CLAY SUMMER SCHOOL ON RESOLUTION, OBERGURGL 2012

EXERCISES HERWIG HAUSER, FIRST WEEK

Monday, June 4th

1.$^\bigcirc$ Show that X defined in $\mathbb{A}^3$ by $27x^2y^3z^2+(x^2+z^2-y^3)^3=0$ is the image of the cartesian product of the cusp $C: x^3-y^2=0$ with itself.

2.$^\bigcirc$ Find a parametrization of $Y=C\times C$ and X.

3.$^\bigcirc$ Blow up X and Y in the origin, describe exactly the geometry of the transforms X' and Y' and produce realistic visualizations.

4.$^\bigcirc$ Blowup X' and Y' in the respective origins.

5.$^\square$ How could the resolution of a cartesian product of two singular varieties be related to or expressed through the resolutions of the two factors?

6.$^\bigcirc$ Let Z be the variety in $\mathbb{A}^3_\mathbb{C}$ given by the equation $(x^2-y^3)^2=(z^2-y^2)^3$. Show that the map $\alpha:\mathbb{A}^3\to\mathbb{A}^3:(x,y,z)\mapsto(\omega^2z^3,\omega yz^2,\omega z^2)$, where $\omega(x,y)=x(y^2-1)+y$, resolves the singularities of Z. What is the inverse image $\alpha^{-1}(Z)$? Produce instructive pictures.

Tuesday, June 5th

1.a$^\bigcirc$ Compare the algebraic varieties X with the property $(X,a)\cong(\mathbb{A}^d,0)$ for all $a\in X$, where $\cong$ stands for biregularly isomorphic Zariski-germs, with those where the isomorphism is just formal. Varieties with the first property are called *plain*. 1.b$^\square$ It can be shown that any smooth and rational surface (i.e., smooth and birationally isomorphic to affine space) is plain [BHSV]. Show directly that X defined in $\mathbb{A}^3$ by $x-(x^2+z^2)y=0$ is plain by exhibiting a local isomorphism at 0 with $\mathbb{A}^2$. 1.c** Is there an algorithm to construct such a local isomorphism for any smooth and rational surface?

2.$^\bigcirc$ Let X be an algebraic variety and $a\in X$ be a point. What is the difference between the concept of regular system of parameters in $\mathcal{O}_{X,a}$ and $\widehat{\mathcal{O}}_{X,a}$?

3.$^\bigcirc$ Revise the Jacobian criterion for smoothness (this is important for understanding the case of characteristic p).

4.$^\bigcirc$ Let X be an algebraic variety. Show that the set of smooth points on X is Zariski open (or revisit in the literature).

5.$^\bigcirc$ Let X be the node in $\mathbb{A}^2$ given by the equation $x^2-y^2-x^3=0$. Show that X has (analytic) normal crossings at the origin but not algebraic normal crossings. Express this in particular through regular parameter systems of $\mathbb{A}^2$ at 0.

Date: June 8, 2012.

6.$^{\bigcirc}$ Express the condition of having algebraic normal crossings at a given point in terms of a regular system of parameters of the local ring of the ambient space.

7.$^{\triangle}$ Formulate and prove the theorem of local analytic triviality in positive characteristic. Compare with Theorem 1 of [?] for the precise statement.

8.$^{\Box}$ Prove that the non-normal crossings locus is closed in X. What about the algebraic non-normal crossings locus?

9.$^{\triangle}$ Let a be a point in X. Can we express how far away X is at a from normal crossings? Find a local invariant that measures the "distance" to normal crossings.

10.$^{\bigcirc}$ Compute the blowup of the Whitney umbrella $X = V(x^2 - y^2 z) \subseteq \mathbb{A}^3$ with center 0.

11.$^{\Box}$ Often, one just wants to blow up in non-normal crossings points, i.e., along centers inside the non normal crossings locus. In example 10 we have seen that this requirement does not lead to a resolution in general. Revise the requirement for the Whitney umbrella in order not to get stuck. *Hint*: Think of algebraic normal crossings.

Wednesday, June 6th

12.$^{\bigcirc}$ Find interesting stratifications for three surfaces and three three-folds.

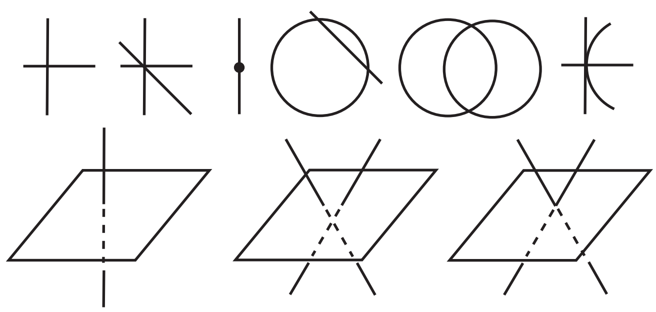
13.$^{\bigcirc}$ Show that any local upper semicontinuous invariant on an algebraic variety induces a stratification.

14.a$^{\bigcirc}$ Show that the iterated singular loci of a variety provide a stratification. 14.b$^{\bigcirc}$ Give an example with four different types of strata. 14.c$^{\Box}$ Associate to a stratification a so called Hasse diagram, i.e., a directed graph, where the nodes correspond to strata and the edges to the adjacency of strata (two strata are called adjacent if one lies in the closure of the other).

