A representation-theoretic proof of the branching rule for Macdonald polynomials

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Introduction

Macdonald polynomials, $U_q(\mathfrak{gl}_n)$, and GT basis

Deducing Macdonald's branching rule

Macdonald polynomials

Macdonald 1988: Introduced $P_\lambda(x; q, t)$ as simultaneous generalization of many special functions, e.g. Schur, Jack, Hall-Littlewood, Heckman-Opdam, ...

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Defined in terms of Macdonald difference operators

$$D_n^r(q,t) = t^{\frac{r(r-n)}{2}} \sum_{|I|=r} \prod_{i \in I, j \notin I} \frac{tx_i - x_j}{x_i - x_j} T_{q,I}$$

with shift operators

$$T_{q,I} = \prod_{i \in I} T_{q,i} \qquad T_{q,i} f(x_1, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n).$$

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For partition λ, $P_\lambda(x;q,t)$ is joint eigenfunction of $D_n^r(q,t)$ with

eigenvalue $e_r(q^\lambda t^\rho)$ leading term x^λ,

where $\rho = \left(\frac{n-1}{2},\ldots,\frac{1-n}{2}\right)$ and e_r elementary sym. polynomial.

Macdonald's branching rule

Theorem (Macdonald)

The Macdonald polynomials satisfy the branching rule

$$P_\lambda(x_1,\dots,x_n;q,t)=\sum_{\mu\prec\lambda}\psi_{\lambda/\mu}(q,t)P_\mu(x_1,\dots,x_{n-1};q,t)x_n^{|\lambda|-|\mu|}$$

for the branching coefficient $\psi_{\lambda/\mu}(q,t)$ *given by*

$$\prod_{1\le i\le j\le \ell(\mu)}\frac{(q^{\mu_i-\mu_j}t^{j-i+1};q)(q^{\lambda_i-\lambda_{j+1}}t^{j-i+1};q)(q^{\lambda_i-\mu_j+1}t^{j-i};q)(q^{\mu_i-\lambda_{j+1}+1}t^{j-i};q)}{(q^{\mu_i-\mu_j+1}t^{j-i};q)(q^{\lambda_i-\lambda_{j+1}+1}t^{j-i};q)(q^{\lambda_i-\mu_j}t^{j-i+1};q)(q^{\mu_i-\lambda_{j+1}}t^{j-i+1};q)}.$$

Implies summation formula

$$P_\lambda(x;q,t)=\sum_{\mu^1\prec\cdots\prec\mu^{n-1}\prec\mu^n=\lambda}\prod_{i=1}^n\psi_{\mu^i/\mu^{i-1}}(q,t)\prod_{i=1}^n x_i^{|\mu^i|-|\mu^{i-1}|}.$$

Notation: $(u;q)=\prod_{n\ge0}(1-uq^n)$; $|\lambda|=\sum_i\lambda_i$;

$\mu\prec\lambda$ interlace if $\lambda_1\ge\mu_1\ge\lambda_2\ge\cdots\ge\mu_{n-1}\ge\lambda_n$.

Branching rule for Schur polynomials

Proposition

The Schur polynomials satisfy the branching rule

$$s_\lambda(x_1, \ldots, x_n) = \sum_{\mu \prec \lambda} s_\mu(x_1, \ldots, x_{n-1}) x_n^{|\lambda| - |\mu|}.$$

- $s_\lambda(x_1, \ldots, x_n)$ is character of $\mathfrak{gl}_n$-irrep. L_λ.
- Branching coefficients of $\mathfrak{gl}_n$-representations are trivial:

$$\mathrm{Res}_{\mathfrak{gl}_n}^{\mathfrak{gl}_{n-1}} L_\lambda = \bigoplus_{\mu \prec \lambda} L_\mu.$$

- $\mathfrak{gl}_n$-branching $\implies$ Schur branching.

Main result

Etingof-Kirillov Jr. 1993: Realized $P_\lambda(x)$ via vector-valued $U_q(\mathfrak{gl}_n)$-characters; new proofs of Macdonald's conjectures.

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Goal: New proof of Macdonald branching via Etingof-Kirillov Jr.

