q-deformed RSKs and Random Polymers
(joint work with Leo Petrov, University of Virginia)

Konstantin Matveev
Harvard University

May 25th, 2015
Robinson-Schensted-Knuth correspondence

\[ n \times t \text{ matrices } A \text{ of nonnegative integers.} \]

\[
\begin{array}{ccc}
A_{n1} & A_{n2} & \cdots & A_{nt} \\
\cdots & & & \\
A_{11} & A_{12} & \cdots & A_{1t}
\end{array}
\]

**bijection**

Ordered pairs \((P, Q)\) of semistandard Young tableaux of the same shape - \(P\) with entries from \(\{1, 2, \ldots, n\}\), \(Q\) with entries from \(\{1, 2, \ldots, t\}\)

\[
\begin{align*}
\omega_j &:= \frac{1}{A_{1j}} \frac{2}{A_{2j}} \cdots \frac{n}{A_{nj}} \\
P &= \leftarrow \omega_1 \leftarrow \omega_2 \cdots \leftarrow \omega_t
\end{align*}
\]

- View \(t\) as time – array evolves.
- Each row insertion – first some interaction among the rightmost particles, which then spreads throughout the array.
- Random entries – random dynamics.
- More precisely, random dynamics of the rightmost particles, which spreads to the array by *deterministic* rule.
Column insertion

Dual operation – column insertion

\[
\omega_j^* := \frac{n}{\lambda_{1j}} \cdot \frac{n-1}{\lambda_{2j}} \cdots \frac{1}{\lambda_{nj}}
\]

\[
P^* = (\omega_1^* \rightarrow \cdots \omega_2^* \rightarrow \omega_1^*)
\]

- Each column insertion – first some interaction among the leftmost particles, which then spreads throughout the array.
- Random entries – random dynamics.
- More precisely, random dynamics of the leftmost particles, which spreads to the array by deterministic rule.
Overview

RSK row/column insertion → **deterministic** lifting of 1D interacting system to 2D interacting system

For some particular distributions of entries – integrable (precise formulas for observables) 1D random interacting particle system.

$q$-deformed integrable 1D random interacting particle systems
$q \rightarrow 1$ limit

$\alpha$-Whittaker process

$q \rightarrow 1$ limit

$q$-deformed RSK row/column insertion → **random** lifting of 1D integrable particle system to 2D random interacting system. This lifting depends on the system.

Evolving with time partition functions of log-gamma random polymers

Geometric RSK column/row insertion evolution for gamma/inverse-gamma distributed entries.
TASEP systems

Particles that try to jump, but are blocked by right neighbor, stay put.

TASEP (continuous time): The $k$-th particle tries to jump to the right independently of others at rate $a_k$.

$q$-TASEP (continuous time): The $k$-th particle tries to jump to the right independently of others at rate $a_k(1 - q^{gap(k)})$, $gap(k) := x_{k-1} - x_k - 1$, $gap(1) := \infty$.

Borodin-Corwin, 2011

In discrete time at each step of dynamics all particles move.

Geometric $q$-TASEP (discrete time): The 1-st particle jumps by $m$ with probability $\frac{a_1 m(\alpha_1,q)_m}{(q;q)_m}$. The $k$-th particle ($k \geq 2$) jumps by $m$ with probability $\frac{a_k m(\alpha_k,q)^{gap(k)}_m(q;q)_m}{(q;q)_m^2(q^gap(k))}$.

Initial condition: $x_k = -k$.

Bernoulli $q$-TASEP (discrete time): Particles update their positions sequentially from right to left and can jump by at most 1. The first particle jumps with probability $\frac{\alpha_1}{1 + \alpha_1}$.

A) If the $k - 1$-st particle has jumped, then the $k$-th particle jumps with probability $\frac{\alpha_k}{1 + \alpha_k}$.

B) If the $k - 1$-st particle hasn’t jumped, then the $k$-th particle jumps with probability $\frac{\alpha_k (1 - q^{gap(k)})}{1 + \alpha_k}$.

Take $\alpha_k := \epsilon a_k$ (Bernoulli) or $\epsilon a_k (1 - q)$ (Geometric), make one step of dynamics occur in time $\epsilon$ and let $\epsilon \to 0$.

Borodin-Corwin, 2013
PushTASEP systems

Particles jump, even if blocked by right neighbor, and push this neighbor.

