Calibrated Submanifolds

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Introduction

A key aspect of mathematics is the study of variational problems. These can vary from the purely analytic to the very geometric. A classic geometric example is the study of geodesics, which are critical points for the length functional on curves. As we know, understanding the geodesics of a given Riemannian manifold allows us to understand some of the ambient geometry, for example the curvature. The higher dimensional analogue would be to study critical points for the volume functional, and we would hope (and it indeed turns out to be the case) that these critical points, called *minimal submanifolds*, encode crucial aspects of the geometry of the manifold.

Just like the geodesic equation, we would expect (and it is true) that minimal submanifolds are defined by a second order partial differential equation. Such equations are very difficult to solve in general, so a key idea is to find a special class of minimal submanifolds, called *calibrated submanifolds*, which are instead defined by a first order partial differential equation. The definition of calibrated submanifolds is motivated by the properties of complex submanifolds in Kähler manifolds, and turns out to be useful in finding minimizers for the volume functional rather than just critical points. However, finding examples outside the classical complex setting turns out to be difficult, leading to important methods coming from a variety of sources, as well as motivating the study of the deformation theory of these objects.

Calibrated submanifolds naturally arise when the ambient manifold has *special holonomy*, including holonomy G_2. In this situation, we would hope that the calibrated submanifolds encode even more, finer, information about the ambient manifold, potentially leading to the construction of invariants. In this setting, there is also a relationship between calibrated submanifolds and gauge theory: specifically, connections whose curvature satisfies a natural constraint determined by the special holonomy group (so-called *instantons*). For these reasons, calibrated submanifolds form a hot topic in current research, especially in the G_2 setting.

1 Minimal submanifolds

We start by analysing the submanifolds which are critical points for the volume functional. Let N be a submanifold (without boundary) of a Riemannian manifold (M,g) and let $F : N \times (-\epsilon,\epsilon) \to M$ be a variation of N with compact support; i.e. $F = \mathrm{Id}$ outside a compact subset $\overline{S}$ of N with S open and $F(p,0) = p$ for all $p \in N$. The vector field $X = \frac{\partial F}{\partial t}|_N$ is called the variation vector field. We have the following definition.

Definition 1.1. N is *minimal* if $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}(F(S,t))|_{t=0} = 0$ for all variations F with compact support $\overline{S}$ (depending on F).

Remark Notice that we do not ask for N to minimize volume: it is only stationary for the volume.

Example. A plane in $\mathbb{R}^n$ is minimal since any small variation will have larger volume.

Example. Geodesics are locally length minimizing, so geodesics are minimal. However, as an example, the equator in $\mathcal{S}^2$ is minimal but not length minimizing since we can deform it to a shorter line of latitude.

For simplicity let us suppose that N is compact. We wish to calculate $\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Vol}(F(N,t))|_{t=0}$. Given local coordinates x_i on N we know that

$$\operatorname{Vol}(F(N,t)) = \int_N \sqrt{\det\left(g\left(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j}\right)\right)}\operatorname{vol}_N .$$

Let $p \in N$ and choose our coordinates x_i to be normal coordinates at p: i.e. so that $\frac{\partial F}{\partial x_i}(p,t) = e_i(t)$ satisfy $g(e_i(0), e_j(0)) = \delta_{ij}$. If $g_{ij}(t) = g(e_i(t), e_j(t))$ then we know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\det(g_{ij}(t))}|_{t=0} = \frac{1}{2}\frac{\sum_i g'_{ii}(t)}{\sqrt{\det(g_{ij}(t))}}|_{t=0} = \frac{1}{2}\sum_i g'_{ii}(0).$$

Now

$$\begin{aligned}\frac{1}{2}\sum_i g'_{ii}(0) &= \frac{1}{2}\sum_i \frac{\mathrm{d}}{\mathrm{d}t} g\left(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_i}\right)|_{t=0}\\ &= \sum_i g(\nabla_X e_i, e_i)\\ &= \sum_i g(\nabla_{e_i} X, e_i) = \operatorname{div}_N(X)\end{aligned}$$

since $[X, e_i] = 0$ (i.e. the t and x_i derivatives commute). Moreover, we see that

$$\operatorname{div}_N(X) = \sum_i g(\nabla_{e_i} X, e_i) = \operatorname{div}_N X^{\mathrm{T}} - \sum_i g(X^\perp, \nabla_{e_i} e_i) = \operatorname{div}_N X^{\mathrm{T}} - g(X, H)$$

(since $\nabla_{e_i} g(X^\perp, e_i) = 0$) where $^{\mathrm{T}}$ and $^\perp$ denote the tangential and normal parts and

$$H = \sum_i \nabla^\perp_{e_i} e_i$$

is the *mean curvature vector*. Overall we have the following.

Theorem 1.2. *The first variation formula is*

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Vol}(F(N,t))|_{t=0} = \int_N \operatorname{div}_N(X)\operatorname{vol}_N = -\int_N g(X,H)\operatorname{vol}_N .$$

We deduce the following.

Definition 1.3. N is a *minimal submanifold* if and only if $H = 0$.

This is a *second order nonlinear PDE*. For a function $f : U \subseteq \mathbb{R}^k \to \mathbb{R}^{n-k}$ we shall see in Problem Sheet 1 that Graph(f) is minimal if and only if

$$\operatorname{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right) = 0.$$

We see that we can write this equation as $\Delta f + Q(\nabla f, \nabla^2 f) = 0$ where Q consists of nonlinear terms (but linear in $\nabla^2 f$). Hence, if we linearise this equation we just get $\Delta f = 0$, so f is harmonic. In other words, the minimal submanifold equation is a nonlinear equation whose linearisation is just Laplace's equation: this is an example of a nonlinear *elliptic* PDE, which we shall discuss further later.

Example. A plane in $\mathbb{R}^n$ is trivially minimal because if X, Y are any vector fields on the plane then $\nabla^\perp_X Y = 0$ as the second fundamental form of a plane is zero.

Example. For curves γ, $H = 0$ is equivalent to the geodesic equation $\nabla_{\dot\gamma}\dot\gamma = 0$.

The most studied minimal submanifolds (other than geodesics) are minimal surfaces in $\mathbb{R}^3$, since here the equation $H = 0$ becomes a scalar equation on a surface, which is the simplest to analyse. In general we would have a system of equations, which is more difficult to study.

Example. The helicoid $M = \{(t\cos s, t\sin s, s) \in \mathbb{R}^3 \,:\, s,t \in \mathbb{R}\}$ is a complete embedded minimal surface, discovered by Meusnier in 1776.

Example. The catenoid $M = \{(\cosh t\cos s, \cosh t\sin s, t) \in \mathbb{R}^3 \,:\, s,t,\in \mathbb{R}\}$ is a complete embedded minimal surface, discovered by Euler in 1744 and shown to be minimal by Meusnier in 1776.

In fact the helicoid and the catenoid are locally isometric, and there is a 1-parameter family of locally isometric minimal surfaces deforming between the catenoid and helicoid.

It took about 70 years to find the next minimal surface, but now we know many examples of minimal surfaces in $\mathbb{R}^3$, as well in other spaces by studying the nonlinear elliptic PDE given by the minimal surface equation. The amount of literature in the area is vast, with key results including the Lawson and Willmore Conjectures, and minimal surfaces have applications to major problems in geometry including the Positive Mass Theorem, the Penrose Conjecture and the Poincaré Conjecture.

2 Introduction to calibrations

As we have seen, minimal submanifolds are extremely important. However there are two key issues.

- Minimal submanifolds are defined by a second order nonlinear PDE system – therefore they are hard to analyse.
- Minimal submanifolds are only critical points for the volume functional, but we are often interested in minima for the volume functional – we need a way to determine when this occurs.

We can help resolve these issues using the notion of calibration, introduced by Harvey–Lawson (1982).

Definition 2.1. A differential k-form η on a Riemannian manifold (M,g) is a *calibration* if

- $\mathrm{d}\eta = 0$ and
- $\eta(e_1,\ldots,e_k) \leq 1$ for all unit tangent vectors $e_1,\ldots,e_k$ on M.

Example. Any form with constant coefficients on $\mathbb{R}^n$ can be rescaled so that it is a calibration with at least one plane where equality holds.

This example shows that there are many calibrations η, but the interesting question is: for which planes $V = \mathrm{Span}\{e_1,\ldots,e_k\}$ does $\eta(e_1,\ldots,e_k) = 1$? More importantly, can we find submanifolds N so that this equality holds on each tangent space? This motivates the next definition.

Definition 2.2. Let η be a calibration k-form on (M,g). An oriented k-dimensional submanifold N of (M,g) is *calibrated* by η if $\eta|_N = \mathrm{vol}_N$, i.e. if for all $p \in N$ we have $\eta(e_1,\ldots,e_k) = 1$ for an oriented orthonormal basis $e_1,\ldots,e_k$ for T_pN.

Example. Any plane in $\mathbb{R}^n$ is calibrated. If we change coordinates so that the plane P is $\{x \in \mathbb{R}^n \,:\, x_{k+1} = \ldots = x_n = 0\}$ then $\eta = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_k$ is a calibration and P is calibrated by η.

