January 68.

Harish-Chandra claims that canonical isomorphism

\[ S(g) / g \rightarrow U(g) / g \]

In the semi-simple case, independent of choice of positive roots.

He chooses \( h \cong g \) and \( \mathfrak{h}^+ , \mathfrak{h}^- \) and defines

\[ \gamma : U(g) \rightarrow U(h) \]

to be \( \beta = e^- \otimes 1 \otimes e^+ \)

followed by \( \mathfrak{h} \rightarrow \mathfrak{h} \)

\[ \mathfrak{h} \rightarrow \mathfrak{h} - \mathfrak{p}H \]

\[ s = \frac{1}{2} \sum_{\alpha \in \mathfrak{p}} \alpha \]

The claim is that the result is independent of the choice of \( \mathfrak{p} \). Maybe obvious by character formula, e.g. \( \chi \) is so arranged that

\[ \chi_\lambda (z) = \langle \beta z , e^\lambda \rangle \]

\[ = \langle \gamma z , e^{\lambda + s} \rangle \]

\[ = \langle \delta z , \frac{\text{det} e^{\lambda + s}}{\text{det} e^s} \dim V_\lambda \rangle \]

\[ = (\delta z) \left( \frac{1}{\dim V_\lambda} \chi \right) (0) \]

which shows independence if done carefully.
2. Classify maximal ideals in $U(\mathfrak{g})$

$U(\mathfrak{g} \times \mathfrak{g})$

Problem: What happens when $\mathfrak{k}$ is nilpotent?

$U(\mathfrak{g}) \otimes_{\mathfrak{k}} 1 \longrightarrow M$

So $M$ is unipotent as a $\mathfrak{k}$ module.

Can you show that $M^2$ is 1-dimensional. Yes, because consider $\text{Hom}(U(\mathfrak{g}), 1)$ this is an injective $\mathfrak{k}$ module of nilpotent as well as $\mathfrak{k}$.

So $\text{Hom}_k(U(\mathfrak{g}) \otimes_{\mathfrak{k}} 1)$ is clearly the injective hull of the trivial rep of $\mathfrak{k}$.

So consider $M \longrightarrow \text{Hom}_k(U(\mathfrak{g}), \text{Hom}(U(\mathfrak{k}), 1))$
Can you establish a general isomorphism

\[ S(\mathfrak{g})^g \cong U(\mathfrak{g})^g ? \]

Idea: An irreducible representation of $$\mathfrak{g}$$ should determine a character on $$\mathbb{Z}$$ and an orbit of $$\mathfrak{g}$$ in $$\mathfrak{g}'$$.

Brauer’s theorem only holds for f.d. Hopf algs.

Generalize your results to when $$k$$ is nilpotent in $$\mathfrak{g}$$.

Let $$\mathfrak{n}$$ be a nilpotent root space and let $$M$$ be a $$\mathfrak{g}$$ module with a vector killed by $$\mathfrak{n}$$, hence $$M$$ is nilpotent under $\mathfrak{n}$.

Let $$b = h + v$$ be the normalizer of $$v$$. Does $$b$$ act on $$M^n$$?

Let $$b \in b$$, $$m \in M^n$$.

$$n b m = \left( [n, b] + b n \right) m = 0$$

Yes.

(2) Let $$N \subseteq M^n$$ be a $$b$$-submodule.

Problem: Can you show that $$U(\mathfrak{g}) N \cap M^n = N$$.
Problems:
1) Calculate the answer for \( sl(2,\mathbb{C}) \) or read Bergman.
2) The H-C procedure consists in inducing characters from the various Cartan subalgebras. Determine whether these are all.
3) Is there a \( \Lambda \) reps of multiplicity 1? - minimal as in PRV.
4) Determine \( \Omega \) and especially \( (\Omega)_{ab} \) if 3) is true.

Assuming 3) calculate the character on \( \Omega \) by means of a canonical map

\[
(\Omega)_{ab} \rightarrow \lambda.
\]

A basic theorem: \( R \) f.d. algebra over \( \mathbb{C} \), \( \Lambda \) an irreducible representation of \( R \) (rec. of finite dim.) \( \chi_R : R \rightarrow \mathbb{C} \) its trace. Claim \( \chi_R \) completely determines \( \Lambda \).

