

Orbifold Landau-Ginzburg theories, Mirror symmetry and Frobenius Structures

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- 1 Physics background
- 2 Gauged TFTs and G -Frobenius algebras
 - G -Frobenius algebras
 - Details
 - Gerbe twisting
- 3 Singularities with Symmetries
 - (Re)construction
 - Singularities
- 4 Newer developments, Outlook and Conclusion
 - Newer developments
 - Summary

References

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Physics background

(2, 2) super-conformal field theory I

- ▶ $N = 2$ super-conformal symmetry for both the left and the right movers. This implies that there are four finite rings which are closed under the operator product.
- ▶ (c, c) , (a, c) , (a, a) and (c, a) .
- ▶ Certain constraints for their eigenvalues with respect to the operators $J_0, \bar{J}_0, L_0, \bar{L}_0$ of the two $N = 2$ super-conformal algebras, which are usually called q, \bar{q}, h and \bar{h} , respectively.
- ▶ It turns out the rings (a, a) and (c, a) can be recovered from (c, c) and (a, c) by charge conjugation.

Physics background

(2, 2) super-conformal field theory II

- ▶ $|\phi\rangle$ is left chiral if $G_{-1/2}^+|\phi\rangle = 0$ or equivalently $h = \frac{q}{2}$. It is called left anti-chiral if $G_{-1/2}^-|\phi\rangle = 0$ or equivalently $h = -\frac{q}{2}$. Right chiral means that $\bar{G}_{-1/2}^+|\phi\rangle = 0$ or equivalently $\bar{h} = \frac{\bar{q}}{2}$, and finally right anti-chiral means that $\bar{G}_{1/2}^-|\phi\rangle = 0$ or equivalently $\bar{h} = -\frac{\bar{q}}{2}$.
- ▶ Thus one confines oneself to study the former two rings. Mirror symmetry as it was originally conceived in physics was an operation which takes one conformal field theory T and produces another conformal field theory \check{T} such that the (c, c) ring of T is isomorphic to the (a, c) -ring of \check{T} and vice versa.

More Physics

Landau–Ginzburg Theory

$N = 2$ theory which is the conformally invariant fixed point of the Lagrangian

$$\mathcal{L} = \int K(X, \bar{X}) d^2 z d^4 \theta + \int (f(z_i) + \text{complex conjugate}) d^2 z d^2 \theta,$$

where f is a quasi-homogeneous function of fractional degree q_i for z_i . This model leads to a trivial (a, c) -ring and a (c, c) -ring which is given by $\mathbb{C}[\mathbf{z}]/J_f$ where $J_f = (f_{z_i})$ is the Jacobian ideal.

Moreover, the bi-degree (q, \bar{q}) for z_i is given by (q_i, q_i) .

(c, c) -ring is the Milnor ring of the singularity.

(a, c) -ring is trivial. (B-Model)

Mathematical model

Mathematical model

In [1] we defined a mathematical model for an orbifold Landau–Ginzburg theory for a singularity f with symmetry group G . It has an (a, c) and a (c, c) ring which will be denoted GM_f and $(GM_f)^\vee$. Both of these rings have a bi-grading.

Duality Spectral Flow

The motivation for the dual bi-grading again comes from the physical interpretation of GM_f in orbifold Landau-Ginzburg theory and the dualization being implemented by the spectral flow operator $\mathcal{U}_{(1,0)}$ [IV] which has the natural charge $(d = \hat{c} = \frac{c}{3}, 0)$.

Mirror Symmetry

Mirror pairs

We will in certain situations produce mirror pairs who will have interchanged (a, c) and (c, c) rings. This includes the ADE and B,F cases which are mirror-self dual.

Remark

This means the mirror dual model can be called a “Landau–Ginzburg A–model”. Following suggestions of Witten: Fan, Jarvis and Ruan constructed a version of an A-model for special types of Landau–Ginzburg potential. The state space they use is the above (a, c) ring of the corresponding LG orbifold model.

