# Commutative Algebra for Singular Algebraic Varieties

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## Worksheet 3: Integral (and finite) extensions of rings

### Integral extensions: the basics

Let $f : A \to B$ be a ring homomorphism. An element $b \in A$ is said to be integral over $A$ if there is a monic polinomial with coeficients in $A$, $p(x) = x^n + a_1 x^{n-1} + \ldots + a_0 \in A[x]$, such that $p(b) = 0$ in $B$. The ring $B$ is said to be *integral over* $A$, if all elements in $B$ are integral over $A$. The ring $B$ is an *integral extension of* $A$ if $B$ is integral over $A$ and $f$ is injective.

**2.** Let $k$ be a field. Show that

$$k[x] \to k[x,y]/\langle x^2 - y^3 \rangle$$

is an integral extension. How may points are in the fiber over $\langle x - 1 \rangle$? How many points does the generic fiber have?

**3.** Let $I \subset A$ be an ideal. Show that $A \to A/I$ is integral. Observe that in this case the map is not injective. In the lectures we will mostly consider integral extensions, namely those where $A \to B$ is an integral and injective.

**4.** Let $I \subset A$ be an ideal, and suppose that $B$ is integral over $A$. Show that $B/IB$ is integral over $A/I$.

**5.** Let $S \subset A$ be a multiplicative set. If $B$ is integral over $A$ then $B_S = B \otimes_A S$ is integral over $A_S$.

### Integral closures

**6.** Show that set of elements of $B$ that are integral over $A$ form a subring. We refer to this subring as the *integral closure of* $A$ *in* $B$ and it is often denoted by $\overline{A}$. If $A = \overline{A}$ then $A$ is said to be *integrally closed in* $B$. When $A$ is a domain, that is integrally closed in its field of fractions, then it is said that $A$ is *normal*. Regular rings are normal.

### Integral extension vs. finite extensions

An $A$-algebra $C$ is said to be *finitely generated over* $A$ if it is finitely generated as an $A$-module.

**7.** If $b \in B$ is integral over $A$, show that $A \to A[b]$ is a finite extension.

**8.** Using the *Determinantal trick* it can be shown that if an $A$-algebra $B$ is a finitely generated $A$-module, then $B$ is an integral extension of $A$. However, not all integral extensions are finite. Show that

$$\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \ldots, \sqrt[p]{2}, \ldots)$$

is an integral extension that is not finite.

### Integral extensions of domains

**9.** Assume that $A \to B$ is an integral extension of domains. Show that $A$ is a field if and only if $B$ is a field.

**10.** Observe that $\mathbb{Q} \to \mathbb{Q}/\langle x^2 - 5x + 6 \rangle$ is an integral extension but $\mathbb{Q}/\langle x^2 - 5x + 6 \rangle$ is not a field.

**Integral extensions and Krull dimension**

Let $A \to B$ be an integral extension.

**11.** Let $n \subset B$ be a maximal ideal. Show that $n \cap A$ is a maximal ideal. *Hint: $n \cap A$ is a prime $\mathfrak{p}$. Now prove that $A/\mathfrak{p} \to B/nB$ is an integral extension of domains.*

**12.** Let $m \subset A$ be a maximal ideal. Show that there is a maximal ideal $n \subset B$ such that $n \cap A = m$. *Hint: The extension $A_m \to B \otimes_A A_m$ is integral. Now use exercise 11.*

**13.** Let $\mathfrak{q}_1 \subset q_2 \subset B$ be prime ideals, and let $\mathfrak{p}_i = \mathfrak{q}_i \cap A$, $i = 1, 2$. Show that $\mathfrak{q}_1 = q_2$ if and only if $\mathfrak{p}_1 = \mathfrak{p}_2$. *Hint: Use that $A_{\mathfrak{p}_1} \to B \otimes_A \mathfrak{p}_1$ is integral plus exercises (11) and (12).*

**14.** Deduce from the previous exercises that $\dim_{\mathrm{Krull}} B \leq \dim_{\mathrm{Krull}} A$.

**15.** *Going-down.* Let $\mathfrak{p}_1 \subset p_2 \subset A$ be prime ideals, and let $\mathfrak{q}_1 \subset B$ be a prime ideal with $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$. Show that there is a prime ideal $\mathfrak{q}_2 \subset B$ contaning $\mathfrak{q}_1$ and such that $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$. *Hint: Use that $A/\mathfrak{p}_1 \to B/\mathfrak{p}_1 B$ is integral.*

