Tracy-Widom asymptotics for a random polymer with gamma-distributed weights

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joint work with Neil O'Connell (Warwick)

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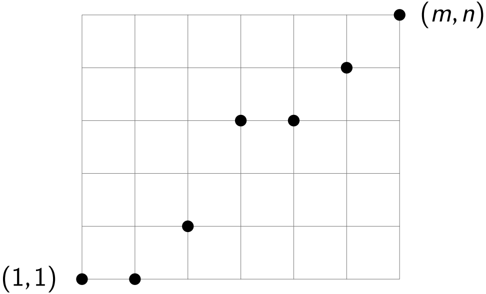
A polymer with gamma weights

- Our model:
  $\Phi_{m,n} = \{(1,j_1),(2,j_1),\ldots,(m,j_m) : 1 \le j_1 \ldots \le j_m \le n\}$



- interested in partition function:

$$Z_{m,n} = \sum_{\pi \in \Phi_{m,n}} \prod_{(j,\ell)\in\phi} g(j,\ell)$$

$g(j,\ell) \sim \text{Gamma}(\gamma)$ i.i.d.

Formula for the Laplace transform

Theorem (O'Connell-O. 2014)

$$\mathbb{E}\left[e^{-s/Z_{m,n}}\right] = \det\left(I + K_{n,r}^{LT}\right)_{L^2(C_{\delta_1})}$$

where

$$K_{n,r}^{LT}(v_1, v_2) = \frac{1}{2\pi i}\int_{\ell_{\delta_2}} \frac{dw}{w - v_2} \frac{\pi}{\sin(\pi(v_1 - w))} \frac{F_s(w)}{F_s(v_1)} \prod_{j=1}^{n} \frac{\Gamma(v_1)}{\Gamma(w)}$$

and $F_s(w) = s^w \prod_{j=1}^{h} \Gamma(\gamma + w)$.

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Here C_{δ_1} is a circle of radius δ_1 around the origin and $\ell_{\delta_2} = \delta_2 + i\mathbb{R}$, for some $0 < \delta_1 < \delta_2 < 1 - \delta_1$.

Tracy-Widom asymptotics

Corollary

Suppose $\frac{m}{n} \to \alpha > 0$ *as* $m, n \to \infty$. *For* γ *sufficiently small,*

$$\lim_{n\to\infty} \mathbb{P}\left\{ \frac{\ln Z_{m,n} - n\mu}{n^{1/3}} \le r \right\} = F_{GUE}\left((\overline{g}/2)^{-1/3}\, r \right)$$

where

$$\mu = \inf_{z>0} \left[(1+\alpha)\psi'(z+\gamma) - \psi'(z) \right].$$

Geometric RSK (Kirillov 2001)

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- Mapping ℓ_{jk} replaces $\begin{pmatrix} W_{j-1,k-1} & W_{j-1,k} \\ W_{j,k-1} & W_{j,k} \end{pmatrix}$ by its image under the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} \frac{bc}{ab+ac} & b \\ c & d(b+c) \end{pmatrix}$$

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- Define now $\pi_i^j = \ell_{ij} \circ \ldots \circ \ell_{i1}$ and for $1 \le i \le h$,

$$R_i = \begin{cases} \pi_1^{n-i+1} \circ \pi_2^{n-i+2} \circ \ldots \circ \pi_i^n & \text{if } i \le n \\ \pi_{i-n+1}^1 \circ \pi_{i-n+2}^2 \circ \ldots \circ \pi_i^n & \text{otherwise} \end{cases}$$

and set $T = R_n \circ \ldots \circ R_1$.