Notice that the calibrated condition is now an algebraic condition on the tangent vectors to N, so being calibrated is a *first order nonlinear PDE*. We shall motivate these definitions further later, but for now we make the following observation.

Theorem 2.3. *Let N be a calibrated submanifold. Then N is minimal and moreover if F is any variation compact support $\overline{S}$ then* $\mathrm{Vol}(F(S,t)) \geq \mathrm{Vol}(S)$*; i.e. N is volume-minimizing.*

Proof. Suppose that N is calibrated by η and suppose for simplicity that N is compact. We will show that N is homologically volume-minimizing.

Suppose that N' is homologous to N. Then,

$$\mathrm{Vol}(N) = \int_N \eta = \int_{N'} \eta \leq \mathrm{Vol}(N')$$

by Stokes' Theorem as $\mathrm{d}\eta = 0$, since because N, N' are homologous there exists a compact manifold K with boundary $-N \cup N'$ and by Stokes' Theorem

$$0 = \int_K \mathrm{d}\eta = \int_{N'} \eta - \int_N \eta.$$

We have the result by the definition of minimal submanifold. □

We conclude this introduction with the following elementary result.

Proposition 2.4. *There are no compact calibrated submanifolds in* $\mathbb{R}^n$.

Proof. Suppose that η is a calibration and N is compact and calibrated by η. Then $\mathrm{d}\eta = 0$ so by the Poincaré Lemma $\eta = \mathrm{d}\zeta$, and hence

$$\mathrm{Vol}(N) = \int_N \eta = \int_N \mathrm{d}\zeta = 0$$

by Stokes' Theorem. □

Although there are many calibrations, having calibrated submanifolds greatly restricts the calibrations you want to consider. The calibrations which have calibrated submanifolds have special significance and there is a particular connection with special holonomy, due to the following observations.

Let G be a holonomy group of a Riemannian metric g on an n-manifold M. Then G acts on the k-forms on $\mathbb{R}^n$, so suppose that η_0 is a G-invariant k-form. We can always rescale η_0 so that $\eta_0|_P \leq \mathrm{vol}_P$ for all oriented k-planes P and equality holds for at least one P. Since η_0 is G-invariant, if P is calibrated then so is $\gamma \cdot P$ for any $\gamma \in G$, which usually means we have quite a few calibrated planes. We know by the *holonomy principle* that we then get a parallel k-form η on M which is identified with η_0 at every point. Since $\nabla\eta = 0$, we have $\mathrm{d}\eta = 0$ and hence η is a calibration. Moreover, we have a lot of calibrated tangent planes on M, so we can hope to find calibrated submanifolds.

3 Complex submanifolds

We would now like to address the question: where does the calibration condition come from? The answer is from *complex geometry*. On $\mathbb{R}^{2n} = \mathbb{C}^n$ with coordinates $z_j = x_j + iy_j$, we have the complex structure J and the distinguished Kähler 2-form

$$\omega = \sum_{j=1}^n \mathrm{d}x_j \wedge \mathrm{d}y_j = \frac{i}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j.$$

More generally we can work with a *Kähler manifold* (M, J, ω). Our first key result is the following.

Theorem 3.1. *On a Kähler manifold* (M, J, ω), $\frac{\omega^k}{k!}$ *is a calibration whose calibrated submanifolds are the complex* k*-dimensional submanifolds: i.e. submanifolds* N *such that* $J(T_pN) = T_pN$ *for all* $p \in N$.

Since $\mathrm{d}\omega^k = k\mathrm{d}\omega \wedge \omega^{k-1} = 0$, the theorem follows immediately from the following result.

Theorem 3.2 (Wirtinger's inequality)**.** *For any unit vectors* $e_1, \ldots, e_{2k} \in \mathbb{C}^n$,

$$\frac{\omega^k}{k!}(e_1, \ldots, e_{2k}) \leq 1$$

with equality if and only if $\mathrm{Span}\{e_1, \ldots, e_{2k}\}$ *is a complex* k*-plane in* $\mathbb{C}^n$.

Before proving this we make the following observation, which is an exercise in Problem Sheet 2.

Lemma 3.3. *If* η *is a calibration and* $*\eta$ *is closed then* $*\eta$ *is a calibration.*

Proof. We see that $|\frac{\omega^k}{k!}|^2 = \frac{n!}{k!(n-k)!}$ and $\mathrm{vol}_{\mathbb{C}^n} = \frac{\omega^n}{n!}$ so $*\frac{\omega^k}{k!} = \frac{\omega^{n-k}}{(n-k)!}$. Hence, by the lemma, it is enough to study the case where $k \leq \frac{n}{2}$.

Let P be any $2k$-plane in $\mathbb{C}^n$ with $2k \leq n$. We shall find a canonical form for P. First consider $\langle Ju, v\rangle$ for orthonormal unit vectors $u, v \in P$. This must have a maximum, so let $\cos\theta_1 = \langle Ju, v\rangle$ be this maximum where $0 \leq \theta_1 \leq \frac{\pi}{2}$.

Suppose that $w \in P$ is a unit vector orthogonal to $\mathrm{Span}\, u, v$. The function

$$f_w(\theta) = \langle Ju, \cos\theta v + \sin\theta w\rangle$$

has a maximum at $\theta = 0$ so $f_w'(0) = \langle Ju, w\rangle = 0$. Similarly we have that $\langle Jv, w\rangle = 0$, and thus $w \in \mathrm{Span}\{u, v, Ju, Jv\}^\perp$.

We then have two cases. If $\theta_1 = 0$ then $v = Ju$ so we can set $u = e_1, v = Je_1$ and see that $P = \mathrm{Span}\{e_1, Je_1\} \times Q$ where Q is a $2(k-1)$-plane in $\mathbb{C}^{n-1} = \mathrm{Span}\{e_1, Je_1\}^\perp$. If $\theta_1 \neq 0$ we have that $v = \cos\theta_1 Ju + \sin\theta_1 w$ where w is a unit vector orthogonal to u and Ju, so we can let $u = e_1$, $w = e_2$ and see that $P = \mathrm{Span}\{e_1, \cos\theta_1 Je_1 + \sin\theta_1 e_2\} \times Q$ where Q is a $2(k-1)$-plane in $\mathbb{C}^{n-2} = \mathrm{Span}\{e_1, Je_1, e_2, Je_2\}^\perp$.

Proceeding by induction we see that we have an oriented basis $\{e_1, Je_1, \ldots, e_n, Je_n\}$ for $\mathbb{C}^n$ so that

$$P = \mathrm{Span}\{e_1, \cos\theta_1 Je_1 + \sin\theta_1 e_2, \ldots, e_{2k-1}, \cos\theta_k Je_{2k-1} + \sin\theta_k e_{2k}\},$$

where $0 \leq \theta_1 \leq \ldots \leq \theta_{k-1} \leq \frac{\pi}{2}$ and $\theta_{k-1} \leq \theta_k \leq \pi - \theta_{k-1}$.

Since we can write $\omega = \sum_{j=1}^n e^j \wedge Je^j$ we see that $\frac{\omega^k}{k!}$ restricts to P to give a product of $\cos\theta_j$ which is certainly less than or equal to 1. Moreover, equality holds if and only if all of the $\theta_j = 0$ which means that P is complex. □

Corollary 3.4. *Compact complex submanifolds of Kähler manifolds are homologically volume-minimizing.*

We know that complex submanifolds are defined by holomorphic functions; i.e. solutions to the Cauchy–Riemann equations, which are a first-order PDE system.

Example. $N = \{(z, \frac{1}{z}) \in \mathbb{C}^2 : z \in \mathbb{C} \setminus \{0\}\}$ is a complex curve in $\mathbb{C}^2$, and thus is calibrated.

Example. An important non-trivial example of a Kähler manifold is $\mathbb{CP}^n$, where the zero set of a system of polynomial equations defines a (singular) complex submanifold.

4 Special Lagrangians

Complex submanifolds are very familiar, but can we find any other interesting classes of calibrated submanifolds? The answer is that indeed we can, particularly when the manifold has special holonomy. We begin with the case of holonomy $\mathrm{SU}(n)$ – so-called *Calabi–Yau manifolds.* The model example for Calabi–Yau manifolds is $\mathbb{C}^n$ with complex structure J, Kähler form ω and holomorphic volume form

$$\Upsilon = \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n,$$

if $z_1, \ldots, z_n$ are complex coordinates on $\mathbb{C}^n$.

Theorem 4.1. *Let M be a Calabi–Yau manifold with holomorphic volume form Υ. Then* $\mathrm{Re}(e^{-i\theta}\Upsilon)$ *is a calibration for any $\theta \in \mathbb{R}$.*

Since $\mathrm{d}\Upsilon = 0$, the result follows immediately from the following result.

