Geometric RSK, Whittaker functions and random polymers

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The longest increasing subsequence problem

For a permutation $\sigma \in S_n$, write

$$L_n(\sigma) = \text{ length of longest increasing subsequence in } \sigma$$

E.g. if $\sigma = 154263$ then $L_6(\sigma) = 3$.

Ulam (1960) What is typical behaviour of $L_n(\sigma)$? Long history ...

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Baik, Deift and Johansson (1999): for each $x \in \mathbb{R}$,

$$\frac{1}{n!}|\{\sigma \in S_n : \ n^{-1/6}(L_n(\sigma) - 2\sqrt{n}) \leq x\}| \to F_2(x),$$

where F_2 is the Tracy-Widom (GUE) distribution from random matrix theory (Tracy and Widom 1994 — limiting distribution of largest eigenvalue of high-dimensional random Hermitian matrix)

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How is this possible?

The Robinson-Schensted correspondence

From the representation theory of S_n,

$$n! = \sum_{\lambda \vdash n} d_\lambda^2$$

where d_λ = number of standard tableaux with shape λ.

A standard tableau with shape $(4, 3, 1) \vdash 8$:

1	3	5	6
2	4	8	
7			

In other words, S_n has the same cardinality as the set of pairs of standard tableaux of size n with the same shape.

The Robinson-Schensted correspondence

Robinson (38): A bijection between S_n and such pairs

$$\sigma \longleftrightarrow (P, Q)$$

Schensted (61):

$$L_n(\sigma) = \text{ length of longest row of } P \text{ and } Q$$

This yields

$$|\{\sigma \in S_n : L_n(\sigma) \leq k\}| = \sum_{\lambda \vdash n,\ \lambda_1 \leq k} d_\lambda^2.$$

The RSK correspondence

Knuth (70): Extends to a bijection between matrices with nonnegative integer entries and pairs of *semi-standard* tableaux of same shape.

A *semistandard tableau* of shape $\lambda \vdash n$ is a diagram of that shape, filled in with positive integers which are *weakly* increasing along rows and strictly increasing along columns.

A semistandard tableau of shape $(5, 3, 1)$:

1	2	2	5	7
3	3	8		
4				

Cauchy-Littlewood identity

This gives a combinatorial proof of the Cauchy-Littlewood identity

$$\prod_{ij}(1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(x) s_\lambda(y),$$

where s_λ are Schur polynomials, defined by

$$s_\lambda(x) = \sum_{\text{sh } P=\lambda} x^P,$$

where $x = (x_1, x_2, \dots)$ and

$$x^P = x_1^{\sharp 1's \ in \ P} x_2^{\sharp 2's \ in \ P} \dots .$$

Cauchy-Littlewood identity

Let $(a_{ij}) \mapsto (P, Q)$ under RSK.

Then $C_j = \sum_i a_{ij} = \sharp\, j$'s in P and $R_i = \sum_j a_{ij} = \sharp\, i$'s in Q.

For $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ we have

$$\prod_{ij} (y_i x_j)^{a_{ij}} = \prod_j x_j^{C_j} \prod_i y_i^{R_i} = x^P y^Q.$$

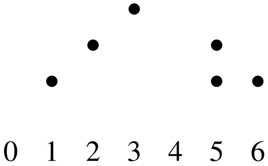
Summing over (a_{ij}) on the left and (P, Q) with sh $P =$ sh Q on the right gives

$$\prod_{ij} (1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

Tableaux and Gelfand-Tsetlin patterns

Semistandard tableaux $\longleftrightarrow$ discrete Gelfand-Tsetlin patterns

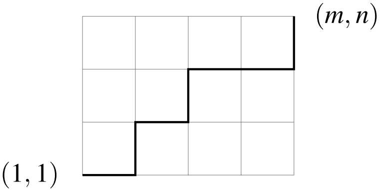
1	1	1	2	2	3
2	2	3	3	3	
3					



The RSK correspondence

If $(a_{ij}) \in \mathbb{N}^{m\times n}$, then length of longest row in corresponding tableaux is

$$M = \max_\pi \sum_{(i,j)\in\pi} a_{ij}$$



Combinatorial interpretation

From the RSK correspondence:

If a_{ij} are independent random variables with $P(a_{ij} \geq k) = (p_i q_j)^k$ then

$$P(M \leq k) = \prod_{ij} (1 - p_i q_j) \sum_{\lambda:\ \lambda_1 \leq k} s_\lambda(p) s_\lambda(q).$$

cf. Weber (79): *The interchangeability of* $\cdot/M/1$ *queues in series.*

Johansson (99): As $n, m \to \infty$, $M \sim$ Tracy-Widom distribution
(and other related asymptotic results)

Geometric RSK correspondence

The RSK mapping can be defined by expressions in the $(\max, +)$-semiring.

Replacing these expressions by their $(+, \times)$ counterparts, A.N. Kirillov (00) introduced a *geometric lifting* of RSK correspondence. It is a bi-rational map

$$T : (\mathbb{R}_{>0})^{n\times n} \to (\mathbb{R}_{>0})^{n\times n}$$

$$X = (x_{ij}) \mapsto (t_{ij}) = T = T(X).$$

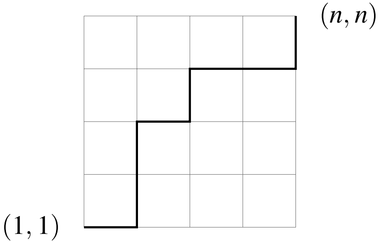
For $n = 2$,

$$\begin{matrix} & x_{21} & \\ x_{11} & & x_{22} \\ & x_{12} & \end{matrix} \quad \mapsto \quad \begin{matrix} & x_{11}x_{21} & \\ x_{12}x_{21}/(x_{12}+x_{21}) & & x_{11}x_{22}(x_{12}+x_{21}) \\ & x_{11}x_{12} & \end{matrix}$$

Geometric RSK correspondence

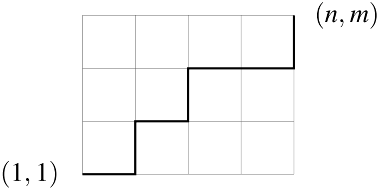
The analogue of the 'longest increasing subsequence' is the matrix element:

$$t_{nn} = \sum_{\phi \in \Pi_{(n,n)}} \prod_{(i,j) \in \phi} x_{ij}$$



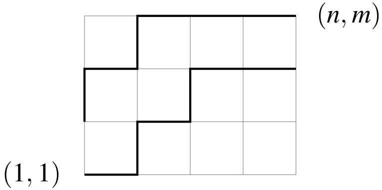
Geometric RSK correspondence

$$t_{nm} = \sum_{\phi \in \Pi_{(n,m)}} \prod_{(i,j) \in \phi} x_{ij}$$



Geometric RSK correspondence

$$t_{n-k+1,m-k+1}\dots t_{nm} = \sum_{\phi\in\Pi^{(k)}_{(n,m)}} \prod_{(i,j)\in\phi} x_{ij}$$



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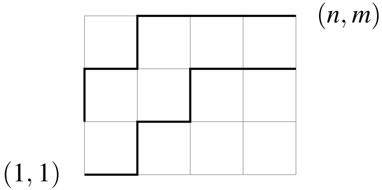
$$T(X)' = T(X')$$



Whittaker functions

- Whittaker functions were first introduced by Jacquet (67). They play an important role in the theory of automorphic forms and also arise as eigenfunctions of the open quantum Toda chain (Kostant 77)

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- In the context of $GL(n,\mathbb{R})$, they can be considered as functions $\Psi_\lambda(x)$ on $(\mathbb{R}_{>0})^n$, indexed by a (spectral) parameter $\lambda \in \mathbb{C}^n$
- The following ‘Gauss-Givental’ representation for Ψ_λ is due to Givental (97), Joe-Kim (03), Gerasimov-Kharchev-Lebedev-Oblezin (06)

Whittaker functions

A *triangle* P with shape $x \in (\mathbb{R}_{>0})^n$ is an array of positive real numbers:

$$P = \begin{array}{ccccccc} & & & z_{11} & & & \\ & & z_{22} & & z_{21} & & \\ & \iddots & & & & \ddots & \\ z_{nn} & & & \cdots\cdots & & & z_{n1} \end{array}$$

with bottom row $z_{n\cdot} = x$.

