

G_2 Geometry

Problem sheet on Chern-Weil theory

Questions 4, 5, 7, 8 and 10 depend on material in Lecture 2. Questions 9, 10 and 11 will largely be familiar to you if you have some background in complex geometry.

- Let E, E' be complex vector bundles on M . Deduce from the Chern-Weil definition that
 - $c_k(E)$ are real cohomology classes,
 - $c_1(E \otimes E') = c_1(E) + c_1(E')$,
 - $c_1(\det E) = c_1(E)$.
- The action of $SU(2)$ on \mathbb{C}^2 induces an action by bundle isomorphisms on the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{C}P^1$. Show that there is a unique $SU(2)$ -invariant connection on $\mathcal{O}(-1)$, and that its curvature equals $\frac{i}{2}$ of the volume form of the standard round metric $(\frac{4|dz|^2}{(1+|z|^2)^2})$ in the stereographic chart $z \mapsto (1:z) \in \mathbb{C}P^1$. Deduce that

$$\int_{\mathbb{C}P^1} c_1(\mathcal{O}(-1)) = -1.$$

- Recall that $R[\lambda_1, \dots, \lambda_n]^{S_n}$, the ring of symmetric polynomials in n variable with coefficients in a ring R , equals the free polynomial algebra $R[\sigma_1, \dots, \sigma_n]$, where the *elementary symmetric polynomials* σ_k are defined by

$$(t - \lambda_1) \cdots (-\lambda_n) = t^n - t^{n-1}\sigma_1 + \cdots + (-1)^n \sigma_n.$$

Show that

$$\begin{aligned} r : \mathcal{P}_{\mathrm{GL}(n)} &\rightarrow \mathbb{C}[\lambda_1, \dots, \lambda_n], \\ P &\mapsto P(\mathrm{diag}(\lambda_1, \dots, \lambda_n)) \end{aligned}$$

is injective, with $r(\Sigma_k) = \sigma_k$. Deduce that $\mathcal{P}_{\mathrm{GL}(n)} \cong \mathbb{C}[\Sigma_1, \dots, \Sigma_n]$.

- Show that the direct sum of $T\mathbb{C}P^n$ with a trivial line bundle on $\mathbb{C}P^n$ is isomorphic to the direct sum of $n+1$ copies of $\mathcal{O}(-1)$. Compute the Chern classes of $\mathbb{C}P^n$.
- Let E, E' be real vector bundles over M .
 - Show that $c_{2k+1}(E_{\mathbb{C}}) = 0$ for all k .
 - Show that $p_k(E \oplus E') = \sum_{i=0}^k p_i(E)p_{k-i}(E')$.

- For $m \in \mathrm{Skew}_{2n}\mathbb{R}$, let $\omega(m) = \sum_{i < j} m_{ij} e^i \wedge e^j \in \Lambda^2(\mathbb{R}^{2n})^*$. Let $\mathrm{Pf} : \mathrm{Skew}_{2n}\mathbb{R} \rightarrow \mathbb{R}$ be the homogeneous degree n polynomial such that

$$\omega(m)^n = \mathrm{Pf}(m) e^1 \wedge \cdots \wedge e^{2n} \in \Lambda^{2n}(\mathbb{R}^{2n})^*.$$

Show that

- $\mathrm{Pf}(g^T m g) = \det(g) \mathrm{Pf}(m)$ for any $g \in \mathrm{GL}(2n, \mathbb{R})$.
- $\mathrm{Pf}(m)^2 = \det(m)$.

- (a) For $a_1, \dots, a_n \in \mathbb{R}$, let $\mathrm{Block}(a_1, \dots, a_n) \in \mathrm{Skew}_{2n}\mathbb{R}$ be the matrix with blocks $\begin{pmatrix} 0 & a_k \\ -a_k & 0 \end{pmatrix}$ along the diagonal. Show that

$$\begin{aligned} r : \mathcal{P}_{O(2n)} &\rightarrow \mathbb{R}[a_1, \dots, a_n], \\ P &\mapsto P(\mathrm{Block}(a_1, \dots, a_n)) \end{aligned}$$

is injective, with image $\mathbb{R}[a_1^2, \dots, a_n^2]^{S_n}$. Deduce that $\mathcal{P}_{O(2n)} = \mathbb{R}[\Sigma_2, \Sigma_4, \dots, \Sigma_{2n}]$.