Theorem 4.2. *On $\mathbb{C}^n$, $|\Upsilon(e_1, \ldots, e_n)| \leq 1$ for all unit vectors $e_1, \ldots, e_n$ with equality if and only if $P = \mathrm{Span}\{e_1, \ldots, e_n\}$ is a Lagrangian plane, i.e. P is an n-plane such that $\omega|_P \equiv 0$.*

Proof. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$ and let P be an n-plane in $\mathbb{C}^n$. There exists $A \in \mathrm{GL}(n, \mathbb{C})$ so that $f_1 = Ae_1, \ldots, f_n = Ae_n$ is an orthonormal basis for P. Then $\Upsilon(Ae_1, \ldots, Ae_n) = \det_{\mathbb{C}}(A)$ so

$$|\Upsilon(f_1, \ldots, f_n)|^2 = |\det{}_{\mathbb{C}}(A)|^2 = |\det{}_{\mathbb{R}}(A)| = |f_1 \wedge Jf_1 \wedge \ldots \wedge f_n \wedge Jf_n| \leq |f_1||Jf_1| \ldots |f_n||Jf_n| = 1$$

with equality if and only if $f_1, Jf_1, \ldots, f_n, Jf_n$ are orthonormal. However, this is exactly equivalent to the Lagrangian condition, since $\omega(u, v) = g(Ju, v)$ so $\omega|_P \equiv 0$ if and only if $JP = P^\perp$. □

Definition 4.3. A submanifold N of M calibrated by $\mathrm{Re}(e^{-i\theta}\Upsilon)$ is called *special Lagrangian* with phase $e^{i\theta}$. If $\theta = 0$ we say that N is simply special Lagrangian. By the previous theorem, we see that N is special Lagrangian if and only if $\omega|_N \equiv 0$ (i.e. N is Lagrangian) and $\mathrm{Im}\,\Upsilon|_N \equiv 0$ (up to a choice of orientation so that $\mathrm{Re}\,\Upsilon|_N > 0$).

Example. Consider $\mathbb{C} = \mathbb{R}^2$ with coordinates $z = x + iy$, complex structure J given by $Jw = iw$, Kähler form $\omega = \mathrm{d}x \wedge \mathrm{d}y = \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z}$ and holomorphic volume form $\Upsilon = \mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$. We want to consider the special Lagrangians in $\mathbb{C}$, which are 1-dimensional submanifolds or curves N in $\mathbb{C} = \mathbb{R}^2$.

Since ω is a 2-form, it vanishes on any curve in $\mathbb{C}$. Hence every curve is $\mathbb{C}$ is Lagrangian. For N to be special Lagrangian with phase $e^{i\theta}$ we need that

$$\mathrm{Re}(e^{-i\theta}\Upsilon) = \cos\theta \mathrm{d}x + \sin\theta \mathrm{d}y$$

is the volume form on N, or equivalently that

$$\mathrm{Im}(e^{-i\theta}\Upsilon) = \cos\theta \mathrm{d}y - \sin\theta \mathrm{d}x$$

vanishes on N. This means that $\cos\theta\partial_x + \sin\theta\partial_y$ is everywhere a unit tangent vector to N, so N is a straight line given by $N = \{(t\cos\theta, t\sin\theta) \in \mathbb{R}^2 : t \in \mathbb{R}\}$ (up to translation), so it makes an angle θ with the x-axis, hence motivating the term "phase $e^{i\theta}$".

Notice that this result is compatible with the fact that special Lagrangians are minimal, and hence must be geodesics in $\mathbb{R}^2$; i.e. straight lines.

Example. Consider $\mathbb{C}^2 = \mathbb{R}^4$. We know that $\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2$. We also know that $\mathrm{Re}\,\Upsilon = \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}y_2 \wedge \mathrm{d}y_1$, which looks somewhat similar. In fact, if we let J' denote the complex structure given by $J'(\partial_{x_1}) = \partial_{x_2}$ and $J'(\partial_{y_2}) = \partial_{y_1}$, then $\mathrm{Re}\,\Upsilon = \omega'$, the Kähler form corresponding the complex structure J'. Hence special Lagrangians in $\mathbb{C}^2$ are complex curves for a different complex structure.

In fact, we have a hyperkähler triple of complex structures J_1, J_2, J_3, where $J_1 = J$ is the standard one and $J_3 = J_1J_2 = -J_2J_1$ so that $J_1 = J_2J_3 = -J_3J_2$ and $J_2 = J_3J_1 = -J_1J_3$, and the corresponding Kähler forms are $\omega = \omega_1$, ω_2, ω_3 which are orthogonal and the same length with $\Upsilon = \omega_2 + i\omega_3$.

This shows we should only consider complex dimension 3 and higher to find new calibrated submanifolds.

Example. $\mathrm{SU}(n)$ acts transitively on the space of special Lagrangian planes with isotropy $\mathrm{SO}(n)$. So any special Lagrangian plane is given by $A \cdot \mathbb{R}^n$ for $A \in \mathrm{SU}(n)$ where $\mathbb{R}^n$ is the standard real $\mathbb{R}^n$ in $\mathbb{C}^n$.

Given $\theta = (\theta_1, \ldots, \theta_n)$ we can define a plane $P(\theta) = \{(e^{i\theta_1}x_1, \ldots, e^{i\theta_n}x_n) \in \mathbb{C}^n : (x_1, \ldots, x_n) \in \mathbb{R}^n\}$ (where we can swap orientation). We see that $P(\theta)$ is is special Lagrangian if and only if $\mathrm{Re}\,\Upsilon|_P = \pm\cos(\theta_1 + \ldots + \theta_n) = 1$ so that $\theta_1 + \ldots + \theta_n \in \pi\mathbb{Z}$. Given any $\theta_1, \ldots, \theta_n \in (0, \pi)$ with $\theta_1 + \ldots + \theta_n = \pi$, there exists a special Lagrangian N (called a *Lawlor neck*) asymptotic to $P(0) \cup P(\theta)$. It is diffeomorphic to $\mathcal{S}^{n-1} \times \mathbb{R}$. By rotating coordinates we have a special Lagrangian with phase i asymptotic to $P(-\frac{\theta}{2}) \cup P(\frac{\theta}{2})$.

The simplest case is when $\theta_1 = \ldots = \theta_n = \frac{\pi}{n}$: here N is called the *Lagrangian catenoid*. When $n = 2$, under a coordinate change the Lagrangian catenoid becomes the complex curve $\{(z, \frac{1}{z}) \in \mathbb{C}^2 : z \in \mathbb{C}\backslash\{0\}\}$ that we saw before. When $n = 3$, the only possibilities for the angles are $\sum_i \theta_i = \pi, 2\pi$, but if $\sum_i \theta_i = 2\pi$ we can rotate coordinates and change the order of the planes so that $P(0) \cup P(\theta)$ becomes $P(0) \cup P(\theta')$ where $\sum_i \theta'_i = \pi$. Hence, given any pair of transverse special Lagrangian planes in $\mathbb{C}^3$, there exists a Lawlor neck asymptotic to their union.

We can find special Lagrangians in Calabi–Yaus using the following easy result.

Proposition 4.4. *Given a Calabi–Yau manifold* (M, ω, Υ) *and* $\sigma : M \to M$ *be such that* $\sigma^2 = \mathrm{Id}$, $\sigma^*(\omega) = -\omega$, $\sigma^*(\Upsilon) = \overline{\Upsilon}$. *Then Fix*$(\sigma)$ *is special Lagrangian.*

Example. Let $X = \{[z_0, \ldots, z_4] \in \mathbb{CP}^4 : z_0^5 + \ldots + z_4^5 = 0\}$ (the *Fermat quintic*) with its Calabi–Yau structure (which exists by Yau's solution of the Calabi conjecture since the first Chern class of X vanishes). Let σ be the restriction of complex conjugation on $\mathbb{CP}^4$ to X. Then the fixed point set of σ, which is the real locus in X, is a special Lagrangian 3-fold (if it is non-empty). (There is a subtlety here: σ is certainly an anti-holomorphic isometric involution for the induced metric on X, but this is *not* the same as the Calabi–Yau metric on X. Nevertheless, it is the case that σ satisfies the conditions of the proposition above.)

Example. There exists a Calabi–Yau metric on $T^*\mathcal{S}^n$ (the Stenzel metric) so that the base $\mathcal{S}^n$ is special Lagrangian. When $n = 2$ this is a hyperkähler metric called the Eguchi–Hanson metric.

5 Constructing calibrated submanifolds

It is easy to construct complex submanifolds in Kähler manifolds algebraically. Constructing other calibrated submanifolds is much more challenging because one needs to solve a nonlinear PDE, even in Euclidean space. There are approaches in Euclidean space and other simple spaces which have involved reducing the problem to ODEs or other simpler problems. For example, we have the following methods, which you can find out more about in Dominic Joyce's book "Riemannian holonomy groups and calibrated geometry".

- (Harvey–Lawson, Haskins, Goldstein, Joyce, L-) Symmetries/evolution equations.
- (Haskins, Carberry, McIntosh) Used integrable systems to study calibrated cones.
- (Bryant, Fox, Joyce, L-) Calibrated cones and ruled smoothings of these cones.
- (Karigiannis–Ionel–Min-Oo/–Min-Oo/–Leung) Vector sub-bundle constructions.
- (Bryant, Ionel, Fox, L-) Classification of calibrated submanifolds satisfying pointwise constraints on their second fundamental form.

