FJRW ring and Mirror Symmetry for Singularities

Marc Krawitz

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Heuristic Overview

Associated to a non-degenerate, quasi-homogeneous polynomial $W\colon \mathbb{C}^N \to \mathbb{C}$ we have:

- Landau-Ginzburg B-Model $\mathscr{Q}_W$ (Milnor ring).
- Landau-Ginzburg A-Model $\mathscr{H}_{W,G}$ [FJR].
  G is an appropriate symmetry group.

- Self Duality for simple (A, D, E) singularities [FJR]:

$$\mathscr{H}_{W,G} \cong \mathscr{Q}_W$$

with some anomaly for D_{odd}.

- Anomaly is resolved by considering duality $W \leftrightarrow W^T$ among singularities so that

$$\mathscr{H}_{W,G} \cong \mathscr{Q}_{W^T}$$

Landau-Ginzburg B-model

For $W\colon \mathbb{C}^N \to \mathbb{C}$, the LG B-model of W is

$$\mathcal{Q}_W := \frac{\mathbb{C}[X_1, \ldots, X_N]}{\langle \partial_1 W, \ldots, \partial_N W \rangle}$$

FJRW A-model

Defined for $W\colon \mathbb{C}^N \to \mathbb{C}$ a *quasi-homogeneous*, *non-degenerate* polynomial having isolated singularity at the origin.
This means:

- There are charges (or weights) $q_1, \ldots, q_N$ so that for $\lambda \in \mathbb{C}$ we have
$$W(\lambda^{q_1} X_1, \ldots \lambda^{q_N} X_N) = \lambda W(X_1, \ldots X_N).$$
- The charges are uniquely determined by this condition. So, for example, $W = xy$ is not non-degenerate.

Fan-Jarvis-Ruan produce

$\overline{\mathscr{M}_{g,k,W}}$ = the moduli-space of genus g, k-pointed, orbifold curves endowed with a W-structure.

- W-structure on an orbicurve is a collection of line-bundles $\mathscr{L}_1, \dots, \mathscr{L}_N$ so that for any monomial $\prod X_i^{\alpha_i}$ of W, $\mathscr{L}_1^{\alpha_1} \otimes \cdots \otimes \mathscr{L}_N^{\alpha_N} \cong K_{\log}$.

For our purposes, the key point is that $\overline{\mathscr{M}_{g,k,W}}$ supports a virtual fundamental cycle $\rightsquigarrow$ Cohomological Field Theory Λ.

A-model state space

The state space for this Coh. Field Theory depends on a choice of admissible group G of diagonal symmetries of W.

$$\mathscr{H}_{W,G} = \bigoplus_{g \in G} \mathscr{H}_g,$$

where

$$\mathscr{H}_g = H^{mid}(\mathrm{Fix}\, g, W_g^{\infty}, \mathbb{C})^G$$

The summands and G-action may be understood via the isomorphism

$$H^{\mathrm{mid}}(\mathrm{Fix}\, g, W_g^{\infty}, \mathbb{C}) \cong \mathscr{Q}_{W|_{\mathrm{Fix}\, g}}\, \omega_{\mathrm{Fix}\, g}.$$

A-model Frobenius Algebra

Three-point, genus-zero correlators in the Coh. Field Theory Λ produce a multiplication on this state space:

$$\alpha \star \beta = \sum_{\mu,\nu} < \alpha, \beta, \mu > \eta^{\mu\nu} \nu,$$

and $\mathscr{H}_{W,G}$ becomes a Frobenius algebra when endowed with the pairing induced by the residue pairing on the summands $\mathscr{H}_g$.

We consider the so-called *invertible potentials*, namely non-degenerate, quasi-homogeneous polynomials

$$W\colon \mathbb{C}^N \to \mathbb{C}$$

with

number of monomials = number of variables

Then

$$W = \sum_{i=1}^{N} c_i X^{a_{i1}} \cdots X^{a_{iN}},$$

and non-degeneracy implies the matrix $A = (a_{ij})$ is invertible. In this case, the variables can be rescaled to absorb the c_i, so the matrix of exponents completely specifies W.