**16.** Conclude that if $A \to B$ is an integral extension, then $\dim_{\mathrm{Krull}} A = \dim_{\mathrm{Krull}} B$.

**17.** Observe that the map
$$\mathrm{spec}(B) \to \mathrm{spec}(A)$$
is surjective.

**Finite extensions: infinitesimal structure of fibers**

**A. Finite extensions of fields**

**18.** Let $K$ be a field, and let $K \to L$ be a finite extension. Show that $L$ is semi-local, and can be expressed as a direct sum of local rings (see the Proposition in worksheet 1). If $L$ is reduced, show that $L$ is isomorphic to a finite direct sum of fields. *Hint: What is the Krull dimension of $L$?*

**19.** Observe that $K \to K[x]/\langle x^2(x+1)^3$ is finite but not reduced. In this case
$$K[x]/\langle x^2(x+1)^3 \simeq K[x]/\langle x^2\rangle \times K[x]/\langle (x+1)^3\rangle$$
is a direct sum of (non-reduced) local rings.

**B. Finite extensions of rings**

Assume $A \to B$ is a finite extension.

**20.** Let $\mathfrak{p} \subset A$ be a prime, and let $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.
i) Prove that $B \otimes_A k(\mathfrak{p})$ is semilocal, and isomorphic to a finite sum of local rings.
ii) Prove that $B \otimes_A A_{\mathfrak{p}}$ is semilocal, and there is a natural bijection of maximal ideals in $B \otimes_A A_{\mathfrak{p}}$ with maximal ideals in $B \otimes_A k(\mathfrak{p})$.

**21.** Conclude that the induced map
$$\mathrm{spec}(B) \to \mathrm{spec}(A)$$
has finite fibres.

### Discussion

Let $A \to B$ be a finite extension, let $\mathfrak{p} \subset A$ be a prime with residue field $k(\mathfrak{p})$, and let $q_1 \ldots, \mathfrak{q}_s \subset B$ be the primes in $B$ dominating $\mathfrak{p}$, with residue fields $k(\mathfrak{q}_i)$ for $i = 1, \ldots, s$. Consider the diagram:

$$\begin{array}{ccccc} B & \longrightarrow & B \otimes_A A_{\mathfrak{p}} & \longrightarrow & B \otimes_A K(\mathfrak{p}) \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & A_{\mathfrak{p}} & \longrightarrow & K(\mathfrak{p}) \end{array}$$

Observe that

$$B \otimes_A K(\mathfrak{p}) \simeq (B \otimes_A K(\mathfrak{p}))_{\mathfrak{q}_1} \oplus \ldots \oplus (B \otimes_A K(\mathfrak{p}))_{\mathfrak{q}_s},$$

where each summand is either a field or a non-reduced ring. Also notice that for $i =, \ldots, s$,

$$(B \otimes_A K(\mathfrak{p}))_{\mathfrak{q}_i} = B_{\mathfrak{q}_i}/\mathfrak{p}B_{\mathfrak{q}_i}.$$

Therefore, the summand $(B \otimes_A K(\mathfrak{p}))_{\mathfrak{q}_i}$ is a field if and only if

xtendido (0.0.1) $$\mathfrak{p}B_{\mathfrak{q}_i} = q_i B_{\mathfrak{q}_i};$$

otherwise, i.e., if

$$\mathfrak{p}B_{\mathfrak{q}_i} \subset q_i B_{\mathfrak{q}_i}$$

is an strict inclusion, the term $(B \otimes_A K(\mathfrak{p}))_{\mathfrak{q}_i}$ is non-reduced. It turns out that (0.0.1) is the interesting situation (see worksheet 4).

**22.** Consider $A = \mathbb{R}[X] \subset B = \mathbb{R}[X, Y]/\langle X^2 + Y^2 - 1\rangle$.

(i) Compute the fiber over $\langle 0 \rangle \subset \mathbb{R}[X]$ and write it as a direct sum of local rings. Do the same at $\langle x - 1 \rangle$. What differences do you find?

(ii) Compute the fiber at generic point of $\mathbb{R}[X]$ and write it as a direct sum of local rings. What happens if we consider the fiber over an algebraic closure of $\mathbb{R}(X)$ by base change?

**27.** Observe that the morphism

$$K[x] \to K[x, y]/\langle xy - 1\rangle$$

is not finite, but the corresponding map

$$\text{spec}\,(K[x, y]/\langle xy - 1\rangle) \to \text{spec}K[x]$$

has finite fibers. However, observe that a suitable localization of $K[x, y]/\langle xy - 1\rangle$ is finite over $K[x]$. This is an exmaple of the so called Zariski's Main Theorem.

### Zariski's Main Theorem

**Quasi-finite morphisms.** A morphism $A \to B$ is said to be *quasi-finite* if spec$(B) \to$ spec$(A)$ has finite fibres. Finite extensions are quasi-finite, but the converse is not true (but almost!).