- Show that $\mathcal{P}_{SO(2n)} = \mathbb{R}[\Sigma_2, \Sigma_4, \dots, \Sigma_{2n-2}, \mathrm{Pf}]$.
- Show that $\mathcal{P}_{O(2n+1)} = \mathcal{P}_{SO(2n+1)} = \mathbb{R}[\Sigma_2, \Sigma_4, \dots, \Sigma_{2n}]$.

8. Let E be a complex vector bundle of rank n , and $E_{\mathbb{R}}$ the underlying real vector bundle of rank $2n$.
- Express $p_k(E_{\mathbb{R}})$ in terms of the Chern classes of E .
 - Show that $e(E_{\mathbb{R}}) = c_n(E)$.
9. (a) Let E be a real vector bundle over M , and $s \in \Gamma(E)$ a transverse section, *i.e.* if s vanishes at $p \in M$ then the composition $D_p s : T_p M \rightarrow T_p E \cong T_p M \oplus E_p \rightarrow E_p$ is surjective. Then $X := s^{-1}(0)$ is a smooth submanifold. Show that the normal bundle of X in M is isomorphic to the restriction $E|_X$.
- (b) Let s be a transverse holomorphic section of a holomorphic vector bundle E over a complex manifold M . Then $X := s^{-1}(0)$ is a complex submanifold of M . Show that the normal bundle of X in M is isomorphic (as a complex vector bundle) to $E|_X$.
- (c) Let M be a complex manifold and $X \subset M$ a complex submanifold. Let U_i be an open cover of M with holomorphic functions $f_i : U_i \rightarrow \mathbb{C}$ such that $X \cap U_i = f_i^{-1}(0)$, and f_i vanishes to order 1 along X . Let $[X]$ be the holomorphic line bundle with trivialisations over U_i such that the transition function from U_i to U_j is multiplication by $\frac{f_j}{f_i}$. Explain why this is well-defined, and show that $[X]$ has a global holomorphic section vanishing to order 1 along X . (Then $c_1([X]) = e([X]) = (\text{Poincaré dual to } X) \in H^2(M)$).
10. Let $p : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ be a homogeneous polynomial such that $D_z p \neq 0$ for all $z \in \mathbb{C}^n$ where $p(z) = 0$. Describe the Chern classes of $X := \{p = 0\} \subset \mathbb{C}P^n$, and calculate the Euler characteristic $\chi(X)$.
11. (a) For an open set $U \subset \mathbb{C}^n$ containing 0, let $\text{Bl}_0 U := \{(z, p) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} : z \in p\}$. For a complex manifold M of dimension n and $x \in M$, show that there is a manifold $\text{Bl}_x M$ with a holomorphic map $\pi : \text{Bl}_x M \rightarrow M$ such that $E := \pi^{-1}(x) \cong \mathbb{C}P^{n-1}$, $\pi|_{\text{Bl}_x M \setminus E}$ is a biholomorphism $\text{Bl}_x M \setminus E \rightarrow M \setminus \{x\}$, and E has a neighbourhood biholomorphic to $\text{Bl}_0 \Delta^n$.
- (b) Show that $\text{Bl}_p M$ is diffeomorphic to $M \# (-\mathbb{C}P^n)$, the connected sum of M and $\mathbb{C}P^n$ with its orientation reversed.
- (c) Show that $H^*(\text{Bl}_p M)$ is a direct sum of $\pi^* H^*(M)$ and the subspace generated by $c_1([E]), \dots, c_1([E])^{n-1}$.
- (d) The canonical line bundle of M is the holomorphic line bundle $K_M := \Lambda^n T^* M$. Show that $K_{\text{Bl}_p M} \cong \pi^* K_M \otimes [E]^{\otimes (n-1)}$.