- Relates Macdonald branching to $U_q(\mathfrak{gl}_n)$-branching.
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Degenerations give:

- Classical: Proof of Jack branching rule (easy)
- Quasiclassical: Proof of Borodin-Gorin integral formula for Heckman-Opdam hypergeometric functions. Input:

$$\mathrm{Tr}|_{L_{\varepsilon^{-1}\lambda}}(-) \text{ for } q = e^{\varepsilon} \qquad \overset{\varepsilon \to 0}{\to} \qquad \int_{\mathcal{O}_\Lambda} - d\mu_\Lambda,$$

where mapping between "$-$" is explicit.

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Deducing Macdonald's branching rule

Etingof-Kirillov Jr. approach

Let L_λ be f.d. $U_q(\mathfrak{gl}_n)$-irrep with h.w. λ. For $k \in \mathbb{Z}_{\geq 0}$, define

$$W_{k-1} = L_{((k-1)(n-1),-(k-1),\ldots,-(k-1))}.$$

Choose $W_{k-1}[0] \simeq \mathbb{C} \cdot w_{k-1}$. For λ, there is unique

$$\Phi^n_\lambda : L_{\lambda+(k-1)\rho} \to L_{\lambda+(k-1)\rho} \otimes W_{k-1}$$

so that $v_{\lambda+(k-1)\rho} \mapsto v_{\lambda+(k-1)\rho} \otimes w_{k-1} + \text{(l.o.t.)}$.

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Theorem (Etingof-Kirillov Jr.)

Macdonald polynomial $P_\lambda(x; q^2, q^{2k})$ is given by

$$P_\lambda(x; q^2, q^{2k}) = \frac{Tr(\Phi_\lambda^n x^h)}{Tr(\Phi_0^n x^h)}.$$

Note: Interpret traces of Φ_λ^n as scalars via $W_{k-1}[0] \simeq \mathbb{C} \cdot w_{k-1}$.

Gelfand-Tsetlin basis

GT basis $\{v_{\boldsymbol{\mu}}\}$ of L_λ indexed by GT patterns subordinate to λ:

$$\boldsymbol{\mu} = \{\mu^1 \prec \mu^2 \prec \cdots \prec \mu^{n-1} \prec \mu^n = \lambda\}.$$

The vector $v_{\boldsymbol{\mu}}$ admits explicit construction. Key properties:

- respects decomposition

$$\mathrm{Res}^{U_q(\mathfrak{gl}_{n-1})}_{U_q(\mathfrak{gl}_n)} L_\lambda = \bigoplus_{\mu \prec \lambda} L_\mu.$$

- satisfies $\mathrm{wt}(v_{\boldsymbol{\mu}}) = \Big(|\mu^n| - |\mu^{n-1}|, \ldots, |\mu^2| - |\mu^1|, |\mu^1|\Big)$.

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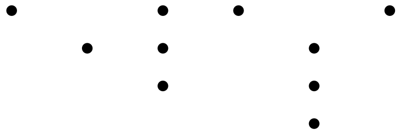
Define $\mathrm{gt}(\mu)$ by $\mathrm{gt}(\mu)^l_i = \mu_i$; $v_{\mathrm{gt}(\mu)}$ is h.w. vector for $U_q(\mathfrak{gl}_{n-1})$.

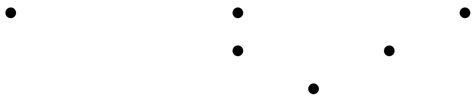
gt(μ) :



$\lambda = (6, 4, 3, 1)$
$\mu = (5, 3, 2)$

Matrix elements in Gelfand-Tsetlin basis

Define $\widetilde{\lambda}_i = \lambda_i - (k-1)(i-1)$ (shift of $\lambda + (k-1)\rho$):

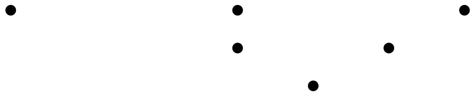


$\widetilde{\lambda} = (5, 2, -1)$
$\widetilde{\mu}^2 = (4, 2)$
$\widetilde{\mu}^1 = (3)$

Ex: $k = 2$, $\lambda = (5, 3, 1)$, $\mu^2 = (4, 3)$, $\mu^1 = (3)$, $\widetilde{\mu}^1 \prec \widetilde{\mu}^2 \prec \widetilde{\lambda}$.

Matrix elements in Gelfand-Tsetlin basis

Define $\widetilde{\lambda}_i = \lambda_i - (k-1)(i-1)$ (shift of $\lambda + (k-1)\rho$):



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Ex: $k = 2$, $\lambda = (5, 3, 1)$, $\mu^2 = (4, 3)$, $\mu^1 = (3)$, $\widetilde{\mu}^1 \prec \widetilde{\mu}^2 \prec \widetilde{\lambda}$.