Gauged TFTs and G -Frobenius algebras

Theorem (Turaev, K)

Topological Field Theories with a finite gauge group G as thought of as projective functors from the rigidified cobordism category of G -bundles to vector spaces are in 1-1 correspondence with G -Frobenius algebras.

Remark

We will need the version [K] which includes a twist by a character χ of G .

G -Frobenius algebras (short definition)

Theorem

A G -Frobenius algebra A is precisely a unital, associative, commutative algebra object in $D(k[G])\text{-Mod}$ together with a metric η which additionally satisfies

$$(S) \quad \rho(v^{-1}) = \chi^{-1} \text{ for a character } \chi \in \text{Hom}(G, k^*)$$

$$(T) \quad \forall c \in A_{[g,h]} \text{ and } l_c \text{ left multiplication by } c:$$

$$\chi_h \text{Tr}(l_c \varphi_h|_{A_g}) = \chi_{g^{-1}} \text{Tr}(\varphi_{g^{-1}} l_c|_{A_h})$$

Here $D(k[G])$ is a quasi-triangular quasi-Hopf algebra and v is the element such that $S^2(u) = vuv^{-1}$.

Remark

We can also treat just Frobenius algebra objects in the category and use $\tau(\phi, g) := \epsilon(\mu(\phi \otimes id)\Delta(v(1_k)|_g))$ where $\Delta(v(1_k)|_g)$ is the bi-degree (g, g^{-1}) part of $\Delta(v(1_k))$ for the trace.

G -Frobenius algebras, long definition

Definition

A G -Frobenius algebra is $\langle G, A, \circ, 1, \eta, \varphi, \chi \rangle$, where

G finite group

A finite dim G -graded k -vector space $A = \bigoplus_{g \in G} A_g$ ¹

\circ a multiplication on A which respects the grading:

$$\circ : A_g \otimes A_h \rightarrow A_{gh}$$

1 a fixed element in A_e —the unit

η non-degenerate bilinear form which respects the G grading

i.e. $\eta|_{A_g \otimes A_h} = 0$ unless $gh = e$.

φ an action of G on A (which will be by algebra automorphisms),

$$\varphi \in \text{Hom}(G, \text{Aut}(A)), \text{ s.t. } \varphi_g(A_h) \subset A_{ghg^{-1}}$$

χ a character $\chi \in \text{Hom}(G, k^*)$

¹ A_e is called the untwisted sector and the A_g for $g \neq e$ are called the twisted sectors.

Axioms for a G Frobenius Algebra

- a) *Associativity* $(a_g \circ a_h) \circ a_k = a_g \circ (a_h \circ a_k)$
- b) *Twisted commutativity* $a_g \circ a_h = \varphi_g(a_h) \circ a_g$
- c) *G Invariant Unit*: $1 \circ a_g = a_g \circ 1 = a_g$ and $\varphi_g(1) = 1$
- d) *Invariance of the metric*: $\eta(a_g, a_h \circ a_k) = \eta(a_g \circ a_h, a_k)$
 - i) *Projective self-invariance of the twisted sectors*
 $\varphi_g|_{A_g} = \chi_g^{-1} id$
 - ii) *G -Invariance of the multiplication*
 $\varphi_k(a_g \circ a_h) = \varphi_k(a_g) \circ \varphi_k(a_h)$
 - iii) *Projective G -invariance of the metric* $\varphi_g^*(\eta) = \chi_g^{-2} \eta$
 - iv) *Projective trace axiom* $\forall c \in A_{[g,h]}$ and l_c left multiplication by c : $\chi_h \text{Tr}(l_c \varphi_h|_{A_g}) = \chi_g^{-1} \text{Tr}(\varphi_g^{-1} l_c|_{A_h})$

The twisted Drinfel'd double

Definition

For a finite group G and an element $\beta \in Z^3(G, k^*)$, the twisted Drinfel'd double $D^\beta(k[G])$ is the quasi-triangular quasi-Hopf algebra whose

