A) Homotopy again — review.

Consider again the ~~elements~~ problem

$$\begin{tikzcd}
{} \arrow[r] & R \arrow[r, "d"] \arrow[d, "1+\delta"] & (\Omega^1 R)_\natural \arrow[r] \arrow[d, "1+\delta"] & R \arrow[r] \arrow[d, "1+\delta"] & {} \\
{} \arrow[r] & (R+\Omega^1 R) \arrow[r, "d"] \arrow[d] & \Omega^1(R+\Omega^1 R)_\natural \arrow[r] \arrow[d] & (R+\Omega^1 R) \arrow[r] \arrow[d] & {} \\
& \Omega^1 R & \Gamma & \Omega^1 R &
\end{tikzcd}$$

$$\Gamma \simeq R\otimes\bar R \oplus \Omega^2 R_\natural / \text{relation}$$

$d = X \qquad \delta = Y.$

$W = d\delta = \delta d$

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[d, "\delta"] & \Omega^1 A \arrow[d, "\delta"] \\
\Omega^1 A \arrow[r] & A\otimes\bar A\otimes A \oplus \Omega^2 A
\end{tikzcd}$$

$d\delta a \qquad da_1\,\delta a_2$

$$da_1\,\delta a_2 + \delta a_1\,da_2 + d\delta a_1\,a_2 - d\delta(a_1a_2) + a_1 d\delta a_2 = 0$$

$$\Gamma = A\otimes\bar A \oplus \Omega^2 A_\natural / \text{relations.}$$

$a_0\,d\delta a_1, \quad a_0\,da_1\,\delta a_2$

$f_1(a_0,a_1) \qquad g_2(a_0,a_1,a_2) \qquad g_2 + \kappa g_2 + bf_1 = 0.$

why $\quad da_2\,a_0\,\delta a_1 \Longleftrightarrow a_0\,\delta a_1\,da_2$ ''

B) Relation must be

$$da_2\, a_0\, \delta a_1 \;\dot{-}\; a_0\, \delta a_1\, da_2$$

$$= da_2\, a_0\, \delta a_1 + a_0\, da_1\, \delta a_2 + a_2 a_0\, d\delta a_1 - a_0\, d\delta(a_1 a_2) + a_0 a_1\, d\delta a_2$$

$$\boxed{\Omega^2 A_\natural \xrightarrow{(b,\ 1+K)} A\otimes\bar{A} \oplus \Omega^2 A_\natural \longrightarrow \Gamma \longrightarrow 0}$$

What are the maps

$$\begin{tikzcd}
 & \Omega^1 A_\natural \arrow[d, "\delta"] & \\
\Omega^1 A \arrow[r, "d"] & (A\otimes\bar{A} \oplus \Omega^2 A_\natural)/\mathrm{rel} \arrow[r, "c"] & \Omega^1 A
\end{tikzcd}$$

$$a_0\, da_1 \mapsto a_0\, d\delta a_1 + \delta a_0\, da_1$$

$$a_0\,\delta a_1 \longmapsto a_0\, d\delta a_1 + da_0\, \delta a_1$$

$$a_0\, d\delta a_1 \longmapsto [a_0, \delta a_1]$$

$$a_0\, da_1\, \delta a_2 \longmapsto [\delta a_2\, a_0, a_1]$$

$$a_0\, d\delta a_1 + da_0\, \delta a_1 \longmapsto [a_0, \delta a_1] + [\delta a_1, a_0] = 0.$$

$$\delta(a_0\, da_1) = a_0\, \delta da_1 + \delta a_0\, da_1$$

$$= a_0\, \delta da_1 + da_1\, \delta a_0 \xmapsto{c} [a_0, \delta a_1] + [\delta a_0, a_1]$$

$$\| $$

$$\delta [a_0, a_1] = \delta c(a_0\, da_1).$$

9) ~~So~~ we now have ~~a clean~~ to find a homotopy when A is free. So let us ~~consider~~

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[d, "\delta"] & \overset{a_0 da_1}{\Omega^1 A_q} \arrow[r, "c"] \arrow[d, "\delta"] \arrow[dl, dashed] & \overset{[a_0,a_1]}{A} \arrow[d, "\delta"] \arrow[dl, dashed] \\
(\Omega^1 A)_q \arrow[r, "d"] & (A\otimes\bar{A} + \Omega^2 A_q)/\mathrm{rel} \arrow[r, "c"] & \Omega^1 A
\end{tikzcd}$$

