A) Certainly we can subtract

$$\nabla - (d + \operatorname{ad} A)$$

$$[(d + \operatorname{ad} A), \iota_X] = L_X + \operatorname{ad} X$$

$$[\nabla, \iota_X] = -A^a \iota_{[X_a, X]}$$

This is a puzzling ~~question~~ – you're comparing ∇ on scalar valued forms with $d + \operatorname{ad} A$ on Lie algebra valued forms.

From old files

$$0 \leftarrow \bar{C}^\lambda(A) \leftarrow C^\lambda(A) \xleftarrow{+1} B \xleftarrow{+1} B_\sigma^{\otimes 2}$$

$$T(A^*) \hookrightarrow T(A^* \oplus \mathbb{C}\varepsilon^*) = T(A^*) \oplus T(A^*)\varepsilon^* T(A^*) \oplus \cdots$$

$$u_t = t^D$$

$$\dot{u}_t = t^{D-1} D$$

$$\int_0^1 dt \, (u_t, \dot{u}_t)_* (x \, dy) = \int_0^1 dt \, (t^D x)(t^{D-1} Dy)$$

$$= \int_0^1 \frac{dt}{t} t^D (x Dy) = \left[\frac{t^D}{D}\right]_0^1 (x Dy)$$

$$= \frac{1-P}{D} (x Dy)$$

where $P = \lim_{t \to 0} t^D$ = projection on the null space

Note $1-$

B) Review and translate

Consider first $T(A^*)$ $\quad T(\bar{A}^*)$ $\quad \nabla$

So how does it work

A has basis $\{X_a\} = \{e, X_i\}$

A^* has dual basis $\{\rho, \theta^i\}$

$$X_a X_b = f^c_{ab} X_c$$

$$\theta = \rho e + \theta^i X_i \quad \in T^1(A^*) \otimes A$$

$$[d\theta] + \theta^2 = 0$$

$$[d\theta] = (d\rho)e + (d\theta^i)X_i \qquad + f^0_{jk}\theta^j\theta^k e$$

$$\theta^2 = \rho^2 e + (\rho\theta^i + \theta^i\rho)X_i + f^i_{jk}\theta^j\theta^k X_i$$

$$d\rho + \rho^2 = -f^0_{jk}\theta^j\theta^k = \omega$$

$$d\theta^i + [\rho, \theta^i] + f^i_{jk}\theta^j\theta^k = 0$$

$$T(A^*) = \mathbb{C}\langle \rho, \theta^i \rangle$$

$$T(\bar{A}^*) = \mathbb{C}\langle \theta^i \rangle \qquad \nabla = d + \operatorname{ad}\rho$$

$$\nabla^2 = \operatorname{ad}\omega$$

What I want to find is a ~~[illegible]~~ surjection.

Wait we have

$$T(A^*) = T(\bar{A}^*) \oplus T(\bar{A}^*)\rho\, T(\bar{A}^*) \oplus \cdots$$

$$= T(\bar{A}^*) * \mathbb{C}[\rho]$$

$$\begin{cases} dx = \nabla x - [\rho, x] & x \in T(\bar{A}^*) \\ d\rho = \omega - \rho^2 \end{cases}$$

C) $\quad T(\bar{A}^*) \hookrightarrow T(A^*)$

induces $\bar{C}^\bullet_\lambda(A) \hookrightarrow C^\bullet_\lambda(A)$. What I want is a map backwards. Wait: I want ~~a homotopy equivalence~~

[redacted]

the ~~inclusion~~ map $\bar{C}^\bullet_\lambda(A) \hookrightarrow C^\bullet_\lambda(A)$ to be a homotopy equivalence ?

$$0 \to C^\lambda(\mathbb{C}) \longrightarrow C^\lambda(A) \longrightarrow\!\!\!\!\to \bar{C}^\lambda(A) \to 0$$

I want a lifting $\bar{C}^\lambda(A) \xhookrightarrow{l} C^\lambda(A)$
~~such that the composition with~~
such that modulo $C^\lambda(\mathbb{C})$ it is compatible with d. Thus $[d,l]: \bar{C}^\lambda(A)^{[1]} \to C^\lambda(\mathbb{C})$

Dually I want a ~~map~~ retraction

$$T(A^*)_\natural \xrightarrow{r} T(\bar{A}^*)_\natural$$

such that $[d,r]$ factors through $T(A^*)_\natural \to \mathbb{C}\langle\rho\rangle_\natural$

$$\begin{tikzcd}
 & C^\lambda(\mathbb{C}) \arrow[d, hook] \\
\bar{C}^\lambda(A) \arrow[ur, dashed, "{[d,l]}"] \arrow[r, dashed, "l"] \arrow[dr, "1"'] & C^\lambda(A) \arrow[d] \\
 & \bar{C}^\lambda(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C_\lambda(\mathbb{C}) \arrow[dl, "{[d,r]}"'] \\
\bar{C}^\bullet_\lambda & C^\bullet_\lambda \arrow[l, "r"'] \arrow[u, two heads] \\
 & \bar{C}_\lambda \arrow[u, hook]
\end{tikzcd}$$

What I have to do this morning is to write up an account of yesterday's calculations concerning $\tilde{A} + \mathbb{C}\varepsilon$.

