§2. The $K$-theory associated to a ring. Let $A$ be a ring with unit but not necessarily commutative.

Let $\text{GL}(A)$ be the infinite general linear group (inductive limit of $\text{GL}_n(A)$) and let $E(A)$ be the subgroup generated by elementary matrices. 

Let $B\text{GL}(A)$ be a classifying space for $\text{GL}(A)$ which is a $\text{CW}$ complex. Since $E(A) \subset \text{GL}(A) = \pi_1 B\text{GL}(A)$ is perfect we may kill it as in the preceding section obtaining a $\text{CW}$ complex $B\text{GL}(A)^+ = B\text{GL}(A)/E(A)$.

We define the algebraic $K$-groups unique up to homotopy type of the ring $A$ by setting

$$K_i A = \pi_i B\text{GL}(A)^+ \quad i \geq 1$$

and taking $K_0 A$ to be the Grothendieck
group of finitely-generated projective $A$-modules.

Since $E(A)$ is normal in $GL(A)$, 1.2(i) shows that $K_1A = \frac{GL(A)}{E(A)}$, so our $K_1$ coincides with that of Bass. By 1.2(ii) there is a cartesian square

$$
\begin{array}{ccc}
BE(A) & \xrightarrow{f'} & BE(A)^+ \\
\downarrow & & \downarrow g \\
BGL(A) & \xrightarrow{f} & BGL(A)^+
\end{array}
$$

where $g$ is the universal covering of $BGL(A)^+$ and where $f'$ induces isomorphisms on homology. Hence by the Hurewicz theorem

$$
K_2A = \pi_2BE(A)^+ = H_2BE(A)^+ = H_2BE(A)
$$

showing that our $K_2$ coincides with the one defined by Milnor [ ].
The following illustrates the relation of our $K$-groups to those of topological $K$-theory. Let $A = \text{Hom}(X, \mathbb{R})$ be the ring of continuous real-valued functions on a compact space $X$ where $\text{Hom}$ denotes the function space with compact-open topology. Then $GL(A) = \varprojlim_{n} \text{Hom}(X, \text{GL}_n(\mathbb{R}))$ is a topological group in a natural way (at least if we work in the category of compactly-generated spaces) and we let $\text{B}^{\text{top}} GL(A)$ as constructed by Segal [1]. In analogy with the definition of $K_1 A$, we define $K_i^{\text{top}} A$ for $i \geq 1$ to be the homotopy groups of a space $\text{B}^{\text{top}} GL(A)^+$. $\text{B}^{\text{top}} GL(A)^+$ endowed with a universal arrow
$f : B^{\text{top}} \text{GL}(A) \to B^{\text{top}} \text{GL}(A)^+ \text{ killing } E(A)$, or more precisely the image of $E(A)$ in $\pi_1 B^{\text{top}} \text{GL}(A)$. But this image is zero since $E(A)$ is contained in the connected component of $\text{GL}(A)$, hence we can take $f$ to be the identity and conclude that

$$K^\text{top}_i \text{Hom}(X, \mathbb{R}) = \lim_{\rightarrow} \pi_{i-1} \text{Hom}(X, \mathbb{R})$$

$$= KO^{-i}(X) \quad i \geq 1.$$
from the algebraic to the topological $K$-groups.

Similar things hold with $C$ instead of $R$.

Suppose now that $A$ is a general ring, and set

$$\tilde{K}(X; A) = [X, \text{BGL}(A)^+]$$

for any pointed connected space $X$. The above example shows that $\text{BGL}(A)^+$ is the analogue in algebraic $K$-theory of the spaces $BO$ and $BU$ of topological $K$-theory, hence it is reasonable to think of an element of $\tilde{K}(X; A)$ as a virtual vector bundle for the ring $A$ over $X$ which is reduced, i.e. restricts over the basepoint to zero.

We are now going to develop the properties of this
following the example of topological algebraic $K$-functor $K$-theory as closely as possible.

Let $\mathcal{P}(A)$ denote the additive category of finitely-generated projective (left) $A$-modules. By a representation of a group $G$ over $A$ we shall mean an object $P$ of $\mathcal{P}(A)$ endowed with a linear action of $G$. By an $A$-vector bundle over a space $X$ we mean a fibre bundle over $X$ with $A$-module structures on the fibres which is locally isomorphic to $X \times P$ where $P$ is an object of $\mathcal{P}(A)$ endowed with the discrete topology.

