

# CMI Summerschool 2012 / Course Schicho

## Definitions

Let  $K$  be a field of characteristic zero.

**Habitat.** A habitat is a nonsingular variety  $W$  together with a finite sequence of divisors  $(E_1, \dots, E_n)$ , such that for each point  $p \in W$ , there is a system  $u$  of regular parameters of the local ring of  $W$  at  $p$  such that for  $i = 1, \dots, n$ , either  $E_i$  does not pass through  $p$  or there is a parameter in  $u$  which serves as a local equation of  $E_i$ ; in addition, we want that these local equations are pairwise distinct (or equivalently, the  $E_i$  are not allowed to have common components). For any habitat, we define an operator  $\Delta$  from the set of ideals on  $W$  to itself, as follows. For  $I \subset K[W]$ ,  $\Delta(I)$  is the ideal generated by  $I$  and all first order partial derivatives of elements in  $I$ .

**Straight subvariety.** A subvariety  $Z$  of a habitat as above is called straight iff it is pure-dimensional, and for every point  $p \in Z$ , there is a system  $u$  of regular parameters as above which additionally satisfies the condition that  $Z$  is locally defined by a subset of  $u$ .

**Blowup.** Let  $Z$  be a straight subvariety of a habitat  $(W, (E_1, \dots, E_n))$ . The blowup along  $Z$  is the habitat  $(W', (E'_1, \dots, E'_n, E_{n+1}))$ , where  $W'$  is the blowup of  $W$  along  $Z$ ,  $E'_i$  is the strict transform of  $E_i$  for  $i = 1, \dots, n$ , and  $E_{n+1}$  is the exceptional divisor.

**Singularity.** A singularity on a habitat as above is a finitely generated Rees algebra  $A = \bigoplus_{i=0}^{\infty} A_i$  over  $A_0 = K[W]$ , i.e. sequence of ideals  $A_i \subset K[W]$  such that  $A_0 = K[W]$  and  $A_i \cdot A_j \subseteq A_{i+j}$  and equality holds for sufficiently large indices  $i, j$ .

We say a singularity is of ideal-type iff there is a  $b > 0$  and ideal  $I$  such that  $A_{nb} = I^n$  for all indices which are multiples of  $b$ , and  $A_i = (0)$  otherwise. These singularities are denoted by  $(I, b)$ .

**Singular Locus.** The singular locus  $\text{Sing}(A)$  of a singularity  $A = \bigoplus_{i=0}^{\infty} A_i$  is the intersection of the zero sets of  $\Delta^i(A_i)$ ,  $i > 0$ .

**Transform.** Let  $Z$  be a straight subvariety in the singular set of  $A$  as above. The transformed singularity on the blowup is  $\bigoplus_{i=0}^{\infty} A'_i$ , where  $A'_i$  is such that  $f^*(A_i) = \text{Ideal}(E_{n+1})^i \cdot A'_i$  for  $i > 0$ .

**Resolution.** A resolution of a singularity is a sequence of singularity-habitat pairs, where the next is the transform of the previous under blowup of a straight

subvariety in the singular locus, such that the last singularity has empty singular locus.

**Monomial Singularity.** A monomial singularity is an ideal-type singularity  $(I, b)$  such that there exist integers  $a_1, \dots, a_n$  such that  $I = \prod_{i=1}^n \text{Ideal}(E_i)^{a_i}$ .

**Equivalence.** Two singularities  $A, B$  are equivalent –  $A \approx B$  – iff their singular loci coincide and, for any sequence of blowups in the center in the singular locus, the singular loci of the transforms of  $A$  and  $B$  coincide.

**Differential Closure.** A singularity  $A = \bigoplus_{i=0}^{\infty} A_i$  is closed iff  $\Delta(A_{i+1}) \subseteq A_i$  for all  $i > 0$ .

The differential closure of a singularity  $A$  is the smallest closed singularity containing  $A$ .