Proof: Let \( N \) be the radical of \( R \). Then as trace of a nilpotent trans is 0 we have

\[
\chi_R(r) = 0 \quad \text{if} \quad r \in N.
\]

so \( \chi_R(N) = 0 \) and \( \chi_R : R/N \rightarrow \mathbb{C} \). By Wedderburn

\[
R/N = \bigoplus R_i \quad \text{where} \quad R_i \text{ are simple. Then} \quad \text{the non-zero on} \quad R_i \text{ and 0 on others}
\]

\( \dim \Lambda \text{ on } R_i \).
Proposition: Let $M$ be an irreducible module, and let $M_1$ be a submodule generated by $M$. Then

$$\text{Hom}_k(M_1, M) \cong \text{Hom}_k(\text{Hom}_k(M_1, M), \text{Hom}_k(M, M))$$

Proof: Clearly 1-1 because if $\varphi : M_1 \to M$ kills every map $A \to M$ over $k$, then $\varphi$ kills generators of $M$, and so $\varphi$ is 0. Conversely, given $\psi : \text{Hom}_k(M_1, M) \to \text{Hom}_k(M, M)$, $\psi$ must be onto by irreducibility of $\text{Hom}_k(A, M)$

$$\text{Hom}_k(M_1, M) \cong \text{Hom}_k(\text{Hom}_k(M_1, M), \text{Hom}_k(M, M))$$

The problem is to show that $(\otimes \otimes \varphi) \text{ Ker } \psi \subset \text{ Ker } \psi$.

However, we know that $\text{ Ker } \psi$ is the largest submodule of $\text{ target } \otimes \otimes \varphi$ which is contained in the subspace disjoint from $1 \otimes \text{ Hom}_k(M, M)$. Thus we have to show that $(\otimes \otimes \varphi) \text{ Ker } \psi$ has no $k$-subreps of type $A$. This will follow if $\text{ Ker } \psi$, has no $A$-reps. But we know that

$$\text{ Hom}_k(A, (\otimes \otimes \varphi) \otimes_k N)$$

so it's clear.
Problem: Construct a canonical map

\[ \Lambda \rightarrow \text{Hom}_k (U(\mathfrak{g}), \Lambda) \]

Two possibilities:

(i) \[ U(\mathfrak{g}) = \mathfrak{g}(\mathfrak{g}) \mathfrak{u}(k) \]

(ii) \[ U(\mathfrak{g}) = \mathfrak{g}(\mathfrak{g}) \mathfrak{u}(k) \]

\[ = U(\mathfrak{g} + k) \cdot U(k) \]

Example: Take \( \Lambda = 1. \) Then want a \( k \) invariant in

\[ \text{Hom}_k (U(\mathfrak{g}), 1) \]

Want an element of \( \mathfrak{g}_1 \)

\[ \mathfrak{g}_1 \rightarrow S(\mathfrak{g}) \xrightarrow{\text{evaluation}} \]

\[ C \]

Canonical linear functional on \( \mathfrak{g}_1 \), homomorphism augmentation.

Therefore I get a canonical element of \( \mathfrak{g}_1 \), which in fact is a character. Same under (i)+(ii).
Mackey coset formula

\[ \text{Hom}_K (L, \tau^* \chi \cdot M) = \text{Hom}_K (L, \text{Hom}_T (G, M)) \]

\[ = \text{Hom} (G, \text{Hom}_T (L, M)) \]

\[ = \Gamma (G \times \text{Hom}_T (L, M)) \]

\[ = \Gamma (K \setminus G / H, K \times H \times \text{Hom}_T (L, M)) \]

A kind of group of cohomological correspondences!