- underlying vector space has the basis $g_{\underline{x}}$ with $x, g \in G$

$$D^\beta(k[G]) = \bigoplus k g_{\underline{x}}$$

- algebra structure is given by $g_{\underline{x}} h_{\underline{y}} = \delta_{g, xhx^{-1}} \theta_g(x, y) g_{\underline{xy}}$

$$\text{where } \theta_g(x, y) = \frac{\beta(g, x, y) \beta(x, y, (xy)^{-1} g(xy))}{\beta(x, x^{-1} g x, y)}$$

- the co-algebra structure is given by

$$\Delta(g_{\underline{x}}) = \sum_{g_1 g_2 = g} \gamma_x(g_1, g_2) g_{1\underline{x}} \otimes g_{2\underline{x}} \text{ where}$$

$$\gamma_x(g_1, g_2) = \frac{\beta(g_1, g_2, x) \beta(x, x^{-1} g_1 x, x^{-1} g_2 x)}{\beta(g_1, x, x^{-1} g_2 x)}$$

The twisted Drinfel'd double

- 4 The Drinfel'd associator Φ is given by

$$\Phi = \sum_{g,h,k \in G} \beta(g, h, k)^{-1} g_e^L \otimes h_e^L \otimes k_e^L$$

- 5 The R matrix is given by

$$R = \sum_{g \in G} g_e^L \otimes \mathbf{1}_g^L, \text{ where } \mathbf{1}_g^L = \sum_{h \in G} h_g^L$$

- 6 The antipode S is given by

$$S(g_X^L) = \frac{1}{\theta_{g^{-1}}(x, x^{-1}) \gamma_x(g, g^{-1})} x^{-1} g_X^L x^{-1}$$

Remark

This is good for gerbe twisting

Remark: Gerbe twisting

Theorem (KPham)

There are universal gerbe twistings for the TFT with finite gauge group which can be understood as follows.

- 0-Gerbes twisting by a character χ*
- 1-Gerbes twisting by discrete torsion*
- 2-Gerbes twisting on the the inertia*

Example

For β a 2-gerbe on the stack $[pt/G]$

$$K_{\text{JKK full}}^{\beta}([pt/G]) \simeq \text{Rep}(D^{\beta}(k[G]))$$

Special GFAs and construction of GFA

The stringy multiplication problem

Given a $D(k[G])$ module $A = \bigoplus_{g \in G} A_g$ which satisfies (S) for χ together with

- 1 A Frobenius algebra structure on each A_g
- 2 Isomorphisms $A_g \simeq A_{g^{-1}}$
- 3 An (cyclic) A_e module structure on each A_g

find the possible G -Frobenius algebra structures on A

Definition

A GFA A is special if each twisted sector is a cyclic module over the untwisted sector.

(Re)construction: algebraic solution

Theorem (K02)

$A := \bigoplus_{g \in G} A_g$ as above, up to an isomorphism of Frobenius algebras on the A_g , then the structures of super G -Frobenius algebra on A are in 1-1 correspondence with triples (\sim, γ, φ) .

- ▶ The function \sim is just a super-sign which determines if 1_g is even or odd.
- ▶ γ is a G -graded, section-independent cocycle compatible with the metric satisfying the condition of supergrading with respect to the natural G action
- ▶ φ is a non-Abelian two cocycle with values in k^* which satisfies the condition of discrete torsion with respect to σ and the natural G action, such that (γ, φ) is a compatible pair.
- ▶ The cocycle γ is a special type of A_e valued group 2-cocycle

Remarks

- 1 γ defines multiplication on the cyclic generators via

$$1_g \circ 1_h := \gamma(g, h) 1_{gh}$$

the extra conditions ensure that the extension of this multiplication using the cyclic A_e -module structures is well defined.

- 2 φ gives the G -action on the generators

$$\rho(g)(1_h) = \varphi(g, h) 1_{ghg^{-1}}$$

- 3 In the singularity case we will alternatively use

$$\sigma(g) := \tilde{g} + |N_g| \pmod{2}, \quad (1)$$

where $|N_g| := \text{codim}(\text{Fix}(g))$ in \mathbb{C}^n . $\sigma \in \text{Hom}(G, \mathbb{Z}/2\mathbb{Z})$

- 4 In many cases the equations for the cocycles allow one to find a unique multiplication up to the twist by discrete torsion ϵ .