**PushTASEP** (continuous time): The \( k \)-th particle jumps to the right at rate \( a_k \) and possibly initiates an avalanche of pushings.

**\( q \)-PushTASEP** (continuous time): The \( k \)-th particle jumps to the right at rate \( a_k \), then it pushes the \( k+1 \)-st particle with probability \( q^{\text{gap}(k)} \), and so on...

\[
\text{gap}(k) := x_{k+1} - x_k - 1
\]

Initial condition: \( x_k = k \)

**Geometric \( q \)-PushTASEP** (discrete time): Particles update their positions sequentially from left to right. Given the move of the \( k \)-th particle by \( m \), the \( k+1 \)-st particle jumps due to the push by \( p \) with probability \( q^{\text{gap}(k)} \frac{q^{\text{gap}(k)}; q^{-1})_{m-p}}{(q^{-1}; q^{-1})_{m-p}} \), and then jumps by \( s \) with probability \( \frac{\alpha_{k+1}(\alpha_{k+1}; q)_{\infty}}{(q; q)_{s}} \).

**Bernoulli \( q \)-PushTASEP** (discrete time): Particles update their positions sequentially from left to right and can jump by at most 1. The first particle jumps with probability \( \frac{\alpha_k}{1+\alpha_k} \). A) If the \( k \)-th particle hasn’t jumped, then the \( k+1 \)-st particle jumps with probability \( \frac{\alpha_{k+1}}{1+\alpha_{k+1}} \).

B) If the \( k \)-th particle has jumped, then the \( k+1 \)-st particle jumps with probability \( \frac{\alpha_{k+1} + q^{\text{gap}(k)}}{1+\alpha_{k+1}} \).

Take \( \alpha_k := \epsilon a_k \) (Bernoulli) or \( \epsilon a_k (1 - q) \) (Geometric), make one step of dynamics occur in time \( \epsilon \) and let \( \epsilon \to 0 \).

\( x \) \hspace{1cm} \( \cdots \) \hspace{1cm} \( x_1 \) \hspace{1cm} \( x_2 \) \hspace{1cm} \( x_3 \) \hspace{1cm} \( \cdots \) \hspace{1cm} \( x_n \)

\( \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \)
RSK row and column insertions

Semistandard Young Tableaux with entries from \( \{1, 2, \ldots, n\} \)

Interlacing particle arrays on \( n \) levels

\[
\lambda^5 = (6, 5, 3, 2, 2) \\
\lambda^4 = (5, 5, 3, 2) \\
\lambda^3 = (5, 3, 2) \\
\lambda^2 = (4, 2) \\
\lambda^1 = (4)
\]

Row insertion of a word \( P \leftarrow \omega \)

Column insertion of a word \( \omega \rightarrow P \)

Several "Pull" waves

Several "Push" waves
RSK dynamics

RSK random dynamics on interlacing arrays

At time $t$ a column of independent random entries $A_{1t}, A_{2t}, \ldots, A_{nt}$ is attached to the table and the word $\omega_t$ is row inserted in the array.

At time $t$ a column of independent random entries $A_{1t}, A_{2t}, \ldots, A_{nt}$ is attached to the table and the word $\omega'_t$ is column inserted in the array.

(row Bernoulli dynamics) ------------ (row geometric dynamics)

Bernoulli distribution:
$P(A_{it} = 0) = \frac{1}{1 + \alpha_i \beta_t}$
$P(A_{it} = 1) = \alpha_i \beta_t \frac{1}{1 + \alpha_i \beta_t}$

Geometric distribution:
$P(A_{it} = k) = (1 - \alpha_i \beta_t)^k \beta_t$ for $k = 0, 1, 2, \ldots$

(column Bernoulli dynamics) ------------ (column geometric dynamics)

Bernoulli distribution:
$P(A_{it} = 0) = \frac{1}{1 + \alpha_i \beta_t}$
$P(A_{it} = 1) = \alpha_i \beta_t \frac{1}{1 + \alpha_i \beta_t}$

Geometric distribution:
$P(A_{it} = k) = (1 - \alpha_i \beta_t)^k \beta_t$ for $k = 0, 1, 2, \ldots$