Denote by $\Delta(x)$ the set of triangles with shape x.

Whittaker functions

Let

$$P = \begin{matrix} & & & z_{11} & & & \\ & & z_{22} & & z_{21} & & \\ & \iddots & & & & \ddots & \\ z_{nn} & & & \cdots & & & z_{n1} \end{matrix}$$

Define

$$P^\lambda = R_1^{\lambda_1} \left(\frac{R_2}{R_1}\right)^{\lambda_2} \cdots \left(\frac{R_n}{R_{n-1}}\right)^{\lambda_n}, \qquad \lambda \in \mathbb{C}^n, \qquad R_k = \prod_{i=1}^k z_{ki}$$

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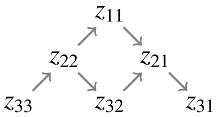
$$\mathcal{F}(P) = \sum_{a \to b} \frac{z_a}{z_b}$$



Whittaker functions

For $\lambda \in \mathbb{C}^n$ and $x \in (\mathbb{R}_{>0})^n$, define

$$\Psi_\lambda(x) = \int_{\Delta(x)} P^{-\lambda} e^{-\mathcal{F}(P)} dP,$$

where $dP = \prod_{1 \le i \le k < n} dz_{ki}/z_{ki}$.

For $n = 2$,

$$\Psi_{(\nu/2,-\nu/2)}(x) = 2K_\nu\left(2\sqrt{x_2/x_1}\right).$$

These are called $GL(n)$-Whittaker functions.

They are the analogue of Schur polynomials in the geometric setting.

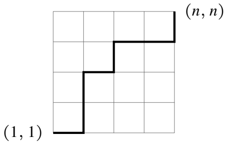
Geometric RSK correspondence

Recall

$$X = (x_{ij}) \mapsto (t_{ij}) = T \quad = \quad \begin{array}{ccccc} & & t_{31} & & \\ & t_{21} & & t_{32} & \\ t_{11} & & t_{22} & & t_{33} \\ & t_{12} & & t_{23} & \\ & & t_{13} & & \end{array}$$

= pair of triangles of same shape $(t_{nn}, \ldots, t_{11})$.

$$t_{nn} = \sum_{\phi \in \Pi_{(n,n)}} \prod_{(i,j)\in\phi} x_{ij}$$



Whittaker measures

Let $a, b \in \mathbb{R}^n$ with $a_i + b_j > 0$ and define

$$\mathbb{P}(dX) = \prod_{ij} \Gamma(a_i + b_j)^{-1} x_{ij}^{-a_i - b_j - 1} e^{-1/x_{ij}} dx_{ij}.$$

Theorem (Corwin-O'C-Seppäläinen-Zygouras 14)

Under $\mathbb{P}$, the law of the shape of the output under geometric RSK is given by the Whittaker measure *on $\mathbb{R}_+^n$ defined by*

$$\mu_{a,b}(dx) = \prod_{ij} \Gamma(a_i + b_j)^{-1} e^{-1/x_n} \Psi_a(x) \Psi_b(x) \prod_{i=1}^{n} \frac{dx_i}{x_i}.$$

Application to random polymers

Corollary

Suppose $a_i > 0$ *for each* i *and* $b_j < 0$ *for each* j*. Then*

$$\mathbb{E}e^{-st_{nn}} = \int_{\iota\mathbb{R}^m} s^{\sum_{i=1}^n (b_i - \lambda_i)} \prod_{ij} \Gamma(\lambda_i - b_j) \prod_{ij} \frac{\Gamma(a_i + \lambda_j)}{\Gamma(a_i + b_j)} s_n(\lambda) d\lambda,$$

where

$$s_n(\lambda) = \frac{1}{(2\pi\iota)^n n!} \prod_{i \neq j} \Gamma(\lambda_i - \lambda_j)^{-1}.$$

If $a_i + b_j = \theta$ for all i, j, this is the log-gamma polymer model introduced by Seppäläinen (2012).

