Ovidiu Savin

Ovidiu Savin's talk reviewed the connection between non-local equations and local equations in one higher dimension. Local PDEs are equations of the form

$$F(u, u_t.u_x, u_{tt}, u_{tx}, u_{xx}, \dots) = 0$$

where in order to check the equation at a point, it is enough to know u in a neighbourhood of the point. By contrast, to check a nonlocal PDE at a point it is necessary to know solution in the whole space.

The talk focused on a simple example, the fractional Laplacian. This has several definitions, but the preferred one is in terms of an extension property that allows the fractional Laplacian in $\mathbb{R}^n$ to be realised in terms of a local operator in $\mathbb{R}^{n+1}$.

Savin began the review by recollecting the definition of a fractional derivative in one dimension. For a 'nice functions' $u : \mathbb{R} \to \mathbb{R}$, D is said to be a *derivative* of u if

a) D is linear in u

b) D is translation invariant, and

c) D has *order* σ under dilation; that is

$$Du_\lambda(x) = \lambda^\sigma Du(\lambda x), \quad \text{where} \quad u_\lambda(x) := u(\lambda x).$$

Up to a constant, there is only one derivative of order 1 that is also *isotropic*, that is, invariant under reflection in the origin. One way to see this is to use translation invariance to show that $(D\cos x) = a\cos x + b\sin x$. Isotropy implies that $b = 0$; and we can choose $a = 1$. It follows that $De^{i\xi x} = |\xi|e^{i\xi x}$, and hence that D is given in general by its action on the Fourier transform of a function by $\widehat{Du} = |\xi|\hat{u}$.

This derivative is 'half' a second-order derivative. It can be defined in other ways. First, as a weighted average of second-order increments: define Du for $\beta \in (-1, 1)$ by

$$Du(x) = \int_0^\infty \frac{u(x+h) + u(x-h) - 2u(x)}{h^{2+\beta}}\, dh. \tag{1}$$

Then D is a derivative of order $1 + \beta$, so we can get the desired operator by taking $\beta = 0$.

Second, one can construct Du by taking the y-derivative at $y = 0$ of the harmonic extension of u to $\mathbb{R}^2_+$. That is $Du(x) = U_y(x, 0)$ where

$$\Delta U = 0 \quad \text{and} \quad U(x, 0) = u(x).$$

In all three approaches, it is clear that D is nonlocal since $Du(x)$ depends on values of u far from x.

The definitions via the Fourier transform and via the weighted average adapt in straightforward ways for other values of σ. The definition in terms of the harmonic extension also generalises, but in a less obvious way. The key is to replace the harmonic extension by the *Caffarelli-Silvestre extension.* Define

$$L_\sigma U : U = U_{xx} + U_{yy} + (1-\sigma)\frac{U_y}{y}.$$

and extend u by taking $L_\sigma u = 0$ and $U(x, 0) = u(x)$. We then get a derivative of order σ by putting $Du(x) = \partial_{y^\sigma} U(x, 0)$.

Formally L_σ can be viewed as the Laplace operator acting on the rotation of U around the x-axis in $2-\sigma$ dimensions. The solutions of $L_\sigma U = 0$ minimize the *energy*

$$\int_{\mathbb{R}^2_+} |\nabla U|^2 y^{1-\sigma}\, \mathrm{d}x\mathrm{d}y\,.$$

This extension was already known in a probabilistic context.

In higher dimensions, one can extend the 'weighted average' construction in (1) to define fractional powers of the Laplacian on $\mathbb{R}^n$. If $u \in C^2$ near x and if $u(y)$ decays more slowly than $|y|^\sigma$ at infinity, then the (nonlocal) definition is

$$\Delta^{\sigma/2}u(x) = c_{n,\sigma}\ \mathrm{P.V.}\int_{\mathbb{R}^n} \frac{u(x+y)-u(x)}{|y|^{n+\sigma}}\,\mathrm{d}y.$$

This operator appears in the theory of stochastic processes as the generator of an isotropic σ-stable Lévy process X_t, with

$$\lim_{t\to 0^+} \frac{E\big(u(x+X_t)\big)-u(x)}{t} = \Delta^{\sigma/2}u(x)\,.$$

(In the case of Brownian motion $\sigma = 2$ and the operator on the right is the Laplacian.) The nonlocal character of the operator reflects the fact that the Lévy process can make discontinuous jumps.The corresponding *fractional heat equation*, $u_t = \Delta^{\sigma/2}$ models *anomalous diffusion*. It arises in financial mathematical and in a wide variety of physical and geometric problems.