Distribution on the array after $t$ steps starting from zero initial conditions is the Schur process given by $P(\lambda) = \frac{\prod_{k=1}^n \prod_{j=1}^t (1 - \alpha_i \beta_j)^k}{\prod_{k=1}^n \prod_{j=1}^t (1 - \alpha_i \beta_j)^k}$ $S_{\lambda^n}(\tilde{\beta}_1, \ldots, \tilde{\beta}_t)$

Distribution on the array after $t$ steps starting from zero initial conditions is the Schur process given by $P(\lambda) = \frac{\prod_{k=1}^n \prod_{j=1}^t (1 - \alpha_i \beta_j)^k}{\prod_{k=1}^n \prod_{j=1}^t (1 - \alpha_i \beta_j)^k}$ $S_{\lambda^n}(\alpha_1, \ldots, \alpha_t)$

Schur function is $S_\lambda = \sum_{T-semistandard tableaux} \prod_{k=1}^n \prod_{j=1}^t x_i^{\text{number of } i's} y_j^{\text{number of } j's}$

$S_\lambda(\tilde{\beta}_1, \ldots, \tilde{\beta}_t)$ is the image of the Schur function under a homomorphism from the ring of symmetric functions to real numbers given by $\sum_{m=1}^\infty x_m \rightarrow (-1)^m \sum_{j=1}^\infty \beta_j$. 
We seek $q$-deformations of these four dynamics.

row Bernoulli $\rightarrow$ $Q_{\text{row}}^q[\hat{\beta}]$  

column Bernoulli $\rightarrow$ $Q_{\text{col}}^q[\hat{\beta}]$

row geometric $\rightarrow$ $Q_{\text{row}}^q[\alpha]$  

column geometric $\rightarrow$ $Q_{\text{col}}^q[\alpha]$
What should the $q$-deformations of these dynamics be?

"Come, listen, my men, while I tell you again
The five unmistakable marks
By which you may know, wheresoever you go,
The warranted genuine Snarks.

Lewis Carroll, "The Hunting of the Snark".

1). (Distribution) The distribution of the array at a particular moment of time should be a $q$-Whittaker process instead of Schur process.

2). (Update) Each dynamics should be of sequential update type.

3). (Degenerations) For $q = 0$ these dynamics should be classical RSK dynamics. In the continuous limit we should recover constructions of Borodin-Petrov (row insertion case) and O’Connell-Pei (column insertion case).

4). (Projection) The leftmost or the rightmost particles of the array should evolve in a marginally markovian manner according to some 1D integrable dynamics.

5). (Naturalness) The dynamics should evolve through ”contiguous interactions” and each progressive wave of movements should travel all the way to the upper level.
### Distribution

skew Schur functions \( S_\lambda/\mu \rightarrow \) skew \( q \)-Whittaker functions \( P_\lambda/\mu \) and \( Q_\lambda/\mu \)

Schur processes \( \rightarrow \) \( q \)-Whittaker processes:

\[
\begin{align*}
\text{Top floor measure in the Schur case } & \quad P(\lambda^n) = \\
& \left( \prod_{i=1}^{n} \Pi_j^{t=1} \frac{1}{1 + \beta_j a_i} \right) \frac{S_\lambda(a_1, \ldots, a_n) S_\lambda(\hat{\beta}_1, \ldots, \hat{\beta}_t)}{S_\lambda(a_1, \ldots, a_n)} \\
\text{Stochastic Links in the Schur case} & \quad \begin{cases} \\
P(\lambda^n \rightarrow \lambda^{n-1}) = \frac{S_{\lambda^{n-1}}(a_1, \ldots, a_{n-1}) S_\lambda(a_1, \ldots, a_n)}{S_{\lambda^{n-1}}(a_1, \ldots, a_{n-1})} \\
\vdots \\
\end{cases} \\
\text{Top floor measure in the } q \text{-Whittaker case } & \quad P(\lambda^n) = \\
& \left( \prod_{i=1}^{n} \Pi_j^{t=1} \frac{1}{1 + \beta_j a_i} \right) \frac{P_\lambda(a_1, \ldots, a_n) Q_\lambda(\hat{\beta}_1, \ldots, \hat{\beta}_t)}{P_\lambda(a_1, \ldots, a_n) Q_\lambda(a_1, \ldots, a_n)} \\
\text{Stochastic Links in the } q \text{-Whittaker case} & \quad \begin{cases} \\
P(\lambda^n \rightarrow \lambda^{n-1}) = \frac{P_{\lambda^{n-1}}(a_1, \ldots, a_{n-1}) P_{\lambda^{n-1}}(a_n)}{P_{\lambda^{n-1}}(a_1, \ldots, a_{n-1})} \\
\vdots \\
\end{cases}
\end{align*}
\]