For simplicity we suppose from now on that $X$ is a pointed connected CW complex. Then
associating to an $A$-vector bundle the natural action of $\pi_1 X$ on the fibre over the basepoint gives an equivalence of the categories of $A$-vector bundles over $X$, with fibrewise $A$-linear maps for morphisms, with the category of representations of $\pi_1 X$ over $A$.

So from now on we use this equivalence to identify $A$-vector bundles and representations of the fundamental group.

Call two representations $E$ and $E'$ of $G$ over $A$ stably isomorphic if there are trivial representations $P$ and $P'$, i.e. objects of $P(A)$ with trivial $G$-action, such that $E \oplus P \cong E' \oplus P'$. 
Denote by \( \text{St}(G;A) \) the stable isomorphism classes and by \( \text{cl}(E) \) the isomorphism class of representations of \( G \) of \( E \). \( \text{St}(G;A) \) inherits an abelian monoid structure from the direct sum operation on representations. We call elements of \( \text{St}(G;A) \) stable representations and elements of \( \text{St}(G;X;A) \) stable vector bundles over \( X \).

Let \( E_n A \) be the subgroup of \( \text{Gl}_n A \) generated by elementary matrices. We claim that by associating to a homomorphism \( G \to \text{Gl}_n A \) the corresponding representation of \( G \) on \( A^n \) gives rise to a bijection

\[
(1) \quad \lim_{n \to \infty} \frac{\text{Hom}_{\text{grs.}}(G,\text{Gl}_n A)}{E_n A} \to \text{St}(G;A).
\]

Indeed, the map is surjective because given any representation \( E \) there is a trivial representation \( P \) such that \( E \oplus P \) is a representation on a free \( A \)-module.
For injectivity suppose \( u, u' \) are two homomorphisms \( G \rightarrow \text{GL}_n A \) giving rise to stably isomorphic representations \( E \) and \( E' \). We must show \( u \) and \( u' \) become conjugate in \( \text{GL}_n A \) by an element of \( E_n A \) for some \( N \geq n \). It will suffice to show \( u \) and \( u' \) are conjugate by an element \( \Theta \) of \( \text{GL}_n A \), because then they will be conjugate in \( \text{GL}_2 N A \) by \( \Theta \Theta^{-1} \), which belongs to \( E_{2n} A \) by the Whitehead lemma \([1]\). We are given that \( E \oplus P \) and \( E' \oplus P' \) are isomorphic.

Adding a further trivial representation we can suppose \( P \) is free.
As \( E \) and \( E' \) are representations of \( A \)-modules, \( A^n \oplus P' \cong A^n \oplus P \cong A^m \) on \( A^n \) we have for some \( m \), hence adding \( A^n \) to both \( P \) and \( P' \) yields an isomorphism \( E \oplus A^m \cong E' \oplus A^m \), giving the desired element \( \Theta \) with \( N = n + m \).

We now define a map

\[
\text{St}(\pi_1 X; A) \longrightarrow [X, \text{BGL}(A)^+] = \tilde{K}(X; A)
\]

(2)

\[\text{cl}(E) \longrightarrow [E] \]

which should be thought of as the map associating to a vector bundle the associated reduced virtual vector bundle. According to (1) an element \( \text{cl}(E) \) of \( \text{St}(\pi_1 X; A) \) determines a homomorphism \( \text{cl}(E) : \pi_1 X \rightarrow \text{Gl}(A) \).
which is unique up to conjugacy by elements of $E(A)$. Composing the map $X \rightarrow BGL(A)$ in $\mathcal{N}_0$ associated to $u$ with the canonical map $f: BGL(A) \rightarrow BGL(A)^+$, which we recall is a quotient for the action of $E(A)$ on $BGL(A)$, we obtain a well-defined map $X \rightarrow BGL(A)^+$ in $\mathcal{N}_0$, which will be denoted by $\eta[u]$.

Unlike topological $K$-theory where stable vector bundles and reduced virtual bundles are the same, at least over finite complexes, the map

(2) in algebraic $K$-theory is not usually an isomorphism, e.g. $X$ simply-connected. Instead we have the following universal property for the arrow
Denote by $\mathcal{C}$ the category of pointed connected finite complexes and homotopy classes of basepoint-preserving maps, and call a functor $F: \mathcal{C} \rightarrow \text{sets}$ representable if it is of the form $F(X) = [X, Z]_0$ for some pointed space $Z$.