However, an important direction which has borne fruit in calibrated geometry and special holonomy recently has been to study the nonlinear PDE head on, especially by perturbative and gluing methods.

We want to solve nonlinear PDE, so how do we tackle this? The idea is to use the linear case to help. Suppose we are on a compact manifold N and recall the theory of linear *elliptic* operators L of order l on N, including:

- the definition of ellipticity of L via the *principal symbol* σ_L (which encodes the highest order derivatives in the operator) being an isomorphism;
- the use of *Hölder spaces* $C^{k,a}$ to give elliptic regularity theory (so-called *Schauder theory*), namely that if $w \in C^{k,a}$ and $Lv = w$ then $v \in C^{k+l,a}$ and there is a universal constant C so that

$$\|v\|_{C^{k+l,a}} \leq C(\|Lv\|_{C^{k,a}} + \|v\|_{C^0})$$

  (and we can drop the $\|v\|_{C^0}$ term if v is orthogonal to $\operatorname{Ker} L$);
- the adjoint operator L^* and that $\sigma_{L^*} = (-1)^l \sigma_L^*$ so that L^* is elliptic if and only if L is elliptic; and
- the Fredholm theory of L, namely that $\operatorname{Ker} L$ (and hence $\operatorname{Ker} L^*$) is finite-dimensional, and we can solve $Lv = w$ if and only if $w \in (\operatorname{Ker} L^*)^\perp$.

We shall discuss this in a model example which we shall use throughout this section.

Example. The Laplacian on functions is given by $\Delta f = \mathrm{d}^*\mathrm{d}f$ which in normal coordinates at a point is given by $f \mapsto -\sum_i \frac{\partial^2 f}{\partial x_i^2}$, so it is a linear second order differential operator. We see that its principal symbol is $\sigma_\Delta(x,\xi)f = -|\xi|^2 f$ which is an isomorphism for $\xi \in T_x^*N \setminus \{0\}$, so Δ is elliptic. We therefore have that if $h \in C^{k,a}(N)$ and $\Delta f = h$ then $f \in C^{k+2,a}(N)$, and we have an estimate

$$\|f\|_{C^{k+2,a}} \leq C(\|\Delta f\|_{C^{k,a}} + \|f\|_{C^0}).$$

We also know that $\Delta^* = \Delta$ and $\operatorname{Ker} \Delta$ is given by the constant functions (since if $f \in \operatorname{Ker} \Delta$ then

$$0 = \langle f, \Delta f\rangle_{L^2} = \langle f, \mathrm{d}^*\mathrm{d}f\rangle_{L^2} = \|\mathrm{d}f\|_{L^2}^2$$

so $\mathrm{d}f = 0$). Hence, we can solve $\Delta f = h$ if and only if h is orthogonal to the constants, i.e. $\int_N h \,\mathrm{vol}_N = 0$.

The minimal submanifold operator $P(f) = -\operatorname{div}(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}})$ is a nonlinear second order operator whose linearisation $L_0 P$ at 0 is Δ. Thus P is a nonlinear elliptic operator at 0. If we linearise P at f_0 we find a more complicated expression depending on f_0, but it is still a perturbation of the Laplacian.

Suppose we are on a compact manifold N and we want to solve $P(f) = 0$ where P is the minimal submanifold operator on functions f. Let us consider regularity for f. We can re-arrange $P(f) = 0$ by taking all of the second derivatives to one side as:

$$R(x, \nabla f(x))\nabla^2 f(x) = E(x, \nabla f(x))$$

where $x \in N$ and $R, E : C^{k+1,a} \to C^{k,a}$. Since $L_0 P = \Delta$ is elliptic and ellipticity is an open condition we know that the operator L_f (depending on f) given by

$$L_f(h)(x) = R(x, \nabla f(x))\nabla^2 h(x)$$

is a *linear* elliptic operator whenever $\|\nabla f\|_{C^0}$ is small, in particular if $\|f\|_{C^{1,a}}$ is sufficiently small. The operator L_f does not have smooth coefficients, but if $f \in C^{k,a}$ then the coefficients $R \in C^{k-1,a}$.

Suppose that $f \in C^{1,a}$ and $\|f\|_{C^{1,a}}$ is small with $P(f) = 0$. Then $L_f(f) = E(f)$ and L_f is a linear *second order* elliptic operator with coefficients in $C^{0,a}$ and $E(f)$ in $C^{0,a}$. So by elliptic regularity we can deduce that $f \in C^{2,a}$. We have gained one degree of regularity, so we can "bootstrap", i.e. proceed by induction and deduce that any $C^{1,a}$ solution to $P(f) = 0$ is smooth.

Example. $C^{1,a}$-minimal submanifolds (and thus calibrated submanifolds) are *smooth*.

Remark More sophisticated techniques can be used to deduce that C^1-minimal submanifolds are real analytic. Notice that elliptic regularity results are *not* valid for C^k spaces, so this result is not obvious.

We can also arrange our simple equation $P(f) = 0$ as $\Delta f + Q(\nabla f, \nabla^2 f) = 0$, where Q is nonlinear but linear in $\nabla^2 f$. If we know that $\int_N P(f)\,\mathrm{vol}_N = 0$, i.e. that $P(f)$ is orthogonal to the constants, then we can always solve $\Delta f_0 = -Q(\nabla f, \nabla^2 f)$. We do know that $\int_N P(f)\,\mathrm{vol}_N = 0$ since P has a divergence form. This means we are in the setting for implementing the Implicit Function Theorem for Banach spaces to conclude that we can always solve $P(f) = 0$ for some f near 0, and f will be smooth by our regularity argument above. In general, we will use the following.

Theorem 5.1 (Implicit Function Theorem)**.** *Let X, Y be Banach spaces, let $U \ni 0$ be open in X, let $P : U \to Y$ with $P(0) = 0$ and $L_0 P : X \to Y$ surjective with finite-dimensional kernel K.*

Then for some U, $P^{-1}(0) = \{u \in U : P(u) = 0\}$ is a manifold of dimension $\dim K$*. Moreover, if we write $X = K \oplus Z$, $P^{-1}(0) = \operatorname{Graph} G$ for some map G from an open set in K to Z with $G(0) = 0$.*

This gives us a way to describe all perturbations of a given calibrated submanifold, as we now see in the special Lagrangian case.

Theorem 5.2 (McLean)**.** *Let N be a compact special Lagrangian in a Calabi–Yau manifold M. Then the moduli space of deformations of N is a smooth manifold of dimension* $b^1(N)$.

Remark One should compare this result to the deformation theory for complex submanifolds in Kähler manifolds. There, one does not get that the moduli space is a smooth manifold: in fact, it can be singular, and one has *obstructions* to deformations. It is somewhat remarkable that special Lagrangian calibrated geometry enjoys a much better deformation theory than this classical calibrated geometry.

Proof. The tubular neighbourhood theorem gives us a diffeomorphism $\exp : S \subseteq \nu(N) \to T \subseteq M$ which maps the zero section to N acting as the identity; in other words, we can write any nearby submanifold to N as the graph of a normal vector field on N. We know that N is Lagrangian, so the complex structure J gives an isomorphism between $\nu(N)$ and TN and the metric gives an isomorphism between TN and T^*N: $v \mapsto g(Jv, .) = \omega(v, .) = \alpha_v$. Therefore any deformation of N in T is given as the graph of a 1-form. In fact, using the Lagrangian neighbourhood theorem, we can arrange that any $N' \in T$ is the graph of a 1-form α, so that if $f_\alpha : N \to N_\alpha$ is the natural diffeomorphism then

$$f_\alpha^*(\omega) = \mathrm{d}\alpha \quad \text{and} \quad -*f_\alpha^*(\operatorname{Im}\Upsilon) = F(\alpha, \nabla\alpha) = \mathrm{d}^*\alpha + Q(\alpha, \nabla\alpha).$$

Hence, N_α is special Lagrangian if and only if $P(\alpha) = (F(\alpha, \nabla\alpha), \mathrm{d}\alpha) = 0$. This means that infinitesimal special Lagrangian deformations are given by closed and coclosed 1-forms, which is the kernel of L_0P.

Since $\operatorname{Im}\Upsilon = 0$ on N we have that $[\operatorname{Im}\Upsilon] = 0$ on N_α, so

$$P : C^\infty(S) \to \mathrm{d}^*(C^\infty(T^*N)) \oplus \mathrm{d}(C^\infty(T^*N)) \subseteq C^\infty(\Lambda^0T^*N \oplus \Lambda^2T^*N).$$

If we let $X = C^{1,a}(T^*N)$, $Y = \mathrm{d}^*(C^{1,a}(T^*N)) \oplus \mathrm{d}(C^{1,a}(T^*N))$ and $U = C^{1,a}(S)$ we can apply the Implicit Function Theorem if we know that

$$L_0P : \alpha \in X \mapsto (\mathrm{d}^*\alpha, \mathrm{d}\alpha) \in Y$$

is surjective, i.e. given $\mathrm{d}\beta + \mathrm{d}^*\gamma \in Y$ does there exist α such that $\mathrm{d}\alpha = \mathrm{d}\beta$ and $\mathrm{d}^*\alpha = \mathrm{d}^*\gamma$? If we let $\alpha = \beta + \mathrm{d}f$ then we need $\Delta f = \mathrm{d}^*\mathrm{d}f = \mathrm{d}^*(\gamma - \beta)$. Since

$$\int_N \mathrm{d}^*(\gamma - \beta)\,\mathrm{vol}_N = \pm\int_N \mathrm{d}*(\gamma - \beta) = 0$$

we can solve the equation for f, and hence L_0P is surjective.