\( Q_\lambda(\hat{\beta}_1, \ldots, \hat{\beta}_t) \) is the image of the \( q \)-Whittaker function under a homomorphism from the ring of symmetric functions to real numbers given by \( \sum_{m=1}^{\infty} x_m \rightarrow (-1)^{k-1}(1 - q^k) \sum_{j=1}^{t} \beta_j^k \).
Sequential update type dynamics:

\[
\lambda^n \rightarrow U^n \quad \lambda^{n-1} \rightarrow U^{n-1} \quad \vdots \quad \lambda^1 \rightarrow U^1
\]

Probability of such transition at time step \( t - 1 \rightarrow t \) is of the form

\[
U_1(\lambda^1 \rightarrow \nu^1 \mid \lambda^2 \rightarrow \nu^2 \mid \lambda^1 \rightarrow \nu^1) \cdots U_n(\lambda^n \rightarrow \nu^n \mid \lambda^{n-1} \rightarrow \nu^{n-1}).
\]

Update on the 1st level update on the 2nd level update on the nth level

\[
\sum_{\nu_k} U_k(\lambda^k \rightarrow \nu^k \mid \lambda^{k-1} \rightarrow \nu^{k-1}) = 1, \quad (\lambda^{k-1}, \lambda^k, \nu^{k-1} \text{ are fixed}),
\]

\[
\sum_{\nu_k} U_k(\lambda^k \rightarrow \nu^k \mid \lambda^{k-1} \rightarrow \nu^{k-1}) = (\alpha_t a_k; q) \infty \beta_{\lambda^k/\mu^{k-1}}(\lambda^k, \nu^k, \nu^{k-1} \text{ are fixed}).
\]

Equations from the condition on distributions:

\[
\psi_{\lambda/\mu} := \prod_{i=1}^{\ell(\mu)} \left( \frac{\lambda_i - \lambda_{i+1}}{\lambda_i - \mu_i} \right) q
\]

\[
\varphi_{\lambda/\mu} := \frac{1}{q \beta_{\lambda - 1/\mu}} \prod_{i=1}^{\ell(\mu)} \left( \frac{\mu_i - \mu_{i+1}}{\mu_i - \lambda_{i+1}} \right) q
\]

\[
\psi'_{\lambda/\mu} := \prod_{i=1}^{\ell(\mu)} \left( \frac{\lambda_i - \lambda_{i+1}}{\lambda_i - \mu_i} \right) q
\]

Systems of linear equations: we are interested only in nonnegative (!) solutions.

\[
\psi_{\lambda/\mu} := \prod_{i=1}^{\ell(\mu)} \left( \frac{\lambda_i - \lambda_{i+1}}{\lambda_i - \mu_i} \right) q
\]
Projections

Rightmost particles (after shift $x_k := \lambda_i^k + k$)  
Leftmost particles (after shift $x_k := \lambda_k^k - k$)

$q$ - row Bernoulli $Q_{\text{row}}^q[\hat{\beta}]$  
Bernoulli $q$-PushTASEP

$q$ - column Bernoulli $Q_{\text{col}}^q[\hat{\beta}]$  
Bernoulli $q$-TASEP

$q$ - row geometric $Q_{\text{row}}^q[\alpha]$  
geometric $q$-PushTASEP

$q$ - column geometric $Q_{\text{col}}^q[\alpha]$  
geometric $q$-TASEP
Continuous time limits I

In $\alpha, \beta \to 0$ limit (time rescaled)

**Row dynamics** $Q_{\text{row}}^\alpha$ and $Q_{\text{row}}^\beta \rightarrow$ construction of *Borodin-Petrov, 2014:*

Rightmost particle at the $j$-th level jumps independently from the others with rate $a_j$. Each move triggers move of one particle on the upper level.
Continuous time limits II

In $\alpha, \beta \rightarrow 0$ limit (time rescaled)

Column dynamics $Q^q_{\text{col}}[\alpha]$ and $Q^q_{\text{col}}[\beta] \rightarrow$ construction of O'Connell-Pei, 2013:

Particle $\lambda^\alpha_j$ jumps independently from the others with rate:
- $a_j(1 - q^{j-1})$ if $k = j$,
- $a_j(1 - q^{j-1})q\sum_{i=k}^{j-1} \lambda^{j-1} - \lambda^i_{i+1}$ if $1 < k < j$,
- $a_jq\sum_{i=1}^{j-1} \lambda^{j-1} - \lambda^i_{i+1}$ if $k = 1$.