There's a canonical choice for the first arrow mod Im c. namely $a_0 da_1 \longmapsto a_0 \delta a_1$

$\boxed{\text{OKAY}}$

$$\begin{tikzcd}
 & a_0 da_1 \arrow[d, mapsto] \arrow[dl] \\
a_0\delta a_1 \arrow[r, mapsto] & a_0 d\delta a_1 + da_0 \delta a_1 \\
 & a_0 \delta d a_1 + \delta a_0 da_1
\end{tikzcd}$$

difference is the map $a_0 da_1 \longmapsto \delta a_0 da_1 - da_0 \delta a_1$

Here we recognize B applied to a 2-cocycle $(a_1, a_2) \longmapsto da_1 \delta a_2$ and we have to write ~~this in the form~~ this cyclic cocycle as a coboundary

What can we say about Γ in the free case a linear fl on it is a pair f_1, g_2 with $bf_1 + (1+K)g_2 = 0$ $\quad bg_2 = 0$. In the free case we can write $g_2 = bh_1$ canonically.

so we get $\quad b(f_1 + (1+K)h_1) = 0$

D) Think it all out using cochains.

Linear fcls^F on Γ are (f_1, g_2) where

$$bf_1 + (1+K)g_2 = 0 \qquad bg_2 = 0.$$

$$f_1(a_0, a_1) = T(a_0\, d\delta a_1) \qquad g_2(a_0,a_1,a_2) = T(a_0\, da_1\, \delta a_2)$$

Can split into invariant and non invariant components.

Inv. $\quad bf_1 + 2g_2 = 0 \qquad$ space = all invariant 1-cochains.

Noninv. $\quad bf_1 = 0 \qquad bg_2 = 0 \qquad$ non-invariant 2-cocycles.

$$\Downarrow \qquad\qquad Kg_2 = -g_2.$$

$$f_1 = 0 \qquad 2g_2 = (1-K)g_2 = bsg_2$$

space of all $h_1(a_0, a_1)$ reduced & symmetric.

Thus $\quad \Gamma \cong \Omega^1 A \quad$ in some funny way.

Let's go over it carefully. The cochains.

$$\begin{array}{ccccc} A & \xrightarrow{d} & \Omega^1 A_\natural & \xrightarrow{c} & A \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ \Omega^1 A & \xrightarrow{d} & \Gamma & \xrightarrow{d} & \Omega^1 A \end{array}$$

Γ is a quotient of the degree 1,1 part of $\Omega\Omega A$

$\cancel{(\Omega\Omega A)^{11}} \qquad \Omega^{11} A$

$A \qquad \Omega^{01} A$

$$\Omega^{11} A \cong A \otimes \bar{A} \otimes A \oplus \Omega^2 A$$

$$a_0\, d\delta a\, a_1 \longleftrightarrow (a_0, a_1, a_2)$$

$\Omega^{10} A \qquad \Omega^{11} A \qquad a_0\, da_1\, \delta a_2 \longleftrightarrow a_0\, da_1\, da_2$

E) A linear fnl T on Γ is a pair

$$f_1(a_0, a_1) = T(a_0\, d\delta a_1)$$
$$g_2(a_0, a_1, a_2) = T(a_0\, da_1\, da_2)$$

$\ni$ $\quad bg_2 = 0 \qquad bf_1 + (1+\kappa) g_2 = 0.$

$$T(a_0 a_1\, d\delta a_2 \quad a_0\, d\delta(a_1 a_2) + a_2 a_0\, d\delta a_1$$
$$+ T(a_0\, da_1\, \delta a_2) + \underbrace{(\kappa g_2)(a_0, a_1, a_2)}$$

$$g_2(a_2, a_0, a_1) - g_2(1, a_2 a_0, a_1)$$
$$= T(a_2\, da_0\, \delta a_1) - g_2(d(a_2 a_0)\delta a_1)$$
$$= -T(da_2\, a_0\, \delta a_1)$$

~~which makes no s~~. Apparently there's a mistake in sign. It should be

$$bf_1 + (1-\kappa) g_2 = 0$$

Let's go over the other maps.