Singularities with Symmetries

General Setup

Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a function with an isolated critical point at 0.

- ▶ We also fix f to be quasi-homogenous with q_i the degree of z_i . $Q = \text{diag}(q_i)$
- ▶ $M_f := \mathbb{C}[[\mathbf{z}]]/J_f$ where J_f is the Jacobian ideal.
- ▶ Using the Grothendieck residue pairing this is a Frobenius algebra.

Singularities with Symmetries

Symmetries

Now also fix $G \subset GL(n, \mathbb{C})$ finite, such that $\forall g \in G, g^*(f) = f$.
Let $Fix(g) \subset \mathbb{C}^n$ be the fixed point set.

- ▶ $A_g := M_{f|_{Fix(g)}}$
- ▶ $\chi = \det(g)$
- ▶ Get data for the Stringy Multiplication Problem.

Euler

$j := \exp(2\pi i Q)$. We call the data (f, G) Euler if $j \in G$. Also
 $J = \langle j \rangle$.

Notice J is always a symmetry, so we can enlarge G if necessary
(quasi-Euler).

Grading

Grading Operators

$Q^{(1)}$ inherent grading in the definition of each A_g . For an element $a_g \in A_g$ is

$$Q(a_g) = Q^{(1)}(a_g) + \frac{1}{2}(s^+(g) + s^-(g)).$$

Standard grading shift

For a G -Frobenius algebra with a choice of linear representation $\rho : G \rightarrow GL_n(k)$, $s_g := \frac{1}{2}(s_g^+ + s_g^-)$ with $s_g^+ := d - d_g$ (d_g the degree of co-unit in A_g) and

$$s_g^- := \frac{1}{2\pi i} \left(\text{tr}(\log(g)) - \text{tr}(\log(g^{-1})) \right) := \frac{1}{2\pi i} \left(\sum_i l_i(g) - \sum_i l_i(g^{-1}) \right)$$

where the $l_i(g)$ are the logarithms of the eigenvalues of $\rho(g)$ using the arguments in $[0, 2\pi)$.

Bi-grading

Definition

Set $\bar{s}_g := \frac{1}{2}(s_g^+ - s_g^-)$. Since $s_g^+ = s_{g-1}^+$ and $s_g^- = -s_{g-1}^-$, it follows that $\bar{s}_g = s_{g-1}$. We define the bi-grading (Q, \bar{Q}) by

$$Q(a_g) := Q(a_g) + s_g \quad \bar{Q}(a_g) := Q(a_g) + \bar{s}_g \quad \text{for } a_g \in A_g$$

Definition

Fix a graded Frobenius algebra A with grading operator Q .

We define its (c, c) -realization $A^{(c,c)}$ to be given by the Frobenius algebra A together with the bi-grading (Q, Q) , i.e. $\bar{Q} = Q$.

We define the (a, c) -realization of A denoted by $A^{(a,c)}$ to be given by the Frobenius algebra A together with the bi-grading $(Q, -Q)$, i.e., $\bar{Q} = -Q$.

(a,c) -ring $(GM_f)^\vee$ via a duality

The dual k -module

Given the G -Frobenius algebra GM_f , its dual G -graded k -module $(GM_f)^\vee$ is defined as

$$(GM_f)^\vee = \bigoplus_{g \in G} \check{A}_g : \check{A}_g = A_{g^{j-1}} = M_f|_{\text{Fix}(g^{j-1})},$$

where j is the group element defining the exponential of the grading operator Q via $\rho(j) = \exp(2\pi iQ)$.

The dual $D(k[G])$ -module $(GM_f)^\vee$

The G -module structure is given by pulling back the action and scaling by χ . In the case of a singularity the character χ is determined by a choice of sign function $\sigma \in \text{Hom}(G, \mathbb{Z}/2\mathbb{Z})$ given by $\chi(g) = (-1)^{\sigma(g)} \det(g)$.