Each move triggers move of one particle on the upper level:
Given update $\lambda = \lambda' \rightarrow \nu' = \nu$ on the predecessor level, to determine update $\lambda = \lambda' \rightarrow \nu' = \nu$ on the $j$-th level first choose **Birth of a new impulse** (with probability $\frac{\beta}{1 + \alpha_j}$): $|\nu' - |\lambda'| + |\lambda'| - |\nu'| | = 1$ and the rightmost particle on the $j$-th level moves \textit{or} **No Birth** (with probability $\frac{1}{1 + \alpha_j}$): $|\nu' - |\lambda'| + |\lambda'| - |\nu'| | = 0$.

Then independent splitting of islands

Let $f_i := \frac{\lambda_i - \hat{\lambda}_i}{1 - q^{\lambda_i - \hat{\lambda}_i}}$, $g_i := 1 - q^{\lambda_i - \hat{\lambda}_i}$.

Except in the case of Birth the rightmost particle on the $j$-th level moves and the island containing the rightmost particle on the $j$-th level (if it exists) splits the first way.
Birth of a new impulse (with probability $\frac{\beta}{1+\beta}$): $|\nu| - |\lambda| + |\lambda^{-1} - |\nu^{-1}| = 1$ or No Birth (with probability $\frac{1}{1+\beta}$): $|\nu| - |\lambda| + |\lambda^{-1} - |\nu^{-1}| = 0$. Then independent detachment of rightmost impulse from each island.

Let $f'_i := \frac{1-\nu^{j-1}}{1-\nu^i-1}$, $g'_i := 1 - q^{\lambda^{-1} - \lambda_i}$.

Probability (r might be 0)

In case of Birth also another particle moves, the leftmost if $r = j - 1$, if $r < j - 1$ - according to:

Let $f'_i := \frac{1-\nu^{j-1}}{1-\nu^i-1}$, $g'_i := 1 - q^{\lambda^{-1} - \lambda_i}$.

Probability (r might be 0)

In case of Birth also another particle moves, the leftmost if $r = j - 1$, if $r < j - 1$ - according to:
Complementation

\[ U_j^q(\lambda \rightarrow \nu | \bar{\lambda} \rightarrow \bar{\nu}) := \left( a_j \beta \right)^{-2(|\lambda| - |\nu| - |\bar{\lambda}| + |\bar{\nu}|) - 1} U_j([-S - \lambda] \rightarrow [S + 1 - \nu] | [-S - \bar{\lambda}] \rightarrow [S + 1 - \bar{\nu}]) \]

for \( S \) large enough.

for \( Q^q_{\text{col}}[\bar{\beta}] \)

\[ U_j^q(\lambda \rightarrow \nu | \bar{\lambda} \rightarrow \bar{\nu}) := \left( a_j \beta \right)^{-2(|\lambda| - |\nu| - |\bar{\lambda}| + |\bar{\nu}|) - 1} U_j([-S - \lambda] \rightarrow [S + 1 - \nu] | [-S - \bar{\lambda}] \rightarrow [S + 1 - \bar{\nu}]) \]

for \( Q^q_{\text{row}}[\bar{\beta}] \)

\[ U_j^q(\lambda \rightarrow \nu | \bar{\lambda} \rightarrow \bar{\nu}) := \left( a_j \beta \right)^{-2(|\lambda| - |\nu| - |\bar{\lambda}| + |\bar{\nu}|) - 1} U_j([-S - \lambda] \rightarrow [S + 1 - \nu] | [-S - \bar{\lambda}] \rightarrow [S + 1 - \bar{\nu}]) \]

\[ [S - \lambda] := S - \lambda_j \geq S - \lambda_{j-1} \geq \cdots \geq S - \lambda_1 \]

\[ \lambda_j \quad \lambda_{j-1} \quad \ldots \quad \lambda_2 \quad \lambda_1 \quad S \]

\[ S - \lambda_1 \quad S - \lambda_2 \quad S - \lambda_{j-1} \quad S - \lambda_j \]