$$\begin{tikzcd} & \cdot \\ \cdot & (f_1, g_2) \arrow[l, "d^t"'] \arrow[u, "\delta^t"'] \end{tikzcd}$$

$$(T'' d)(a_0, a_1) = T'' d(a_0\, \delta a_1)$$
$$= T''(a_0\, d\delta a_1 + da_0\, \delta a_1)$$
$$= f_1(a_0, a_1) + (sg)(a_0, a_1)$$

$$\boxed{T'' d = f_1 + s g_2}$$

$$\boxed{T''\delta = f_1 - \lambda s g_2}$$

$$\delta(a_0\, da_1) = \delta a_0\, da_1 + a_0\, d\delta a_1$$
$$(T''\delta)(a_0, a_1) = f_1(a_0, a_1) + (sg_2)(a_1, a_0)$$

F) Next what gives?

$$\cancel{\tau c(a_0 d\delta a_1) = \tau([a_0, \delta a_1])}$$

$$T^{10}_c(a_0\, d\delta a_1) = T^{10}([a_0, \delta a_1])$$

$$= T^{10}(a_0 \delta a_1 - \delta(a_1 a_0) + a_1 \delta a_0)$$

$$1 \qquad = T^{1,0}(a_0, a_1) - (KT^{10})(a_0, a_1)$$

$$\boxed{(T^{10}_c)(a_0 d\delta a_1) = \{(1-K)T^{10}\}(a_0, a_1)}$$

$$(T^{10}_c)(a_0\, da_1\, \delta a_2) = T^{10}([\delta a_2\, a_0, a_1])$$

$$= T^{10}(\delta a_2\, a_0 a_1 - a_1 \delta a_2 a_0)$$

$$= T^{10}(\delta(a_2 a_0 a_1) - a_2 \delta(a_0 a_1) - a_1 \delta(a_2 a_0) + a_1 a_2 \delta a_0)$$

$$= T^{10}(1, a_2 a_0 a_1) - T^{10}(a_2, a_0 a_1) - T^{10}(a_1, a_2 a_0) + T^{10}(a_1 a_2, a_0)$$

$$\cancel{+T^{10}(a_2 a_0, a_1) - T^{10}(a_0, a_1 a_2)}$$

$$+ \underbrace{T^{10}(1, a_1 a_2 a_0)} - \underbrace{T^{10}(1, a_1 a_2 a_0)}$$

$$= (KT^{10})(a_0 a_1, a_2) + (KT^{10})(a_2 a_0, a_1) - (KT^{10})(a_0, a_1 a_2)$$

$$\cancel{T^{10}_c = bKT^{10}}$$

$$\boxed{(T^{10}_c)(a_0\, da_1\, \delta a_2) = (bKT^{10})(a_0, a_1, a_2)}$$

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, "\delta"] & \Omega^1 A_\natural \arrow[r, "c"] \arrow[d, "\delta"] & A \arrow[d, "\delta"] \\ \Omega^1 A \arrow[r, "d"'] & \Gamma \arrow[r, "c"] & \Omega^1 A \end{tikzcd}$$

Denote a linear fnl on by $\quad \tau \quad T$

$\psi \qquad \phi \qquad \psi$

$\psi = (\psi_1, \psi_2)$ where

$$\psi_1(a_0, a_1) = \psi(a_0\, d\delta a_1)$$

$$\psi_2(a_0, a_1, a_2) = \psi(a_0\, da_1\, \delta a_2)$$

basic reln. $\quad b\psi_1 + (1-\kappa)\psi_2 = 0$

$$(\tau c)_1(a_0, a_1) = \tau([a_0, a_1]) = (b\tau_0)(a_0, a_1)$$

$$(Td)_0(a_0) = T(1, a_0) = (sT_1)(a_0)$$

$$\begin{aligned}(\psi d)_1(a_0, a_1) &= \psi(d(a_0 \delta a_1)) \\ &= \psi(a_0\, d\delta a_1) + \psi(da_0\, \delta a_1) \\ &= \psi_1(a_0, a_1) + (s\psi_2)(a_0, a_1)\end{aligned}$$

$$\begin{aligned}(\psi \delta)_1(a_0, a_1) &= \psi(\delta(a_0\, da_1)) \\ &= \psi(a_0\, d\delta a_1) + \psi(da_1\, \delta a_0) \\ &= \psi_1(a_0, a_1) + (\lambda s\psi_2)(a_0, a_1)\end{aligned}$$