Details on the dual G -action

Pull back twisted action

If we denote the G -action on \check{A} by $\check{\varphi}$, then using the k -module isomorphism $M : A_g \rightarrow A_{gj^{-1}}$

$$\check{\varphi}(g)(\check{a}_h) := \chi(g)M\varphi(g)M^{-1}(\check{a}_h) \in \check{A}_{hgh^{-1}}, \quad \text{for } \check{a}_h \in \check{A}_g;$$

or if we denote $M(a) =: \check{a}$ and fix $\sigma \in \text{Hom}(G, \mathbb{Z}/2\mathbb{Z})$, then for $\check{a} \in \check{A}_h$

$$\check{\varphi}(g)(\check{a}) := (-1)^{\sigma(g)} \det(g)(\varphi(g)(a)) \in \check{A}_{ghg^{-1}}$$

Remark

For $\check{a}_h = M(a1_{hj^{-1}}) \in \check{A}_h$

$$\begin{aligned} \check{\varphi}(g)(\check{a}_h) &= \check{\varphi}(g)M(a1_{hj^{-1}}) \\ &= \epsilon(g, hj^{-1})(-1)^{\sigma(g)(\sigma(h)+\sigma(j)+1)} \det(g|_{\text{Fix}(hj^{-1})})M(a1_{ghg^{-1}j^{-1}}) \end{aligned}$$

The dual bi-grading

New bi-grading

If A was initially graded by the operator $Q^{(1)}$ and $Q(a_g) = Q^{(1)}(a_g) + s_g$, then set $\check{s}_g := s_{gj-1} - d$ and $\check{\bar{s}}_g := \bar{s}_{gj-1}$, where we recall that d is the degree of the G -Frobenius algebra. We define a bi-grading on \check{A} by

$$\check{Q}(\check{a}) = Q^{(1)}(a) + \check{s}_g \quad \check{\bar{Q}} := Q^{(1)}(a) + \check{\bar{s}}_g \quad \text{for } \check{a}_g \in \check{A}_g. \quad (2)$$

Remark

An Euler G -Frobenius algebra $A = \langle G, A, \circ, 1, \eta, \varphi, \chi, j \rangle$ naturally gives rise to a triple $\langle A, j, \chi \rangle$ and thus determines a dual $D(k[G])$ -module with a non-degenerate pairing and a bi-grading. This transformation is a duality on triples.

A mirror theorem

Theorem (K '03)

Let f be one of the simple singularities A_n, D_n, E_6, E_7 and E_8 or a Pham singularity with coprime powers, let J be the exponential grading operator and $\Gamma := \langle j \rangle$ with $\rho(j) = J$.

- ▶ There is a projectively unique, maximally non-degenerate, degenerate Γ -Frobenius algebra structure of degree j on $(\Gamma M_f)^\vee$.
- ▶ Moreover the invariants of the Γ -Frobenius algebra ΓM_f are one-dimensional and yield the Frobenius algebra A_1 , while the invariants of the $(\Gamma M_f)^\vee$ are isomorphic as a bi-graded Frobenius algebra to $M_f^{(a,c)}$.
- ▶ In short, the A, D, E singularities are mirror self dual in the sense that $((\Gamma M_g)^\vee)^\Gamma$ is the mirror dual.

M_f	restriction	G	σ	GM_f^G	$((GM_f)^\vee)^G$
A_n		$\mathbb{Z}/(n+1)\mathbb{Z}$	0	A_1	A_n
A_{2n-1}		$\mathbb{Z}/(n+1)\mathbb{Z}$	1	A_1	B_n
A_{2n-1}	n odd for dual	$\mathbb{Z}/2\mathbb{Z}$	0	B_n	$I_2(4)$
A_{2n-1}		$\mathbb{Z}/2\mathbb{Z}$	1	D_{n+1}	A_1
A_{2n-1}		$\mathbb{Z}/n\mathbb{Z}$	0	$I_2(4)$	B_n
D_{n+1}		$\mathbb{Z}/(2n\mathbb{Z})$	0	A_1	A_{2n-1}
D_{n+1}	n even	$\mathbb{Z}/(2n\mathbb{Z})$	1	A_1	D_{n+1}
D_{n+1}	n odd	$\mathbb{Z}/n\mathbb{Z}$	0	A_1	D_{n+1}
D_{n+1}	n even	$\mathbb{Z}/n\mathbb{Z}$	0	$I_2(4)$	B_n
D_{n+1}		$\mathbb{Z}/2\mathbb{Z}$	0	B_n	$I_2(4)$
D_{n+1}	n odd for dual	$\mathbb{Z}/2\mathbb{Z}$	1	A_{2n-1}	$I_2(4)$
$A_{k_1-1} \otimes \cdots \otimes A_{k_n-1}$	k_i coprime	$\mathbb{Z}/k_1\mathbb{Z} \times \cdots \times \mathbb{Z}/k_n\mathbb{Z}$	0	A_1	$A_{k_1-1} \otimes \cdots \otimes A_{k_n-1}$
E_6		$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	0	A_1	E_6
E_7		$\mathbb{Z}/9\mathbb{Z}$	0	A_1	E_7
E_8		$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$	0	A_1	E_8