Reverse direction
**q - Geometric Row Construction** $Q^q_{\text{ROW}}[\alpha]$

"impulse allocation among neighbors"

$q$-deformed beta distribution on $\{0, 1, \ldots, y\}$ –

with $P(s) = \eta_{q,\mu,\nu}(s \mid y) := \mu^s \frac{(q/\nu)^s(q/\nu)^{y-s}}{(q)^y}$

---

Given update $\bar{\lambda} = \lambda^{j-1} \to \nu^{j-1} = \bar{\nu}$ on the $j-1$-st level, determine update $\lambda = \lambda^j \to \nu^j = \nu$ on the $j$-th level first independently for each $1 \leq k \leq j-1$ with probability $\eta_q^{-1,0,\nu}\bar{\lambda}_k - \bar{\lambda}_k - d$.

Then the leftmost particle on the $j$-th level makes additional jump $\ell$ with probability $\eta_{q,a_j,\alpha;0}(\ell \mid \infty) = (a_j \alpha; q)^{a_j \alpha} (q)^{\ell}$

---

![Diagram showing geometric row construction](image-url)
q - Geometric Column Construction $Q_{\text{row}}^q[\alpha]$

"immediate vs. reserve pushing"

Given update $\lambda = \lambda^j \rightarrow \nu^{j-1} = \bar{\nu}$ on the $j-1$-st level, determine update $\lambda = \lambda^j \rightarrow \nu^j = \nu$ on the $j$-th level first by successively updating positions of particles from left to right.

$\lambda_k$, voluntary movement $x$ \hspace{1cm} immediate push $y$ \hspace{1cm} push from the Stabilization Fund $z$

Where $S$ is the current value of the Stabilization Fund.

After such step the Stabilization Fund is updated

$S \rightarrow S + \bar{\nu}_k - \hat{\lambda}_k - y - z$.

For $k = 1$ just take $y = \bar{\nu}_1 - \hat{\lambda}_1$, $z$ - current value of the Stabilization Fund.
**Limit** \( q \to 1 \)

\[
q = e^{−\epsilon}, a_j = e^{−\theta_j \epsilon}, \alpha_t = e^{−\hat{\theta}_t \epsilon}; \quad \hat{\theta}_j, \theta_t > 0, \epsilon \to 0^+.
\]

Gamma distribution \( X \sim \Gamma(\theta) \) – \( P(X \in dx) = \frac{1}{\Gamma(\theta)}x^{\theta−1}e^{−x}dx; \) \( P(X^{-1} \in dx) = \frac{1}{\Gamma(\theta)}x^{−\theta−1}e^{−1/x}dx. \)

**Geometric \( q \)-PushTASEP**

\[
x_1 x_2 x_3 \ldots x_n
\]

- Rescale: \( x_j(t) − j =: (t + j - 1)\epsilon^{−1} \log \epsilon^{−1} + \epsilon^{−1} \log R^1_j(t, \epsilon) \)

**Log-gamma polymer**

*Seppäläinen, 2012*

\[
R^1_j(t) := \sum_{\pi: (1,1)\rightarrow (t,j)} \prod_{(s,i)\in \pi} a^j_s
\]

- \( \pi \) – up/right lattice paths

|   |   |   |   | \( a^j_1 \) |   | \( a^j_2 \) |   | \( a^j_3 \) |   | \( \ldots \) | \( a^j_n \) |
|---|---|---|---|----|---|----|---|----|---|---|
| \( a^1_1 \) | \( a^1_2 \) | \( \ldots \) | \( a^1_n \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( a^m_1 \) | \( a^m_2 \) | \( \ldots \) | \( a^m_n \) |

**Strict-weak polymer**

*Corwin - Seppäläinen-Shen; O’Connell-Ortmann, 2014*

\[
L^1_j(t) := \sum_{\pi: (0,1)\rightarrow (t,j)} \prod_{(s,i)\in \pi} a^j_s
\]