Application to random polymers

Using the above integral formula, Borodin, Corwin and Remenik (2013) have shown that for $\theta < \theta^*$ (for technical reasons)

$$\frac{\log t_{nn} - c(\theta)n}{d(\theta)n^{1/3}} \xrightarrow{dist} F_2.$$

The constant $c(\theta) = -2\Psi(\theta/2)$ and bound on fluctuation exponent $\chi < 1/3$ were established earlier by Seppäläinen (2012).

Combinatorial approach

Recall: $X = (x_{ij}) \mapsto (t_{ij}) = T(X) = (P, Q)$.

The following is a refinement of the previous theorem.

Theorem (O'C-Seppäläinen-Zygouras 14)

- *The map* $(\log x_{ij}) \to (\log t_{ij})$ *has Jacobian* ± 1
- *For* $\nu, \lambda \in \mathbb{C}^n$,

$$\prod_{ij} x_{ij}^{\nu_i + \lambda_j} = P^\lambda Q^\nu$$

- *The following identity holds:*

$$\sum_{ij} \frac{1}{x_{ij}} = \frac{1}{t_{11}} + \mathcal{F}(P) + \mathcal{F}(Q)$$

This theorem (a) 'explains' the appearance of Whittaker functions and (b) extends to models with symmetry.

Analogue of the Cauchy-Littlewood identity

It follows that

$$\prod_{ij} x_{ij}^{-\nu_i-\lambda_j} e^{-1/x_{ij}} \frac{dx_{ij}}{x_{ij}} = P^{-\lambda} Q^{-\nu} e^{-1/t_{11}-\mathcal{F}(P)-\mathcal{F}(Q)} \prod_{ij} \frac{dt_{ij}}{t_{ij}}.$$

Integrating both sides gives, for $\Re(\nu_i + \lambda_j) > 0$:

Corollary (Stade 02)

$$\prod_{ij} \Gamma(\nu_i + \lambda_j) = \int_{\mathbb{R}_+^n} e^{-1/x_n} \Psi_\nu(x) \Psi_\lambda(x) \prod_{i=1}^n \frac{dx_i}{x_i}.$$

This is equivalent to a Whittaker integral identity which was conjectured by Bump (89) and proved by Stade (02). The integral is associated with Archimedean L-factors of automorphic L-functions on $GL(n,\mathbb{R}) \times GL(n,\mathbb{R})$.

Local moves

Proof of second theorem uses new description of the gRSK map T as a composition of a sequence of 'local moves' applied to the input matrix

$$\begin{array}{ccccc} & & x_{31} & & \\ & x_{21} & & x_{32} & \\ x_{11} & & x_{22} & & x_{33} \\ & x_{12} & & x_{23} & \\ & & x_{13} & & \end{array}$$

This description is a re-formulation of Noumi and Yamada's (2004) geometric row insertion algorithm.

Local moves

The basic move is:

b

a d

c

Local moves

The basic move is:

$$b$$

$$\frac{bc}{ab+ac} \qquad bd+cd$$

$$c$$

Local moves

This can be applied at any position in the matrix:

$$\begin{array}{ccccc} & & c & & \\ & b & & f & \\ a & & e & & i \\ & d & & h & \\ & & g & & \end{array}$$

Local moves

This can be applied at any position in the matrix:

$$\begin{array}{ccccc} & & c & & \\ & b & & {\color{red} f} & \\ a & & e & & i \\ & d & & h & \\ & & g & & \end{array}$$

Local moves

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$$
\begin{array}{ccccc}
 & & c & & \\
 & \dfrac{ce}{bc+be} & & cf+ef & \\
a & & e & & i \\
 & d & & h & \\
 & & g & &
\end{array}
$$

Local moves

This can be applied at any position in the matrix:

$$
\begin{array}{ccccc}
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 & b & & f & \\
a & & e & & i \\
 & d & & \color{red}{h} & \\
 & & g & &
\end{array}
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$$
\begin{array}{ccccc}
 & & c & & \\
 & b & & f & \\
a & & e & & i \\
 & \dfrac{eg}{de+dg} & & eh+gh & \\
 & & g & &
\end{array}
$$