**Proposition 2.1:** The arrow \( (2) \) is a universal morphism of functors from $X \mapsto \text{St}(\pi_1X; A)$ to a representable functor.

**Proof:** Let $A_5$ be the alternating group on 5 letters and embed it in the natural way in $\text{GL}_5A$. Let $N$ be the normal subgroup of $\text{GL}(A)$ generated by $A_5$. 

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"OKAY, necessary to add diagram of representable functors..."
Then $N$ contains all even permutation matrices, hence $\text{GL}(A) / N$ is abelian since given $x, y \in \text{GL}(A)$ there is an even permutation matrix $p$ such that $x$ commutes with $pyp^{-1}$. Thus $N$ contains $(\text{GL}(A), \text{GL}(A)) = E(A)$, hence $N = E(A)$ since $A_5 = (A_5, A_5) \subseteq E(A)$. Consequently $B_{\text{GL}(A)^+} = \text{BGL}(A) / A_5$.

If we take $B_{\text{GL}(A)}$ to be the realization of the semi-simplicial set $\tilde{W}(\text{GL}(A))$ (see [1]) then $B_{A_5}$ can be regarded as a sub-complex of
Let $\pi_1 Y_0 = A_5$ be a simple non-abelian group so by attaching one 2-cell and one 3-cell we can construct an embedding $f: Y_0 \to Y_0^+$ inducing an isomorphism on homology with $Y_0^+$ a finite simply-connected pointed complex, i.e. $Y_0^+ = Y_0 / A_5$ in the notation of §1. Then we can take $BGL(A)^+ = BGL(A) \cup_{Y_0} Y_0^+$ because the inclusion map $f: BGL(A) \to BGL(A)^+$ satisfies (i) and (ii) of 1.2. Let $\{Y_\nu\}$ be the lattice of finite sub-complexes of $BGL(A)$ containing $Y_0$ and set $Y_\nu^+ = Y_\nu \cup_{Y_0} Y_0^+$. Then $BGL(A) = \bigcup Y_\nu$ and $BGL(A)^+ = \bigcup Y_\nu^+$. 
so for any $X$ in $\mathcal{C}$ we have

$$[X, BGL(A)^0] = \varinjlim_{\nu} [X, Y^\nu]_0,$$

$$[X, BGL(A)^+] = \varinjlim_{\nu} [X, Y^{+\nu}]_0.$$

Using ( ) we have

$$St(\pi X, A) = \text{Hom}_{ypo.} (\pi \pi X, GL(A))/E(A)$$

$$= [X, BGL(A)]_0/E(A).$$

Therefore if $Z$ is a pointed space

$$\text{Hom}(\pi X, [?, Z]_0)$$

$$= \text{Hom}([?, BGL(A)]_0/E(A), [?, Z]_0)$$

$$= \text{Hom}([?, BGL(A)]_0/A_5, [?, Z]_0)$$

$$= \varinjlim_{\nu} [Y^\nu, Z]_0^{A_5}$$

$$= \varinjlim_{\nu} [Y^{+\nu}, Z]_0$$

$$= \text{Hom}([?, BGL(A)^+]_0, [?, Z]_0).$$
where \( \text{Hom} \) denotes morphisms in the category of functors from \( \mathcal{C} \) to sets. The proposition follows.

We use this proposition to extend operations on stable representations to operations on \( \tilde{\mathcal{R}}(x, A) \). For example if \( u : A \to A' \) is a ring homomorphism, base extension by \( u : E \to A' \otimes_A E \) gives rise to a natural transformation \( u^* : \mathcal{S}(\pi; A) \to \mathcal{S}(\pi; A') \).

\[
\begin{align*}
\text{map} \; \mathcal{S}(\pi; A') & \to \tilde{\mathcal{R}}(\pi; A') \quad \text{we obtain a } \tilde{\mathcal{R}}(\pi; A) \rightarrow \tilde{\mathcal{R}}(\pi; A') \\
 \text{natural transformation from } \mathcal{S}(\pi; A) \text{ to a representable } \int \mathcal{X} \text{ which by } 2.1 \text{ is obtained from a unique } \\
 \text{natural transformation } u^* : \tilde{\mathcal{R}}(\pi; A) \to \tilde{\mathcal{R}}(\pi; A')
\end{align*}
\]
and by 2.1 there is a unique natural transformation \( u^*: \tilde{\mathcal{K}}(\cdot; A) \to \tilde{\mathcal{K}}(\cdot; A') \) such that the square