H)

$$(\phi c)_1(a_0,a_1) = \phi\, c(a_0\, d\delta a_1)$$
$$= \phi[a_0,\delta a_1]$$
$$= \phi(a_0\delta a_1) - \phi(\delta a_1\, a_0)$$
$$= \phi(a_0\delta a_1) - \phi(\delta(a_1a_0)) + \phi(a_1\delta a_0)$$
$$= \phi_1(a_0,a_1) - (\kappa\phi_1)(a_0,a_1)$$

$$(\phi c)_2(a_0,a_1,a_2) = \phi c(a_0\, da_1\,\delta a_2)$$
$$= \phi c(\delta a_2\, a_0\, da_1)$$
$$= \phi([\delta a_2 a_0, a_1])$$
$$= \phi(\delta a_2 a_0 a_1 - a_1\delta a_2 a_0)$$
$$\cancel{= \phi c(a_0 d(a_1)\delta a_2) - a_0a_1\, d\delta a_2)}$$
$$\cancel{= \phi([a_0, a_1\delta a_2]) - \phi([a_0a_1, \delta a_2])}$$

$$= \phi\{\delta(a_2a_0a_1) - a_2\delta(a_0a_1) - a_1\delta(a_2a_0) + a_1a_2\delta a_0\}$$
$$= \underbrace{\phi_1(1,a_2a_0a_1) - \phi_1(a_2,a_0a_1)}_{(\kappa\phi_1)(a_0a_1,a_2)} - \phi_1(a_1,a_2a_0) + \phi_1(a_1a_2,a_0)$$
$$\phi_1(1,a_1a_2a_0) - \phi_1(1,a_1a_2a_0)$$
$$+ (\kappa\phi_1)(a_2a_0,a_1) - (\kappa\phi_1)(a_0,a_1a_2)$$

$$\boxed{\therefore (\phi c)_2 = b\kappa\phi_1} \qquad \boxed{(\phi c)_1 = (1-\kappa)\phi_1}$$

I/ Formulas

$$\begin{array}{ccccc} & & T \in (\Omega^1 A_{\mathcal{I}})^* & & \\ \xleftarrow{d^t} & \psi \in \Gamma^* & \xleftarrow{c^t} & \phi \in (\Omega^1 A)^* \end{array}$$

$$\boxed{\begin{aligned}(\psi d)_1 &= \psi_1 + s\psi_2 \\ (\psi \delta)_1 &= \psi_1 \cancel{\pm} \kappa s \psi_2\end{aligned}}$$

$$\boxed{\begin{aligned}(\phi c)_1 &= (1-\kappa)\phi_1 \\ (\phi c)_2 &= b\kappa\phi_1\end{aligned}}$$

$$\begin{aligned}(\phi c d)_1 &= (\phi c)_1 + s(\phi c)_2 \\ &= (1-\kappa)\phi_1 + sb\kappa\phi_1 = (bs + sb + sb\kappa)\phi_1 \\ &= (bB + Bb)\phi_1 = 0\end{aligned}$$

$$a_0\, d\delta a_1 \overset{d}{\longmapsto} a_0\, d\delta a_1 + da_0\,\delta a_1 \overset{c}{\longmapsto} \underbrace{[a_0, \delta a_1] + [\delta a_1, a_0]}_{=0}$$

$$\begin{aligned}(\psi d c)_1 &= (1-\kappa)(\psi d)_1 = (1-\kappa)(\psi_1 + s\psi_2) \\ &= (bs+sb)(\psi_1) + s\underbrace{bs}_{(1-\kappa)}\psi_2 \\ &= bB\psi_1 \cancel{\pm} \cancel{s(1-\kappa)\psi_2} + \cancel{s(1-\kappa)\psi_2} \\ &= -Bb\psi_1 = -B(1-\kappa)\psi_2 = 0.\end{aligned}$$

$$\begin{aligned}(1-\kappa)\psi_1 + s\underbrace{(1-\kappa)\psi_2}_{-b\psi_1} &= (bs \cancel{+sb})\psi_1 = bB\psi_1 \\ &= -Bb\psi_1 = B(1-\kappa)\psi_2 = 0\end{aligned}$$

J)

$$(\psi dc)_2 = bK(\psi d)_1$$

$$= bK(\psi_1 + s\psi_2)$$

$$= K(K-1)\psi_2 + \underbrace{bs}_{1-K}{}^{K}\psi_2 = 0$$

$$\boxed{b\psi_1 + (1-K)\psi_2 = 0}$$

$$\cancel{\psi(\delta a_2\, a_0\, da_1) \overset{?}{=} \psi(a_0 da_1 \delta}$$

$$\psi(da_2\, a_0\, \delta a_1) = \psi(a_0\, \delta a_1\, da_2)$$

$$\psi(d(a_2 a_0)\, \delta a_1) - \psi(a_2\, da_0\, \delta a_1) \qquad \psi(-a_0\, da_1\, \delta a_2 - a_0\, a_1\, d\delta a_2 + a_2\, d\delta(a_1 a_2) - a_0\, a_1\, d\delta a_2)$$

$$\left(\psi_2(a_2, a_0, a_1) + \psi_2(t, a_2 a_0, a_1)\right) - (K\psi_2)(a_0, a_1, a_2) \qquad (-b\psi_1 - \psi_2)(a_0\, da_1\, \delta a_2)$$