Take special case where there is a single coset, i.e. \( K \) acts transitively on \( G / H \) and assume that \( T = K \cdot H \). Then get

\[ \text{Hom}_G (j^* L, \tau^* M) = \text{Hom}_T (j^* L, \tau^* M) \]

Note that
Example: \( \Lambda = 1 \). Want distributions \( \varphi \) on \( G \) bivariant under \( K \) with support in \( K \). Then get a bivariant differential operator \( D \) on \( G/K \) by

\[
(Df) \circ \pi = \varphi \circ (f \circ \pi)
\]

bivariance under \( K \) means

\[
\delta_k \ast \varphi = k^{-1} \circ \varphi
\]

\[
\varphi \ast \delta_k = \varphi \circ k^{-1}
\]

Proof: for functions

\[
(\delta_k \ast \varphi)_g = \int \delta_k(gx^{-1}) \varphi_x \quad = \quad \varphi_{k^{-1}g} = k^{-1} \circ \varphi_g
\]

Conjecture: \( \Omega_\Lambda \) is the subalgebra of distributions on \( G \) with values in \( \text{Hom}(\Lambda, \Lambda) \) which satisfy

(i) bivariant under \( k \):

\[
\delta_k \ast \varphi = k^{-1} \circ \varphi
\]

\[
\varphi \ast \delta_k = \varphi \circ k^{-1}
\]

(ii) have support in \( K \).

Conjecture: \( \Omega_\Lambda \cong S(\mathfrak{o})^W \otimes \text{Hom}_M(\Lambda, \Lambda) \)

\[ \Rightarrow \text{one gets characters only when } \Lambda \text{ contains a } M. \]
Two problems

1. Irreducibility of induced representations with dominant weight vector: \( U(g) \otimes_{\tilde{\mathfrak{g}}} \Lambda \to \text{Hom}_F(U(g), \lambda_0) \)

2. Action of \( k \) on the resulting \( \lambda \) module:

\[
\text{Hom}_F \left( \mathfrak{n}, \text{Hom}_F(U(g), \lambda_0) \right)
\]

\[
\cong \text{Hom}_F(U(g), \text{Hom}(\mathfrak{n}, \lambda_0))
\]

Now we shall assume that \( q_j = k + b_- \) or equivalently that \( K_0 \) acts transitively on \( \mathfrak{g}/B_- \). In the complex case this is true. In the principal series \( \mathfrak{b}_+ = m + \alpha + \tau \) is parabolic so \( k + \alpha + \tau = q_j \Rightarrow q_j = k + b_+ \Rightarrow q_j = k + b_- \).

The general case not clear.
\[ \text{Hom}_g(U(2) \otimes \Lambda, U(2) \otimes \Lambda_2) \cong [U(2) \otimes \text{Hom}_\Lambda(\Lambda, \Lambda_2)]^W \]

Question: Is there an analogue of the isomorphism \( U(2)^\otimes = S(2)^\otimes \), i.e., is there an isomorphism to

\[ \text{Hom}_g(U(2) \otimes \Lambda, U(2) \otimes \Lambda_2) \]

where \( g = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) semi-direct product.

Thus is there a correspondence between irreducible representations of the homogeneous Lorentz group and irreducible reps of the group of Euclidean motions? Is there any correspondence between these two groups? Things are parametrized by pairs \((k_0, c)\) where \( k_0 \) is an integer and \( c \) a positive number. The same is true for the latter by Mackey's theory.

For \( sl(2, \mathbb{R}) \), \( g = \mathbb{R} \times \mathbb{R} \) Euclidean motions in the plane.

\( \text{(solvable - not Heisenberg)} \)
in the complex case!! and determine when principal series are irreducible.