- \( \pi \) – up-and-right/right lattice paths

|   |   |   |   | \( a^j_1 \) |   | \( a^j_2 \) |   | \( a^j_3 \) |   | \( \ldots \) | \( a^j_n \) |
|---|---|---|---|----|---|----|---|----|---|---|
| \( a^1_1 \) | \( a^1_2 \) | \( \ldots \) | \( a^1_n \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( a^m_1 \) | \( a^m_2 \) | \( \ldots \) | \( a^m_n \) |

**Rescale**

- \( x_j(t) + j =: (t + j + 1)\epsilon^{−1} \log \epsilon^{−1} - \epsilon^{−1} \log L^1_j(t, \epsilon) \)

*CSS, 2014*

**Log-gamma polymer**

*M.-Petrov, 2015*
**Array limit $q \to 1$**

$r_{j,k}(t,\epsilon)$ - position of the $k$-th particle from the right on the $j$-th level after $t$ steps of $Q^q_{\text{row}[\alpha]}$ dynamics.

$l_{j,k}(t,\epsilon)$ - position of the $k$-th particle from the left on the $j$-th level after $t$ steps of $Q^q_{\text{col}[\alpha]}$ dynamics.

<table>
<thead>
<tr>
<th>$r_{j,k}(t,\epsilon)$</th>
<th>$l_{j,k}(t,\epsilon)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>For $t \geq k$</td>
<td>For $t \geq j - k + 1$</td>
</tr>
<tr>
<td>$r_{j,k}(t,\epsilon) =$</td>
<td>$l_{j,k}(t,\epsilon) =$</td>
</tr>
<tr>
<td>$(t + j - 2k + 1)\epsilon^{-1} \log \epsilon^{-1} + \epsilon^{-1} \log \hat{R}_{j,k}(t,\epsilon)$</td>
<td>$(t - j + 2k - 1)\epsilon^{-1} \log \epsilon^{-1} - \epsilon^{-1} \log \hat{L}_{j,k}(t,\epsilon)$</td>
</tr>
</tbody>
</table>

$R_{j,k}^l(t)$ - sum over $k$-tuples of nonintersecting paths from $(1,1), \ldots, (1,k)$ to $(t,j-k+1), \ldots, (t,j)$ in the setting of log-gamma polymer.

$\hat{R}_{j,k}(t) := R_{j,k}^l(t)/R_{j,k}^l(t)$

Then

$\hat{R}_{j,k}(t,\epsilon) \to \hat{R}_{j,k}(t)$

$L_{j,k}^l(t)$ - sum over $k$-tuples of nonintersecting paths from $(0,1), \ldots, (0,k)$ to $(t,j-k+1), \ldots, (t,j)$ in the setting of strict-weak polymer.

$\hat{L}_{j,k}(t) := L_{j,k}^l(t)/L_{j,k}^l(t)$

Then

$\hat{L}_{j,k}(t,\epsilon) \to \hat{L}_{j,k}(t)$

![Diagram](image1.png)

![Diagram](image2.png)
Geometric / (de)tropical RSK row insertion

Geometric row insertion
(detropicalization via \((\max, +) \rightarrow (+, x)\))

\[ a = a(0) \]
\[ z_1 \rightarrow z'_1 \]
\[ a(1) \]
\[ z_2 \rightarrow z'_2 \]
\[ a(2) \]
\[ \vdots \]
\[ a(n-1) \]
\[ z_n \rightarrow z'_n \]

\[ \lambda \]
\[ \xrightarrow{a} \]
\[ b \]

\[ \nu^j = \sum_{i=k}^j \lambda^i a^i \cdots a^j \]
\[ b^j = a^j \frac{\lambda^j \nu^j - 1}{\lambda^{j-1} \nu^j} \]
Geometric / (de)tropical column insertion

\[ L_k^j(t) \]

Geometric column insertion
(detropicalization via \((\min, +) \rightarrow (+, \times)\))

\[
\begin{align*}
\nu^k &= a^k \lambda^k \\
\nu^j &= \lambda^j a^j + \lambda^{j-1} \text{ for } k < j \leq n \\
b^j &= a^j \frac{\lambda^j \nu^{j-1}}{\lambda^{j-1} \nu^j} \text{ if } \lambda^j > 0, = a^j \nu^{j-1} \text{ if } \lambda^j = 0 \\
\text{and } \lambda^{j-1} > 0 \text{ and } = a^j \text{ if } \lambda^{j-1} = 0
\end{align*}
\]