

G_2 Geometry

Problem sheet 2 – Classifying spaces

1. Show that $S^\infty := \cup_{n \geq 0} S^n$ (with weak topology) is contractible. (*Hint:* Consider the shift map $S^\infty \rightarrow S^\infty$ that maps $(x_1, \dots, x_n) \in S^{n+1}$ to $(0, x_1, \dots, x_n) \in S^{n+2}$.)
2. Find a pair of smooth manifolds M_1, M_2 with a homotopy equivalence $f : M_1 \rightarrow M_2$ such that $w_k(M_1) \neq f^*w_k(M_2)$ for some k .
3. Let M^n be a closed smooth manifold. For $i_1 + \dots + i_k = n$, the (i_1, \dots, i_k) Stiefel-Whitney number of M is

$$\int_M w_{i_1}(M) \cdots w_{i_k}(M) \in \mathbb{Z}_2.$$

Show that if $M = \partial X$ for some compact smooth X^{n+1} then all Stiefel-Whitney numbers of M vanish.

4. Compute the Stiefel-Whitney classes of $\mathbb{R}P^n$.
 - (a) For which n is $\mathbb{R}P^n$ orientable and spin?
 - (b) Show that $\mathbb{R}P^n$ is the boundary of some smooth compact $n + 1$ -manifold if and only if n is odd.
5. Deduce from the characterisation of the Chern and Stiefel-Whitney classes given in the lectures that if $E \rightarrow X$ is a complex vector bundle with underlying real bundle $E_{\mathbb{R}}$, then

$$\begin{aligned} w_{2k}(E_{\mathbb{R}}) &= c_k(E) \pmod{2} \in H^{2k}(M; \mathbb{Z}_2), \\ w_{2k+1}(E_{\mathbb{R}}) &= 0 \in H^{2k+1}(M; \mathbb{Z}_2) \end{aligned}$$

6. (a) Let $U_{\pm} := S^n \setminus \{(\pm 1, 0, \dots, 0)\}$, and $\pi : U_+ \cap U_- \rightarrow S^{n-1}$ the projection to the equator, $\pi(x_1, \dots, x_n) := \frac{(x_2, \dots, x_n)}{|(x_2, \dots, x_n)|}$. Given $g : S^{n-1} \rightarrow G$, define a G -bundle $P \rightarrow S^n$ with “clutching function” g by requiring transition function $U_+ \cap U_- \rightarrow G$ between trivialisations over U_{\pm} to be $g \circ \pi$. Show that this induces a bijection

$$\pi_{n-1}G \rightarrow \{\text{isomorphism classes of } G\text{-bundles over } S^n\}.$$

- (b) Deduce the existence of such a bijection from the existence of a universal bundle EG instead.
7. (a) Show that there are exactly two isomorphism classes of $SO(3)$ -bundles over S^2 .
(b) Show that $\text{Bl}_p \mathbb{C}P^2$ is diffeomorphic to the total space of an S^2 fibre bundle over S^2 .
(c) Show that $\text{Bl}_p \mathbb{C}P^2$ is not homotopy equivalent to $S^2 \times S^2$. (*Hint:* Compare the intersection forms.)
 8. For any cochain complex C^* of free abelian groups, the short exact sequence $0 \rightarrow C^* \otimes \mathbb{Z}_2 \rightarrow C^* \otimes \mathbb{Z}_4 \rightarrow C^* \otimes \mathbb{Z}_2 \rightarrow 0$ induces a long exact sequence

$$\cdots \rightarrow H^i(C; \mathbb{Z}_2) \rightarrow H^i(C; \mathbb{Z}_4) \rightarrow H^i(C; \mathbb{Z}_2) \xrightarrow{\beta} H^{i+1}(C; \mathbb{Z}_2) \rightarrow \cdots .$$

β is called the *Bockstein map* of $0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 0$.

- (a) If C^* has a graded ring structure that descends to $H^*(C)$, show that

$$\beta(ab) = \beta(a)b + a\beta(b)$$

for any $a, b \in H^*(C; \mathbb{Z}_2)$.

- (b) For a topological space X , the *first Steenrod square* $Sq^1 : H^i(X; \mathbb{Z}_2) \rightarrow H^{i+1}(X; \mathbb{Z}_2)$ is defined to equal the Bockstein map. Show that Sq^1 is a cohomology operation, *i.e.* for any $f : X \rightarrow Y$ and $a \in H^i(Y; \mathbb{Z}_2)$ we have

$$Sq^1(f^*a) = f^*(Sq^1a) \in H^{i+1}(X; \mathbb{Z}_2).$$

- (c) Show that

$$Sq^1a = a^2 \in H^2(X; \mathbb{Z}_2)$$

for any $a \in H^1(X; \mathbb{Z}_2)$.