In this case \( \mathfrak{g} = \mathfrak{m} + \mathfrak{t} \) is connected and \( W = \text{Weyl group of } \mathfrak{g} \). Everything can be done in the \( k \) framework.

i.e. Have to calculate

\[
[U(\mathfrak{g}) \otimes \text{Hom}_M(\Lambda_1, \Lambda_2)]^W = U(\mathfrak{g})^W
\]

\[ W = \text{ordinary Weyl group.} \]

\[
[U(\mathfrak{h}) \otimes \Lambda^k]^W
\]

Choose a hom. \( \chi : U(\mathfrak{h})^W \rightarrow \mathbb{C} \). Let \( U(\mathfrak{g})_\chi \) be the ring \( U(\mathfrak{g}) \otimes \mathbb{C} \). Assume the generic case, then \( U(\mathfrak{g})_\chi \otimes \mathbb{C} = \text{the product of fields in } \text{Hom}(W, \mathbb{C}) \).

\[ U(\mathfrak{g})_\chi \cong \text{Hom}(W, \mathbb{C}) \] So

\[
\mathbb{C}_\chi \otimes \left[ U(\mathfrak{g})_\chi \otimes \text{Hom}_M(\Lambda_1, \Lambda_2) \right]^W = \left[ U(\mathfrak{g})_\chi \otimes \text{Hom}_M(\Lambda_1, \Lambda_2) \right]^W
\]

\[ = \left[ \text{Hom}(W, \text{Hom}_M(\Lambda_1, \Lambda_2)) \right]^W
\]

\[ = \text{Hom}_M(\Lambda_1, \Lambda_2) \]

So we recover Brubaker's theory when \( \mathfrak{g} \) is not a wall.
The idea is that the category is now equivalent to sheaves on $\mathcal{F}'$, as follows. Conjecturally

$$[U(\mathcal{F}) \otimes \text{Hom}_M(\Lambda_1, \Lambda_2)]^W \cong \text{Hom}_{\mathcal{K}}(S(\mathcal{F}) \otimes \Lambda_1, S(\mathcal{F}) \otimes \Lambda_2).$$

This doesn't hold water because of the integer conditions:

$$[U(\mathcal{F}) \otimes \text{Hom}(\Lambda_1, \Lambda_2)]^W \cong [S(\mathcal{F}) \otimes \text{Hom}(\Lambda_1, \Lambda_2)]^W$$

$\mathcal{S}(\mathcal{F})$

NO NO NO. You get contradiction because you cannot see the integer conditions from this point of view. Nothing in your argument prevents you from applying same argument to get an isomorphism with the situation where integer conditions do not occur.
Idea: In passing from $\delta$ to $\gamma$ involves these integer fudge factors like going from $\mathbb{Z}$ to $\mathbb{C}^2$.

Conjectural situation: Have established a map

$$\text{Hom}_g(U(2) \otimes \Lambda_1, U(2) \otimes \Lambda_2) \to U(\mathbb{C}) \otimes \text{Hom}(\Lambda_1, \Lambda_2)$$

namely apply functor $U(\mathbb{C}) \otimes U(\Lambda + \rho)$

This map is compatible with composition and we want to determine its image. Idea is to make $N_\Lambda = \text{mergelage of}$

$\text{Sim}_K$ act on $U(\mathbb{C}) \otimes \text{Hom}(\Lambda_1, \Lambda_2)$ in a reasonable way so that

the image is the invariants of the action.
Go back to Iwasawa decomp.

Fix $\Lambda$ problem is given an irreducible $\Omega_\Lambda$ module.

Suppose $\mathcal{V}$ is an irreducible.

Let $\mathcal{V}$ be an irreducible representation of $\mathcal{M}$, let $\lambda \in \mathfrak{a}$, $\lambda(H_i) \geq 0$ all $i$. Define the principal series rep $\Pi_{\lambda, \mathcal{V}}$.

Idea somehow is to have $k$ structure

$$\bigoplus_{\lambda} \Lambda \otimes \text{Hom}(\Lambda, \mathcal{V})$$

hence if $\lambda$ occurs with mult $k$ means that $\text{Hom}_M(\Lambda, \mathcal{V})$ is $k$ dimensional. Does this always happen?