Local moves

This can be applied at any position in the matrix:

$$\begin{array}{ccccc} & & c & & \\ & b & & f & \\ a & & e & & i \\ & d & & h & \\ & & \color{red}{g} & & \end{array}$$

Local moves

This can be applied at any position in the matrix:

$$\begin{matrix} & & c & & \\ & b & & f & \\ a & & e & & i \\ & d & & h & \\ & & dg & & \end{matrix}$$

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Local moves

Start with:

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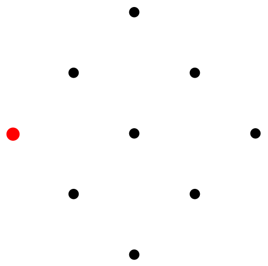
Local moves

Apply the local moves in the following order:



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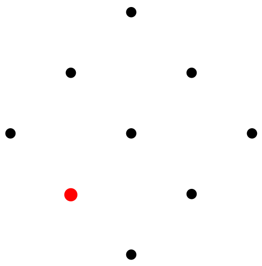

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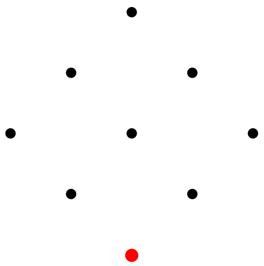

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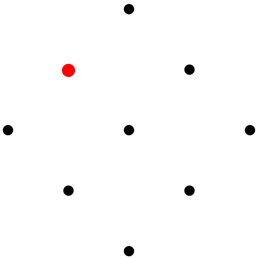

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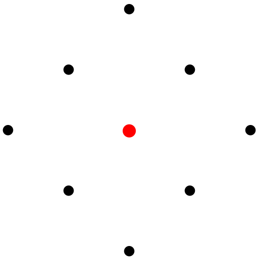

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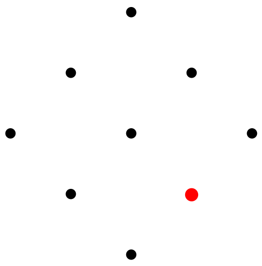

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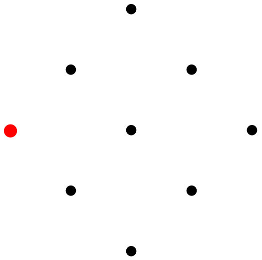

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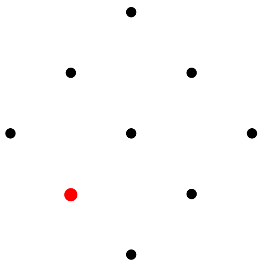

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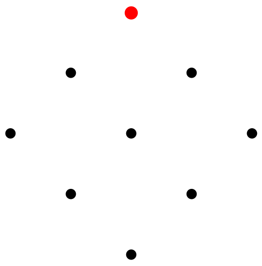

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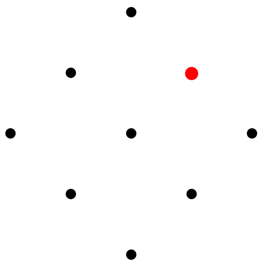

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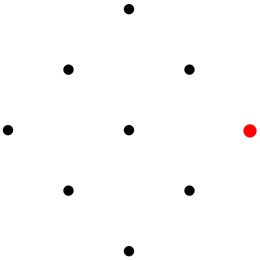

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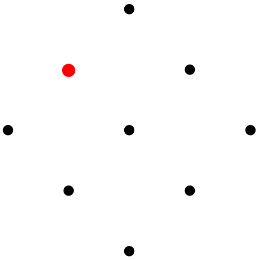

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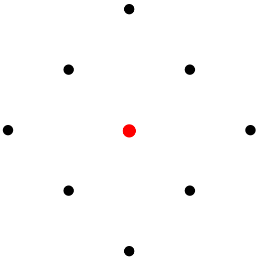

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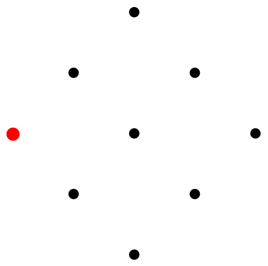


Local moves

To arrive at:

$$
\begin{array}{ccccc}
 & & t_{31} & & \\
 & t_{21} & & t_{32} & \\
t_{11} & & t_{22} & & t_{33} \\
 & t_{12} & & t_{23} & \\
 & & t_{13} & &
\end{array}
$$

Combinatorial approach

Recall: $X = (x_{ij}) \mapsto (t_{ij}) = T(X) = (P, Q)$.