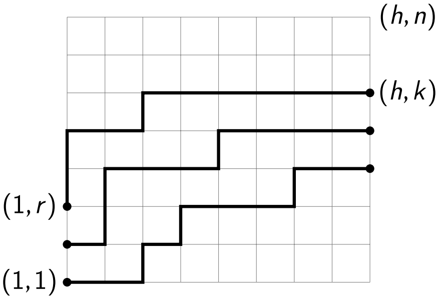
RSK and LPP

- Replacing $(+, \times)$ by $(\max, +)$ yields 'classical' RSK algorithm
- Can think of this as 'zero temperature' analogue
- Zero-temperature version of polymer: last passage percolation
- Johansson (2000): zero-temperature version of log-gamma polymert can be written in terms of RSK
- Corwin–O'Connell–Seppäläinen–Zygouras: use geometric RSK for the log-gamma polymer

gRSK and the polymer

If $T(W) = (t_{ij})_{i,j}$ then for $1 \leq k \leq n$ and $1 \leq r \leq h \wedge k$,

$$t_{h-r+1,k-r+1} \dots t_{hk} = \sum_{(\pi_1,\dots,\pi_r) \in \Pi_{h,k}^{(r)}} \prod_{(i,j) \in \pi_1 \cup \dots \cup \pi_r} w_{ij},$$



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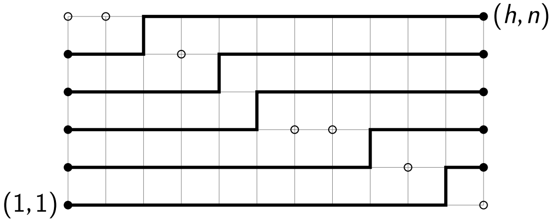

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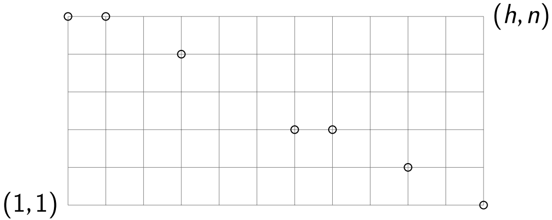

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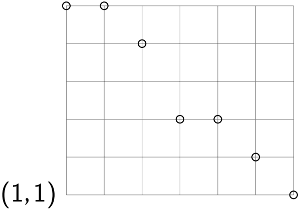

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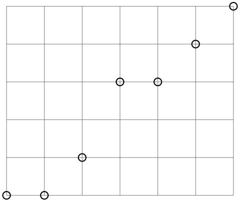

- Hence if $h = m + n + 1$ and W is defined by $g_{j,\ell} = \frac{1}{w_{j+\ell-1,n-j+1}}$ then

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$$= \int_{(i\mathbb{R})^n} \prod_{i,j=1}^{n} \Gamma(-\lambda_j) \prod_{j=1}^{n} \frac{s^{\lambda_j} \prod_{i=1}^{h} \Gamma(\gamma+\lambda_j)}{s^{a_j} \prod_{i=1}^{h} \Gamma(\gamma)}\, s_n(\lambda) d\lambda$$
where s_n is the density of the Sklyanin measure
$$s_n(\lambda) = \frac{1}{(2\pi i)^n\, n!} \prod_{i<j=1}^{n} \frac{1}{\Gamma(\lambda_i - \lambda_j)}.$$

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- Uses Plancherel theorem for GL(n)-Whittaker functions

Fredholm determinant representation

Formula not suitable for asymptotics. Instead:

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Uses Cauchy's theorem and the identity $\det(I+AB) = \det(I+BA)$.

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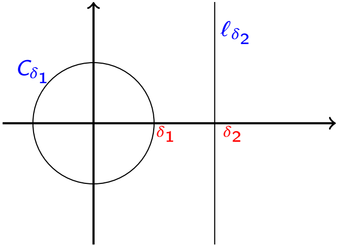

- Choose $s = e^{n\mu - rn^{1/3}}$. Then, setting $Z_n = Z_{\lfloor \alpha n \rfloor, n}$:

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- By steepest descent analysis, right hand side converges to $F_{\mathrm{GUE}}((\overline{g}/2)^{-1/3}r)$.

Zero-temperature limit

- As $\gamma \longrightarrow 0$,

$$-\gamma Z_{m,n} \longrightarrow f_{m,n} := \min_{\phi \in \Phi_{m,n}} \sum_{(i,j)\in\phi} w(i,j)$$

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- Also $f_{m,n}$ has the same law as the smallest eigenvalue in the Laguerre ensemble