Let $c(\boldsymbol{\mu}, \lambda)$ be diagonal matrix element of $v_{\boldsymbol{\mu}}$ in shift of Φ^n_λ:

$$\widetilde{\Phi}^n_\lambda = \Phi^n_\lambda \otimes \mathrm{id}_{(\det)^{-\frac{(k-1)(n-1)}{2}}} : L_{\widetilde{\lambda}} \to L_{\widetilde{\lambda}} \otimes W_{k-1}.$$

Proposition

Have $c(\boldsymbol{\mu}, \lambda) = 0$ *unless* $\widetilde{\boldsymbol{\mu}} = \{\widetilde{\mu}^1 \prec \widetilde{\mu}^2 \prec \cdots \prec \widetilde{\mu}^n = \widetilde{\lambda}\}$, *and*

$$c(\widetilde{\boldsymbol{\mu}}, \lambda) = \prod_{i=1}^{n-1} c(\mathrm{gt}(\widetilde{\mu}^i), \mu^{i+1}).$$

Matrix elements in Gelfand-Tsetlin basis

Theorem (S.)

The diagonal matrix element of $v_{\mathrm{gt}(\widetilde{\mu})}$ in $\widetilde{\Phi}^n_\lambda$ is

$$c(\mathrm{gt}(\widetilde{\mu}), \lambda) = \frac{\prod_{a=1}^{k-1} D_{n-1,q^{2\bar{\mu}}}(q^{2a}; q^{-2}, q^{2(k-1)})\Delta^{k-1}(\mu', \lambda)}{\Delta_2^{k-1}(\lambda)\Delta_1^{k-1}(\mu)}.$$

Notations: $\bar{\mu}_i = \mu_i - k(i-1)$, $\mu'_i = \mu_i + k - 1$, and

$$D_{n-1,q^{2\bar{\mu}}}(u; q^2, t^2) = \sum_{r=0}^{n-1} (-1)^{n-1-r} u^{n-1-r} D^r_{n-1,q^{2\bar{\mu}}}(q^2, t^2)$$

is difference operator of $\prod_i (Y_i - u)$. Branching coefficient

$$\psi_{\lambda/\mu}(q^2, q^{2k}) = \frac{\Delta^{k-1}(\mu, \lambda)}{\Delta_1^{k-1}(\mu)\Delta_2^{k-1}(\lambda)} = \frac{\prod_{i \geq j}[\bar{\lambda}_j - \bar{\mu}'_i + k - 1]_{k-1} \prod_{i<j}[\bar{\mu}'_i - \bar{\lambda}_j - 1]_{k-1}}{\prod_{i \leq j}[\bar{\mu}'_i - \bar{\mu}'_j + k - 1]_{k-1} \prod_{i<j}[\bar{\lambda}_i - \bar{\lambda}_j - 1]_{k-1}}.$$

Deducing Macdonald's branching rule

Theorem (Etingof-Kirillov Jr.)

The Macdonald polynomial $P_\lambda(x; q^2, q^{2k})$ *is given by*

$$P_\lambda(x; q^2, q^{2k}) = \frac{Tr(\Phi^n_\lambda x^h)}{Tr(\Phi^n_0 x^h)} = \frac{Tr(\widetilde{\Phi}^n_\lambda x^h)}{Tr(\widetilde{\Phi}^n_0 x^h)}$$

Recall: $\widetilde{\Phi}^n_\lambda : L_{\widetilde{\lambda}} \to L_{\widetilde{\lambda}} \otimes W_{k-1}$ and $\widetilde{\lambda}_i = \lambda_i - (k-1)(i-1)$

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Recall: $\widetilde{\Phi}^n_\lambda : L_{\widetilde{\lambda}} \to L_{\widetilde{\lambda}} \otimes W_{k-1}$ and $\widetilde{\lambda}_i = \lambda_i - (k-1)(i-1)$

- GT basis gives sum over patterns subordinate to $\widetilde{\lambda}$.
- Branching rule gives sum over patterns subordinate to λ
- Need to convert between the two

