Functional Transcendence via Groups and Galois Theories

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Functional Transcendence around Ax-Schanuel

Oxford, 30 September 2014

Theorem: (Hölder, 1887) The Gamma function defined by

$$\Gamma(x+1) - x\Gamma(x) = 0$$

satisfies no polynomial differential equation that is, there is no nonzero polynomial

$$P(x, y, y', y'', \ldots, y^{(n)}) \in \mathbb{C}[x, y, y', \ldots, y^{(n)}, \ldots]$$

such that $P(x, \Gamma(x), \Gamma'(x), \ldots \Gamma^{(n)}(x)) = 0$.

Theorem: (Hölder, 1887) The Gamma function defined by $\Gamma(x+1) - x\Gamma(x) = 0$ satisfies no polynomial differential equation.

Theorem: (Hardouin; van der Put; H.-S.) Let $b(x) \in \mathbb{C}(x)$. The equation

$$u(x+1) - b(x)u(x) = 0.$$

has a meromorphic solution that is differentially algebraic over $\mathbb{C}(x)$ if and only if there exists a nonzero homogeneous linear differential polynomial $L(Y)$ with coefficients in $\mathbb{C}$ such that

$$L(\frac{u'(x)}{u(x)}) = g(x)$$

for some $g(x) \in \mathbb{C}(x)$, and so

$$L(\frac{b'(x)}{b(x)}) = g(x+1) - g(x).$$

<u>Ex:</u> For $\Gamma(x)$, $L(\frac{1}{x}) = g(x+1) - g(x)$???

Group Theory

$\Downarrow$ Galois Theory

Form of Functional Dependencies
if they occur

- Galois Theory of Linear Differential Equations
- Galois Theory of Linear Differential Equations with Continuous Parameters
- Galois Theory of Linear Differential Equations with Discrete Parameters/Action
- Galois Theory of Linear Difference Equations with Continuous or Discrete Parameters/Action

Picard-Vessiot (PV) Theory

$$\frac{dY}{dx} = A(x)Y \quad A \in \mathrm{M}_n(\mathbb{C}(x))$$

Galois group = the group of transformations of Y that preserve all algebraic relations among x, Y and the derivatives of Y.

Formally: $Y = (y_{i,j})$, $y_{i,j}$ analytic near $x = x_0$, $\det Y \neq 0$,

$K = \mathbb{C}(x) \subset \mathbb{C}(x)(y_{1,1}, \ldots, y_{n,n}) = E$, PV-Extension

- E is closed under $\partial = \frac{d}{dx}$
- $\mathrm{Gal}(E/K) = \{\sigma \mid \sigma = K\text{-autom. of } E, \sigma\partial = \partial\sigma\}$

$$\forall \sigma \in \mathrm{Gal}(E/K),\ \partial(\sigma Y) = A(\sigma Y)$$
$$\Rightarrow \exists\ C_\sigma \in \mathrm{GL}_n(\mathbb{C}) \text{ s.t. } \sigma Y = Y \cdot C_\sigma$$

- $\mathrm{Gal}(E/K) \subset \mathrm{GL}_n(\mathbb{C})$ is *Zariski closed*
- *Galois Correspondence:*

  $H^{\text{Zariski closed } \subset \mathrm{Gal}(E/K)} \Leftrightarrow F^{\text{Diff. field, } k \subset F \subset E}$

Picard-Vessiot (PV) Theory - Examples

Ex. 1: $\frac{dy}{dx} = \frac{1}{2x} y$

$$K = \mathbb{C}(x), \quad E = K(x^{\frac{1}{2}}), \quad \mathrm{Gal}(E/K) = \mathbb{Z}/2\mathbb{Z} \subset \mathrm{GL}_1(\mathbb{C})$$

Ex. 2: $\frac{dy}{dx} = \frac{t}{x} y$

$$\mathrm{Gal}(E/K) = \begin{cases} \mathbb{Z}/q\mathbb{Z} & \text{if } t = p/q, (p,q) = 1 \\ \mathbb{C}^* = \mathrm{GL}_1(\mathbb{C}) & \text{if } t \notin \mathbb{Q} \end{cases}$$

Ex. 3: $\frac{d^2y}{dx^2} - x^{-1} \frac{dy}{dx} + (1 - \nu^2 x^{-2}) y = 0$ $\quad \mathrm{Gal}(E/K) = \mathrm{SL}_2(\mathbb{C})$