D) Today Febr. 6 I want to write up calculations concerning $\tilde{A} \oplus \mathbb{C}\varepsilon$, D etc. Need to summarize.

$$
\begin{array}{ccccc}
A & \longleftarrow & \tilde{A} \oplus \mathbb{C}\varepsilon & \longrightarrow & \mathbb{C}[\varepsilon] \\
e & & e & & 0 \\
X_i & & X_i & & 0 \\
0 & & e^{\perp} & & 1 \\
0 & & \varepsilon & & \varepsilon
\end{array}
$$

dual basis to $e, X_i, e^{\perp}, \varepsilon$?

$\rho, \theta^i, \chi, -\varphi$

χ augm. $\quad \chi(e) = 0$, $\chi(1) = 1$

Not [redacted] clear.

First

$$\theta = \rho e + \theta^i X_i + \chi e^{\perp} - \varphi \varepsilon$$

$$
\begin{aligned}
0 = d'\theta + \theta^2 = {} & (d'\rho)e + (d'\theta^i)X_i + (d'\chi)e^{\perp} - (d'\varphi)\varepsilon \\
& + \rho^2 e + [\rho, \theta^i]X_i + \chi^2 e^{\perp} \\
& + f^i_{jk}\theta^j\theta^k X_i + f^0_{jk}\theta^j\theta^k e - [\chi, \varphi]\varepsilon
\end{aligned}
$$

$$d'\rho + \rho^2 \overset{-\omega}{+ f^0_{jk}\theta^j\theta^k} = 0$$

$$d'\theta^i + [\rho, \theta^i] + f^i_{jk}\theta^j\theta^k = 0$$

$$d'\chi + \chi^2 = 0$$

$$d'\varphi + [\chi, \varphi] = 0$$

$$d''\rho = d''\theta_i = 0$$

$$d''\chi = \varphi, \quad d''\varphi = 0.$$

E) (Feb 6) But we want the horizontal space

$$T\left((\tilde{A}\oplus\mathbb{C}\varepsilon/\mathbb{C})^*\right)$$

I want to introduce $\alpha = \chi - \rho$

$$\chi_t = t\chi + (1-t)\rho = \rho + t\alpha$$

$$\alpha(e^\perp) = \chi(e^\perp) - \rho(e^\perp) = 1.$$

$$\alpha(e) = -1.$$

$\alpha(1) = 0$

$\therefore$ α horizontal

$$d'\alpha = d'\chi - d'\rho$$

$$= -\chi^2 + \rho^2 - \omega$$

$$= -\cancel{\rho^2} - [\rho,\alpha] - \alpha^2 + \cancel{\rho^2} - \omega$$

$$\boxed{\begin{aligned} d'\alpha + [\rho,\alpha] &= -\alpha^2 - \omega \\ d''\alpha &= \varphi \end{aligned}}$$

$$d''\alpha = \overset{\varphi}{d''\chi} - \overset{0}{d''\rho}$$

$$\nabla = d + \operatorname{ad}\rho$$

using ρ as connection form.

$$\nabla\alpha = \varphi - \alpha^2 - \omega$$

So

$$T\left((\tilde{A}\oplus\mathbb{C}\varepsilon/\mathbb{C})^*\right) = \mathbb{C}\langle \theta^i, \alpha, \nabla\alpha\rangle$$

where $\nabla\alpha = \varphi - \alpha^2 - \omega$

$$\cancel{\nabla(\nabla\alpha) = -[\nabla\alpha,\alpha] = -[\varphi - \alpha^2 - \omega, \alpha]}$$

$$\cancel{= -[\varphi,\alpha] + [\omega,\alpha]}$$

$$\nabla^2\alpha = \nabla\varphi - [\nabla\alpha,\alpha]$$

$$\cancel{= d\varphi + [\rho,\varphi] \quad [d\alpha] \quad [[\rho,\alpha],\alpha]}$$

$$\nabla(\varphi - \alpha^2) \qquad ?$$

F) Alternatives

$$T((\tilde{A} \oplus \mathbb{C}\varepsilon/\mathbb{C})^*) \xrightarrow{\sim} T((A + \mathbb{C}\varepsilon)^*)$$

$$\mathbb{C}\langle \theta^i, \alpha, \nabla\alpha \rangle \qquad \mathbb{C}\langle \theta^i, \rho, \varphi \rangle$$

$$\theta^i \longmapsto \theta^i$$

$$\alpha \longmapsto -\rho$$

$$\nabla_0 \alpha = d\alpha + [\rho, \alpha]$$

$$\nabla_t \alpha = d\alpha + [\rho + t\alpha, \alpha]$$

$$\nabla_{t_1} \alpha \longmapsto \varphi - \rho^2 - \omega + 2t\rho^2$$

$$\varphi - \alpha^2 - \omega + 2t\alpha^2$$

Let's leave the DG alg + cochain picture and try to make some sense out of the DG coalg side. Mainly concerned with

$$\begin{array}{ccc} T(A) & \longleftarrow & T(\tilde{A} \oplus \mathbb{C}\varepsilon) \\ \downarrow & & \downarrow \\ T(\bar{A}) & \longleftarrow & T(\tilde{A} \oplus \mathbb{C}\varepsilon/\mathbb{C}) \end{array}$$

What should you examine?????

