WORKSHOP ON SHRINKING TARGET PROPERTIES

J. S. ATHREYA, A. GHOSH AND D. KLEINBOCK

1. Scientific Goals

Let (X, μ) be a probability space, and G be a group acting on X by measure-preserving transformations. Let $\mathcal{A}$ be a family of measurable subsets of X. A natural question arising from a variety of arithmetic and geometric situations, is to understand the set (or the measure of the set) of points $x \in X$ whose orbits intersect a family of sets from $\mathcal{A}$ in a prescribed manner. In particular, many applications involve families of sets which are in some appropriate sense, "shrinking". This phenomenon at its broadest, is what is referred to as the STP of the G action. To be more precise, let $\{g_n\}_{n\in\mathbb{N}} \subset G$ be a sequence of group elements.

Definition. *We say that $\mathcal{A}$ satisfies the STP for the sequence $\{g_n\}_{n=0}^{\infty}$ if for any sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$*

$$\mu(\{x \in X : g_n x \in A_n \text{ infinitely often}\}) = \begin{cases} 1 & \sum_{n=0}^{\infty} \mu(A_n) = \infty \\ 0 & \text{otherwise} \end{cases}$$

In recent years, there has been a flurry of activity in the study of STP's, and their relationships to geometry and number theory. Most of this activity falls into one of two schools:

1.1. **Algebraic group actions.** The influential paper of D. Y. Kleinbock and G. A. Margulis [16] provides very general conditions for a family of neighborhoods of a cusp in a finite-volume symmetric space to satisfy the shrinking target property with respect to a sequence of Cartan elements in a (real) Lie group. Using this, they obtained a generalization of D. Sullivan's [20] logarithm law for geodesic excursions into the cusp of hyperbolic manifolds. Another important feature of this paper was the further advancement of the unified treatment (as initiated by Sullivan) of two seemingly disparate subjects-the aforementioned logarithm laws, and the Khintchine-Groshev theorem in Diophantine approximation. Indeed, both these subjects are manifestations of the shrinking target property in different contexts and Kleinbock-Margulis used the mixing property of semisimple actions to approach them. Moreover, the dynamical approach leads to stronger results in the Diophantine setting, than can be obtained using classical tools. For instance, this paper of Kleinbock-Margulis also resolves a conjecture of Skriganov regarding multiplicative Diophantine approximation.

Inspired by this paper, J. S. Athreya, A. Ghosh and A. Prasad, in joint work-in-progress [1], have extended its results to the action of semi-simple algebraic groups over local fields of positive characteristic, proving analogous diophantine approximation result for function fields, and deriving logarithm laws for (quotients of) buildings. These logarithm laws generalize the work of Hersonsky-Paulin [13] on logarithm laws on trees to the context of more general ultrametric spaces.

Other results and work-in-progress in this direction comes from the work of J. S. Athreya and G. A. Margulis [2], in which they consider excursions into the cusp of unipotent flows on the space of lattices. This work focuses on geometric methods to explicitly obtain lower bounds on the depth of excursions, rather than using representation theory to obtain more general Borel-Cantelli laws as in [16, 1].

It is very likely that there will be further applications of this fertile circle of ideas. Here we mention two beginning projects with wide applications. A natural question is to replace the "point at infinity" in the aforementioned quotients with a point inside a given geometrical object (say, a hyperbolic manifold) and a sequence of shrinking neighborhoods around this point. Such questions have been studied in special cases by several authors, most recently by F.Maucourant [19] who considers it for quotients of hyperbolic manifolds. A natural question would be to obtain generalizations to symmetric spaces, in fact such a generalization would have several important number theoretic implications.

Another class of interesting Diophantine problems which lend themselves to this framework involve Diophantine approximation on varieties and manifolds. Using quantitative nondivergence results for unipotent trajectories, D.Kleinbock and G.Margulis have proved several long standing conjectures in the theory of Diophantine approximation on manifolds [15, 14]. A natural variation on this question however, is much less understood. We consider a real algebraic variety which has an abundance of rational points, and ask for a characterization of points on this variety which are well approximated by these rational points. It turns out that this problem has a natural STP interpretation and can be approached using dynamical methods.