Table: Since all groups are cyclic, and $\epsilon \equiv 0$, $\text{Hom}(G, \mathbb{Z}/2\mathbb{Z}) = e$ or $\text{Hom}(G, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ defining, the entry in the column σ . The conditions for the duals are the conditions to be quasi-Euler.

Orbifold mirror philosophy

Philosophy

Let T be a $N = 2$ theory (which for us at the moment means Frobenius algebra) and let $H \subset G$ be symmetry groups with H normal in G , then

$$(T/H)^H \simeq (((T/H)^H / (G/H))^{\vee})^{(G/H)}.$$

Theorem (Mirror self-duality for D, B, F)

The orbifold mirror philosophy produces mirror pairs for the self-dual cases listed in Table 1, with the group $G = \Gamma$ being the group generated by the exponential grading operator and $H = e$.

The orbifold mirror philosophy holds for the case of

$T = A_{2n-1}$, $G = \mathbb{Z}/(2n\mathbb{Z})$, $H = \mathbb{Z}/2\mathbb{Z}$, with n odd, and the choice of $\sigma = 1$ for $\mathbb{Z}/(2n\mathbb{Z})$ which restricts to $\sigma = 1$ for $\mathbb{Z}/2\mathbb{Z}$ and $\sigma = 0$ for $G/H = \mathbb{Z}/n\mathbb{Z}$.

Theorem

An extended orbifold mirror philosophy holds for the entries in Table 2 and produces the additional mirror pairs $((B_n, I_2(4)), (I_2(4), B_n))$ and $((F_4, I_2(4)), (I_2(4), F_4))$.

T	G	H	$K = G/H$	$\left(\begin{array}{c} (T/H)^H, \\ ((T/H)^\vee)^H \end{array} \right)$	$\left(\begin{array}{c} (T/H)/(K)^K, \\ (((T/H)/K)^\vee)^K \end{array} \right)$
A_{2n-1} n odd	$\mathbb{Z}/(2n\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$	(D_{n+1}, A_1)	(A_1, D_{n+1})
A_{2n-1}	$\mathbb{Z}/(2n\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$	$(B_n, I_2(4))$	$(I_2(4), B_n)$
D_{n+1} n even	$\mathbb{Z}/(2n\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$	$(B_n, I_2(4))$	$(I_2(4), B_n)$
E_6	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	$e \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$(F_4, I_2(4))$	$(I_2(4), F_4)$

Table: Mirror pairs from orbifold mirror philosophy

The case E_7

Recall that for $E_7 : x^3 + xy^3$, we have the following degrees

$$q_1 = q_x = \frac{1}{3}, q_2 = q_y = \frac{2}{9}, d = \frac{8}{9}.$$

Fix $\zeta_9 := \exp(2\pi i \frac{1}{9})$, then the E_7 singularity has the exponential grading operator $J = \exp(2\pi i Q)$

$$J = \begin{pmatrix} \zeta_9^3 & 0 \\ 0 & \zeta_9^2 \end{pmatrix}.$$

This operator generates a subgroup $\langle J \rangle \subset GL(n, \mathbb{C})$, which is isomorphic to $\mathbb{Z}/9\mathbb{Z}$. We fix a generator j of $\mathbb{Z}/9\mathbb{Z}$ and regard the representation $\rho : \mathbb{Z}/9\mathbb{Z} \rightarrow GL(n, \mathbb{C})$ as given by $\rho(j) = J$.