Deducing Macdonald's branching rule

Compute trace in GT basis:

$$\begin{aligned}\mathrm{Tr}(\widetilde{\Phi}^n_\lambda x^h) &= \sum_{\widetilde{\mu}^1 \prec \cdots \prec \widetilde{\mu}^{n-1} \prec \widetilde{\lambda}} c(\mathrm{gt}(\widetilde{\mu}^0), \mu^1) \cdots c(\mathrm{gt}(\widetilde{\mu}^{n-1}), \lambda) \prod_i x_i^{|\widetilde{\mu}^i| - |\widetilde{\mu}^{i-1}|} \\ &= \sum_{\widetilde{\mu} \prec \widetilde{\lambda}} c(\mathrm{gt}(\mu), \lambda) x_n^{|\lambda| - |\mu| - (k-1)(n-1)} P_\mu(\underline{x}; q^2, q^{2k}) \mathrm{Tr}(\widetilde{\Phi}^{n-1}_0 x^h).\end{aligned}$$

Notation: $\underline{x} = (x_1, \ldots, x_{n-1})$. Recall

$$c(\widetilde{\boldsymbol{\mu}}, \lambda) = \prod_{i=1}^{n-1} c(\mathrm{gt}(\widetilde{\mu}^i), \mu^{i+1}).$$

Deducing Macdonald's branching rule

Strategy of proof (cont'd):

- Explicit formula $\mathrm{Tr}(\widetilde{\Phi}_0^n x^h) = \frac{\prod_{s=1}^{k-1}\prod_{i<j}(x_i - q^{2s}x_j)}{(x_1\cdots x_n)^{(k-1)(n-1)}}$ yields

$$\frac{\mathrm{Tr}(\widetilde{\Phi}_\lambda^n x^h)}{\mathrm{Tr}(\widetilde{\Phi}_0^n x^h)} = \frac{(x_1\cdots x_{n-1})^{k-1}}{\prod_{s=1}^{k-1}\prod_{i=1}^{n-1}(x_i - q^{2s}x_n)} \sum_{\widetilde{\mu}\prec\widetilde{\lambda}} c(\mathrm{gt}(\widetilde{\mu}),\lambda) x_n^{|\lambda|} P_\mu(\underline{x}/x_n; q^2, q^{2k})$$

Deducing Macdonald's branching rule

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- Recall $c(\mathrm{gt}(\widetilde{\mu}),\lambda) = \frac{\prod_{a=1}^{k-1} D_{n-1,q^{2\bar{\mu}}}(q^{2a}; q^{-2}, q^{2(k-1)})\Delta^{k-1}(\mu',\lambda)}{\Delta_2^{k-1}(\lambda)\Delta_1^{k-1}(\mu)}$.

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Strategy of proof (cont'd):

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- Recall $c(\mathrm{gt}(\widetilde{\mu}),\lambda) = \frac{\prod_{a=1}^{k-1} D_{n-1,q^{2\bar{\mu}}}(q^{2a};q^{-2},q^{2(k-1)})\Delta^{k-1}(\mu',\lambda)}{\Delta_2^{k-1}(\lambda)\Delta_1^{k-1}(\mu)}$.
- Apply summation by parts to $D_{n-1,q^{2\bar{\mu}}}(q^{2a};q^{-2},q^{2(k-1)})$:

$$P_\lambda(x;q^2,q^{2k}) = \frac{x_n^{(k-1)(n-1)}}{\prod_{s=1}^{k-1}\prod_{i=1}^{n-1}(x_i - q^{2s}x_n)}\sum_{\widetilde{\mu}\prec\widetilde{\lambda}}\frac{\Delta^l(\mu',\lambda)}{\Delta_2^l(\lambda)\Delta_1^l(\mu)}x_n^{|\lambda|}$$

$$\prod_{a=1}^{l}\mathrm{Ad}_{\Delta_1^l(\mu)}D_{n-1,q^{2\bar{\mu}}}(q^{2a};q^{-2},q^{2(k-1)})^\dagger P_{\mu'}(\underline{x}/x_n;q^2,q^{2k}).$$

- If $\widetilde{\mu}\prec\widetilde{\lambda}$ but not $\mu'\prec\lambda$, $\Delta^l(\mu,\lambda)$ vanishes.

