CLAY SUMMER SCHOOL ON RESOLUTION, OBERGURGL 2012

EXERCISES HERWIG HAUSER, SECOND WEEK

Monday, June 11th

39.○ Compute the blowup of the curve $x^2 = y$ in the origin.

40.○ Compute the following blowups:

- $\mathbb{A}^2$ in the center $(x, y)(x, y^2)$.
- $\mathbb{A}^3$ in the center (x, yz), respectively $(x, yz)(x, y)(x, z)$.
- $\mathbb{A}^3$ in the center $(x^2 + y^2 - 1, z)$.

Use both affine charts and coordinate rings to show that the resulting varieties are smooth or singular.

41.□ Interpret the blowup of $\mathbb{A}^3$ in $(x, yz)(x, y)(x, z)$ as a composition of blowups in smooth centers.

42.○ Show that the blowup of $\mathbb{A}^n$ along a coordinate subspace Z equals the cartesian product of the point-blowup in a transversal subspace V of $\mathbb{A}^n$ of complementary dimension (with respect to Z) with the identity map on Z.

43.□ Show that the ideals $(x_1, \ldots, x_n)$ and $(x_1, \ldots, x_n)^m$ define the same blowup of $\mathbb{A}^n$ when taken as center. *Hint*: Think before you start to compute, or, better, don't compute.

44.○ Let E be a normal crossings subvariety of $\mathbb{A}^n$ and let Z be a subvariety of $\mathbb{A}^n$ such that $E \cup Z$ also has normal crossings (the union being defined by the union of ideals). Show that the inverse image of E under the blowup of $\mathbb{A}^n$ along Z is again normal crossings.

45.○ Show by an example that the assumption on Z in exercise 44 cannot be dropped in general.

46.○ Compute the strict transform of the ideal $X = V(x^2 - y^3, xy - z^3) \subseteq \mathbb{A}^3$ under the blowup of the origin.

47.○ Determine the total transform, weak transform and strict transform of $X = V(x^2 - y^3, z^3) \subseteq \mathbb{A}^3$ under the blowup of the origin. Point out their geometric differences.

48.□ Blow up $\mathbb{A}^3$ along the curve $y^2 - x^3 + x = z = 0$ and compute the strict transform of the lines $x = z = 0$ and $y = z = 0$.

49.□ Show that any line through 0 inside the cone $X = V(x^2 + y^2 - z^2) \subseteq \mathbb{A}^3$ is not a Cartier divisor of X. Compute the blowup of the cone along any such line.

Date: June 16, 2012.

Tuesday, June 12th

49.$^\square$ (mandatory) Let Z be a smooth center in $\mathbb{A}^n$, $\pi : \widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ the induced blowup and a' a point of $\widetilde{\mathbb{A}}^n$ mapping to a point $a \in Z$. Show that it is possible to choose local formal coordinates at a so that the center is a coordinate subspace, and a' the origin of one of the affine charts. Is this also possible with local algebraic coordinates, i.e., with an RPS of $\mathcal{O}_{\mathbb{A}^n,a}$?

50.a$^\square$ (mandatory) Let $X = V(f)$ be a hypersurface in $\mathbb{A}^n$ and Z a regular closed subvariety which is contained in the top locus T of X (T is the locus of points where f attains its maximal order). Let $\pi : \widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ be the blowup along Z and let $X' = V(f')$ be the strict transform of X. Show that for points $a \in Z$ and $a' \in E = \pi^{-1}(Z)$ with $\pi(a') = a$ the inequality $\mathrm{ord}_{a'} f' \le \mathrm{ord}_a f$ holds. 50.b$^\square$ Do the same for arbitrary ideals.

51.$^\square$ Find an example of a variety X for which the dimension of the singular locus increases under blowup of a closed smooth center Z that is contained in the top locus of X.

52.$^\bigcirc$ Determine the stratification by order of the following varieties. If the smallest stratum is smooth, blow it up. Produce pictures of X and $\widetilde{X}$ and describe the geometric changes.