$\nu \notin \mathbb{Z} + \frac{1}{2}$

E = PV-extension of $\mathbb{C}(x)$

$$\dim_{\mathbb{C}} \mathrm{Gal}(E/\mathbb{C}(x)) = \mathrm{tr.deg.}_{\mathbb{C}(x)} E$$

Ex. 3(bis): $\frac{d^2y}{dx^2} - x^{-1}\frac{dy}{dx} + (1 - \nu^2 x^{-2})y = 0$ $\quad \mathrm{Gal}(E/K) = \mathrm{SL}_2(\mathbb{C})$

$\nu \notin \mathbb{Z} + \frac{1}{2}$

$K = \mathbb{C}(x)(Y_\nu, Y'_\nu, J_\nu, J'_\nu)$ $\qquad \dim_{\mathbb{C}} \mathrm{SL}_2(\mathbb{C}) = 3 \Rightarrow \mathrm{tr.\ deg.}_{\mathbb{C}(x)} E = 3$

$Y_\nu J'_\nu - Y'_\nu J_\nu \in \mathbb{C}(x)$ and Y_ν, Y'_ν, J_ν alg. indep. over $\mathbb{C}(x)$.

K a $\frac{d}{dx}$- differential field of functions meromorphic in domain $\mathcal{G} \subset \mathbb{C}$.
$a_1, \ldots, a_n \in K$

Proposition: $\int a_1, \ldots, \int a_n$ are algebraically dependent over K iff $\exists c_1, \ldots c_n \in \mathbb{C}$ and $g \in K$ such that

$$g' = c_1 a_1 + \ldots + c_n a_n.$$

In particular, for $a \in K$, $\int a$ is algebraic over K iff $\exists g \in K$ s.t $g' = a$.

Proposition: $e^{\int a_1}, \ldots, e^{\int a_n}$ are alg. dep. over K iff
$\exists g \in K,\ m_1, \ldots, m_n \in \mathbb{Z}$ not all zero s.t.

$$\frac{g'}{g} = m_1 a_1 + \ldots + m_n a_n.$$

In particular, for $a \in K$, $e^{\int a}$ is algebraic over K iff $\exists g \in K, n \neq 0 \in \mathbb{Z}$ s.t. $\frac{g'}{g} = na$.

Galois Theory of Linear Differential Equations with Continuous Parameter

$$\partial_x Y = A(t,x)Y \quad A \in \mathrm{M}_n(\mathbb{C}(t,x)), \;\; \partial_x = \frac{\partial}{\partial x}$$

Galois group = the group of transformations of Y that preserve all algebraic relations among x, Y, and the $\{\partial_x, \partial_t\}$-derivatives of Y.

Formally: $Y = (y_{i,j}),\;\; y_{i,j}$ analytic near $x = x_0, t = t_0$, det $Y \neq 0$,

$K = \mathbb{C}(t,x) \subset \mathbb{C}(t,x)(y_{1,1}, \ldots, y_{n,n}, \partial_t y_{1,1}, \ldots, \partial_t^m y_{i,j,\ldots}) = E$

∂_t-PV-Extension

- E is closed under ∂_x, ∂_t. Note: $(Y_t)_x = (Y_x)_t = A_t Y + AY_t$
- ∂_t-Gal(E/K) $= \{\sigma \mid \sigma = k\text{-autom.}, \sigma\partial = \partial\sigma \;\; \partial \in \{\partial_x, \partial_t\}\}$

$$\forall \sigma \in \partial_t\text{-Gal}(E/K),\; \partial_x(\sigma Y) = A(\sigma Y)$$
$$\Rightarrow \exists\; C_\sigma \in \mathrm{GL}_n(\mathbb{C}(t)) \text{ s.t. } \sigma Y = Y \cdot C_\sigma$$

Ex.4: $\frac{\partial y}{\partial x} = \frac{t}{x} y$

$$K = \mathbb{C}(t,x), \quad E = K(y = x^t, \frac{\partial y}{\partial t} = (\log x)x^t) = K(x^t, \log x),$$

$$\sigma \in \partial_t\text{-Gal}(E/K) \Rightarrow \begin{cases} \sigma(x^t) = ux^t \Rightarrow u \in \mathbb{C}(t) \\ \sigma(\log x) = \log x + \frac{\partial_t u}{u} \Rightarrow \partial_t(\frac{\partial_t u}{u}) = 0 \end{cases}$$

$$\begin{aligned} \partial_t\text{-Gal}(E/K) &= \{(u(t)) \in \mathrm{GL}_1(\mathbb{C}(t)) \mid \partial_t(\frac{\partial_t u}{u}) = 0\} \\ &= \mathbb{C}^* \end{aligned}$$

$$E^{\partial_t\text{-Gal}(E/K)} = K(\log x) \neq K$$

PROBLEM: $\mathrm{GL}_n(\mathbb{C}(t))$ is not big enough.

