \Sigma$. Then
\[
s_{\alpha} (\lambda + \alpha) - \alpha = \lambda - (\ell + 1) \alpha
\]

Hand hope: If $\exists \alpha \in \Sigma \ni 2\frac{\lambda(H_\alpha)}{\alpha(H_\alpha)} = \ell \geq 0$, then $U(\mathfrak{g}) \otimes \lambda$ is reducible, and the annihilator of the irreducible rep. with dominant wt $\lambda$ is not generated by the maximal ideal in the center.

This is true for $\mathfrak{sl}(2, \mathbb{R})$, since $\alpha(H_\alpha) \leq \frac{1}{2}$ so we have trouble with $\lambda \geq \frac{1}{2}$.

\[
\lambda(H) = \frac{\ell}{2}
\]

Geometric Ideas: We know that $U(\mathfrak{g}) \otimes \lambda$ is irreducible $\iff$ $U(\mathfrak{g}) \otimes \lambda \rightarrow \text{Hom}_\mathbb{C}(U(\mathfrak{g}), \lambda)$ is injective. Think of these sections of the bundle $G \times \lambda \rightarrow G/B$ now using the finite integrality, define a differential equation. The idea is that if $\forall \alpha \exists \ell \geq 0$ then I know that the induced bundle is holom. $+$ has...
sections. Note that if we have a zero it is easy because then get a larger group.