CLAY SUMMER SCHOOL ON RESOLUTION, OBERGURGL 2012

EXERCISES HERWIG HAUSER, THIRD WEEK

Monday, June 18th

76.○ According to the statement and the inductive proof of our main resolution theorem for ideals of polynomials in zero characteristic, compute the respective blowups given by the algorithm. First determine the factorization $J = M \cdot I$ with M the exceptional principal monomial ideal. Don't consider all points of the exceptional divisor; instead, choose just one distinguished point, e.g., the origin of an affine chart. Compute there again the local invariant and the exceptional locus in order to determine, locally at the chosen point, the next center of blowup (it should contain the chosen point). Neglect the other points of the exceptional divisor.

- $W = A^2$, $E = \emptyset$, $J = I = (x^2y^3)$.
- $W = A^2$, $E = V(x)$, $J = (x^2y^3)$.
- $W = A^2$, $E = V(xy)$, $J = (x^2y^3)$.
- $W = A^3$, $E = V(x + z^2)$, $J = (x^2y^3)$.
- $W = A^3$, $E = V(y + z)$, $J = (x^3 + (y + z)z^2)$.
- $W = A^3$, $E = \emptyset$, $J = I = (x^2 + yz)$.
- $W = A^3$, $E = \emptyset$, $J = I = (x^3 + y^2z^2)$.
- $W = A^3$, $E = \emptyset$, $J = I = (x^2 + xy^2 + y^5)$.
- $W = A^3$, $E = \emptyset$, $J = I = (x^2 + xy^3 + y^5)$.
- $W = A^3$, $E = V(x + y^2)$, $J = (x^3 + (x + y^2)y^4z^5)$.
- $W = A^3$, $E = \emptyset$, $J = I = (x^2 + y^2z^2(y + z))$.
- $W = A^3$, $E = V(yz)$, $J = (x^2 + y^2z^2(y + z))$.
- $W = A^4$, $E = V(yz)$, $J = (x^2 + y^2z^2(y + w))$.

77.○ Check in 76. to what extent you used characteristic zero.

78.a□ Consider $I = (x^2 + y^7 + yz^4)$ in $\mathbb{A}^3$, with $E = \emptyset$. Consider the sequence of three point blowups with the following centers. First the origin of $\mathbb{A}^3$, then the origin of the y-chart, then the origin of the z-chart. On the last blowup, consider the midpoint between the origin of the y- and the z-chart. Show that I is resolved at this point if the characteristic is zero or > 2.

78.b△ What happens in 78. if the characteristic is equal to 2? Describe in detail the behaviour of the first two components of the resolution invariant under these three local blowups.

79.○ Determine for all characteristics the maximal order of the coefficient ideal of $I = (x^3 + 5y^3 + 3(x^2y^2 + xy^4) + y^6 + 7y^7 + y^9 + y^{10})$ in smooth local hypersurfaces V at 0.

80.a○ Find the defining ideal for the parametrized curve $(t^{32}, t^7, t^{19}, t^{15})$ in $\mathbb{A}^4$. 80.b□ What is the local embedding dimension at 0?

Date: June 25, 2012.

Tuesday, June 19th

81.a○ Consider $f = x^2 + y^3z^3 + y^7 + z^7$. What would be your preferred center of blowup? Why? 81.b○ Show that $V : x = 0$ is a local hypersurface of weak maximal contact, i.e., V maximizes the order of the corresponding coefficient ideal J_-. 81.c○ How does the order of J_- behave under the chosen blowup at the points where the order of f has remained constant? Was the choice of center correct? Compute the transform of the factorization $J_- = M_- \cdot I_-$.

82.○ Consider $f = x^4 + y^4 + z^6$, $g = x^4 + y^4 + z^{10}$ and $h = xy + z^{10}$ under point blowup. Determine the points where the order of f has remained constant. *Hint*: Take care of the characteristic.

83.a○ Show that $f = x^2 + yz^3 + zw^3 + y^7w$ has in characteristic 2 top locus $T(f)$ equal to the parametrized curve $(t^{32}, t^7, t^{19}, t^{15})$ in $\mathbb{A}^4$.

83.b□ Show that f is not contained in the square of the ideal defining the parametrized curve $(t^{32}, t^7, t^{19}, t^{15})$. *Remark*: This signifies that the order of f along this curve has to be really computed in the localization of $K[x, y, z, w]$ with respect to the ideal of the curve. This phenomenon is related to the difference between powers of ideals and symbolic powers of ideals, see Zariski-Samuel.

84.□ Show that $f = x^2 + yz^3 + zw^3 + y^7w$ admits in characteristic 2 at the point 0 no local smooth hypersurface of permanent maximal contact (i.e., whose successive strict transforms contain all points where the order of f has remained constant in any sequence of blowups with smooth centers inside the top locus).

85.□ Consider $f = x^2 + y^7 + yz^4$ in characteristic 2. Show that f admits at the point 0 no local smooth hypersurface whose transform under the sequence of blowups of 78. has weak maximal contact with the transform of f at the midpoint chosen after the third blowup.

86.a○ Let c be an integer ≥ 2, and $[a, b] \subset \mathbb{N}$ an interval. Determine the number of points in $[a, b] \cap c \cdot \mathbb{N}$ according to the residues $\overline{a}$ and $\overline{b}$ of a and b modulo c.

86.b□ For $r, s, k \in \mathbb{N}$, let Δ be the subset of $\mathbb{N}^2$ defined by $\Delta = ((r, s) + \mathbb{N}^2) \cap \{(a, b) \in \mathbb{N}^2,\ a + b \leq k\}$. For $c \geq 2$, evaluate the cardinality of $\Delta \cap c \cdot \mathbb{N}^2$ according to the values of r, s, k.