Therefore $P^{-1}(0)$ is a manifold of dimension $\dim\operatorname{Ker} L_0P = b^1(N)$ by Hodge theory. Moreover, if $P(\alpha) = 0$ then N_α is special Lagrangian, hence minimal and since $\alpha \in C^{1,a}$ we deduce that α is in fact smooth. □

Example. The special Lagrangian $\mathcal{S}^n$ in $T^*\mathcal{S}^n$ has $b^1 = 0$ and so is rigid.

Observe that if we have a special Lagrangian T^n in M then $b^1(T^n) = n$ and its deformations locally foliate M, so we can hope to find special Lagrangian torus fibrations. This cannot happen in compact manifolds without singular fibres, but still motivates the SYZ conjecture in Mirror Symmetry. The deformation result also motivates the following theorem.

Theorem 5.3 (Bryant). *Every compact oriented real analytic Riemannian 3-manifold can be isometrically embedded in a Calabi–Yau 3-fold as the fixed point set of an involution.*

Another well-known way to get a solution of a linear PDE from two solutions is simply to add them. However, for a nonlinear PDE $P(v) = 0$ this will not work. Intuitively, we can try to add two solutions to give us a solution v_0 for which $P(v_0)$ is small. Then we may try to perturb v_0 by v to solve $P(v+v_0) = 0$.

Geometrically, this occurs when we have two calibrated submanifolds N_1, N_2 and then glue them together to give a submanifold N which is "almost" calibrated, then we deform N to become calibrated. If the two submanifolds N_1, N_2 are glued using a very long neck then one can imagine that N is almost the disjoint union of N_1, N_2 and so close to being calibrated. If instead one scales N_2 by a factor t and then glues it into a singular point of N_1, we can again imagine that as t becomes very small N resembles N_1 and so again is close to being calibrated. These two examples are in fact related, because if we rescale the shrinking N_2 to fixed size, then we get a long neck between N_1 and N_2 of length of order $-\log t$. However, although these pictures are appealing, they also reveal the difficulty in this approach: as t becomes small, N becomes more "degenerate", giving rise to analytic difficulties which are encoded in the geometry of N_1, N_2 and N.

These ideas are used extensively in geometry, and particularly successfully in calibrated geometry (e.g. Haskins–Kapouleas, Joyce, Y.-I. Lee, L-, Pacini). A particular simple case is the following, which we will describe to show the basic idea of the gluing method.

Theorem 5.4. *Let N be a compact connected 3-manifold and let $i : N \to M$ be a special Lagrangian immersion with tranverse self-intersection points in a Calabi–Yau manifold M. Then there exist embedded special Lagrangians N_t such that $N_t \to N$ as $t \to 0$.*

Remark One might ask about the sense of convergence here: for definiteness, we can say that N_t converges to N in the sense of currents; that is, if we have any compactly supported 3-form χ on M then $\int_{N_t}\chi \to \int_N \chi$ as $t \to 0$. However, all sensible notions of convergence of submanifolds will be true in this setting.

Proof. At each self-intersection point of N the tangent spaces are a pair of transverse 3-planes, which we can view as a pair of tranverse special Lagrangian 3-planes P_1, P_2 in $\mathbb{C}^3$. Since we are in dimension 3, we know that there exists a (unique up to scale) special Lagrangian Lawlor neck L asymptotic to $P_1 \cup P_2$. We can then glue tL into N near each intersection point to get a compact embedded (if we glue in a Lawlor neck for every self-intersection point) submanifold $S_t = N\#tL$. We can also arrange that S_t is Lagrangian, i.e. that it is a Lagrangian connect sum.

Now we want to perturb S_t to be special Lagrangian. Since S_t is Lagrangian, by the deformation theory we can write any nearby submanifold as the graph of a 1-form α, and this graph will be special Lagrangian if and only if (using the same notation as in our deformation theory discussion)

$$P_t(\alpha) = (-*f_\alpha^*(\operatorname{Im}\Upsilon), f_\alpha^*(\omega)) = 0.$$

Since S_t is Lagrangian but not special Lagrangian we have that

$$f_\alpha^*(\omega) = \mathrm{d}\alpha \quad \text{and} \quad -*f_\alpha^*(\operatorname{Im}\Upsilon) = P_t(0) + \mathrm{d}_t^*\alpha + Q_t(\alpha, \nabla\alpha)$$

where $P_t(0) = -*\operatorname{Im}\Upsilon|_{S_t}$ and $\mathrm{d}_t^* = L_0 P_t$, which is a perturbation of the usual d^* since we are no longer linearising at a point where $P_t(0) = 0$. By choosing $\alpha = \mathrm{d}f$, we then have to solve

$$\Delta_t f = -P_t(0) - Q_t(\nabla f, \nabla^2 f)$$

where Δ_t is a perturbation of the Laplacian.

For simplicity, let us suppose that Δ_t is the Laplacian on S_t. The idea is to view our equation as a fixed point problem. We know that if we let $X^k = \{f \in C^{k,a}(N) : \int_N f \operatorname{vol}_N = 0\}$ then $\Delta_t : X^{k+2} \to X^k$ is an isomorphism so it has an inverse G_t. We know by our elliptic regularity result that there exists a constant $C(\Delta_t)$ such that

$$\|f\|_{C^{k+2,a}} \leq C(\Delta_t)\|\Delta_t f\|_{C^{k,a}} \Leftrightarrow \|G_t h\|_{C^{k+2,a}} \leq C(\Delta_t)\|h\|_{C^{k,a}}$$

for any $f \in X^{k+2}$, $h \in X^k$.

We thus see that $P_t(f) = 0$ for $f \in X^{k+2}$ if and only if

$$f = G_t(-P_t(0) - Q_t(f)) = F_t(f).$$

The idea is now to show that F_t is a contraction sufficiently near 0 for all t small enough. Then it will have a (unique) fixed point near 0, which will also be smooth because it satisfies $P_t(f) = 0$ and hence defines a special Lagrangian as the graph of $\mathrm{d}f$ over S_t.

We know that $F_t : X^{k+2} \to X^{k+2}$ with

$$\|F_t(f_1) - F_t(f_2)\|_{C^{k+2,a}} = \|G_t(Q_t(f_1) - Q_t(f_2))\|_{C^{k+2,a}} \leq C(\Delta_t)\|Q_t(f_1) - Q_t(f_2)\|_{C^{k,a}}.$$

Since Q_t and its first derivatives vanish at 0 we know that

$$\|Q_t(f_1) - Q_t(f_2)\|_{C^{k,a}} \leq C(Q_t)\|f_1 - f_2\|_{C^{k+2,a}}(\|f_1\|_{C^{k+2,a}} + \|f_2\|_{C^{k+2,a}}).$$

Hence, F_t is a contraction on $\overline{B_{\epsilon_t}(0)} \subseteq X^{k+2}$ if we can choose ϵ_t so that

$$2C(\Delta_t)\|P_t(0)\|_{C^{k,a}} \leq \epsilon_t \leq \frac{1}{2C(\Delta_t)C(Q_t)}.$$

(This also proves our earlier Implicit Function Theorem result by hand since there $P_t(0) = P(0) = 0$ so we just need to take ϵ_t small enough.) In other words, we need that

- $P_t(0)$ is small, so S_t is "close" to being calibrated and is a good approximation to $P_t(f) = 0$;
- $C(\Delta_t), C(Q_t)$, which are determined by the linear PDE and geometry of N, L and S_t, are well-controlled as $t \to 0$.

The statement of the theorem is then that there exists t sufficiently small and ϵ_t so that the contraction mapping argument works.

This is a delicate balancing act since as $t \to 0$ parts of the manifold are collapsing, so the constants $C(\Delta_t), C(Q_t)$ above (which depend on t) can and typically do blow-up as $t \to 0$. To control this, we need to understand the Laplacian on N, L and S_t and introduce "weighted" Banach spaces so that tL gets rescaled to constant size (independent of t), and S_t resembles the union of two manifolds with a cylindrical neck (as we described earlier). It is also crucial to understand the relationship between the kernels and cokernels of the Laplacian on the *non-compact* N (without the intersection pts), L and compact S_t: here is where connectedness is important so that the kernel and cokernel of the Laplacian is 1-dimensional. □

Remark In more challenging gluing problems it is not possible to show that the relevant map is a contraction, but rather one can instead appeal to an alternative fixed point theorem (e.g. Schauder fixed point theorem) to show that it still has a fixed point.

6 Associative and coassociative submanifolds

We now want to introduce our calibrated geometry associated with G_2 holonomy. The first key result is the following.