Kreuzer-Skarke Classification

Kreuzer and Skarke show that invertible potentials are (decoupled) sums of three types of potentials:

Fermat

$$W = X^a$$

Loop

$$W = X_1^{a_1} X_2 + X_2^{a_2} X_3 + \cdots + X_{N-1}^{a_{N-1}} X_N + X_N^{a_N} X_1$$

Chain

$$W = X_1^{a_1} X_2 + X_2^{a_2} X_3 + \cdots + X_{N-1}^{a_{N-1}} X_N + X_N^{a_N}$$

This classification is useful, because under decoupled sums,

the B-model (Milnor ring) and the A-model (FJRW ring with maximal symmetry group)

both decompose as tensor products of the decoupled rings.

Berglund-Hübsch Transposition

Berglund and Hübsch suggest a duality among invertible singularities:

$$\begin{tikzcd}
W \arrow[d] & W^T \\
A_W \arrow[r, "\text{transpose}"'] & A_W^T \arrow[u]
\end{tikzcd}$$

Note that this operation respects the Kreuzer-Skarke classification.

Statement of Theorem

Theorem (K-)

Let $W\colon \mathbb{C}^N \to \mathbb{C}$ be a non-degenerate quasi-homogeneous singularity, with maximal diagonal symmetry group G. Then

$$\mathscr{H}_{W,G}^{FJRW} \cong \mathscr{Q}_{W^T}.$$

ADE Self-duality?

Fan-Jarvis-Ruan prove a self-duality theorem for simple singularities and symmetry group generated by the exponential grading operator J.
For A_n, D_n (n odd) and E_6, E_7, E_8, the maximal diagonal symmetry group is generated by J.
Obviously self-dual under B-H transposition:

- $A_n = X^{n+1}$
- $E_6 = X^3 + Y^4$
- $E_8 = X^3 + Y^5$
- $E_7 = X^3 + XY^3 \longleftrightarrow X^3Y + Y^3$

On the other hand, D_n (n odd) is the anomalous case in the FJR duality theorem:

$$\mathscr{H}^{FJRW}_{D_n,\langle J\rangle} \cong \mathscr{Q}_{A_{2n-3}}$$

In the BH transposition formalism,

$$D_n = X^{n-1} + XY^2 \longleftrightarrow X^{n-1}Y + Y^2$$

It easy so see that the Milnor ring of D_n^T is the same as the Milnor ring of $A_{2n-3} = X^{2n-2}$.
This resolves the self-duality anomaly.

Idea of proof

- A_n case dealt with in FJR corresponds to *Fermat* singularities.
- Rows of the matrix A_W correspond to monomials in W. Easy fact: columns of A^{-1} give the phases of symmetries of W.

  i.e. if the k^{th} column of A^{-1} is

  $$\rho_k = \left(\varphi_1^{(k)}, \quad \ldots \quad, \varphi_N^{(k)}\right)^T,$$

  then

  $$\left(\exp(2\pi i\varphi_1^{(k)}) \quad , \ldots , \quad \exp(2\pi i\varphi_N^{(k)})\right) \in (\mathbb{C}^*)^N$$

  is a diagonal symmetry of W.

- Substituting $\exp(2\pi i \rho_k)$ for $\overline{X}_k$ in the monomials of W^T yields the trivial automorphism of W.
- So, relations in G reflect relations in $\mathscr{Q}_{W^T}$, and the map

$$\mathscr{H}_{W,G} \to \mathscr{Q}_{W^T}$$

acting on generators via

$$1_{\mathscr{H}_{\rho_k J}} \mapsto \overline{X}_k$$

extends to an isomorphism of Frobenius algebras.