- Cross: $xyz = 0$.
- Whitney umbrella: $x^2 - yz^2 = 0$.
- Kolibri: $x^3 + x^2z^2 - y^2 = 0$.
- Xano: $x^4 + z^3 - yz^2 = 0$.
- Cusp & Plane: $(y^2 - x^3)z = 0$.

53.a$^\bigcirc$ Consider for a given RPS $x, y_1, \dots, y_n$ on $\mathbb{A}^n$ a polynomial of order c at 0,

$$g(x, y_1, \dots, y_n) = x^c + \sum_{i=0}^{c-1} g_i(y)x^i.$$

Express the order condition and the top locus in terms of the orders of the coefficients g_i.

53.b$^\bigcirc$ Let $\widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ be the blowup of $\mathbb{A}^n$ in 0 and let a' be the origin of the y_1-chart of $\widetilde{\mathbb{A}}^n$. Compute the transforms g^* and $g' := g^{st} = g^\curlyvee$ of g.

53.c$^\bigcirc$ Let V be the hypersurface $x = 0$ of $\mathbb{A}^n$ and let $V' \to V$ be the induced blowup. Assume that $\mathrm{ord}_{a'} g' = \mathrm{ord}_a g$ at the origin a' of the y_1-chart. Compare the total transform of the ideal $J_V(g)$ of $K[V]$ generated by the powers $g_i^{c!/c-i}$ with the respective ideal $J_{V'}(g')$.

54.a$^\bigcirc$. Let $\pi : (\widetilde{\mathbb{A}}^n, a') \to (\mathbb{A}^n, a)$ be the local blowup of $\mathbb{A}^n$ with center a, considered at a point $a' \in E$. Let be given an RPS $x_1, \dots, x_n$ at a. Determine the coordinate changes in $(\mathbb{A}^n, a)$ which make the chart expression of π monomial.

54.b$^\square$ Determine the formal automorphisms of $(\widetilde{\mathbb{A}}^n, a')$ which commute with π.

54.c$^\square$ Let $x_1, \dots, x_n$ be local coordinates at a so that the local blowup is monomial with respect to them. Which formal automorphisms of $(\widetilde{\mathbb{A}}^n, a')$ preserve this monomiality?

55.$^\triangle$ In the situation of exercise 53.c, show that the maximum of the order of $J_V(g)$ over all RPS of $\widehat{\mathcal{O}}_{\mathbb{A}^n,0}$ is attained (it might be ∞). *Hint*: Artin approximation theorem.

56.$^\triangle$ Show that this maximum, when taken at any point of the stratum T of $\mathbb{A}^n$ where g has order c, defines an upper semicontinuous function on T.

Wednesday, June 13th

57.□ Can you do exercise 49 in presence of a normal crossings divisor E to which Z is transversal (i.e., $E \cup Z$ defined by the product of ideals has normal crossings)?

58.□ Define the coefficient ideal $J_V(g)$ of a polynomial g in V from 53.c with respect to any RPS of $\mathcal{O}_{W,a}$ without passing to the completion by taking derivatives and working only in the local rings.

59.○ Determine in all characteristics the points of $\widetilde{\mathbb{A}}^2$ where the strict transform of $g = x^4 + kx^2y^2 + y^4 + 3y^7 + 5y^8 + 7y^9$ under the blowup of $\mathbb{A}^2$ at 0 has order 4, for any $k \in \mathbb{N}$.

60.○ Same as 59 for $g = xy + y^4 + 3y^7 + 5y^8 + 7y^9$.

61.○ Compute the coefficient ideal of $g = x^5 + y^4x^2 + y^k$ in $x = 0$, respectively $y = 0$, according to the value of $k \geq 5$.

62.□ Assume that, for a given RPS $x_1, \ldots, x_n$ in $\mathcal{O}_{W,a}$, the coefficient ideal of a polynomial g as in 53.c in V defined by $x_n = 0$ is a principal monomial ideal. Show that there is a sequence of monomial blowups (i.e., taking only coordinate subspaces as centers and considering always the origins of the affine charts) which eventually makes the order of g drop.

Thursday, June 14th

63.○ Compute in the following situations the coefficient ideal of I in W at a with respect to the given formal RPS x, y, z and the hypersurface V. Determine in each case whether the order of the coefficient ideal is maximal. If not, find a coordinate change which maximizes it.