Such problems have attracted some attention recently, being at the interface of dynamics, number theory and algebraic and arithmetic geometry; see for instance the recent paper of C.Drutu [7] on the one hand and the papers of Gorodnik, Maucourant and Oh [11] and Gorodnik, Oh and Shah [12] who use dynamical methods to resolve difficult and long-standing problem on the distribution of rational points on varieties. The above questions are being discussed by A.Ghosh, A.Gorodnik, D.Kleinbock and N.Shah in various subsets and share the common theme of being very interesting arithmetic manifestations of the shrinking target property.

1.2. **Rotations, Interval Exchange Maps, and Teichmuller flows:** The above mentioned work relates closely to general shrinking target properties for *hyperbolic* dynamical systems have been studied by N. Chernov and D. Y. Kleinbock [5] and D. Dolgopyat [6], and there are many fascinating questions remaining in this less algebraic, more dynamical context. In common with the work of Kleinbock-Margulis [16], these results rely strongly on rapid (i.e., exponential) mixing (or, equivalently, exponential decay of matrix coefficients) of the associated dynamical system.

Even in the case of unipotent flows and actions, some partial results can be obtained from the polynomial rate of mixing of the flows (and decay of matrix coefficients of the representations). One might then think that (some sort of) mixing is a prerequisite for the existence of shrinking target properties.

That this is not the case is seen, by the results of J.Kurzweil [17] and more recently, B. Fayad [8] and J. Tseng [21], on irrational rotations of the circle and special flows over such rotations, which, while ergodic, are not even weak-mixing. A fascinating property of their results is the close relation of the diophantine properties and continued fraction expansion of the rotation angle and the type of shrinking target properties one can obtain.

A very natural generalization of circle rotations are interval exchange transformations (i.e.t.'s), and an immediate question is what kind of shrinking target properties an i.e.t. satisfies. Just as the diophantine properties and continued fraction expansion of the rotation angle play a large role in the dynamics in the case of circle rotations, one expects the diophantine properties of the i.e.t. to play a major role.

In fact, this relates closely to the subject of dynamics and flows on Teichmuller spaces: the continued fraction expansion of a number can be obtained by studying the recurrence properties of an associated geodesic on the modular surface, and similarly, important ergodic and diophantine properties of an i.e.t. can be gleaned from the properties of the associated Teichmuller geodesic, and in particular, of the *Kontsevich-Zorich cocycle* (see, for example, [9] for a good introduction) along this geodesic trajectory. Some conjectures in this direction have been made by J. S. Athreya and C. Ulcigrai, in discussions at the Clay Summer School in Pisa. There is some initial work in the study of shrinking target properties for i.e.t.'s in the work of Galotolo-Kim [10]

Finally, Teichmuller dynamics themselves are a rich subject for the study of shrinking target properties. H. Masur [18] proved a logarithm law for Teichmuller geodesic flow on the moduli space of quadratic differentials, and J. .S. Athreya and Y. Minsky [3] prove a similar result for Teichmuller horocycle flow on strata of quadratic differentials. These results, plus the recent proof of exponential mixing of the Teichmuller geodesic flow by A. Avila, S. Gouezel, and J.-C. Yoccoz [4], raise many questions about what other shrinking target properties one can obtain for these flows.

1.3. **Future directions.** As the number of works-in-progress mentioned above indicate, there are many fascinating questions in this field. Similarly, the number of different (and predominantly young) people pursuing these questions, and the variety of applications, suggest their centrality and vitality. The purpose of this workshop is to bring these researchers together for a weekend, so that they can exchange and develop new ideas, and new directions for further research.