\[
\begin{align*}
S^+(\pi_1^*; A) & \longrightarrow S^+(\pi_1^*; A') \\
\downarrow & \downarrow \\
\tilde{\mathcal{K}}(\cdot; A) & \longrightarrow \tilde{\mathcal{K}}(\cdot; A')
\end{align*}
\]

commutes. Similarly if \( A' \) is a finitely-generated projective \( A \)-module there is a "restriction of scalars" map

\[
u_*: \tilde{\mathcal{K}}(\cdot; A) \to \tilde{\mathcal{K}}(\cdot; A').\]

To handle binary operations, we note that the product of the maps \( \pi_1^* \) for

\[
\pi_1^*: \tilde{\mathcal{K}}(\cdot; A) \times \tilde{\mathcal{K}}(\cdot; A) \to \tilde{\mathcal{K}}(\cdot; A).\]
Applying 2.1 to the product $A \times A'$ of two rings we see that
\[
\text{St}(\pi_1; A) \times \text{St}(\pi_1; A') \to \tilde{K}(\cdot; A) \times \tilde{K}(\cdot; A')
\]
is a universal map to a representable functor on $C$, because this arrow is isomorphic to the arrow (2) for $A \times A'$, since there is an isomorphism in $\mathcal{H}_0$

\[
BGL(A \times A')^+ \cong \left\{ BGL(A) \times BGL(A') \right\}^+
\]

\[
\cong BGL(A)^+ \times BGL(A)^+
\]

by 1.5. Consequently any binary, tertiary, etc. operation on stable bundles extends uniquely to the $\tilde{K}$-functor. For example the sum operation on
\[
\text{St}(\pi_1; A)
\]
extends to define an abelian monoid structure on
\( \tilde{K}(\cdot; A) \), the associativity of the extension resulting from the uniqueness. But one sees quite easily by induction on the number of cells of the finite complex \( X \) that the monoid \( \tilde{K}(X; A) \) is in fact an abelian group, so we have proved

Proposition 2.2: There is a unique abelian group structure on \( \tilde{K}(\cdot; A) \) such that the canonical arrow \( \pi_1(\cdot; A) \to \tilde{K}(\cdot; A) \) is a homomorphism of monoids.

Another way of saying that \( \tilde{K}(\cdot; A) \) has a monoid structure is to say that \( BGL(A)^+ \) is a weak H-space. One consequence is that \( BGL(A)^+ \) is a simple space, i.e., the fundamental group acts
trivially on \([X, BGL(A)^+]\) for all \(X\) in \(C\), or equivalently that
\[ [X, BGL(A)^+]_0 = [X, BGL(A)^+] \]

Another consequence is a formula for the groups \(K_i A \otimes \mathbb{Q}\) in terms of the rational homology of \(GL(A)\).

**Proposition 2.3:** The Hurewicz homomorphism for \(BGL(A)^+\) induces an isomorphism of the Hopf algebra \(H_x(BGL(A), \mathbb{Q})\).

This follows from the theorem of Milnor and Moore (LG, appendix).
This follows from a theorem of Milnor and Moore (Ⅵ, appendix), or more precisely from their argument which one can check works for weak H-spaces.

Remainder 2.4: Actually $BGL(A)^+$ is a homotopy commutative and associative H-space, in fact it is an infinite loop space as we shall prove later, by relating it to Segal's theory [Ⅵ]. On a more elementary level one can define the product on $BGL(A)^+$ to be the one induced by the embedding of $GL(A) \times GL(A)$ into $GL(A)$ which one obtains from enumerating $N^+ \sqcup N^+$, where $N^+$ is the
set of positive integers. To prove the unitarity, associativity, and commutativity one can the following lemma which we state without proof.

**Lemma:** Let \( u : \text{GL}(A) \rightarrow \text{GL}(A) \) be the embedding associated to an injection of \( \mathbb{N}^+ \) into \( \mathbb{N}^+ \). Then the triangle in \( \mathbb{N} \)

\[
\begin{array}{c}
\text{BGL}(A) \\
\downarrow_{\text{B}(u)} \\
\text{BGL}(A)
\end{array} \quad \xrightarrow{f} \quad \begin{array}{c}
\text{BGL}(A) \\
\downarrow_{\text{BGL}(A)^+}
\end{array}
\]

is commutative.