- $a = 0 \in \mathbb{A}^1$, x, $V : x = 0$, $I = (x)$ and $I = (x + x^2)$.
- $a = 0 \in \mathbb{A}^2$, x, y, $V : x = 0$, $I = (x)$, $I = (x + y^2)$, $I = (y + x^2)$, $I = (xy)$.
- $a = (1, 0, 0) \in \mathbb{A}^3$, x, y, z, $V : y + z = 0$, $I = (x^2)$, $I = (xy)$, $I = (x^3 + z^3)$.
- $a = 0 \in \mathbb{A}^3$, x, y, z, $V : x = 0$, $I = (xyz)$, $I = (x^2 + y^3 + z^5)$.
- $a = 0 \in \mathbb{A}^2$, x, y, $V : x = 0$, $I = (x^2 + y^4, y^4 + x^2)$.

64.○ (Characteristic 0) Blow up in each of the above examples the origin and determine the points of the exceptional divisor Y' where the order of the weak transform $I^\curlyvee$ of I has remained constant. Verify at these points the commutativity of the descent to the coefficient ideal with the blowup when taking the controlled transform of the coefficient ideal.

65.□ Show that, in characteristic zero, the locus $T(I)$ of maximal order of an ideal I of W, taken locally at a point a, is contained in a local smooth hypersurface V through a, and that this hypersurface maximizes the order of the coefficient ideal.

66.□ Show that the maximum order of the coefficient ideal of an ideal I in W at a point a can be realized in an RPS of the local ring $\mathcal{O}_{W,a}$ without passing to the completion.

67.○ Let I_1 and I_2 be ideals of $\mathbb{A}^n$ of order c_1 and c_2 at 0. Take the blowup of $\mathbb{A}^n$ at zero. Show that the weak transform of $I_1^{c_2} + I_2^{c_1}$ is the sum of the weak transforms of $I_1^{c_2}$ and $I_2^{c_1}$.

68.○ Let V be a hypersurface and $x_n, \ldots, x_1$ a formal RPS with $V : x_n = 0$ maximizing the order of the coefficient ideal of an ideal I in W at a point a. Take a blowup with smooth center contained in the top locus $T(I)$ of I. Show that the strict transform $V' = V^{st}$ of V contains all points of Y' where the order of I has remained constant.

Friday, June 15th

69.○ Start the resolution process for the three surfaces x^2+y^2z, $x^2+y^3+z^5$ and $x^3+y^4z^5+z^{11}$.

70.a□ Prove with all details the embedded resolution of plane curves in characteristic zero, using the induction presented in the class. 70.b△ Then try the case of positive characteristic.

71.○ Prove in all generality in characteristic zero the commutation of the descent to the coefficient ideal with the transform under blowup at the points where the order of the original ideal has remained constant. *Hint*: Work with formal coordinates so that the local blowup is given by a monomial substitution of the variables.

72.△ Find a threefold X in positive characteristic for which the top locus $T(X)$ cannot be embedded locally in a smooth hypersurface. *Remark*: A first example of this phenomenon was given by Narasimhan, a student of Abhyankar.

73.a□ Find the two gaps in the proof of the resolution theorem for ideals in characteristic zero. *Hint*: Investigate the transversality of the chosen center Z_- in V with respect to E, and also the inclusion of $Z_- \subset T(I_-)$ in $T(I) = T(J_-, c)$ (which is required to make the induction work).

73.b△ Fill the gaps in the proof by modifying suitably the ideal I before applying the descent to the coefficient ideal J_-. *Hint*: Look at the definition of the transversality and companion ideals in the paper with Santiago Encinas.

74.□ Let X be a surface in three-space, and T its top locus. Assume T is singular at a, and let X' be the blowup of X in a. Determine the top locus of X'. *Hint*: You may want to look at Zariski's 1944 Annals paper on the reduction of three-folds.

75.△ Find a surface X in positive characteristic and a sequence of point blowups starting at $a \in X$ so that some of the points above a where the order of the weak transforms of X remains constant eventually leave the transforms of any local smooth hypersurface passing through a. *Hint*: You may use 72. or cook up your own example.