Deducing Macdonald's branching rule

By Macdonald symmetry identity

$$\prod_{a=1}^{k-1} \mathrm{Ad}_{\Delta_1^{k-1}(\mu)} D_{n-1,q^{2\bar{\mu}}}(q^{2a}; q^{-2}, q^{2(k-1)})^{\dagger} P_{\mu'}(\underline{x}/x_n; q^2, q^{2k}) = \prod_{s=1}^{k-1} \prod_{i=1}^{n-1} (x_i/x_n - q^{2s}) P_{\mu'}(\underline{x}/x_n; q^2, q^{2k}).$$

Plug in to get

$$P_\lambda(x; q^2, q^{2k}) = \frac{x_n^{(k-1)(n-1)}}{\prod_{s=1}^{k-1} \prod_{i=1}^{n-1} (x_i - q^{2s} x_n)} \sum_{\mu' \prec \lambda} \frac{\Delta^l(\mu', \lambda)}{\Delta_2^l(\lambda) \Delta_1^l(\mu)} x_n^{|\lambda|} \prod_{a=1}^{l} \mathrm{Ad}_{\Delta_1^l(\mu)} D_{n-1,q^{2\bar{\mu}}}(q^{2a}; q^{-2}, q^{2(k-1)})^{\dagger} P_{\mu'}(\underline{x}/x_n; q^2, q^{2k}) = \sum_{\mu' \prec \lambda} \frac{\Delta^l(\mu', \lambda)}{\Delta_2^l(\lambda) \Delta_1^l(\mu)} x_n^{|\lambda| - |\mu'|} P_{\mu'}(\underline{x}; q^2, q^{2k}).$$

Summary

This talk:

- Macdonald branching from $U_q(\mathfrak{gl}_n)$-branching
- Proof via explicit matrix elt. computation in GT basis
- Limits: Jack branching, Heckman-Opdam integral formula

Question: Probabilistic intepretation?

References:

- P. Etingof and A. Kirillov Jr. Macdonald's polynomials and representations of quantum groups. `arXiv:hep-th/9312103`
- Y. S. A representation-theoretic proof of the branching rule for Macdonald polynomials. `arXiv:1412.0714`
- Y. S. A new integral formula for the Heckman-Opdam hypergeometric functions. `arXiv:1406.3772`

Computing diagonal matrix elements I

Key term in $c(\mathrm{gt}(\widetilde{\mu}), \lambda)$ is

$$\prod_{a=1}^{k-1} D_{n-1, q^{2\bar{\mu}}}(q^{2a}; q^{-2}, q^{2(k-1)}) \Delta^{k-1}(\mu', \lambda).$$

For symmetric $p(Y_i^a)$, define

$$\mathrm{Res}(p)(Y_i) = p(q^{2-k}Y_1, \ldots, q^{k-2}Y_1, \ldots, q^{2-k}Y_n, \ldots, q^{k-2}Y_n).$$

Let $D_p^{n(k-1)}$ and $D_{\mathrm{Res}(p)}^n$ be corresponding Macdonald operators.

Theorem (S.)

The following diagram commutes:

$$\begin{array}{ccc}
\mathbb{C}[X_i^a]^{S_{n(k-1)}} & \xrightarrow{\ \ Res\ \ } & \mathbb{C}[X_i]^{S_n} \\
\Big\downarrow D_p^{n(k-1)}(q^{-2(k-1)}, q^2) & & \Big\downarrow D^n_{Res(p)}(q^{-2}, q^{2(k-1)}) \\
\mathbb{C}[X_i^a]^{S_{n(k-1)}} & \xrightarrow{\ \ Res\ \ } & \mathbb{C}[X_i]^{S_n}
\end{array}$$

Computing diagonal matrix elements II

Theorem (S.)

The following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{C}[X_i^a]^{S_{n(k-1)}} & \xrightarrow{\quad Res \quad} & \mathbb{C}[X_i]^{S_n} \\
\Big\downarrow {\scriptstyle D_p^{n(k-1)}(q^{-2(k-1)}, q^2)} & & \Big\downarrow {\scriptstyle D^n_{Res(p)}(q^{-2}, q^{2(k-1)})} \\
\mathbb{C}[X_i^a]^{S_{n(k-1)}} & \xrightarrow{\quad Res \quad} & \mathbb{C}[X_i]^{S_n}
\end{array}
$$

Proof: Use DAHA, Cherednik Fourier transform

Apply to $p(Y_i^a) = \prod_{i,a}(Y_i^a - u)$ to get

$$
\prod_{a=1}^{k-1} D_{n,q^{2\tilde{\mu}_i}}(q^{2a}; q^{-2}, q^{2(k-1)}) \circ \text{Res}
$$

$$
= \text{Res} \circ D_{n(k-1),q^{2\mu_i^a}}(q^k; q^{-2(k-1)}, q^2).
$$