Worked Example: 'self-duality' of E_7.

$$E_7 = X^3 + XY^3$$

- Aut. group $G = \mathbb{Z}/9\mathbb{Z}$ generated by $J = (e^{2\pi i \frac{3}{9}}, e^{2\pi i \frac{2}{9}})$.
- $\text{Fix}\, J^k = \begin{cases} \mathbb{C}^2 & \text{if } k = 0 \\ \mathbb{C} & \text{if } k = 3, 6 \\ \{0\} & \text{else.} \end{cases}$
- Corresponding Milnor Ring (invariants in red)

$$\begin{cases} \mathbb{C}dx \wedge dy\{1, x, x^2, y, xy, x^2y, y^2\} & \text{if } k = 0 \\ \mathbb{C}dx\{1, x\} & \text{if } k = 3, 6 \\ \mathbb{C} & \text{else.} \end{cases}$$

Worked Example: 'self-duality' of E_7.

$$E_7 = X^3 + XY^3$$

- Matrix $A = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix} \rightsquigarrow A^{-1} = \begin{pmatrix} \frac{1}{3} & 0 \\ -\frac{1}{9} & \frac{1}{3} \end{pmatrix}$.
- $\rho_1 = (e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{-1}{9}})$, and $\rho_2 = (1, e^{2\pi i \frac{1}{3}})$.
- Note relations $\rho_1^3 \rho_2 = \mathrm{id}$ and $\rho_2^3 = \mathrm{id}$ correspond to monomials of transposed potential $E_7^T = \overline{X}^3 \overline{Y} + \overline{Y}^3$.
- Note $G = \{\rho_1^{\alpha_1} \rho_2^{\alpha_2} \mid 0 \leq \alpha_1, \alpha_2 \leq 2\}$, and $J = \rho_1 \rho_2$.

Worked Example: ‘self-duality’ of E_7.

- Note $G = \{\rho_1^{\alpha_1}\rho_2^{\alpha_2} \mid 0 \leq \alpha_1, \alpha_2 \leq 2\}$.
- Fix $\rho_1^{\alpha_1}\rho_2^{\alpha_2} = \begin{cases} \mathbb{C}^2 & \text{if } \alpha_1 = 0 = \alpha_2 \\ \mathbb{C} & \text{if } \alpha_1 = 0 \neq \alpha_2 \\ \{0\} & \text{else.} \end{cases}$
- So invariants generating FJRW state space are $y^2 dx \wedge dy \in \mathscr{H}_{\rho_1^0\rho_2^0}$ and $1 \in \mathscr{H}_{\rho_1^{\alpha_1}\rho_2^{\alpha_2}}$ for $\alpha_1 \neq 0$.
$\rightsquigarrow \dim \mathscr{H}_{E_7,G} = 7$.

Worked Example: ‘self-duality’ of E_7.

On the other hand,

$$E_7^T = \overline{X}^3\overline{Y} + \overline{Y}^3$$

has a 7 dimensional Milnor ring:

$$\mathscr{Q}_{E_7^T} = \mathbb{C}\{1, \overline{X}, \overline{X}^2, \overline{Y}, \overline{Y}^2, \overline{XY}, \overline{XY}^2\}.$$

i.e. the Milnor ring is generated (as a vector space) by $\overline{X}^2\overline{Y}^0$ and $\overline{X}^{\alpha_1}\overline{Y}^{\alpha_2}$ with $\alpha_1 \neq 2$.

Worked Example: ‘self-duality’ of E_7.

These parallel descriptions of the FJRW state space of E_7 and the Milnor ring of E_7^T yield a bijection of vector spaces.
A finer analysis, including computations of three-point correlators, shows that the map

$$\mathscr{H}_{E_7,G} \longrightarrow \mathscr{Q}_{E_7^T}$$

generated by

$$1_{\rho_1 J} \mapsto \overline{X} \qquad \textit{and} \qquad 1_{\rho_2 J} \mapsto \overline{Y}$$

induces the isomorphism claimed in the theorem.

Generalization?

So we know what to do with the maximal symmetry group on the A-model side:

$$\mathscr{H}_{W,G_{max}} \cong \mathscr{Q}_{W^T}$$

What about proper admissible A-model subgroups $G \subset G_{\text{max}}$ containing J?
An ideal statement would be

$$\mathscr{H}_{W,G} \cong \mathscr{Q}_{W^T,G^T}$$

Problems

- What is G^T?
- How do we orbifold the Milnor-Ring?

Dual Group

We define the dual group in terms of the generators ρ_k and $\overline{\rho}_k$ of the maximal symmetry groups G_W and G_{W^T}.