SOLUTION: Replace $\mathbb{C}(t)$ with $k = \partial_t$-differentially closed containing $\mathbb{C}(t)$. $\mathrm{GL}_n(k)$ IS big enough.

∂_t-Picard-Vessiot (∂_t-PV) Theory

$$\partial_x Y = A(x)Y \quad A \in \mathrm{M}_n(k(x)), \;\; \partial_x = \frac{\partial}{\partial x}, \;\; \mathbb{C}(t) \subset k = \partial_t\text{-diff. closed.}$$

Galois group = the group of transformations of Y that preserve all algebraic relations among x, Y, and the $\{\partial_x, \partial_t\}$-derivatives of Y.

Formally: $Y = (y_{i,j})$, det $Y \neq 0$, formal solution

$$K = k(x) \subset k(x)(y_{1,1}, \ldots, y_{n,n}, \partial_t y_{1,1}, \ldots, \partial_t^m y_{i,j,\ldots}) = E$$

∂_t-PV-Extension

- E is closed under ∂_x, ∂_t
- ∂_t-Gal(E/K) $= \{\sigma \mid \sigma = K\text{-autom.}, \sigma\partial = \partial\sigma \;\; \partial \in \{\partial_x, \partial_t\}\}$

$$\forall \sigma \in \partial_t\text{-Gal}(E/K), \; \partial_x(\sigma Y) = A(\sigma Y)$$
$$\Rightarrow \exists \; C_\sigma \in \mathrm{GL}_n(k) \text{ s.t. } \sigma Y = Y \cdot C_\sigma$$

- ∂_t-Gal(E/K) $= \{\sigma \mid \sigma = K$-autom.$, \sigma\partial = \partial\sigma \;\; \partial \in \{\partial_x, \partial_t\}\}$

$$\forall\sigma \in \partial_t\text{-Gal}(E/K),\ \partial_x(\sigma Y) = A(\sigma Y)$$
$$\Rightarrow \exists\ C_\sigma \in \mathrm{GL}_n(k) \text{ s.t. } \sigma Y = Y \cdot C_\sigma$$

- ∂_t-Gal$(E/kK) \subset \mathrm{GL}_n(k)$ is a Linear Differential Algebraic Group

  i.e., a Kolchin closed subgroup of $\mathrm{GL}_n(k)$

- ∂_t-Gal(E/K) is Zariski dense in Gal.

- Galois Correspondence:

$$H^{\text{Kolchin closed } \subset\ \partial_t\text{-Gal}(E/K)} \Leftrightarrow F^{\text{Diff. field, } K \subset F \subset E}$$

Ex.4: $\frac{\partial y}{\partial x} = \frac{t}{x} y$

$$K = k(x), \quad E = (\log x)x^t) = K(x^t, \log x),$$

$$\partial_t\text{-Gal}(E/K) = \{(u(t)) \in \mathrm{GL}_1(k) \mid \partial_t(\tfrac{\partial_t u}{u}) = 0\}$$
$$= \{(u(t)) \in \mathrm{GL}_1(k) \mid uu_{tt} - (u_t)^2 = 0\}$$

Linear Differential Algebraic Groups (LDAGs)

k - a ∂_t-differential field.

$G \subset \mathrm{GL}_n(k)$, entries satisfy algebraic DEs with respect to t

- $G \subsetneq \mathbb{G}_a(k_0) = (k, +) = \{\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in k\}$

  $\Rightarrow$ $G = \{\alpha \mid L(\alpha) = 0\}$ where $L \in k[\frac{\partial}{\partial t}]$

- $G \subsetneq \mathbb{G}_m(k) = (k^*, \times) = \mathrm{GL}_1(k)$

  $$\Rightarrow \ G = \begin{cases} \mathbb{Z}/n\mathbb{Z} \\ \{\alpha \neq 0 \mid L(\frac{\partial_t(\alpha)}{\alpha}) = 0\} \text{ where } L \in k[\frac{\partial}{\partial t}] \end{cases}$$

- G a Zariski dense subset of $H(k)$, a simple algebraic group

  $\Rightarrow$ G is conjugate to $H(k^{\partial_t})$.