Consider $\text{Hom}_M(\mathcal{K}, \mathcal{V})$

**S5.**

$K/M \approx$ reg orbit in $\mathfrak{a}$

Basic question: $G_\mathcal{M}$ real!
\[(U(\omega) \otimes_k \Lambda) \otimes_{\Lambda} \Lambda = (\oplus_{\Lambda_1} U(\omega)^W \otimes \Lambda_1 \otimes \text{Hom}_M(\Lambda, \Lambda)) \]

\[\otimes \text{Hom}_M(\mu \otimes \Lambda) \]

\[U(\omega)^W \otimes \text{Hom}_M(\Lambda, \Lambda).\]

\[= \oplus_{\Lambda_1} \Lambda_1 \otimes \text{Hom}_M(\Lambda, \Lambda) \otimes \text{Hom}_M(\mu \otimes \Lambda)\]

\text{Variance is wrong.}

New defn

\[\Omega_\Lambda = \text{Hom}_R(\Lambda_2, U(\omega) \otimes_k \Lambda) = \text{End}_\omega(\Lambda) \otimes (U(\omega) \otimes_k \Lambda)\]

\[\Omega_\Lambda \rightarrow U(\omega)^W \otimes \text{End}_M(\Lambda).\]

\[N = \text{Hom}_M(\Lambda_2, M) \otimes \text{is a right } \Omega_\Lambda \text{ module,}\]

\[= \text{Hom}_M(\Lambda_2, \mu) \text{ where } U(\omega)^W \text{ acts as it should.}\]

\[N \otimes (U(\omega) \otimes_k \Lambda)\]

\[\Omega_\Lambda \]

\[\text{Hom}_M(\Lambda_2, \mu) \otimes \bigoplus_{\Lambda_1} U(\omega) \otimes \text{Hom}_M(\Lambda, \Lambda)\]

\[\oplus_{\Lambda_1} \Lambda_1 \otimes \left[ \text{Hom}_M(\Lambda_2, \mu) \otimes \text{Hom}_M(\Lambda, \Lambda) \right].\]
Conclude that

\[ N \otimes_{\mathbb{Z}} (U(g) \otimes_k \Lambda) \cong \bigoplus \Lambda, \otimes \text{Hom}(\Lambda, \mu) \]

as \( K \)-modules.

In particular, we get \( \text{Hom}_K(\Lambda, \Lambda) \) acting which must determine when irreducible.

Problem: Consider with \( N \) coming from \( \text{Hom}_k(\Lambda, \mu) \), then calculate that in some degree

\[ \text{Hom}_k(\Lambda, N \otimes_{\mathbb{Z}} (U(g) \otimes_k \Lambda)) \cong \text{Hom}_m(\Lambda, \mu). \]

We have to determine when this is irreducible over \( \text{Hom}_k(\Lambda, \Lambda) \). Looks like it always is except that we haven't analyzed how char on \( S(\omega)^W \) interferes.
II. Suppose $\Omega_A$ known. Then is

$$N \otimes_{\Omega_A} (u(y) \otimes_k \Lambda)$$

irreducible.

I have to calculate the right $\Omega_A$ module

$$\text{Hom}_K(\Lambda, N \otimes_{\Omega_A} (u(y) \otimes_k \Lambda))$$

$$\downarrow S$$

$$N \otimes_{\Omega_A} \text{Hom}_K(\Lambda, u(y) \otimes_k \Lambda)$$

$$\downarrow$$

$$u(y) \otimes_k \Lambda$$

$$\Lambda \rightarrow N \otimes_{\Omega_A} (u(y) \otimes_k \Lambda)$$

If $i$ increases then

$$N \otimes \text{Hom}(\ ) \otimes_{\Omega_A}$$

But in any case have only to decide on irred of

$$N \otimes \text{End}_y(\phi; \Lambda) \text{Hom}_y(\phi; \Lambda, \phi; \Lambda)$$

as a \text{End}_y(\phi; \Lambda) module. Depends only on the

formula for these rings to calculate.
Conjecture: $\Omega_{\Lambda} = \text{End}_{g}(W_{\alpha}\otimes_{\Lambda} N)$

$\cong S(W) \otimes \text{Hom}_{M}(\Lambda, \Lambda)$.