This is also the maximal symmetry group $G_{max} = \langle \Lambda \rangle$

$$\Lambda = \begin{pmatrix} \zeta_9^3 & 0 \\ 0 & \zeta_9^{-1} \end{pmatrix}$$

and $J = \Lambda^7$.

The $\mathbb{Z}/9\mathbb{Z}$ -graded k -module $(\mathbb{Z}/9\mathbb{Z})M_f$

The representation is given by $\rho(j^i) = \begin{pmatrix} \zeta_9^{3i} & 0 \\ 0 & \zeta_9^{2i} \end{pmatrix}$.

$g \in \mathbb{Z}/9\mathbb{Z}$	f_g	M_{f_g}	d_g	$\nu_1(g)$	$\nu_2(g)$	s_g^+	s_g^-	s_g	\bar{s}_g
$e = j^0$	$x^3 + xy^3$	E_7	$\frac{8}{9}$	0	0	0	0	0	0
j^1	0	A_1	0	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{8}{9}$	—	0	$\frac{8}{9}$
j^2	0	A_1	0	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{5}{9}$	—	$\frac{5}{9}$	$\frac{4}{9}$
j^3	x^3	A_2	$\frac{1}{3}$	0	$\frac{1}{9}$	$\frac{8}{9}$	—	$\frac{8}{9}$	$\frac{1}{9}$
j^4	0	A_1	0	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{5}{9}$	—	$\frac{5}{9}$	$\frac{2}{9}$
j^5	0	A_1	0	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{2}{9}$	—	$\frac{2}{9}$	$\frac{4}{9}$
j^6	x^3	A_2	$\frac{1}{3}$	0	$\frac{1}{9}$	$\frac{2}{9}$	—	$\frac{2}{9}$	$\frac{4}{9}$
j^7	0	A_1	0	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{8}{9}$	—	$\frac{8}{9}$	$\frac{2}{9}$
j^8	0	A_1	0	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{1}{9}$	—	$\frac{1}{9}$	$\frac{4}{9}$

Lemma

The elements of bi-degree (q, q) of $(\mathbb{Z}/9\mathbb{Z})E_7$ are exactly the elements in the untwisted sector A_e .

The G -action

For $\mathbb{Z}/9\mathbb{Z}$ we have $\epsilon \equiv 1$ and $\sigma \equiv 0$, so the G -action is given by

$$\varphi_{j^i, j^k} = \begin{cases} 1 & \text{if } k = 0 \\ \zeta_9^{-2i} & \text{if } k \in \{3, 6\} \\ \zeta_9^{-5i} & \text{else} \end{cases}$$

and the character is

$$\chi(j^i) = \zeta_9^{5i}.$$

Lemma

The $\mathbb{Z}/9\mathbb{Z}$ -invariants of the only compatible $D(k[\mathbb{Z}/9\mathbb{Z}])$ -module structure are given by the unit 1_e .

The dual bi-grading

The dual grading is given by

	0	1	2	3	4	5	6	7	8
\check{S}_{j^i}	0	$-\frac{8}{9}$	$-\frac{8}{9}$	$-\frac{1}{3}$	$-\frac{4}{9}$	$-\frac{2}{9}$	$-\frac{2}{3}$	$-\frac{7}{9}$	$-\frac{5}{9}$
\check{S}_{j^i}	0	0	$\frac{8}{9}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{5}{9}$

The elements of bi-degree $(-q, q)$ are

$$\langle \check{1}_e, y^2 \check{1}_j, \check{1}_{j^2}, \check{1}_{j^3}, \check{1}_{j^5}, \check{1}_{j^6}, \check{1}_{j^8} \rangle.$$