Applications

$K = \{\partial_x, \partial_t\}$ – differential field

$k = K^{\partial_x} = \{c \in K \mid \partial_x(c) = 0\}$ differentially closed

$a \in K$.

Proposition: Let $y_1, \ldots, y_n$ satisfy $\partial_x y_1 = a_1, \ldots, \partial_x y_n = a_n$. The elements $y_1, \ldots, y_n$ are differentially dependent over K iff there is a linear differential operator w.r.t. $\frac{\partial}{\partial_t}$ L with coefficients in k and $g \in K$ such that

$$L(a_1, \ldots, a_n) = \partial_x g.$$

Ex.5: The *incomplete Gamma function* $\gamma(t, x) = \int_0^x s^{t-1} e^{-s} ds$ satisfies

$$\frac{\partial \gamma}{\partial x} = x^{t-1} e^{-x}$$

and $\gamma, \frac{\partial \gamma}{\partial t}, \frac{\partial^2 \gamma}{\partial^2 t}, \ldots$ are alg. ind. over $K = k(x, \log x, x^{t-1} e^{-x})$.

Galois Theory of Linear Differential Equations with Discrete Parameter/Action

K = a field with a derivation δ and *endomorphism* σ, $\delta\sigma = \sigma\delta$.

Ex.6: $K = \mathbb{C}(x, \alpha),\ \delta = \frac{d}{dx},\ \sigma(\alpha) = \alpha + 1$. The Bessel function $J_\alpha(x)$ satisfies

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0, \text{ and}$$

$$xJ_{\alpha+2}(x) - 2(\alpha + 1)J_{\alpha+1}(x) + xJ_\alpha(x) = 0.$$

Ex.7: $K = \mathbb{C}(x),\ \delta = \frac{d}{dx},\ \sigma(x) = x + 1$.

If $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are lin. indep. solns. of

$$\frac{d^2 y}{dx^2} - xy = 0$$

then $\mathrm{Ai}(x), \mathrm{Bi}(x), \delta(\mathrm{Ai}(x))$ are σ-independent over k, i.e., $\mathrm{Ai}(x), \mathrm{Bi}(x), \delta(\mathrm{Ai}(x)), \mathrm{Ai}(x+1), \mathrm{Bi}(x+1), \delta(\mathrm{Ai}(x+1)), \mathrm{Ai}(x+2), \ldots$ are alg. ind. over $\mathbb{C}(x)$.

Ex.8: $K = \mathbb{C}(x),\ \delta = \frac{d}{dx},\ \sigma(x) = x + 1.$

$$\frac{d^2y}{dx^2} = \frac{1}{2x}y$$

σ-PV Extension: $E = \mathbb{C}(x, \sqrt{x}, \sqrt{x+1}, \sqrt{x+2}, \ldots)$

Naive σ-PV Galois group: $G = \{id, \phi\}, \phi(\sqrt{x}) = -\sqrt{x}$.

PROBLEM:

$$E^G = \mathbb{C}(x, \sqrt{x}\sqrt{x+1}, \ldots, \sqrt{x+i}\sqrt{x+j}, \ldots) \neq \mathbb{C}(x).$$

Cannot be fixed by going to a larger field containing $K^\delta = \mathbb{C}$

σ-PV Theory: A functorial approach

Work of C. Hardouin, L. Di Vizio, M. Wibmer.

$K = \delta\sigma$-field
$k = K^\delta = \{c \in K \mid \delta c = 0\}$

$$\delta Y = AY, \quad A \in \mathrm{gl}_n(K) \tag{1}$$

Definition: A σ-PV extension for (1) is a $\delta\sigma$-field E s.t.

1. $\exists Y \in \mathrm{GL}_n(E)$ s.t. $\delta Y = AY$ and $E = K\langle Y \rangle_\sigma = k(Y, \sigma Y, \sigma^2 Y \ldots)$
2. $E^\delta = K^\delta = k$

$R = K[Y \frac{1}{\det Y}]_\sigma$ is called the σ-PV ring.