The following is a refinement of the previous theorem.

Theorem (O'C-Seppäläinen-Zygouras 14)

- *The map* $(\log x_{ij}) \to (\log t_{ij})$ *has Jacobian* ± 1
- *For* $\nu, \lambda \in \mathbb{C}^n$,

$$\prod_{ij} x_{ij}^{\nu_i + \lambda_j} = P^\lambda Q^\nu$$

- *The following identity holds:*

$$\sum_{ij} \frac{1}{x_{ij}} = \frac{1}{t_{11}} + \mathcal{F}(P) + \mathcal{F}(Q)$$

This theorem (a) 'explains' the appearance of Whittaker functions and (b) extends to models with symmetry.

Symmetric input matrix

Symmetry properties of gRSK:

$$T(X') = T(X)'$$

$$X \mapsto (P,Q) \quad \Longleftrightarrow \quad X' \mapsto (Q,P).$$

$$X = X' \quad \Longleftrightarrow \quad P = Q$$

Theorem (O'C-Seppäläinen-Zygouras 14)

The restriction of T to symmetric matrices is volume-preserving.

Symmetric input matrix

The analogue of the Cauchy-Littlewood identity in this setting is:

Corollary

Suppose $s > 0$ and $\Re\lambda_i > 0$ for each i. Then

$$\int_{(\mathbb{R}_{>0})^n} e^{-sx_1}\Psi^n_{-\lambda}(x)\prod_{i=1}^{n}\frac{dx_i}{x_i} = s^{-\sum_{i=1}^n \lambda_i}\prod_i \Gamma(\lambda_i)\prod_{i<j}\Gamma(\lambda_i+\lambda_j).$$

This is equivalent to a Whittaker integral identity which was conjectured by Bump-Friedberg (90) and proved by Stade (01).

Symmetric input matrix

Corollary

Let $\alpha_i > 0$ *for each i and define*

$$\mathbb{P}_\alpha(dX) = Z_\alpha^{-1} \prod_i x_{ii}^{-\alpha_i} \prod_{i<j} x_{ij}^{-\alpha_i-\alpha_j} e^{-\frac{1}{2}\sum_i \frac{1}{x_{ii}} - \sum_{i<j} \frac{1}{x_{ij}}} \prod_{i\le j} \frac{dx_{ij}}{x_{ij}}.$$

Then

$$\mathbb{P}_\alpha(sh\ P \in dx) = c_\alpha^{-1} e^{-1/2x_n} \Psi_\alpha^n(x) \prod_i \frac{dx_i}{x_i},$$

where

$$c_\alpha = \prod_i \Gamma(\alpha_i) \prod_{i<j} \Gamma(\alpha_i + \alpha_j).$$

Application to ‘symmetrised’ random polymer (reflecting boundary conditions)

Formally, this yields the integral formula:

$$\mathbb{E}_\alpha e^{-st_{nn}} = \int s^{-\sum_i \lambda_i} \prod_i \frac{\Gamma(\lambda_i)}{\Gamma(\alpha_i)} \prod_{i,j} \Gamma(\alpha_i + \lambda_j) \prod_{i<j} \frac{\Gamma(\lambda_i + \lambda_j)}{\Gamma(\alpha_i + \alpha_j)} s_n(\lambda) d\lambda$$

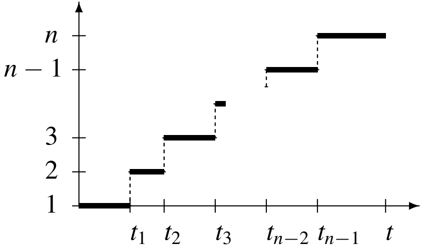
for appropriate vertical contours which stay to the right of zero.