$\therefore$ the rule is $\quad b\psi_1 + \psi_2 = K\psi_2$

$$\boxed{b\psi_1 + (1-K)\psi_2 = 0}$$

$$b(1-K) + (1-K)bK = 0.$$

K)

$$\begin{tikzcd}
 & \tau_0 \arrow[l] & T_1 \arrow[l, "s=B"'] & \tau_0 \arrow[l, "b"'] & \arrow[l, "s=B"'] \\
 & \phi_1 \arrow[l] \arrow[u, "s=B"] & \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \arrow[l, "(1\ \frac{1}{2}B)"'] \arrow[u, "(1-\frac{1}{2}B)"] & \phi_1 \arrow[l, "\begin{pmatrix}\frac{1}{2}bB\\ b\end{pmatrix}"'] \arrow[u, "B"] & \arrow[l]
\end{tikzcd}$$

$$b\psi_1 + (1-K)\psi_2 = 0 \qquad \text{Apply } P$$

$$b P\psi_1 + 0 = 0$$

$$b P\psi_2 = 0.$$

$$\underline{P(1-K) =} \ bB + Psb = bB + \tfrac{1}{2}Bb = \tfrac{1}{2}bB \text{ on } \Omega'$$

$$\underline{1-K + KsbK}$$

Take
$$\begin{tikzcd}
& -B\psi_2 \\
0 & \begin{pmatrix}-\frac{1}{2}B\psi_2\\ \psi_2\end{pmatrix} \arrow[l] \arrow[u]
\end{tikzcd}$$

$$\psi(\delta a_2\, a_0\, da_1) = \psi(a_0\, da_1, \delta a_2)$$

$$\psi(\delta(a_2 a_0)\, da_1) - \psi(a_2\, \delta a_0\, da_1) \qquad -\psi(a_0\, \delta a_1, da_2)$$

$$-\psi(a_0\, \delta d a_1, a_2)$$

$$+\psi(a_0\, \delta d(a_1 a_2))$$

$$-(K\psi_2)(a_0, a_1, a_2) \qquad -\psi(a_0 a_1, \delta d a_2)$$

$$= -\psi_2 - b\psi_1$$

K)

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[d, "\delta"] & \Omega^1 A_\natural \arrow[r, "\subset"] \arrow[d, "\delta"] & A \arrow[d, "\delta"] \\
\Omega^1 A \arrow[r] & \Gamma \arrow[r, "\subset"] & \Omega^1 A
\end{tikzcd}$$

$$\begin{tikzcd}
{} & \tau_0 \arrow[l] & T_1 \arrow[l, "s"'] & \tau_0 \arrow[l, "b"'] & {} \arrow[l, "s"'] \\
{} & \phi_1 \arrow[l] \arrow[u, "s"] & \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \arrow[l, "(1\ \ s)"'] \arrow[u, "(1\ -\cancel{K}s)"] & \phi_1 \arrow[l, "\begin{pmatrix}1-K\\ bK\end{pmatrix}"'] \arrow[u, "S"] & {} \arrow[l, "(1\ \ s)"']
\end{tikzcd}$$

$$1-K+\underbrace{\lambda s b K}_{\substack{K^2 sb\,\cancel{K}\\ \shortparallel\\ 1}} = bs$$

$$\begin{pmatrix}1-K\\ bK\end{pmatrix}\begin{pmatrix}1 & s\end{pmatrix} = \begin{pmatrix}1-K & (1-K)s\\ bK & bKs\end{pmatrix}$$

$$bsT_1 = bBT_1 = -BbT_1 = 0.$$

$$\cancel{(1-K)(\psi_1+s\psi_2)}$$
$$bK(\psi_1+s\psi_2) = \cancel{Kb\psi_1 + \underbrace{bs}_{1-K-sb}K\psi_2}$$

Now that I have the maps straight what about the homotopy. Invariant part.