Theorem. If K^δ is algebraically closed, then there exists a σ-PV extension for (1). If K^δ is σ-closed and E_1, E_2 are σ-PV extensions then for some $\ell > 0$ E_1 and E_2 are isomorphic as $\delta\sigma^\ell$ extensions of K.

$K = \delta\sigma$-field
$k = K^{\delta} = \{c \in K \mid \delta c = 0\}$
$E = K\langle Y\rangle_{\sigma}$ a σ-PV extension for $\sigma Y = AY$
$R = K[Y, \frac{1}{\det Y}]_{\sigma}$, the σ-PV ring.

The σ-PV Galois Group σ-Gal(E/K) is the functor

$$\sigma\text{-Gal}(E/K) : k\text{-}\sigma\text{-algebras} \longrightarrow \text{groups}$$

given by

$$\sigma\text{-Gal}(E/K)(S) := \text{Aut}^{\delta\sigma}(R \otimes_k S / K \otimes_k S)$$

for every k-σ-algebra S where the action of δ on S is trivial and σ is an endomorphism of S.

Theorem. σ-Gal(E/K) is representable by a finitely σ-generated k-σ-algebra, in fact by $(R \otimes_K R)^{\delta}$.

Ex.8(bis): $K = \mathbb{C}(x),\ \delta = \frac{d}{dx},\ \sigma(x) = x + 1$. $K^\delta = \mathbb{C}$. $\quad \delta y = \frac{1}{2x} y$

σ-PV extension: $E = \mathbb{C}(x)\langle\sqrt{x}\rangle_\sigma = \mathbb{C}(x, \sqrt{x}, \sqrt{x+1}, \ldots) = R$.

σ-PV Galois Group: Let $G = \sigma\text{-Gal}(E/K)$

Let SEQ $= \{(c(0), c(1), c(2), \ldots) \mid c(i) \in \mathbb{C}\} = \prod_{i=1}^{\infty} \mathbb{C}$.

- For $c \in$ SEQ, $\sigma(c) := (c(1), c(2), \ldots)$.
- $k = \mathbb{C} \subset$ SEQ via $\alpha \in \mathbb{C} \leftrightarrow (\alpha, \alpha, \ldots) \in$ SEQ.

$G(\text{SEQ}) = \text{Aut}^{\delta\sigma}(R \otimes_{\mathbb{C}} \text{SEQ}/K \otimes_{\mathbb{C}} \text{SEQ}) = \{(\pm 1, \pm 1, \ldots)\} \subset \text{GL}_1(S)$.
Note: $R \cap (R \otimes_{\mathbb{C}} \text{SEQ})^{G(\text{SEQ})} = K$

In general, $G(S) = \{g \in \text{GL}_1(S) \mid g^2 = 1\}$.
G is represented by $\mathbb{C}\{y\}_\sigma/(y^2 - 1)_\sigma$

σ-Algebraic Groups

A σ-algebraic group over a difference field k is a functor G

$$G : k\text{-}\sigma\text{-algebras} \longrightarrow \text{groups}$$

which is representable by a finitely σ-generated k-σ-algebra.

Ex.9: Any linear algebraic group can be considered as a σ-algebraic group.

Ex.10: G a σ-closed subgroup of $\mathbb{G}_a$: There is a set $\mathcal{L}$ of linear homogeneous σ-polynomials s.t.

$$G(S) = \{g \in \mathbb{G}_a(S) \mid L(g) = 0 \text{ for all } L \in \mathcal{L}\}$$

Ex.11: H a σ-closed subgroup of $\mathbb{G}_m$: There is a set $\mathcal{M}$ of multiplicative polynomials s.t.

$$H(S) = \{g \in \mathbb{G}_m(S) \mid M(g) = 1 \text{ for all } M \in \mathcal{M}\}$$

A multiplicative polynomial is a σ-polynomial that is a product of monomials $\sigma^i(y)$.

Galois Correspondence

$K = \delta\sigma$-field
$k = K^\delta = \{c \in k \mid \delta c = 0\}$
$E = K\langle Y\rangle_\sigma$ a σ-PV extension for $\delta Y = AY$
$R = K[Y, \frac{1}{\det Y}]_\sigma$, the σ-PV ring.