Theorem 6.1. *Let (M, φ) be a G_2 manifold (so φ is a closed and coclosed definite 3-form). Then φ and $*\varphi$ are calibrations.*

This follows from the definition of G_2 manifold and Problem Sheet 2. Let us look at the calibrated planes and start with φ.

If u, v, w are unit vectors in $\mathbb{R}^7 \cong \operatorname{Im} \mathbb{O}$ (the imaginary octonions), then $\varphi(u, v, w) = \langle u \times v, w \rangle = 1$ if and only if $w = u \times v$, so $P = \operatorname{Span}\{u, v, w\}$ is a copy of $\operatorname{Im} \mathbb{H}$ in $\operatorname{Im} \mathbb{O}$, so $\operatorname{Span}\{1, u, v, w\}$ is an associative subalgebra of $\mathbb{O}$. Moreover, if we define a vector-valued 3-form χ on $\mathbb{R}^7$ by $\chi(u, v, w) = [u, v, w] = u(vw) - (uv)w$, known as the associator, we see that P is associative if and only if $\chi|_P \equiv 0$ (as was shown on Problem Sheet 3). Hence we call the calibrated planes *associative*. In general on a G_2 manifold we can define a 3-form χ with values in TM using the pointwise formula.

For $*\varphi$ we see that $*\varphi|_P = \mathrm{vol}_P$ for a plane P if and only if $\varphi|_{P^\perp} = \mathrm{vol}_{P^\perp}$. Hence the planes calibrated by $*\varphi$ are the orthogonal complements of the associative planes, so we call them *coassociative*. We also see from Problem Sheet 3 that P is coassociative if and only if $\varphi|_P \equiv 0$.

We thus can define our calibrated submanifolds.

Definition 6.2. Submanifolds calibrated by φ are called *associative* 3-folds. Moreover, N is associative if and only if $\chi|_N \equiv 0$.

Submanifolds calibrated by $*\varphi$ are called *coassociative* 4-folds. Moreover, N is coassociative if and only if $\varphi|_N \equiv 0$.

A simple way to get associative and coassociative submanifolds is by using known geometries.

Proposition 6.3. *Let $x_1, \ldots, x_7$ be coordinates on $\mathbb{R}^7$ and let $z_j = x_{2j} + ix_{2j+1}$ be coordinates on $\mathbb{C}^3$ so that $\mathbb{R}^7 = \mathbb{R} \times \mathbb{C}^3$.*

(a) $N = \mathbb{R} \times S \subseteq \mathbb{R} \times \mathbb{C}^3$ is associative/coassociative if and only if S is a complex curve/special Lagrangian 3-fold with phase $-i$.

(b) $N \subseteq \{0\} \times \mathbb{C}^3$ is associative/coassociative if and only if N is a special Lagrangian 3-fold/complex surface.

Proof. We can write

$$\varphi = \mathrm{d}x_1 \wedge \omega + \operatorname{Re} \Omega \quad \text{and} \quad * \varphi = \frac{1}{2}\omega^2 - \mathrm{d}x_1 \wedge \operatorname{Im} \Upsilon.$$

For associatives, we see that $\varphi|_{\mathbb{R}\times S} = \mathrm{d}x_1 \wedge \mathrm{vol}_S$ if and only if $\omega|_S = \mathrm{vol}_S$ and $\varphi|_N = \operatorname{Re} \Upsilon|_N$ for $N \subseteq \mathbb{C}^3$. For coassociatives, we see that $*\varphi|_{\mathbb{R}\times S} = dx_1 \wedge \mathrm{vol}_S$ if and only if $-\operatorname{Im} \Upsilon|_S = \mathrm{vol}_S$ and $*\varphi|_N = \frac{1}{2}\omega^2|_N$ for $N \subseteq \mathbb{C}^3$. □

We can also produce examples in G_2 manifolds with an isometric involution.

Proposition 6.4. *Let (M,φ) be a G_2 manifold with an isometric involution $\sigma \neq \mathrm{id}$ such $\sigma^*\varphi = \pm\varphi$. Then Fix$(\sigma)$ is an associative/coassociative submanifold in M.*

We also have explicit examples of associatives and coassociatives.

Example. The first explicit examples of associatives in $\mathbb{R}^7$ not arising from other geometries are given by L- from symmetry/evolution equation considerations.

The first explicit non-trivial examples of coassociatives in $\mathbb{R}^7$ are given in Harvey–Lawson. There are two dilation families: one which has one end asymptotic to a cone C on a non-round $\mathcal{S}^3$, and one which has two ends asymptotic to $C \cup \mathbb{R}^4$. The cone C was discovered earlier by Lawson–Osserman and was the first example of a volume-minimizing submanifold which is not smooth (it is Lipschitz but not C^1).

Example. In the Bryant–Salamon holonomy G_2 metric on the spinor bundle of $\mathcal{S}^3$, the base $\mathcal{S}^3$ is associative. In the Bryant–Salamon holonomy G_2 metric on $\Lambda^2_+T^*\mathcal{S}^4$ and $\Lambda^2_+T^*\mathbb{CP}^2$, the base $\mathcal{S}^4$ and $\mathbb{CP}^2$ are coassociative.

We now want to understand deformations of associatives and coassociatives, from which perturbation/gluing results will follow. We begin with associatives.

Notice that if P is an associative plane, $u \in P$ and $v \in P^\perp$ then $u \times v \in P^\perp$ since $\varphi(w,u,v) = g(w, u\times v) = g(v, w\times u) = 0$ for all $w \in P$ since $w \times u \in P$. Thus, if N is associative, cross product gives a (Clifford) multiplication $m : C^\infty(T^*N \otimes \nu(N)) \to C^\infty(\nu(N))$ (viewing tangent vectors as cotangent vectors via the metric), hence using the normal connection $\nabla^\perp : C^\infty(\nu(N)) \to C^\infty(T^*N \otimes \nu(N))$ on $\nu(N)$ we get a linear operator

$$\not{D} = m \circ \nabla^\perp : C^\infty(\nu(N)) \to C^\infty(\nu(N)).$$

We call $\not{D}$ the *Dirac operator*. We see that its principal symbol is given by $\sigma_{\not{D}}(x,\xi)v = \xi \times v$, so $\not{D}$ is elliptic, and we also have that $\not{D}^* = \not{D}$.

Remark Since a 3-manifold is always spin, we have a spinor bundle $\mathbb{S}$ on N, a connection $\nabla : C^\infty(\mathbb{S}) \to C^\infty(T^*M \otimes \mathbb{S})$ (lift of the Levi-Civita connection) and we have Clifford multiplication $m : C^\infty(T^*M \otimes \mathbb{S}) \to C^\infty(\mathbb{S})$ given by $m(\xi, v) = \xi \cdot v$. Hence we have a composition $\not{D} = m \circ \nabla : C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$, which is a first order linear differential operator called the Dirac operator. Locally it is given by $\not{D}v = \sum_i e_i \cdot \nabla_{e_i} v$, so we have that $\sigma_{\not{D}}(\xi, v) = \xi \cdot v$. Hence $\not{D}$ is elliptic. Moreover $\not{D}$ is self-adjoint.

In fact, it is possible to see that the complexified normal bundle $\nu(N) \otimes \mathbb{C} = \mathbb{S} \otimes V$ for a $\mathbb{C}^2$-bundle V over N, so that the Dirac operator on $\nu(N)$ is just a "twist" of the usual Dirac operator on $\mathbb{S}$.

Consider a compact associative N. We know that $\exp_v(N) = N_v$, which is the graph of v, is associative for a normal vector field v if and only if $*\exp_v^*(\chi) \in C^\infty(TM|_N)$ is 0. In fact, it turns out that $F(v) = *\exp_v^*(\chi) \in C^\infty(\nu(N))$ since N is associative and

$$L_0F(v) = *\mathrm{d}(v \lrcorner \chi) = \not{D}v.$$

Here L_0F is not typically surjective so we cannot apply our Implicit Function Theorem, except when $\operatorname{Ker}\not{D} = \operatorname{Ker}\not{D}^* = \{0\}$. Instead, we have the following by applying results from Problem Sheet 3, since we know that $\operatorname{index}\not{D} = \dim\operatorname{Ker}\not{D} - \dim\operatorname{Ker}\not{D}^* = 0$.

Theorem 6.5 (McLean)**.** *The expected dimension of the moduli space of deformations of a compact associative 3-fold N in a G_2 manifold is 0 and infinitesimal deformations of N are given by the kernel of $\not{D}$ on $\nu(N)$. Moreover, if $\operatorname{Ker}\not{D} = \{0\}$ then N is rigid.*

Remark The dimension of the kernel of $\not{D}$ typically depends on the metric on N rather than just the topology, so it is usually difficult to determine.

Example. For the associative $N = \mathcal{S}^3$ in $\mathbb{S}(\mathcal{S}^3)$, $\nu(N) = \mathbb{S}(\mathcal{S}^3)$ so $\not{D}$ is just the usual Dirac operator. A theorem of Lichnerowicz states that $\operatorname{Ker}\not{D} = \{0\}$ as $\mathcal{S}^3$ has positive scalar curvature so N is rigid.