Application to irreducibility:

Let $N$ be an irreducible right $\Omega_{\Lambda}$ module. From the conjecture, there is a hom $f: S(W) \rightarrow B$ and an irreducible $\text{Hom}_{M}(\Lambda, \Lambda)$ module structure on $N$ such that

$\iota_{n}(\varphi \circ \psi) \varphi = f(\varphi) \varphi \psi \quad \varphi \in S(W), \quad n \in N$

Let $\mu \in \text{algebraic group reductive in } K$ so

$\Lambda \cong \bigoplus_{\mu} \mu \otimes \text{Hom}(\mu, \Lambda)$

where $\mu$ runs over the irreducible reps of $M$. Hence $N$ irreducible over $\text{Hom}_{M}(\Lambda, \Lambda)$ if $\text{Hom}_{M}(\Lambda, \mu)$ with composition actions. Look at induced module $N \otimes (W_{\alpha}\otimes_{\Lambda} N)$

as a $k$ module.
To introduce a formalization of $M$, the following occurs

\[ \phi \in \text{Hom}(V, M) \]

... and conclude the results. Where

\[ N = \text{Hom}(V, V) \]

... or auxiliary cases or... say a

\[ \left( V \otimes (B) \right) \otimes N \]

... or conclude the results. Where

\[ \left( V \otimes (B) \right) \otimes N \]

... or conclude the results. Where

\[ \left( V \otimes (B) \right) \otimes N \]

... or conclude the results. Where
\[ P \times \Lambda^2 \leftarrow A \times \Lambda^2 \]

Claim goes onto each \( K \) orbit \( \checkmark \)

\[ K/P \times \Lambda^2 \leftarrow \varphi \leftarrow A \times \Lambda^2 \]

for regular orbits: \[ WA \times M \Lambda^2 \]

\[ K \setminus P \leftarrow W \setminus A \]

\[ \varphi(a \times \lambda^2) = \varphi(a_i \times \lambda_i) \quad \text{i.e.} \quad \exists k \in K \]

\[ ka k^{-1}, k \lambda = a_{i'}, \lambda_i \]

Let \( N \subset K \) be the normalizer of \( A \). Then as \( a, a_i \) are generic it follows that \( k \in N \).

So our orbit space is \( N \setminus A \times \Lambda \).

But \( N = W \times M \) ? \( \text{[No]} \).

\[ \begin{array}{cccc}
0 & \rightarrow & M & \rightarrow & N & \rightarrow & W & \rightarrow & 0 \\
\end{array} \]

\( \text{exact seq.} \)

\[ \left[ U(\alpha) \otimes \text{Hom}(A, \Lambda) \right]^N \]

con
Theorem: \( \text{Hom}_g(U(\mathfrak{g}) \otimes_k \Lambda_1, U(\mathfrak{g}) \otimes_k \Lambda_2) \cong [U(\mathfrak{g}) \otimes \text{Hom}(\Lambda_1, \Lambda_2)]^N \)

where \( N = \text{normalizer of } \alpha \text{ in } K \).

Exact sequences:
\[
\emptyset \to M \to N \to W \to 1
\]

So
\[
[U(\mathfrak{g}) \otimes \text{Hom}(\Lambda_1, \Lambda_2)]^N \cong [U(\mathfrak{g}) \otimes \text{Hom}_M(\Lambda_1, \Lambda_2)]^W
\]

There may a good reason for \( W \) to act trivially on \( \text{Hom}_M(\Lambda_1, \Lambda_2) \).

(e.g. in the complex case)
\[
\begin{array}{ccc}
W & \to & W \\
\Delta & \downarrow & \downarrow \Delta \\
W \times W & \to & W \times \mathfrak{h} \\
0 & \to & \mathfrak{h} \times \mathfrak{h} & \to & \mathfrak{h} \times \mathfrak{h} & \to & 0 \\
\end{array}
\]

\[
\{ (\omega_1, \omega_2) \mid (\omega_1, \omega_2) \mathfrak{h} = \omega_1 \mathfrak{h} + \omega_2 \mathfrak{h} \}
\]

usually Weyl grp. of \( K \).

Let \( W \) act on \( \text{Hom}_k(\Lambda_1, \Lambda_2) \).

Interesting action here

So action is not trivial

Check carefully