87.□ Define the pth order derivative of polynomials in $K[x_1, \dots, x_n]$ for K a field of characteristic p.

Wednesday, June 20th

88.a○ Consider $f = x^3 + y^2z$ in $W = \mathbb{A}^3$ at the origin, with exceptional locus $E = V(yz)$, over a field of characteristic 3. Show that under the point blowup the monomiality of the coefficient ideal (y^2z) in $V : x = 0$ is not preserved. *Hint*: You will have to look at all points of the exceptional divisor, not just the origins of the affine charts.

88.b○ Show that this phenomenon is not a serious obstruction to make the order of f drop.

88.c□ Characterize all $f = x^p + y^az^b$ with $E = V(yz)$ in characteristic p which lose monomiality under point blowup.

88.d□ Do this in any number of variables and with f of order p^e.

89.○ Consider surfaces of the form $f = x^c + y^a z^b \cdot g(y,z)$ where $y^a z^b$ is considered as exceptional monomial. Assume that $a + b + \mathrm{ord}_0\, g \geq p$. Give three examples where the order of g at 0 is not maximal, and indicate the coordinate change which makes it maximal.

90.a○ Consider surfaces of the form $f = x^c + y^a z^b \cdot g(y,z)$ where $y^a z^b$ is considered as exceptional monomial. Assume that $a + b + \mathrm{ord}_0\, g \geq p$. Compute the strict = weak transform $f' = x^c + y^{a'} z^{b'} \cdot g'(y,z)$ of f under point blowup at points where the order of f has remained equal to p.

90.b□ Find three examples where the order of g' is not maximal over all local coordinate choices.

91.□ Let $G(x)$ be a polynomial in one variable over a field K of characteristic p, of degree d and order k at 0. Let $t \in K$, and consider the equivalence class $\overline{G}$ of $K(x+t)$ in $K[x]/K[x^p]$ (i.e., consider $K(x+t)$ modulo pth power polynomials. What is the maximal order of $\overline{G}$ at 0. Describe all examples where this maximum is achieved.

92.a○ Let $(W', a') \to (W, a)$ be a sequence of local blowups in smooth centers such that a' lies in the intersection of n exceptional components where n s the dimension of W at a. Let $f \in \mathcal{O}_{W,a}$ and assume that the characteristic is zero. Show that the order of f must have dropped between a and a'.

92.b□ Show the same in positive characteristic.

93.a□ Let $f = x^c + g(y_1, \ldots, y_m) \in K[[x, y_1, \ldots, y_m]]$ with $\mathrm{ord}_0\, g \geq c$. Show that there exists a formal coordinate change maximizing the order of g.

93.b□ Let now $f = x^c + g(y_1, \ldots, y_m) \in K[x, y_1, \ldots, y_m]$ with $\mathrm{ord}_0\, g \geq c$. Does there exist a local coordinate change maximizing the order of g? *Remark*: A local coordinate change is an automorphism of $K[x,y]_{(x,y)}$.

94.○ Let f be a polynomial in n variables $x_1, \ldots, x_n$ of order c at 0. Assume that the ground field is infinite. Show that there exists a linear coordinate change α so that $\alpha(f)(0, \ldots, x_n)$ is a polynomial of order c at 0. *Remark*: Polynomials f with $f(0, \ldots, x_n) \neq 0$ of order c at 0 are called *x_n-regular of order c* at 0. This is the classical assumption for the Weierstrass preparation theorem for convergent or formal power series.

Thursday, June 21th

95.□ Construct a surface of order p^5 for which the residual order increases under blowup.

96.○ Show that in $f = x^p + y^p z$ with $E = \emptyset$ the residual order along the (closed) points of the z-axis is not equal to its value at the generic point.

97.□ Prove the geometric description via lattice points of simplices of the arithmetic inequality for the residues of the exceptional multiplicities in the kangaroo theorem.

98.□ Let $y_1, \ldots, y_m$ be fixed coordinates, and consider a homogeneous polynomial $G(y) = y^r \cdot g(y)$ with $r \in \mathbb{N}^m$ and $g(y)$ homogeneous of degree k. Let $G^+(y)$ be the polynomial obtained from G by the linear coordinate change $y_i \to y_i + y_m$ for $i = 1, \ldots, m-1$. Show that the order of G^+ along the y_m-axis is at most k.

99.$^\square$ Express the statement of 98. through the invertibility of a matrix of multinomial coefficients.

Friday, June 22th

100.$^\square$ Prove the resolution of plane curves in arbitary characteristic by using the order and the residual order as the resolution invariants.

101.$^\square$ Let $(W', a') \to (W, a)$ be the composition of two monomial point blowups of $W = \mathbb{A}^2$ with respect to coordinates y, z. The first is the blowup of $\mathbb{A}^2$ with center 0 considered at the origin of the y-chart, the second has as center the origin of the y-chart and is considered at the origin of the z-chart. Show that the order of the weak transform $g'(y, z)$ at a' of any non zero polynomial $g(y, z)$ in W is at most the half of the order of $g(y, z)$ at a.

102.$^\bigcirc$ Consider $G(y, z) = y^r z^s \sum_{i=0}^k \binom{k+r}{i+r} y^i (tz - y)^{k-i}$. Compute for $t \in K^*$ the polynomial $G^+(y, z) = G(y + tz, z)$ and its order with respect to y modulo pth power polynomials.

103.$^\square$ Determine all homogeneous polynomials $G(y, z) = y^r z^s g(y, z)$ so that $G^+(y, z)$ has order $k + 1$ with respect to y modulo pth power polynomials.

104.$^\triangle$ Find a new proof for the embedded resolution of surfaces in three-space.