> **Theorem.** *If $f : A \to B$ is quasi-finite, then $f$ factorizes through some ring $C$, $A \to C \to B$ such that $A \to C$ is finite and $C \to B$ is an open immersion; i.e., $f$ is a finite extension $A \to C$ followed by a localization at some element $c \in C$.*

### Noether's Normalization Lemma

**28.** Let $k$ be a field, let $A = k[x, y, z]$, let $B = k[x, y, z, t]/\langle xy - tz\rangle$. Observe that the natural inclusion $A \to B$ induces the projection

$$\text{spec}(B) \to \text{spec}(A)$$

in the first three coordinates.

(a) Show that $A \to B$ is not a finite extension. *Hint: What is the fiber over* $(0, 0, 0)$*?*

(b) Consider the change of variables:

$$x' = x + t; y' = y + t; z' = z - 2t; t' = t.$$

This change of variables induces an isomorphism on $k[x, y, z, t]$. What is the image of $\langle xy - tz \rangle$ by this change of variables? Denote by $I$ this new ideal.

(c) Show that $k[x', y', z'] \to k[x', y', z, t']/I$ is a finite extension.

(d) Give a geometric interpretation of (b) and (c).

(e) Is the change of variables in (b) unique with the property stated in (d)?

**Noether's Normalization Lemma.** *Let $k$ be a field and let $B$ be a $k$-algebra of finite type. Then there there are algebraic independent elements $y_1, \ldots, y_r \in B$ so that*

$$k[y_1, \ldots, y_r] \subset B$$

*is a finite extension.*

**29.** Observe that the Lemma says that any affine $d$-dimensional algebraic variety can be seen as a finite cover of an affine $d$-dimensional $k$-space $\mathbb{A}^d_k$.

**30.** Although the Lemma holds for an arbitrary field $k$, here we will give a proof when $k$ is infinite.

(a) Since $B$ is a $k$-algebra of finite type, there are elements $b_1, \ldots, b_s \in B$ so that $B = k[b_1, \ldots, b_s]$. Given variables $x_1, \ldots, x_s$ there is a $k$-algebra surjection

$$\begin{array}{rccc} \varphi : & k[x_1, \ldots, x_s] & \longrightarrow & B \\ & x_i & \longmapsto & b_i. \end{array}$$

(b) Suppose first that $\ker(\varphi) = (0)$. What can you say about $B$?

If $\ker(\varphi) \neq (0)$, we will prove the Lemma by induction on the number of generators $b_1, \ldots, b_s$.

(c) Let $p(x_1, \ldots, x_s) \in \ker(\varphi)$ be a non-zero polynomial. Show that there are elements $\alpha_1, \ldots, \alpha_{s-1} \in k$ so that after the change of variables

$$x_1' = x_1 + \alpha_1 x_s;\ x_2' = x_2 + \alpha_2 x_s;\ \ldots, x_{s-1}' = x_1 + \alpha_{s-1} x_s$$

the expression

$$p(x_1', x_2', \ldots, x_{s-1}', x_s')$$

contains a term of the form $\alpha \cdot x^l$ for some non-zero $\alpha \in k$.

(d) Set $b_i' = b_i + \alpha_i b_i$. Deduce from (c) that $k[b_1', \ldots, b_{s-1}'] \subset B$ is a finite extension.

(e) Conclude the proof of the Lemma using an inductive argument.

### An example of an integral extension with non-finite fibers

**31.** Let $\mathbb{G}$ be the integral closure of $\mathbb{Z}$ in $\mathbb{C}$. Observe that $\mathbb{Z} \to \mathbb{G}$ is integral. Hence, for any prime $\mathfrak{p} \in \mathbb{Z}$, the map

$$\mathbb{F}_{\mathfrak{p}} \to \mathbb{G} \otimes_{\mathbb{Z}} \mathbb{F}_{\mathfrak{p}}$$

is integral. Note that any irreducible (monic) polynomial over $\mathbb{F}_{\mathfrak{p}}$ can be lifted to an irreducible (monic) polinomial $p(x) \in \mathbb{Z}[x]$, and that $\mathbb{Z} \to \mathbb{Z}[x]/\langle p(x) \rangle \subset \mathbb{G}$. There are irreducible polinomials of all degrees over $\mathbb{F}_{\mathfrak{p}}$, so in particular, for any prime $\mathfrak{q} \in \mathbb{Z}$ there is a degree $q$ extension over $\mathbb{F}_{\mathfrak{p}}$. So there are infinitely many distinct fields $\mathbb{F}_{\mathfrak{p}^q}$ (with no relation of containment) containing $\mathbb{F}_{\mathfrak{p}}$. Thus the fiber

$$\mathbb{F}_{\mathfrak{p}} \to \mathbb{G} \otimes_{\mathbb{Z}} \mathbb{F}_{\mathfrak{p}}$$

has infinitely many points.