Example. Corti–Haskins–Nordström–Pacini construct rigid associative $\mathcal{S}^1 \times \mathcal{S}^2$s in compact holonomy G_2 twisted connected sums.

For coassociatives, the deformation theory is much better behaved, like for special Lagrangians.

Theorem 6.6 (McLean)**.** *Let N be a compact coassociative in a* G_2 *manifold (or just a 7-manifold with closed* G_2 *structure). The moduli space of deformations of N is a smooth manifold of dimension $b^2_+(N)$.*

Proof. Since N is coassociative the map $v \mapsto v \lrcorner \varphi = \alpha_v$ defines an isomorphism from $\nu(N)$ to a rank 3 vector bundle on N, which is $\Lambda^2_+ T^*N$, the 2-forms on N which are self-dual (so $*\alpha = \alpha$). We can therefore view nearby submanifolds to N as graphs of self-dual 2-forms.

We know that $N_v = \exp_v(N)$ is coassociative if and only if $\exp_v^*(\varphi) = 0$. We see that

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp^*_{tv}(\varphi)|_{t=0} = \mathcal{L}_v \varphi = \mathrm{d}(v \lrcorner \varphi) = \mathrm{d}\alpha_v.$$

Hence nearby coassociative N' to N are given by the zeros of $P(\alpha) = \mathrm{d}\alpha + Q(\alpha, \nabla\alpha)$. Moreover, since $\varphi = 0$ on N, $[\varphi] = 0$ on N' and hence $P : C^\infty(\Lambda^2_+ T^*N) \to \mathrm{d}(C^\infty(\Lambda^2 T^*N))$.

Here P is not elliptic, but $L_0 P = \mathrm{d}$ has finite-dimensional kernel, the closed self-dual 2-forms, since $\mathrm{d}\alpha = 0$ implies that $\mathrm{d}^*\alpha = - * \mathrm{d} * \alpha = 0$ so α is harmonic, and $L_0 P$ has injective symbol so it is overdetermined elliptic, which means that elliptic regularity still holds. Another way to deal with this is to consider $F(\alpha, \beta) = P(\alpha) + \mathrm{d}^*\beta$ for β a 4-form. Now $F^{-1}(0)$ is the disjoint union of $P^{-1}(0)$ and multiples of the volume form, as exact and coexact forms are orthogonal. Moreover, $L_0 F(\alpha, \beta) = \mathrm{d}\alpha + \mathrm{d}^*\beta$ is now elliptic. Overall, we can apply our standard Implicit Function Theorem if we know that

$$\mathrm{d}(C^{k+1,a}(\Lambda^2_+ T^*N)) = \mathrm{d}(C^{k+1,a}(\Lambda^2 T^*N)).$$

This is true because by Hodge theory if α is a 2-form, we can write $\alpha = \mathrm{d}^*\beta + \gamma$ for a 3-form β and a closed form γ, so $\mathrm{d}\alpha = \mathrm{d}\mathrm{d}^*\beta = \mathrm{d}(\mathrm{d}^*\beta + *\mathrm{d}^*\beta)$ and $\mathrm{d}^*\beta + \mathrm{d}^*\beta$ is self-dual. □

Example. The $\mathcal{S}^4$ and $\mathbb{CP}^2$ in the Bryant–Salamon metrics on $\Lambda^2_+ T^*\mathcal{S}^4$ and $\Lambda^2_+ T^*\mathbb{CP}^2$ have $b^2_+ = 0$ and so are rigid.

For a K3 surface and T^4 we have $b^2_+ = 3$, so we can hope to find coassociative K3 and T^4 fibrations of compact G_2 manifolds. There is a programme for constructing a coassociative K3 fibration (with singular fibres) by Kovalev. Towards completing this programme, L- constructed the first examples of coassociative 4-folds with conical singularities in compact holonomy G_2 twisted connected sums.

Again, we have a similar isometric embedding result for coassociative 4-folds, motivated by the deformation theory result.

Theorem 6.7 (Bryant)**.** *Any compact oriented real analytic Riemannian 4-manifold whose bundle of self-dual 2-forms is trivial can be isometrically embedded in a* G_2 *manifold as the fixed points of an isometric involution.*

7 Cayley submanifolds

We now discuss our final class of calibrated submanifolds.

Theorem 7.1. *On a* $\mathrm{Spin}(7)$ *manifold (M, Φ), Φ is a calibration.*

This is immediate from the definition of Spin(7) manifold and Problem Sheet 2. We can thus define our calibrated submanifolds.

Definition 7.2. The submanifolds calibrated by Φ are called Cayley submanifolds.

Remark The name Cayley submanifolds is because of the relation between the submanifolds and the octonions or Cayley numbers $\mathbb{O}$.

We can relate Cayley submanifolds to all of the other calibrated geometries we have seen.

Proposition 7.3. *(a) Complex surfaces and special Lagrangian 4-folds in $\mathbb{C}^4$ are Cayley in $\mathbb{R}^8 = \mathbb{C}^4$.*

(b) Write $\mathbb{R}^8 = \mathbb{R} \times \mathbb{R}^7$. Then $\mathbb{R} \times S$ is Cayley if and only if S is associative in $\mathbb{R}^7$ and $N \subseteq \mathbb{R}^7$ is Cayley in $\mathbb{R}^8$ if and only if N is coassociative in $\mathbb{R}^7$.

Proof. (a) is immediate from the formula $\Phi = \frac{1}{2}\omega^2 + \operatorname{Re}\Upsilon$, since complex surfaces are calibrated by $\frac{1}{2}\omega^2$, special Lagrangians are calibrated by $\operatorname{Re}\Upsilon$, Υ vanishes on complex surfaces and ω vanishes on special Lagrangians.

(b) follows immediately from the formula $\Phi = \mathrm{d}x_1 \wedge \varphi + *\varphi$. □

We can also use an isometric involution to construct Cayley submanifolds as in our previous calibrated geometries.

Proposition 7.4. *Let (M, Φ) be a* Spin(7) *manifold and let $\sigma \neq \mathrm{id}$ be an isometric involution with $\sigma^*\Phi = \Phi$. Then Fix(σ) is Cayley submanifold.*

Example. The first interesting explicit examples of Cayleys in $\mathbb{R}^8$ not arising from other geometries were given by L- and are asymptotic to cones.

Example. The base $\mathcal{S}^4$ in the Bryant–Salamon holonomy Spin(7) metric on $\mathbb{S}_+(\mathcal{S}^4)$ is Cayley.

We now discuss deformations of a compact Cayley N, for which we need some discussion of algebra related to Spin(7). Since $\Lambda^2(\mathbb{R}^8)^*$ is 28-dimensional and the 21-dimensional Lie algebra of Spin(7) sits inside the space of 2-forms, we must have a distinguished 7-dimensional subspace Λ^2_7 of 2-forms on $\mathbb{R}^8$. So what is this subspace? Let $u, v \in \mathbb{R}^8$. Then we can construct a 2-form $u \wedge v$, viewing u, v as cotangent vectors. We can also construct a 2-form from u, v by considering $\Phi(u, v, ., .)$. It is then true that

$$\Lambda^2_7 = \{u \wedge v + \Phi(u, v, ., .) \,:\, u, v \in \mathbb{R}^8\}.$$

When P is a Cayley plane and $u, v \in P$ are orthogonal we see that $\Phi(u, v, ., .) = *(u \wedge v)$ so that $u \wedge v + \Phi(u, v, ., .)$ is self-dual. Since $\Lambda^2_+ P^*$ is 3-dimensional, we see that there must be a 4-dimensional space E of 2-forms on P such that $\Lambda^2_7|_P = \Lambda^2_+ P^* \oplus E$. Moreover, if $u \in P$ and $v \in P^\perp$ then $m(u, v) = u \wedge v + \Phi(u, v, ., .) \in E$ and the map $m : P \times P^\perp \to E$ is surjective.

Now let us move to a Cayley submanifold N in a Spin(7) manifold (M, Φ). On M we have a rank 7 bundle Λ^2_7 of 2-forms and we have that $\Lambda^2_7|_N = \Lambda^2_+ T^*N \oplus E$ for some rank 4 bundle E over N. The map m above defines a (Clifford) multiplication $m : C^\infty(T^*N \otimes \nu(N)) \to C^\infty(E)$ (viewing tangent vectors as cotangent vectors via the metric), and thus using the normal connection $\nabla^\perp : C^\infty(\nu(N)) \to C^\infty(T^*N \otimes \nu(N))$ we get a linear first order differential operator

$$\not{D}_+ = m \circ \nabla^\perp : C^\infty(\nu(N)) \to C^\infty(E).$$

Again this an elliptic operator called the *positive Dirac operator*, but it is not self-adjoint: its adjoint is the negative Dirac operator from E to $\nu(N)$.

Remark If N is spin, the spinor bundle $\mathbb{S}$ splits as $\mathbb{S}_+ \oplus \mathbb{S}_-$, and the Dirac operator $\not{D}$ splits into $\not{D}_\pm$ from $\mathbb{S}_\pm$ to $\mathbb{S}_\mp$ so that $\not{D}(v_+, v_-) = (\not{D}_- v_-, \not{D}_+ v_+)$. Hence $\not{D}^* = \not{D}$ says that $\not{D}_\pm^* = \not{D}_\mp$.