Theorem: The σ-PV Galois group is a σ-algebraic group. The map

$$F \mapsto \sigma\text{-Gal}(E/F)$$

is a bijection between the set of intermediate $\delta\sigma$-fields $K \subset F \subset E$ and σ-closed subgroups of σ-Gal(E/F).

Ex.8(bis): $K = \mathbb{C}(x),\ \delta = \frac{\partial}{\partial x},\ \sigma(x) = x + 1.\ K^\delta = \mathbb{C}.\quad \delta y = \frac{1}{2x} y$

$$F = \mathbb{C}(x, \sqrt{x+1}, \sqrt{x+2}, \ldots) \subset \mathbb{C}(x, \sqrt{x}, \sqrt{x+1}, \sqrt{x+2}, \ldots)$$

$$\Rightarrow \sigma\text{-Gal}(K/F)(S) = \{g \in \mathrm{GL}_1(S) \mid g^2 = 1, \sigma(g) = 1\}.$$

Application: Discrete Integrability

Definition: K a $\delta\sigma$-field, $A \in \mathrm{gl}_n(K),\ d \in \mathbb{Z}_{>0}$. We say

$$\delta Y = AY$$

is σ^d-integrable if $\exists B \in \mathrm{GL}_n(K)$ such that the system

$$\begin{aligned} \delta Y &= AY \\ \sigma^d Y &= BY \end{aligned}$$

is compatible, i.e.

$$\delta B + BA = \sigma^d(A)B.$$

Ex.6(bis): Bessel Equation is σ-integrable

$$x^2\delta^2 y + x\delta y + (x^2 - \alpha^2)y = 0, \text{ and}$$

$$xJ_{\alpha+2}(x) - 2(\alpha+1)J_{\alpha+1}(x) + xJ_\alpha(x) = 0.$$

$K = \mathbb{C}(x), \quad \delta = \frac{d}{dx}, \quad \sigma(x) = x + 1$

$$\delta Y = AY, \quad A \in \mathrm{gl}_n(K) \tag{2}$$

$E = \sigma$-PV extension. Assume that the (usual) PV-Galois group is SL_n. Then

- $G = \sigma\text{-Gal}(E/\mathbb{C}(x))$ is Zariski dense in SL_n.
- If $\sigma\text{-dim.}_{\mathbb{C}(x)} E < n^2 - 1$ then $G \neq \mathrm{SL}_n$.
- If G is a proper σ-closed and Zariski-dense subgroup of SL_n then G is conjugate to a subgroup of $\mathrm{SL}_n^{\sigma^d} = \{g \in \mathrm{SL}_n \mid \sigma^d(g) = g\}$ for some $d \in \mathbb{Z}_{>0}$.
- If G is conjugate to a subgroup of $\mathrm{SL}_n^{\sigma^d}$ then (2) is σ^d-integrable.

Theorem: If $\sigma\text{-dim.}_{\mathbb{C}(x)} E < n^2 - 1$, then for some $d \in \mathbb{Z}_{>0}$, the equation

$$\delta B + BA = \sigma^d(A)B$$

has a solution $B \in \mathrm{GL}_n(\mathbb{C}(x))$.

Airy Equation: $\mathrm{Ai}(x), \mathrm{Bi}(x), \delta(\mathrm{Ai}(x))$ are σ-independent.

Galois Theory of Linear Difference Equations with Parameters

Continuous Parameters: (Hardouin/Singer)

Theorem: (Hölder, 1887) The Gamma function defined by

$$\Gamma(x+1) - x\Gamma(x) = 0$$

satisfies no polynomial differential equation.

Theorem: (Ishizaki, 1998) If $a(x), b(x) \in \mathbb{C}(x)$ and $z(x) \notin \mathbb{C}(x)$ satisfies

$$z(qx) = a(x)z(x) + b(x)$$

and is meromorphic on $\mathbb{C}(x)$, then $z(x)$ is not differentially algebraic over $\mathcal{G}(x)$, where $\mathcal{G}$ is the field of q-periodic functions.

Also differential independence of solutions of certain q-hypergeometric functions.

Discrete Parameters/Actions: (Ovchinnikov/Wibmer)

Theorem: The Gamma Function satisfies no difference equation with respect to $x \mapsto x + c, c \notin \mathbb{Q}$ over $M_{<1}$, the field of meromorphic functions f whose Nevanlinna characteristic satisfies $T(f, r) = o(r)$.