15.a$^{\bigcirc}$ Show that the order of a hypersurface, the dimension and the Hilbert-Samuel function of a variety, the embedding-dimension of a variety and the number of irreducible components are local invariants, i.e., invariant under local formal isomorphisms, and determine whether they are upper or lower semicontinuous. 15.b$^{\Box}$ How do these invariants behave under localization?

Thursday, June 7th

16.$^{\bigcirc}$ Determine if the following varieties have normal crossings.



17.○ Do the varieties defined by the following equations have normal crossings (at the origin)? Note that the question can be considered over different fields!
a. $x^2 + y^2 = 0$
b. $x^2 - y^2 = 0$
c. $x^2 + y^2 + z^2 = 0$
d. $x^2 + y^2 + z^2 + w^2 = 0$

18.○ The same question as in 4. for the following varieties:
a. $xy(x - y) = 0$
b. $xy(x^2 - y) = 0$
c. $(x - y)z(z - x) = 0$

19.○ Draw a picture of $(x - y^2)(x - z)z = 0$.

20.○ Blow up $\mathbb{A}^2$ in 0 and compute the inverse image of $x^2 + y^2 = 0$, $xy = 0$ and $x(x - y^2) = 0$ and also their strict transform.

21.○ Blow up $\mathbb{A}^3$ in 0 and compute the inverse image of $x^2 + y^2 = z^2$.

22.○ Blow up $\mathbb{A}^2$ in the point $(0, 1)$. What is the inverse image of the lines $x + y = 0$ and $x + y = 1$?

23.○ Compute the chart transposition map for the blowup of $\mathbb{A}^3$ in the z-axis.

24.□ Blow up $\mathrm{Spec}(\mathbb{Z}[x])$ in the ideal (p, x). What do you observe in contrast to the blowup of $\mathrm{Spec}(K[x, y])$ in the ideal $\langle x, y\rangle$?

25.○ Blow up the cone $x^2 + y^2 = z^2$ in one of its lines.

26.○ Blow up $\mathbb{A}^3$ in the circle $x^2 + (y + 2)^2 = 1$ and in the elliptic curve $y^2 = x^3 - x$.

Challenges, Friday, June 6th

The following research problems are for the more advanced students, having already a good knowledge of blowups. Some of the problems have no complete or satisfactory answer yet. Occasionally, the statements have first to be made precise. Try to give very conceptual and systematic answers.

27.△ [FW] Consider the blowup $\widetilde{\mathbb{A}}^n$ of $\mathbb{A}^n$ in a monomial ideal I. Show that $\widetilde{\mathbb{A}}^n$ may be singular. What types of singularities will occur? Find a natural saturation procedure $I \rightsquigarrow \overline{I}$ so that the blowup of $\mathbb{A}^n$ in $\overline{I}$ is smooth and equal to a (natural) resolution of $\widetilde{\mathbb{A}}^n$.

28.△ [Le, Sp] Let N be an integral convex polyhedron in $\mathbb{R}^n_+$, i.e., the positive convex hull $N = \mathrm{conv}(S + \mathbb{R}^n_+)$ of a finite set S of points in $\mathbb{N}^n$. Player A chooses a subset J of $\{1, \ldots, n\}$, then player B chooses an element $j \in J$. After these moves, S is replaced by the set S' of points α defined by $\alpha'_i = \alpha_i$ if $i \neq j$ and $\alpha'_j = \sum_{k\in J} \alpha_k$, giving rise to a new polyhedron N'. Player A has won if after finitely many rounds the polyhedron has become an orthant $\alpha + \mathbb{R}^n_+$. Player B can never win, but only prevent player A from winning.

a. Show that player A has a winning strategy.

b. If you used induction on the dimension n, try to eliminate this argument from your proof.

29.$^\triangle$ The Nash modification of a variety is the closure of the graph of the map which associates to each smooth point its tangent space, taken as an element of a suitable Grassmanian. Refine this construction by allowing quadratic approximations of the variety at its smooth points.

30.$^\triangle$ [Ha1] Let I be an ideal in $K[x_1, \ldots, x_n]$ and denote for $a \in \mathbb{A}^n$ by I_a the respective ideal in the completion of the local ring of $\mathbb{A}^n$ at a. Choose a regular system of parameters of $\widehat{O}_{\mathbb{A}^n,a}$. For a monomial order $<_\varepsilon$ on $\mathbb{N}^n$ denote by $\mathrm{in}_a(I) = \mathrm{in}(I_a)$ the initial ideal of I_a with respect to $<_\varepsilon$ in the chosen RPS, and by $\mathrm{gin}_a(I)$ the generic initial ideal.

a. Define an ordering on the set of monomial ideals so that $\mathrm{gin}_a(I)$ is the minimal initial ideal $\min_a(I) = \min_x\{\mathrm{in}_a(I)\}$ over all choices of RPS.

b. Show that $\min_a(I)$ is upper semicontinuous when the point a varies.

c. Compare the induced stratification of $\mathbb{A}^n$ with the stratification by the Hilbert-Samuel function of I.

d. Let Z be a smooth center inside a stratum of the $\mathrm{in}(I)$ stratification, and consider the induced blowup $\widetilde{\mathbb{A}}^n \to \mathbb{A}^n$. Show that $\min_a(I)$ does not increase for all points a of Z.

31.$^\triangle$ [Ha2] Let K be an algebraically closed field of characteristic $p > 0$. Develop a significant notion of resolution for elements of the quotient of rings $K[x,y]/K[x^p, y^p]$. Then prove that such a resolution always exists.

References

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