Definition

For $G \subset G_W$ containing J,

$$G^T := \left\{ \prod \overline{\rho}_k^{r_k} \text{ such that } \underline{r} A^{-1} \underline{\alpha}^T \in \mathbb{Z} \text{ for all } \prod \rho_k^{\alpha_k} \in G \right\}$$

Remarks

- Note that dual group elements correspond to (unprojected) invariant monomials.
  More precisely:

$$\prod \rho_k^{\alpha_k} \textit{ preserves } \prod X_k^{r_k}$$

if and only if

$$\prod \overline{\rho}_k^{r_k} \textit{ preserves } \prod \overline{X}_k^{\alpha_k}$$

- This definition agrees with the original duality $G_W \leftrightarrow \{\mathrm{id}\}$.
- The definition also agrees with the physical intuition that $\langle J \rangle$ should be dual to the SL subgroup of G_{W^T}.

Orbifold B-models

Now that we have a dual group, we need an orbifold Milnor ring for W^T.

- Following Intriligator-Vafa, state space

$$\mathscr{Q}_{W,G} = \bigoplus_{g \in G} \mathscr{Q}^{G}_{W|_{\mathrm{Fix}\, g}}$$

with determinant-twisted G-action.

- Rational bi-grading

$$J^g_{\pm} = \pm \sum_{i:\theta^g_i \neq 0} (\theta^g_i - \tfrac{1}{2}) + \sum_{i:\theta^g_i = 0} (q_i - \tfrac{1}{2})$$

- Multiplication should preserve G-grading and $\mathbb{Q}$-grading.

B-model Multiplication

Following Kaufmann, need cocycle γ defined by

$$1_g \star 1_h = \gamma_{g,h} 1_{gh}$$

Guided by $\mathbb{Q}$-grading, consider

$$\gamma_{g,h} = \begin{cases} \mathrm{hess}\, W|_{I_{g,h}} & \textit{if each } X_i \textit{ is stabilized by } g, h \textit{ or } gh \\ 0 & \textit{else} \end{cases}$$

where $I_{g,h}$ is the locus on which gh acts trivially while g and h act non-trivially.

Generalizing Kaufmann's results for ADE and Pham singularities with co-prime powers to the case of invertible potentials, we have:

Theorem (K-.)

For $G \subset SL(N, \mathbb{C})$, this multiplication endows the state space $\mathcal{Q}_{W,G}$ with the structure of a Frobenius Algebra.

The condition $G \subset SL(N, \mathbb{C})$ is required for the multiplication to be defined on G invariants.
This is dual to the requirement that admissible groups on the A-model side must contain J.

Generalize LG Mirror Symmetry

Returning to Landau-Ginzburg Mirror Symmetry, we have

Theorem (K-.)

For $W : \mathbb{C}^N \to \mathbb{C}$ an invertible potential of Loop type and $G \subseteq G_W$ containing J,

$$\mathscr{H}_{W,G} \cong \mathscr{Q}_{W^T,G^T}$$

- I have a conjectural isomorphism for Chain type.
- Specific cases can be treated, including a reformulation of FJR's self-duality for D_n (n even)

$$\mathscr{H}_{D_n,\langle J\rangle} \cong \mathscr{Q}_{D_n} \qquad \leftrightsquigarrow \qquad \mathscr{H}_{D_n,\langle J\rangle} \cong \mathscr{Q}_{D_n^T,\mathbb{Z}/2\mathbb{Z}}$$

Strange Duality

In the case D_n, (n even), we observed that

$$\mathscr{H}_{D_n,\langle J\rangle} \cong \mathscr{Q}_{D_n^T,\mathbb{Z}/2\mathbb{Z}} \cong \mathscr{Q}_{D_n}$$

We observe a similar phenomenon among the 14 exceptional singularities, which exhibit strange duality $W \leftrightarrow W^{SD}$.
In this case,

$$\mathscr{H}_{W,\langle J\rangle} \cong \mathscr{Q}_{W^T,\langle J\rangle^T} \cong \mathscr{Q}_{W^{SD}}$$