It turns out that there exists a $\mathbb{C}^2$-bundle V on N so that $\nu(N) \otimes \mathbb{C} = \mathbb{S}_+ \otimes V$, $E \otimes \mathbb{C} = \mathbb{S}_- \otimes V$ and $\not{D}_+$ on $\nu(N)$ is a "twist" of the usual positive Dirac operator. However, not every 4-manifold is spin, so we cannot always make this identification.

On $\mathbb{O}$ there exists a 4-fold cross product, whose real part gives Φ and whose imaginary part we call τ. Perhaps unsurprisingly, a plane P is calibrated by Φ if and only if $\tau|_P = 0$. We can extend τ to a Spin(7) manifold, except that we need a rank 7 vector bundle on M in which τ takes values: we have one, namely Λ^2_7. So we have that a submanifold N in a Spin(7) manifold is Cayley if and only if $\tau \in C^\infty(\Lambda^4 T^*M; \Lambda^2_7)$ vanishes on N.

Now suppose that N is a compact Cayley 4-fold. Then the zeros of the equation $F(v) = *\exp_v^*(\tau)$ for $v \in C^\infty(\nu(N))$ define Cayley deformations (as the graph of v). We know that F takes values in $\Lambda^2_7|_N = \Lambda^2_+ T^*N \oplus E$ and it turns out that

$$L_0 F(v) = *\mathrm{d}(v \lrcorner \tau) = \not{D}_+ v$$

since N is Cayley. So, we potentially have a problem because F does not necessarily take values only in E (and in general it will not just take values in E). However, the Cayley condition on N means that $F(v) = 0$ if and only $P(v) = \pi_E F(v) = 0$, where π_E is the projection onto E. Then the operator $P : C^\infty(\nu(N)) \to C^\infty(E)$ and $L_0 P = \not{D}_+$ is elliptic.

Again, we cannot say that $L_0 P$ is surjective, so we have the following using the results of Problem Sheet 3.

Theorem 7.5 (McLean). *The expected dimension of the moduli space of deformations of a compact Cayley 4-fold N in a $\mathrm{Spin}(7)$ manifold is $\mathrm{ind}\,\not{D}_+ = \dim\mathrm{Ker}\,\not{D}_+ - \dim\mathrm{Ker}\,\not{D}_+^*$ with infinitesimal deformations given by $\mathrm{Ker}\,\not{D}_+$ on $\nu(N)$. Moreover,*

$$\mathrm{ind}\,\not{D}_+ = \frac{1}{2}\sigma(N) + \frac{1}{2}\chi(N) - [N].[N],$$

where $\sigma(N) = b^2_+(N) - b^2_-(N)$ (the signature of N), $\chi(N) = 2b^0(N) - 2b^1(N) + b^2(N)$ (the Euler characteristic of N) and $[N].[N]$ is the self-intersection of N, which is the Euler number of $\nu(N)$.

Example. For the Cayley $N = \mathcal{S}^4$ in $\mathbb{S}_+(\mathcal{S}^4)$, $\nu(N) = \mathbb{S}_+(\mathcal{S}^4)$ and $\not{D}_+$ is the usual positive Dirac operator. Again, since N has positive scalar curvature, we see that $\mathrm{Ker}\,\not{D}_\pm = \{0\}$ so N is rigid.

8 The angle theorem

We now discuss a very natural and elementary problem in Euclidean geometry where calibrations play a major, and perhaps unexpected, role.

If one takes two lines in $\mathbb{R}^2$ intersecting transversely, then their union is never length-minimizing. A natural question to ask is: does this persist in higher dimensions? In other words, when is the union of two transversely intersecting n-planes in $\mathbb{R}^{2n}$ volume-minimizing? Two such planes are determined by the n angles between them as follows.

Lemma 8.1. *Let P, Q be oriented n-planes in $\mathbb{R}^{2n}$. There exists an orthonormal basis $e_1, \ldots, e_{2n}$ for $\mathbb{R}^{2n}$ such that $P = \mathrm{Span}\{e_1, \ldots, e_n\}$ and*

$$Q = \mathrm{Span}\{\cos\theta_1 e_1 + \sin\theta_1 e_{n+1}, \ldots, \cos\theta_n e_n + \sin\theta_n e_{2n}\}$$

where $0 \leq \theta_1 \leq \ldots \leq \theta_{n-1} \leq \frac{\pi}{2}$ and $\theta_{n-1} \leq \theta_n \leq \pi - \theta_{n-1}$. These angles are called the characterising angles *of P, Q.*

Proof. The proof is very similar to the argument in Wirtinger's inequality. Choose unit $e_1 \in P$ and maximise $\langle e_1, u_1\rangle$ for $u_1 \in Q$, and let $e_{n+1} \in P^\perp$ be defined by $u_1 = \cos\theta_1 e_1 + \sin\theta_1 e_{n+1}$. Now choose $e_2 \in P \cap e_1^\perp$ and maximise $\langle e_2, u_2\rangle$ for $u_2 \in Q \cap u_1^\perp$, then proceed by induction. □

If the characterising angles of P, Q are $\theta_1, \ldots, \theta_n$ and then the characterising angles of $P, -Q$ are $\psi_1, \ldots, \psi_n$ where $\psi_j = \theta_j$ for $j = 1, \ldots, n-1$ and $\psi_n = \pi - \theta_n$.

The idea is that the union of $P \cup Q$ is area-minimizing if $P, -Q$ are not too close together.

Theorem 8.2 (Angle Theorem, Lawlor–Nance). *Let P, Q be oriented transverse n-planes in $\mathbb{R}^{2n}$ and let $\psi_1, \ldots, \psi_n$ be the characterising angles between $P, -Q$. Then $P \cup Q$ is volume-minimizing if and only if $\psi_1 + \ldots + \psi_n \geq \pi$.*

Notice that this criteria is impossible to fulfill in 1 dimension.

Proof. We will sketch the proof which involves calibrations in a fundamental way in both directions.

First if $P \cup Q$ does not satisfy the angle condition, we can choose coordinates by the lemma above so that $P = P(-\frac{\psi}{2})$ and $-Q = P(\frac{\psi}{2})$ where $\psi = (\psi_1, \ldots, \psi_n)$ and $P(\psi) = \{(e^{i\psi_1}x_1, \ldots, e^{i\psi_n}x_n) : (x_1, \ldots, x_n) \in \mathbb{R}^n\}$ as given earlier. We know that we have a special Lagrangian Lawlor neck N asymptotic to $P(-\frac{\psi'}{2}) \cup P(\frac{\psi'}{2})$ for any ψ' where $\sum_{i=1}^n \psi'_i = \pi$. The claim is then that since $\sum \psi_i < \pi$ we can find ψ' so that $\sum \psi'_i = \pi$ and $N \cap P(\pm\frac{\psi'}{2})$ is non-empty (in fact, an ellipsoid). This is actually a way to characterise N. Hence since N is calibrated by $\mathrm{Im}\,\Upsilon$ and $\mathrm{Im}\,\Upsilon|_{P\cup Q} < \mathrm{vol}_{P\cup Q}$, $P \cup Q$ cannot be volume-minimizing by the usual Stokes' Theorem argument for calibrated submanifolds. (Further details concerning this half of the proof of the angle theorem can be found through the questions on Problem Sheet 4.)

Second if $P \cup Q$ does satisfy the angle condition, then (by choosing coordinates so that $P = \mathbb{R}^n$ and Q is in standard position) we claim that it is calibrated by a so-called *Nance calibration*:

$$\eta(u_1, \ldots, u_n) = \mathrm{Re}\left((\mathrm{d}x_1 + u_1 \mathrm{d}y_1) \wedge \ldots \wedge (\mathrm{d}x_n + u_n \mathrm{d}y_n)\right)$$

where $u_1, \ldots, u_n \in \mathcal{S}^2 \subseteq \mathrm{Im}\mathbb{H}$. If $u_m = i$ for all m then $\eta = \mathrm{Re}\,\Omega$. This is a calibration by the question on Problem Sheet 2 on torus forms and moreover we know $P(\theta)$ is calibrated by $\eta(u)$ if and only if

$$\prod_{j=1}^{n} (\cos\theta_j + \sin\theta_j u_j) = 1.$$

We then just need to find the u_j determined by θ_j. Notice that the condition that $\psi_1 + \ldots + \psi_n \geq \pi$ if and only if $\theta_n \leq \theta_1 + \ldots + \theta_{n-1}$. If we write $\cos\theta_j + \sin\theta_j u_j = w_j \overline{w}_{j+1}$ where $w_{n+1} = w_1$ and w_j are unit imaginary quaternions then the product condition is satisfied and we just need $\langle w_j, w_{j+1} \rangle = \cos\theta_j$, which is equivalent to finding n points on the unit 2-sphere so that $d(w_j, w_{j+1}) = \theta_j$, where $\theta_n \leq \theta_1 + \ldots + \theta_{n-1}$. This is possible, by consider an n-sided spherical polygon. □