Bertrand Toën

Bertrand Toën began by posing a fundamental question in algebraic geometry: given p homogeneous polynomials F_i in $n+1$ complex variables, is it possible to read off the topology of the algebraic variety

$$X := \{(x_0, \ldots, x_n) \,|\, F_i(x) = 0\} \subset \mathbb{P}^n_{\mathbb{C}},$$

from the F_is? In low dimensions, there are two well known cases in which it is possible:

- when $n = p = 1$, X is a finite set with cardinality equal to $\deg(F_1)$ (counting according to multiplicity);
- when $n = 2$, $p = 1$ and X is smooth, X is a Riemann surface of genus $(d-1)(d-2)/2$, where $d = \deg(F_1)$.

More generally, in all dimensions, the Euler characteristic is given by

$$\chi(X) = \sum_{p,q} (-1)^{p+q} \dim H^p(X, \Omega^q_X), \tag{1}$$

where Ω^q_X is the sheaf of holomorphic q-forms. The right-hand side can be found purely in terms of the F_i. Eqn (1) is important because it relates a topological invariant on the left to an algebraic invariant on the right. The formula follows directly from the Hodge decomposition

$$H^i(X, \mathbb{Q}) \otimes \mathbb{C} \simeq \bigoplus_{p+q=i} H^p(X, \Omega^q_X)$$

but it also has an independent proof which works also in a more general setting, by combining the Gauss-Bonnet theorem with the Hirzebruch-Riemann-Roch formula. These also hold over other fields. So, for example, for a variety defined by homogenous polynomials over any algebraically closed field k, one has

$$\chi(X) := \sum_i (-1)^i \dim H^i_{\text{et}}(X, \mathbb{Q}_\ell) = \sum_{p,q} \dim H^p(X, \Omega^q_X), \tag{2}$$

where the $H^i_{\text{et}}(X, \mathbb{Q}_\ell)$ are Grothendieck's ℓ-adic cohomology groups and the sheaf cohomology groups on the right are defined by using the Zariski topology. This is a special case of the *trace formula*: if f is an algebraic endomorphism of X, then

$$\sum_i (-1)^i \text{Trace}\big(f : H^i_{\text{et}}(X, \mathbb{Q}_\ell)\big) = [\Gamma_f . \Delta_X],$$

where the right-hand side is the intersection number of the graph Γ_f with the diagonal $\Delta_X \subset X \times X$. The trace formula reduces to (2) when f is the identity.

Toën explained the extension of these ideas to families of algebraic varieties, which tend to degenerate to varieties with singularities. For example, compactified moduli spaces, where the compactification is achieved by adding singular varieties at infinity; or, in arithmetic geometry, in the case of varieties defined over number fields, a variety can have bad (i.e. singular) reductions modulo some primes in the ring of integers.

In a general setting, one studies such degenerations by considering a family of algebraic varieties defined by homogeneous polynomials with coefficients in the ring of functions on a parameter space S. For each $s \in S$, the polynomials determine an algebraic variety $X_s \subset \mathbb{P}^n_{k(s)}$, where $k(s)$ is the residue field of s. The particular examples of interest are where S is a formal "small" disk, such as the spectrum of $\mathbb{C}[[t]]$, $k[[t]]$ for an algebraically closed field k, or the p-adic integers $\mathbb{Z}_p$.

In all these cases, $S = \operatorname{Spec} A$ for some discrete valuation ring A. There are then just two points in S, the *special point* $o \in S$ where $k := k(s) = A/m$ is the residue field; and the *generic point* $\eta \in S$ where $K := k(\eta) = \operatorname{Frac}(A)$ is the fraction field. Corresponding to these, we have two algebraic varieties: the special fibre $X_{\overline{k}}$ and the generic fibre $X_{\overline{K}}$ (the bar denotes algebraic closure).

The *Variational problem* is to understand the change in topology from $X_{\overline{K}}$ to $X_{\overline{k}}$. In particular, to find an algebraic construction that yields the difference in Euler characteristics.

When $X \to S$ is a submersion, both varieties are smooth, the topology is constant, and difference is zero. So the interesting question is what happens when $X_{\overline{K}}$ is smooth, but $X_{\overline{k}}$ may be singular. In the characteristic zero case ($A = \mathbb{C}[[t]]$), a formula of Milnor's give the difference in terms of the cohomology of the twisted de Rham complex. For higher characteristic, it was conjectured by Deligne and Bloch that

$$\chi(X_{\overline{k}}) - \chi(X_{\overline{K}}) = [\Delta_X \,.\, \Delta_X]_0 + \operatorname{Sw}(X_{\overline{K}})$$

(Bloch's formula). Toën's colour-coding uses blue for topological objects, red for algebraic objects, and brown for arithmetic objects. The first term on the right is Bloch's *localized intersection number*, which counts the singularities. The second is the *Swan conductor*. The formula is known to be true in a number of special cases, but is open in the mixed characteristic case (for example $A = \mathbb{Z}_p$).

With this background, Toën turned to his new approach to establishing the conjecture, by looking at it from the point of view non-commutative geometry. In this language, the degenerate case $X_{\overline{\eta}} = 0$ is the commutative case. The general case involves a *quantum parameter*, a choice of a uniformizer $\pi \in A$ and thence a coordinate on S.

Toën's strategy is to seek a non-commutative variety X_π such that $\chi_\pi = \chi(X_{\overline{k}}) - \chi(X_{\overline{K}})$ and apply a non-commutative trace-formula for χ_π. Realizing this involves adopting a very broad definition under which a non-commutative variety over a commutative ring k is simply a k-linear category. There are great many examples, in particular, for any k-algebra, the differential graded category $\mathcal{D}(A)$ of complexes of A-modules. Indeed at first sight the definition seems too weak: there is no true geometry, no topology and no points. But there is a good notion of differential forms, defined by using the Hochschild complex; and, recently, a good notion ℓ-adic cohomology that allows a definition of Euler characteristic.

The non-commutative variety X_π is defined for the family $X \subset \mathbb{P}^n \times S$ by introducing the notion of matrix factorisation for a uniformiser π of A. This is a pair of of vector bundles E_0, E_1 on X together with two morphisms

$$E_0 \xrightarrow{\partial} E_1 \xrightarrow{\partial} E_0$$

such that ∂^2 is multiplication by π. Then X_π is the (dg-)category of matrix factorisations. By assembling all these ingredients, Toën and Vezzosi have established a new case in which Bloch's formula is true. The general case, in which Swan's conductor appears, remains under investigation