The dual $\mathbb{Z}/9\mathbb{Z}$ -action

The dual $\mathbb{Z}/9\mathbb{Z}$ -action is given by

$$\check{\varphi}_{j^i, j^k} = \begin{cases} \zeta_9^{5i} & \text{if } k = 1 \\ \zeta_9^{3i} & \text{if } k \in \{4, 7\} \\ 0 & \text{else.} \end{cases}$$

Lemma

The $\mathbb{Z}/9\mathbb{Z}$ -invariants of the dual $((\mathbb{Z}/9\mathbb{Z})E_7)^\vee$ are given by $\langle \check{1}_e, y^2 \check{1}_j, \check{1}_{j^2}, \check{1}_{j^3}, \check{1}_{j^5}, \check{1}_{j^6}, \check{1}_{j^8} \rangle$; they are all of diagonal bi-degree, and their degrees are $(0, 0), (-\frac{4}{9}, \frac{4}{9}), (-\frac{8}{9}, \frac{8}{9}), (-\frac{1}{3}, \frac{1}{3}), (-\frac{2}{9}, \frac{2}{9}), (-\frac{2}{3}, \frac{2}{3}), (-\frac{5}{9}, \frac{5}{9})$. The pairing, the bi-grading, and the group grading are the same as those of the anti-chiral realization of E_7 under the association $\check{1}_e \mapsto 1, \check{1}_j \mapsto y^2, \check{1}_{j^2} \mapsto x^2 y, \check{1}_{j^3} \mapsto x, \check{1}_{j^5} \mapsto y, \check{1}_{j^6} \mapsto x^2, \check{1}_{j^8} \mapsto xy$, so that E_7 is self dual.

The duality

By inspecting the grading and group grading we have

Proposition

There is a unique maximally degenerate G -Frobenius structure of charge j on $((\mathbb{Z}/9\mathbb{Z})E_7)^\vee$ whose invariants form the (a, c) -realization of E_7 . Hence $((\mathbb{Z}/9\mathbb{Z})E_7)^\vee$ is the mirror dual to E_7 .

Newer developments I: Results from [K'07]

Special actually not so special:

We showed that the co-cycle formalism applies in other cases as well. One example is the deRham case. Furthermore we showed we gave a trivialization of the co-cycles using the JKK orbifold K -theory description of the obstruction bundle. [K '07].

Application to singularities

This works for singularities as well. Get a geometric solution. We have a theory of Chern classes for the obstruction bundles. This defines a geometric multiplication (see [K '07]).

Newer developments II – Work with A. Libgober.

A first test in our LG/CY-correspondence

Orbifold Euler characteristic and super trace agree.

Extending to deformations

We use an orbifold version of the Milnor fibration to get an orbifold version Saito theory. Classical theory by Dubrovin, Manin, Hertling, Givental, Barannikov, Sabbah, ...

Theorems/Conjectures

Theorem

*There is a canonical multiplication $M_{f_{m_1}} \otimes M_{f_{m_2}} \rightarrow M_{f_{m_1 m_2}}$.
Moreover there is even the structure of a G -Frobenius algebra on GM_f .*

Claim

Given a pair (f, G) as above there is a theory of a miniversal unfolding of this pair which gives rise to a pointed G -Frobenius manifold whose fiber over the special point is the G -Frobenius algebra of the previous conjecture.

Summary: older results ('03)

- ▶ Good algebraic theory for orbifold LG-models.
 (a, c) ring $(G_{Ab}M_f)^\vee)^{G_{Ab}}$ is used by [FJR] as state space.
 G_{Ab} is a maximal Abelian subgroup dictated by the form of the polynomial.
- ▶ Mirror symmetry on the level of $D(k[G])$ modules with bi-grading.
- ▶ Stringy multiplication reduces to an algebraic co-cycle problem.
- ▶ Solved in many cases. Highly non-trivial e.g. E_7 case
- ▶ A and B models for ADE, BF models including mirror self-duality.

Summary: newer results ('07) and in progress

- ▶ Geometrically defined multiplication for orbifold Landau–Ginzburg models. ('07)
- ▶ Work in progress will extend to Frobenius manifolds.
- ▶ Can also twist by gerbes and discrete torsion.

The End

Thank you!