In particular, here are some specific questions/conjectures we hope to discuss over the course of the weekend, in addition to all the above-mentioned work-in-progress:

(1) What shrinking target properties can we prove for i.e.t.'s? What is the precise relationship between them and the integrability properties of the Kontsevich-Zorich cocycle?
(2) What shrinking target properties can we prove for the actions of products of linear algebraic groups over local fields (i.e., S-algebraic groups)?
(3) What is the excursion behavior into cusps of stable/unstable manifolds for various (partially) hyperbolic systems? In particular, for the action of horospherical subgroups on G/Γ, and spaces of measured foliations in Teichmuller spaces.
(4) In many of the above results, the shrinking targets are neighborhoods of infinity, i.e., contained in the cusp of G/Γ or moduli space. What shrinking target properties can be proved for sets which are contained in the compact part of these spaces?

(5) A meta-question: in all of the above results, either the base dynamical system or an associated *renormalization* system exhibits hyperbolic behavior (and in particular, rapid mixing). Does this always need to be the case for a dynamical system to have shrinking target properties?

References

[1] J. S. Athreya, A. Ghosh, and A. Prasad, in preparation.
[2] J. S. Athreya and G. A. Margulis, *Logarithm laws for horocyclic flows*, preprint.
[3] J. S. Athreya and Y. Minsky, in preparation.
[4] A. Avila, S. Gouezel, J. C. Yoccoz, *Exponential mixing for the Teichmller flow*, Publ. Math. Inst. Hautes tudes Sci. No. 104 (2006), 143–211.
[5] N. Chernov, D. Y. Kleinbock, *Dynamical Borel-Cantelli lemmas for Gibbs measures*, Israel J. Math. 122 (2001), 1–27.
[6] D. Dolgopyat, *Limit theorems for partially hyperbolic systems*, Trans. Amer. Math. Soc. 356 (2004), no. 4, 1637–1689 (electronic).
[7] C. Drutu, *Diophantine approximation on rational quadrics*, Math. Ann. 333 (2005), no. 2, 405–469.
[8] B. Fayad, *Mixing in the absence of the shrinking target property*, Bull. London Math. Soc. 38 (2006), no. 5, 829–838.
[9] G. Forni, *On the Lyapunov exponents of the Kontsevich-Zorich cocycle*, Handbook of dynamical systems. Vol. 1B, 549–580, Elsevier B. V., Amsterdam, 2006.
[10] S. Galotolo and D. Kim, *The dynamical Borel-Cantelli lemma and the waiting time problems*, preprint.
[11] A. Gorodnik, F. Maucourant and H. Oh, *Manin's conjecture on rational points of bounded height and adelic mixing*, preprint.
[12] A. Gorodnik, H. Oh and N. Shah *Integral points on symmetric varieties and Satake compactifications*, preprint.
[13] S. Hersonsky and F. Paulin, *A logarithm law for automorphism groups of trees*, Arch. Math. (Basel) 88 (2007), no. 2, 97–108.
[14] D. Kleinbock, *Extremal subspaces and their submanifolds*, Geom. Funct. Anal. 13 (2003), no. 2, 437–466.
[15] D. Kleinbock and G. A. Margulis, *Flows on homogeneous spaces and Diophantine approximation on manifolds*, Ann. of Math. (2) 148 (1998), no. 1, 339–360.
[16] D. Y. Kleinbock and G. A. Margulis, *Logarithm laws for flows on homogeneous spaces*, Invent. Math. 138 (1999), no. 3, 451–494.
[17] J. Kurzweil, *On the metric theory of inhomogeneous Diophantine approximations*, Studia Math. **15** (1955), 84–112.
[18] H. Masur, *Logarithmic law for geodesics in moduli space*, Mapping class groups and moduli spaces of Riemann surfaces (Gttingen, 1991/Seattle, WA, 1991), 229–245, Contemp. Math., 150, Amer. Math. Soc., Providence, RI, 1993.
[19] F. Maucourant, *Dynamical Borel-Cantelli lemma for hyperbolic spaces*, Israel J. Math. 152 (2006), 143–155.
[20] D. Sullivan, *Disjoint spheres, approximation by quadratic numbers and the logarithm law for geodesics*, Acta Mathematica 149 (1982), 215–237.
[21] J. Tseng, *On circle rotations and the shrinking target properties*, preprint.