What seems to happen is that a choice of ρ leads to b'_ρ on $T(\bar{A})$ and $T(\tilde{A} \oplus \mathbb{C}\varepsilon)$

Dually

$$T(\bar{A}^*) = \mathbb{C}\langle \theta^i \rangle \hookrightarrow T(\overline{\tilde{A} \oplus \mathbb{C}\varepsilon}^*) = \mathbb{C}\langle \theta^i, \alpha, \varphi \rangle$$

$$d + \mathrm{ad}\,\rho \qquad \varphi = \omega + \nabla\alpha + \alpha^2$$

Also have this deformation retraction based on D.

Notice that a choice of connection is needed to get ∇, but that these algs are intrinsic.

G) What I should calculate carefully is things in terms of the isom.

$$T(\overline{\tilde{A}\oplus\mathbb{C}\varepsilon}) \xleftarrow{\ \sim\ } T(A\oplus\mathbb{C}\varepsilon)$$

$$\begin{tikzcd}
 & \tilde{A}\oplus\mathbb{C}\varepsilon \arrow[dl] & \\
\overline{\tilde{A}\oplus\mathbb{C}\varepsilon} & & A\oplus\mathbb{C}\varepsilon \arrow[ll, "\sim"'] \arrow[ul, hook']
\end{tikzcd}$$

nonunital algebra

e, X_i, ε

What do I want?

The point is that there is a family of connections on $\tilde{A}\oplus\mathbb{C}\varepsilon$

basis $\quad e, X_i, e^{\perp}, -\varepsilon \qquad d\varepsilon = e^{\perp}$

dual b. $\quad \rho, \theta^i, \chi, \varphi$

a connection sends 1 to 1, so we have

$$\chi_t = t\chi + (1-t)\rho = \rho + t(\underbrace{\chi-\rho}_{\alpha})$$

$$1, X_i, e^{\perp}, -\dot{\varepsilon} \longleftarrow e, X_i, -\varepsilon$$

$$\rho, \theta^i, \alpha, \varphi$$

I am getting nowhere again.

Game is to calculate, get control of,

$$\begin{tikzcd}
\tilde{C}^{\lambda}(\tilde{A}\oplus\mathbb{C}\varepsilon) & C^{\lambda}(\overline{\tilde{A}\oplus\mathbb{C}\varepsilon}) \arrow[l, two heads] & C^{\lambda}(A\oplus\mathbb{C}\varepsilon) \arrow[l, hook'] \arrow[ll, bend left, "\simeq"]
\end{tikzcd}$$

H) Point: You get a diffl on $T(\tilde{A} \oplus \mathbb{C}\varepsilon)$ from a choice of connection.

Two choices. $\rho: \tilde{A} \oplus \mathbb{C}\varepsilon \longrightarrow A \xrightarrow{f} \mathbb{C}$ is compatible with internal diffl $d\varepsilon = e^+$ so it leads to a b'_ρ compatible with this d''.
However b'_ρ has curvature.
Other choice is the augmentation $\chi: \tilde{A} \oplus \mathbb{C}\varepsilon \to \mathbb{C}$. This is a homomorphism so $b_\chi'^2 = 0$ but ~~horizontal~~ χ does not commute with d, hence vertical and horizontal derivations do not commute.

The homotopy is best seen in the former setting.

Question: Can we get an further with an S operator on $\bar{C}^\lambda(A)$ associated to ρ.

$$0 \to \bar{C}^\lambda(A) \longrightarrow \bar{A} \otimes T(\bar{A}) \xrightarrow{1-\lambda} \bar{T}(\bar{A}) \longrightarrow \underset{\bar{C}^\lambda(A)}{\underset{\shortparallel}{\bar{T}(\bar{A})_\natural}} \longrightarrow 0$$

Did you learn anything?

$$\begin{tikzcd}
0 \arrow[r] & \bar{A}^{\otimes n} \arrow[r] & A \otimes \bar{A}^{\otimes n} \arrow[r] & \bar{A}^{\otimes n+1} \arrow[r] & 0 \\
0 \arrow[r] & A^{\otimes n} \arrow[r] \arrow[u] & \tilde{A} \otimes A^{\otimes n} \arrow[r] \arrow[u] & A^{\otimes n+1} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

What you are doing is to split the bottom sequence in some way.

$$\begin{pmatrix} 1 & 0 \\ +\iota_\rho & 1 \end{pmatrix} \begin{pmatrix} b & 1-\lambda \\ & -b' \end{pmatrix} \begin{pmatrix} 1 & \cdot \\ -\iota_\rho & 1 \end{pmatrix} = \begin{pmatrix} b_\rho & 1-\lambda \\ -\iota_\omega(1+\lambda) & -b'_\rho \end{pmatrix}$$