The Dirichlet problem for the classical Laplacian has a probabilistic interpretation in terms of Brownian motion in the unit ball B_1. If g is a given function on the boundary ∂B_1, then the expected value $u(x)$ of g at the exit point when the process starts at x satisfies

$$\Delta u = 0 \text{ in } B_1, \quad u(x) = g(x) \text{ on } \partial B_1$$

The same is true for the fractional Laplacian and the Lévy process, except that boundary data must be given on the complement of B_1 rather than just on its boundary, to allow for the possibility of jumps from inside the ball to the outside. So the second equation above is replaced by $u(x) = g(x)$ on $\mathbb{R}^n\backslash B_1$.

So how does one address existence, uniqueness and regularity questions for such problems? In the case of the classical elliptic boundary value problem for the Laplacian Δ, the essential tools are the following.

1) Energy methods: the solution minimizes $\int_{B_1} |\nabla u|^2\,\mathrm{d}x$ subject to $u = g$ on ∂B_1.
2) The maximum principle: if $u \le v$ on ∂B_1 and $\Delta u \ge \Delta v$ then $u \le v$ in B_1
3) Holder estimates: $||u||_{C^\alpha(B_{1/2})} \le C||u||_{L^\infty(B_1)}$.

By iterating the last estimate one can successively improve statements about the smoothness of u. In the case of linear equations with rough coefficients, such a *Harnack inequality* serves the same purpose. For nonlinear equations with rough coefficients, such as the *Bellman equation* in control theory, the approach can be adapted by taking derivatives to obtain linear equations.

For the fractional Laplacian, the key is to pass to a local equation, Laplace's equation, in one extra variable. That is to consider the problem

$$\Delta U = 0 \quad \text{in } \mathbb{R}^{n+1}\backslash B_1\times\{0\}, \qquad U = g \quad \text{on } \mathbb{R}^n\times\{0\}\backslash B_1\,.$$

This, together with boundary regularity ($U \in C^{1/2} \implies u \in C^{1/2}$), allows conclusions about the regularity of u. Apart from the need to take care over the requirement to specify data outside the unit ball, the energy methods and the maximum principle extend in a straightforward way to the fractional Laplacian without the need to extend to $\mathbb{R}^{n+1}$.

More recent developments cover the case of nonlocal equations with rough coefficients and integro-differential equations with measurable kernels. There is a Harnack inequality in the latter case that reduces to the standard one in the limiting local case.

Finally, Savin turned to the *obstacle problem* for $\Delta^{1/2}$: for some given function φ, the problem is to find $u \geq \varphi$ such that

$$\Delta^{1/2} u \leq 0, \qquad \Delta^{1/2} u = 0 \quad \text{in } \{u > \varphi\}\,.$$

This can be reformulated in $\mathbb{R}^{n+1}$ in terms of the extension U and the standard Laplacian as an obstacle problem with a 'thin' obstacle:

$$\Delta U \leq 0 \quad \text{in } B_1, \qquad U(x,0) \geq \varphi(x), \qquad \Delta U = 0 \ \text{ outside } \{U = \varphi\} \cap \{x_{n+1} = 0\}\,.$$

The regularity problems for U and the free boundary have been addressed in this case by using a monotonicity formula to explore the behaviour of U near the free boundary $\partial\{u > \phi\}$. However the formula is very closely tied to the specific geometric properties of the Laplacian and unit ball. A new approach by Caffarelli and others avoids the use of the monotonicity formula, and so may be used more generally.