Semi-discrete polymer and integrable systems

Discrete KPZ equation (cf. TASEP)

$$dX_1 = dB_1, \qquad dX_i = dB_i - e^{X_i - X_{i-1}} dt, \quad i \geq 2.$$

Polymer interpretation: for appropriate initial conditions,
$X_n(t) =$ log partition function of semi-discrete random polymer.



O'C-Yor 01 (combines ideas from queueing theory and mathematical finance)

Connection to quantum Toda / Whittaker functions

Quantum Toda chain

$$H = \Delta - 2\sum_{i=1}^{n-1} e^{x_{i+1}-x_i} \qquad H\psi_\lambda = \Big(\sum_i \lambda_i^2\Big)\psi_\lambda \qquad \psi_\lambda(x) = \Psi_{-\lambda}(e^x)$$

Theorem (O'C 12)

If $(B_1, \ldots, B_n)$ is a BM with drift λ then, for appropriate initial conditions, the process $X_n(t),\ t > 0$ has the same law as the first coordinate of a diffusion process in $\mathbb{R}^n$ with generator

$$L_\lambda = \tfrac{1}{2}\psi_\lambda(x)^{-1}\Big(H - \sum \lambda_i^2\Big)\psi_\lambda(x) = \tfrac{1}{2}\Delta + \nabla \log \psi_\lambda \cdot \nabla.$$

Yields (determinantal) formulae for law of $X_n(t)$.

Generalises earlier result of Matsumoto-Yor 99, which corresponds to $n = 2$.

Geometric RSK (in continuous time)

Biane, Bougerol, O'C 05 (cf. Kirillov 00):

Let $\eta : [0, \infty) \to \mathbb{R}^n$ be smooth (or Brownian) with $\eta(0) = 0$.

Define $b(t)$ in upper triangular matrices by

$$\dot{b} = \epsilon(\dot{\eta})b, \qquad b(0) = Id.,$$

where

$$\epsilon(\lambda) = \begin{pmatrix} \lambda_1 & 1 & 0 & \dots & 0 \\ 0 & \lambda_2 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & \lambda_{n-1} & 1 \\ 0 & & \dots & & \lambda_n \end{pmatrix}.$$

Geometric RSK (in continuous time)

Consider the 'principal' minors

$$\Delta_k = \det \left[b_{ij} \right]_{1 \le i \le k,\ n-k+1 \le j \le n}, \qquad 1 \le k \le n$$

and define $x_i = \log(\Delta_i / \Delta_{i-1})$, where $\Delta_0 = 1$.

Theorem (O'C 12, 14)

(a) If $\eta(t) = B(t) + \lambda t$ then $x(t)$ is a diffusion with generator L_λ.

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(a) If $\eta(t) = B(t) + \lambda t$ then $x(t)$ is a diffusion with generator L_λ.
(b) If $\eta(t) = \lambda t$ then $x(t)$ is a solution to the classical Toda flow with constants of motion (Lax matrix eigenvalues) $\lambda_1, \dots, \lambda_n$.

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Theorem (O'C 12, 14)

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The above construction extends naturally to rational and hyperbolic Calogero-Moser systems. In all cases, the quantum systems (in imaginary time) are obtained by adding noise to the constants of motion in particular representations of the classical systems.

References

N. O'Connell. Stochastic Bäcklund transformations. arxiv:1403.5709. To appear in Seminaires de Probabilités, special volume for Marc Yor.

N. O'Connell. Geometric RSK and the Toda lattice. *Illinois J. Math.*, 2014.

N. O'Connell, T. Seppäläinen, N. Zygouras. Geometric RSK correspondence, Whittaker functions and symmetrized random polymers. *Inventiones Math.*, 2014.

I. Corwin, N. O'Connell, T. Seppäläinen, N. Zygouras. Tropical Combinatorics and Whittaker functions. *Duke Math. J.*, 2014.

N. O'Connell. Directed polymers and the quantum Toda lattice. *Ann. Probab.* 40 (2012) 437–458.