Lectures courses by Daniel G Quillen

A. Topics in K-Theory and Cyclic Cohomology, Michaelmas Term 1989

69 pages of notes. The lecture course is concerned with the fundamental construction of cyclic cohomology, and covers the following topics. Ideals in a free algebra and cyclic cohomology of R/I. Cuntz's proof of the exact sequences. Operations on cochains. The doubly periodic complex and cyclic cochain complex. The bar construction. Curvature of a one cochain. Bianchi's identity. Characterising traces on RA. Cyclic cohomology of A as cohomology groups. Hochschild cohomology and it meaning in low dimensions. Connes definition of τ The bimodule of differential forms over A. Universal derivations. Traces on RA. Constructing the complex and double complex. Properties of RA, QA and ΩA. Superalgebras. $*$ product. Fredholm modules and QA. Karoubi's operators. Normalised Hochschild cochains. Applications to Fredholm modules. Homology for Fredholm modules and supertraces on Q. Inner superalgebras. Connes–Cuntz formula.

B. Topics in K-theory and cyclic homology, Hilary Term 1989

68 pages of notes. The lecture notes are concerned with index theory and Fredholm modules for operator families over manifolds. The topics include: cyclic homology classes. Currents on manifolds. Cochains with the b and λ operations. The fundamental class. The double complex for b and b'. The cup product on cochains. Differential graded algebras. Traces and almost homomorphisms. Bianchi's identity. Index theory on the circle: Toeplitz operators. Index of Fredholm operators. Lefschetz formula. Cyclic 1-cocycles. Abstract GNS construction. Generalized Stinespring theorem. The GNS algebra. Algebraic analogue of GNS. Cuntz algebra from free products; the superalgebra envelope. Fredholm modules. Gradings. DGA and Ω_A. The Fedosov product. Supertraces and derivations. The β and d double complex. Connections and characteristic classes. The de Rham class. The bar construction with Hochschild differential b. Connes bicomplex. Connes's long exact sequence. Bar construction and Connes S operator. Chain and cochain versions of Connes's bicomplex. Connes cyclic bicomplex. Homotopy and total differentials. Vector valued traces, vector bundles over manifolds and connections. Closed currents. Duhamel's principle. Superconnections. Graded and ungraded bundles. The index theorem via Getzler calculus.

C. Topics in Cyclic Cohomology and K-theory, Trinity Term 1992

125 pages of notes. The lecture course is concerned with the analogy between de Rham (co)homology for commutative algebras over $\mathbf{C}$ and cyclic homology theory for noncommutative algebras, and considers the following topics. Smooth algebraic varieties; noncommutative analogue of smoothness. Quasi fee algebras. Resolutions of b and b'; Hochschild homology. Laplace operator and spectral resolution of Karoubi operator. Connes's B operator. Connes's exact sequence. Connes–Tsygan double complex. Periodic cyclic homology. Negative cyclic homology. Hodge filtration. Models for cyclic theory. The Fedosov product. Harmonic decomposition. Cartan homotopy formula for complex. Connes notion of a connection. Quasi free extensions of algebras. Goodwillie's theorem. Curvature and Yang–Mills connections. The torsion of the connection. The exponential map.

D. Cyclic Homology II: Cyclic cohomology and Karoubi Operators, Hilary Term 1991

125 pages of notes. The lecture course is concerned with cyclic homology and traces and considers the following topics. The differential graded algebra of noncommutative differential forms. The Karoubi operator and the analogue of Hodge theory. Connes B operator, and the Greens operator. The Hodge decomposition. Augmented algebras. Morita equivalence of algebras. Noncommutative harmonic forms. Hochschild homology and cyclic homology. The double complex and cyclic homology. Spectral sequences. Connes Tsygan bicomplex. Connes exact sequence. Reduced Hochschild homology. Universal properties of tensor algebra and free algebra. The Fedosov product. Cuntz's algebra. Filtrations with respect to ideals and products. Traces on RA. Bianchi's identities. Characterisations of traces. Karoubi's operator on cochains. Cohomology formulas for cochains. From $(IA)^n$-adic traces to odd cyclic cohomology. Intermission: the analogue of the de Rham complex in noncommutative geometry. The Lefschetz, Atiyah–Hodge and Grothendieck theorem on nonsingular maximal ideal spaces. The smooth algebra is defined via the lifting process for nilpotent extensions. Periodic cyclic homology, homology of smooth and commutative algebras. Quasi free algebras and lifting. Analogue of Zariski–Grothendieck. Universal differential algebra for RA; passage to linear functionals. The complex $X(RA)^*$. The noncommutative analogues of nonsingular varieties. Connes' connections, and Chern character classes. Splitting of connection sequence. Connections on $\Omega^1 R$. Fedosov's construction. Poisson structures on manifolds. Weyl algebras and commutative algebras. Index theorems on $\mathbf{R}^n$. Fedosov product and the Stone–von Neumann relations.

Editor's remark. The lecture notes were taken during lectures at the Mathematical Institute on St Giles in Oxford. There have been subsequent corrections, by whitening out writing errors. The pages are numbered, but there is no general numbering system for theorems and definitions. For the most part, the results are in consecutive order, although in one course the lecturer interrupted the flow to present a self-contained lecture on a topic to be developed further in the subsequent lecture course. The note taker did not record dates of lectures, so it is likely that some lectures were missed in the sequence. The courses typically start with common material, then branch out into particular topics. Quillen seldom provided any references during lectures, and the lecture presentation seems simpler than some of the material in the papers.

- D. Quillen, Cyclic cohomology and algebra extensions, *K-Theory* **3**, 205–246.
- D. Quillen, Algebra cochains and cyclic cohomology, *Inst. Hautes Etudes Sci. Publ. Math.* **68** (1988), 139–174.
- J. Cuntz and D. Quillen, Cyclic homology and nonsingularity, *J. Amer. Math. Soc.* **8** (1995), 373–442.

Commonly used notation

k a field, usually of characteristic zero, often the complex numbers

A an associative unital algebra over k, possibly noncommutative

$\bar{A} = A/k$ the algebra reduced by the subspace of multiples of the identity

$\Omega^n A = A \otimes (\bar{A} \otimes \ldots \otimes \bar{A})$

$\omega = a_0 da_1 \ldots da_n$ an element of $\Omega^n A$

$\Omega A = \oplus_{n=0}^{\infty} \Omega^n A$ the universal algebra of abstract differential forms

e an idempotent in A

d the formal differential (on bar complex or tensor algebra)

b Hochschild differential

b', B differentials in the sense of Connes's noncommutative differential geometry

λ a cyclic permutation operator

K the Karoubi operator

$\circ$ the Fedosov product

G the Greens function of abstract Hodge theory

N averaging operator

P the projection in abstract Hodge theory

D an abstract Dirac operator

∇ a connection

I an ideal in A

V vector space

M manifold

E vector bundle over manifold

τ a trace

$T(A) = \oplus_{n=0}^{\infty} A^{\otimes n}$ the universal tensor algebra over A

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